BBM 205 Discrete Mathematics
Hacettepe University
http://web.cs.hacettepe.edu.tr/∼bbm205

Lecture 10b: Expectation and Important Distributions

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Resources:
Kenneth Rosen, "Discrete Mathematics and App."
http://www.eecs70.org/

Random Variables: Definitions

Definition

A random variable, X, for a random experiment with sample space Ω is a function $X : \Omega \to \Re$.

Thus, $X(\cdot)$ assigns a real number $X(\omega)$ to each $\omega \in \Omega$.

Definitions

(a) For $a \in \Re$, one defines

$$X^{-1}(a) := \{\omega \in \Omega \mid X(\omega) = a\}.$$

(b) For $A \subset \Re$, one defines

$$X^{-1}(A) := \{\omega \in \Omega \mid X(\omega) \in A\}.$$

(c) The probability that $X = a$ is defined as

$$Pr[X = a] = Pr[X^{-1}(a)].$$

(d) The probability that $X \in A$ is defined as

$$Pr[X \in A] = Pr[X^{-1}(A)].$$

(e) The distribution of a random variable X, is

$$\{(a, Pr[X = a]) : a \in \mathscr{A}\},$$

where $\mathscr{A}$ is the *range* of X. That is, $\mathscr{A} = \{X(\omega), \omega \in \Omega\}$.

Expectation - Definition

Definition: The **expected value** (or mean, or expectation) of a random variable X is

$$E[X] = \sum_a a \times Pr[X = a].$$

Theorem:

$$E[X] = \sum_\omega X(\omega) \times Pr[\omega].$$

An Example

Flip a fair coin three times.

$\Omega = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$.

$X =$ number of H's: $\{3, 2, 2, 2, 1, 1, 1, 0\}$.

Thus,

$$\sum_{\omega} X(\omega) Pr[\omega] = \{3+2+2+2+1+1+1+0\} \times \frac{1}{8}.$$

Also,

$$\sum_{a} a \times Pr[X = a] = 3 \times \frac{1}{8} + 2 \times \frac{3}{8} + 1 \times \frac{3}{8} + 0 \times \frac{1}{8}.$$

Win or Lose.

Expected winnings for heads/tails games, with 3 flips?
Recall the definition of the random variable X:
$\{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\} \to \{3, 1, 1, -1, 1, -1, -1, -3\}$.

$$E[X] = 3 \times \frac{1}{8} + 1 \times \frac{3}{8} - 1 \times \frac{3}{8} - 3 \times \frac{1}{8} = 0.$$

Can you ever win 0?

Apparently: expected value is not a common value, by any means.

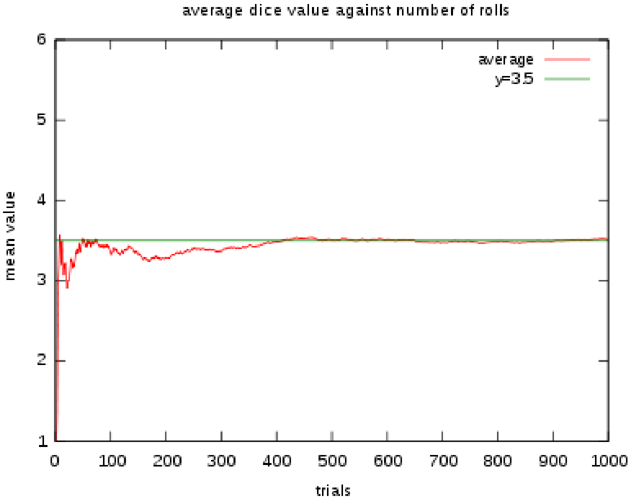
The expected value of X is not the value that you expect!
It is the average value per experiment, if you perform the experiment many times:

$$\frac{X_1 + \cdots + X_n}{n}, \text{ when } n \gg 1.$$

The fact that this average converges to $E[X]$ is a theorem:
the Law of Large Numbers. (See later.)

Law of Large Numbers

An Illustration: Rolling Dice



Indicators

Definition
Let A be an event. The random variable X defined by

$$X(\omega) = \begin{cases} 1, & \text{if } \omega \in A \\ 0, & \text{if } \omega \notin A \end{cases}$$

is called the indicator of the event A.

Note that $Pr[X = 1] = Pr[A]$ and $Pr[X = 0] = 1 - Pr[A]$.

Hence,

$$E[X] = 1 \times Pr[X = 1] + 0 \times Pr[X = 0] = Pr[A].$$

This random variable $X(\omega)$ is sometimes written as

$$1\{\omega \in A\} \text{ or } 1_A(\omega).$$

Thus, we will write $X = 1_A$.

Linearity of Expectation

Theorem: Expectation is linear

$$E[a_1X_1+\cdots+a_nX_n]=a_1E[X_1]+\cdots+a_nE[X_n].$$

Proof:

$$\begin{aligned}
&E[a_1X_1+\cdots+a_nX_n]\\
&=\sum_{\omega}(a_1X_1+\cdots+a_nX_n)(\omega)Pr[\omega]\\
&=\sum_{\omega}(a_1X_1(\omega)+\cdots+a_nX_n(\omega))Pr[\omega]\\
&=a_1\sum_{\omega}X_1(\omega)Pr[\omega]+\cdots+a_n\sum_{\omega}X_n(\omega)Pr[\omega]\\
&=a_1E[X_1]+\cdots+a_nE[X_n].
\end{aligned}$$

□

Note: If we had defined $Y=a_1X_1+\cdots+a_nX_n$ has had tried to compute $E[Y]=\sum_y yPr[Y=y]$, we would have been in trouble!

Using Linearity - 1: Pips (dots) on dice

Roll a die n times.

X_m = number of pips on roll m.

$X = X_1 + \cdots + X_n$ = total number of pips in n rolls.

$$\begin{aligned} E[X] &= E[X_1 + \cdots + X_n] \\ &= E[X_1] + \cdots + E[X_n], \text{ by linearity} \\ &= nE[X_1], \text{ because the } X_m \text{ have the same distribution} \end{aligned}$$

Now,

$$E[X_1] = 1 \times \frac{1}{6} + \cdots + 6 \times \frac{1}{6} = \frac{6 \times 7}{2} \times \frac{1}{6} = \frac{7}{2}.$$

Hence,

$$E[X] = \frac{7n}{2}.$$

Note: Computing $\sum_x xPr[X = x]$ directly is not easy!

Using Linearity - 2: Random assignments Example

Hand out assignments at random to n students.

$X =$ number of students that get their own assignment back.

$X = X_1 + \cdots + X_n$ where
$X_m = 1\{$student m gets his/her own assignment back$\}$.

One has

$$\begin{aligned}
E[X] &= E[X_1 + \cdots + X_n] \\
&= E[X_1] + \cdots + E[X_n], \text{ by linearity} \\
&= nE[X_1], \text{ because all the } X_m \text{ have the same distribution} \\
&= nPr[X_1 = 1], \text{ because } X_1 \text{ is an indicator} \\
&= n(1/n), \text{ because student 1 is equally likely} \\
&\qquad\qquad \text{to get any one of the } n \text{ assignments} \\
&= 1.
\end{aligned}$$

Note that linearity holds even though the X_m are not independent.

Note: What is $Pr[X = m]$? Tricky ....

Using Linearity - 3: Binomial Distribution.

Flip n coins with heads probability p. X - number of heads

Binomial Distibution: $Pr[X = i]$, for each i.

$$Pr[X = i] = \binom{n}{i} p^i (1-p)^{n-i}.$$

$$E[X] = \sum_i i \times Pr[X = i] = \sum_i i \times \binom{n}{i} p^i (1-p)^{n-i}.$$

Uh oh. ... Or... a better approach: Let

$$X_i = \begin{cases} 1 & \text{if } i\text{th flip is heads} \\ 0 & \text{otherwise} \end{cases}$$

$E[X_i] = 1 \times Pr[\text{"heads"}] + 0 \times Pr[\text{"tails"}] = p.$

Moreover $X = X_1 + \cdots X_n$ and

$E[X] = E[X_1] + E[X_2] + \cdots E[X_n] = n \times E[X_i] = np.$

Calculating $E[g(X)]$

Let $Y = g(X)$. Assume that we know the distribution of X.

We want to calculate $E[Y]$.

Method 1: We calculate the distribution of Y:

$$Pr[Y = y] = Pr[X \in g^{-1}(y)] \text{ where } g^{-1}(x) = \{x \in \Re : g(x) = y\}.$$

This is typically rather tedious!

Method 2: We use the following result.

Theorem:

$$E[g(X)] = \sum_x g(x) Pr[X = x].$$

Proof:

$$\begin{aligned} E[g(X)] &= \sum_\omega g(X(\omega)) Pr[\omega] = \sum_x \sum_{\omega \in X^{-1}(x)} g(X(\omega)) Pr[\omega] \\ &= \sum_x \sum_{\omega \in X^{-1}(x)} g(x) Pr[\omega] = \sum_x g(x) \sum_{\omega \in X^{-1}(x)} Pr[\omega] \\ &= \sum_x g(x) Pr[X = x]. \end{aligned}$$

□

An Example

Let X be uniform in $\{-2, -1, 0, 1, 2, 3\}$.

Let also $g(X) = X^2$. Then (method 2)

$$\begin{aligned} E[g(X)] &= \sum_{x=-2}^{3} x^2 \frac{1}{6} \\ &= \{4+1+0+1+4+9\}\frac{1}{6} = \frac{19}{6}. \end{aligned}$$

Method 1 - We find the distribution of $Y = X^2$:

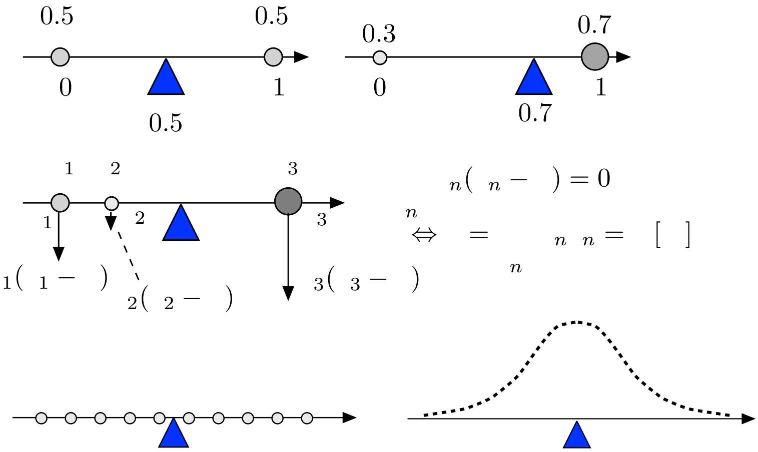
$$Y = \begin{cases} 4, & \text{w.p. } \frac{2}{6} \\ 1, & \text{w.p. } \frac{2}{6} \\ 0, & \text{w.p. } \frac{1}{6} \\ 9, & \text{w.p. } \frac{1}{6}. \end{cases}$$

Thus,

$$E[Y] = 4\frac{2}{6} + 1\frac{2}{6} + 0\frac{1}{6} + 9\frac{1}{6} = \frac{19}{6}.$$

Center of Mass

The expected value has a *center of mass* interpretation:



Monotonicity

Definition
Let X, Y be two random variables on Ω. We write $X \leq Y$ if $X(\omega) \leq Y(\omega)$ for all $\omega \in \Omega$, and similarly for $X \geq Y$ and $X \geq a$ for some constant a.

Facts
(a) If $X \geq 0$, then $E[X] \geq 0$.
(b) If $X \leq Y$, then $E[X] \leq E[Y]$.

Proof
(a) If $X \geq 0$, every value a of X is nonnegative. Hence,

$$E[X] = \sum_a aPr[X = a] \geq 0.$$

(b) $X \leq Y \Rightarrow Y - X \geq 0 \Rightarrow E[Y] - E[X] = E[Y - X] \geq 0.$

□

Example:

$$B = \cup_m A_m \Rightarrow 1_B(\omega) \leq \sum_m 1_{A_m}(\omega) \Rightarrow Pr[\cup_m A_m] \leq \sum_m Pr[A_m].$$

Summary

Random Variables

- A random variable X is a function $X : \Omega \to \Re$.
- $Pr[X = a] := Pr[X^{-1}(a)] = Pr[\{\omega \mid X(\omega) = a\}]$.
- $Pr[X \in A] := Pr[X^{-1}(A)]$.
- The distribution of X is the list of possible values and their probability: $\{(a, Pr[X = a]), a \in \mathscr{A}\}$.
- $E[X] := \sum_a a Pr[X = a]$.
- Expectation is Linear.

Indicator Random Variable

Definition
Let A be an event. The random variable X defined by

$$X(\omega) = \begin{cases} 1, & \text{if } \omega \in A \\ 0, & \text{if } \omega \notin A \end{cases}$$

is called the indicator of the event A.

Note that $Pr[X = 1] = Pr[A]$ and $Pr[X = 0] = 1 - Pr[A]$.

Hence,

$$E[X] = 1 \times Pr[X = 1] + 0 \times Pr[X = 0] = Pr[A].$$

This random variable $X(\omega)$ is sometimes written as

$$1\{\omega \in A\} \text{ or } 1_A(\omega).$$

Thus, we will write $X = 1_A$.

Uniform Distribution

Roll a six-sided balanced die. Let X be the number of pips (dots). Then X is equally likely to take any of the values $\{1, 2, \dots, 6\}$. We say that X is *uniformly distributed* in $\{1, 2, \dots, 6\}$.

More generally, we say that X is uniformly distributed in $\{1, 2, \dots, n\}$ if $Pr[X = m] = 1/n$ for $m = 1, 2, \dots, n$.
In that case,

$$E[X] = \sum_{m=1}^{n} mPr[X = m] = \sum_{m=1}^{n} m \times \frac{1}{n} = \frac{1}{n}\frac{n(n+1)}{2} = \frac{n+1}{2}.$$

Geometric Distribution

Let's flip a coin with $Pr[H] = p$ until we get H.

For instance:

$$\begin{aligned}\omega_1 &= H, \text{ or} \\ \omega_2 &= T\ H, \text{ or} \\ \omega_3 &= T\ T\ H, \text{ or} \\ \omega_n &= T\ T\ T\ T\ \cdots\ T\ H.\end{aligned}$$

Note that $\Omega = \{\omega_n, n = 1, 2, \ldots\}$.

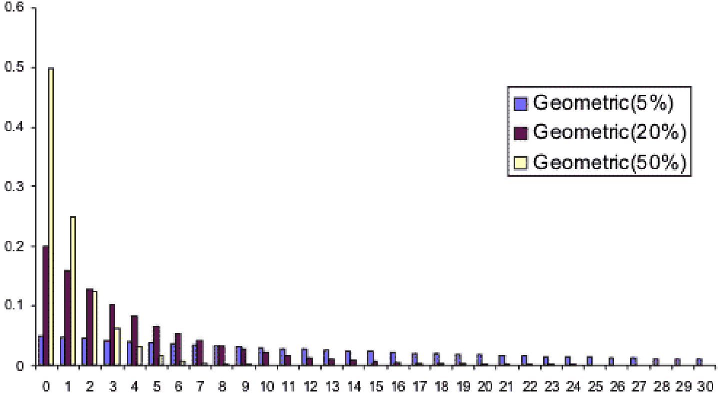
Let X be the number of flips until the first H. Then, $X(\omega_n) = n$.

Also,

$$Pr[X = n] = (1-p)^{n-1}p,\ n \geq 1.$$

Geometric Distribution

$$Pr[X = n] = (1 - p)^{n-1} p, n \geq 1.$$



Geometric Distribution

$$Pr[X = n] = (1-p)^{n-1}p, n \geq 1.$$

Note that

$$\sum_{n=1}^{\infty} Pr[X_n] = \sum_{n=1}^{\infty} (1-p)^{n-1}p = p\sum_{n=1}^{\infty}(1-p)^{n-1} = p\sum_{n=0}^{\infty}(1-p)^n.$$

Now, if $|a| < 1$, then $S := \sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$. Indeed,

$$\begin{aligned} S &= 1 + a + a^2 + a^3 + \cdots \\ aS &= \quad\;\; a + a^2 + a^3 + a^4 + \cdots \\ (1-a)S &= 1 + a - a + a^2 - a^2 + \cdots = 1. \end{aligned}$$

Hence,

$$\sum_{n=1}^{\infty} Pr[X_n] = p\,\frac{1}{1-(1-p)} = 1.$$

Geometric Distribution: Expectation

$$X =_D G(p), \text{ i.e., } Pr[X = n] = (1-p)^{n-1}p, n \geq 1.$$

One has

$$E[X] = \sum_{n=1}^{\infty} nPr[X = n] = \sum_{n=1}^{\infty} n(1-p)^{n-1}p.$$

Thus,

$$\begin{aligned}
E[X] &= p + 2(1-p)p + 3(1-p)^2p + 4(1-p)^3p + \cdots \\
(1-p)E[X] &= (1-p)p + 2(1-p)^2p + 3(1-p)^3p + \cdots \\
pE[X] &= p + (1-p)p + (1-p)^2p + (1-p)^3p + \cdots \\
&\qquad \text{by subtracting the previous two identities} \\
&= \sum_{n=1}^{\infty} Pr[X = n] = 1.
\end{aligned}$$

Hence,

$$E[X] = \frac{1}{p}.$$

Coupon Collectors Problem.

Experiment: Get coupons at random from n until collect all n coupons.
Outcomes: $\{123145..., 56765...\}$
Random Variable: X - length of outcome.

$E[X]$=?

Time to collect coupons

X-time to get n coupons.

X_1 - time to get first coupon. Note: $X_1 = 1$. $E(X_1) = 1$.

X_2 - time to get second coupon after getting first.

Pr[“get second coupon”|“got ~~milk~~ first coupon”] $= \frac{n-1}{n}$

$E[X_2]$? Geometric ! ! ! $\implies E[X_2] = \frac{1}{p} = \frac{1}{\frac{n-1}{n}} = \frac{n}{n-1}$.

Pr[“getting ith coupon|“got $i-1$rst coupons”] $= \frac{n-(i-1)}{n} = \frac{n-i+1}{n}$

$E[X_i] = \frac{1}{p} = \frac{n}{n-i+1}, i = 1, 2, \ldots, n$.

$$\begin{aligned} E[X] &= E[X_1] + \cdots + E[X_n] = \frac{n}{n} + \frac{n}{n-1} + \frac{n}{n-2} + \cdots + \frac{n}{1} \\ &= n(1 + \frac{1}{2} + \cdots + \frac{1}{n}) =: nH(n) \approx n(\ln n + \gamma) \end{aligned}$$

Review: Harmonic sum

$$H(n) = 1 + \frac{1}{2} + \cdots + \frac{1}{n} \approx \int_1^n \frac{1}{x} dx = \ln(n).$$

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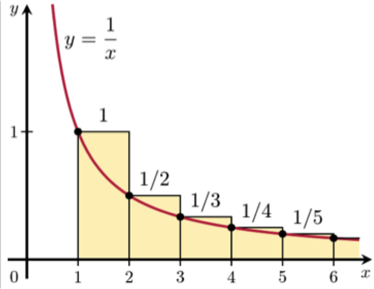


A good approximation is

$H(n) \approx \ln(n) + \gamma$ where $\gamma \approx 0.58$ (Euler-Mascheroni constant).

Geometric Distribution: Memoryless

Let X be $G(p)$. Then, for $n \geq 0$,

$$Pr[X > n] = Pr[\text{ first } n \text{ flips are } T] = (1-p)^n.$$

Theorem

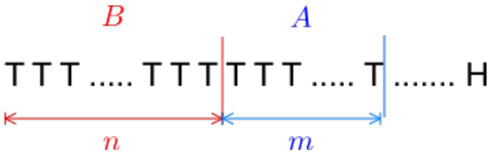
$$Pr[X > n+m | X > n] = Pr[X > m], m, n \geq 0.$$

Proof:

$$\begin{aligned} Pr[X > n+m | X > n] &= \frac{Pr[X > n+m \text{ and } X > n]}{Pr[X > n]} \\ &= \frac{Pr[X > n+m]}{Pr[X > n]} \\ &= \frac{(1-p)^{n+m}}{(1-p)^n} = (1-p)^m \\ &= Pr[X > m]. \end{aligned}$$

Geometric Distribution: Memoryless - Interpretation

$$Pr[X > n+m | X > n] = Pr[X > m], m, n \geq 0.$$



$$Pr[X > n+m | X > n] = Pr[A|B] = Pr[A] = Pr[X > m].$$

The coin is memoryless, therefore, so is X.

Geometric Distribution: Yet another look

Theorem: For a r.v. X that takes the values $\{0,1,2,\ldots\}$, one has

$$E[X] = \sum_{i=1}^{\infty} Pr[X \geq i].$$

[See later for a proof.]

If $X = G(p)$, then $Pr[X \geq i] = Pr[X > i-1] = (1-p)^{i-1}$.

Hence,

$$E[X] = \sum_{i=1}^{\infty} (1-p)^{i-1} = \sum_{i=0}^{\infty} (1-p)^{i} = \frac{1}{1-(1-p)} = \frac{1}{p}.$$

Expected Value of Integer RV

Theorem: For a r.v. X that takes values in $\{0, 1, 2, \ldots\}$, one has

$$E[X] = \sum_{i=1}^{\infty} Pr[X \geq i].$$

Proof: One has

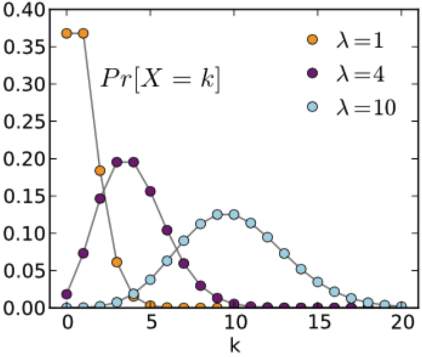
$$\begin{aligned}
E[X] &= \sum_{i=1}^{\infty} i \times Pr[X = i] \\
&= \sum_{i=1}^{\infty} i\{Pr[X \geq i] - Pr[X \geq i+1]\} \\
&= \sum_{i=1}^{\infty} \{i \times Pr[X \geq i] - i \times Pr[X \geq i+1]\} \\
&= \sum_{i=1}^{\infty} \{i \times Pr[X \geq i] - (i-1) \times Pr[X \geq i]\} \\
&= \sum_{i=1}^{\infty} Pr[X \geq i].
\end{aligned}$$

□

Poisson

Experiment: flip a coin n times. The coin is such that $Pr[H] = \lambda/n$.
Random Variable: X - number of heads. Thus, $X = B(n, \lambda/n)$.

Poisson Distribution is distribution of X "for large n."



Poisson

Experiment: flip a coin n times. The coin is such that $Pr[H] = \lambda/n$.
Random Variable: X - number of heads. Thus, $X = B(n, \lambda/n)$.
Poisson Distribution is distribution of X "for large n."
We expect $X \ll n$. For $m \ll n$ one has

$$\begin{aligned}
Pr[X = m] &= \binom{n}{m} p^m (1-p)^{n-m}, \text{ with } p = \lambda/n \\
&= \frac{n(n-1)\cdots(n-m+1)}{m!} \left(\frac{\lambda}{n}\right)^m \left(1 - \frac{\lambda}{n}\right)^{n-m} \\
&= \frac{n(n-1)\cdots(n-m+1)}{n^m} \frac{\lambda^m}{m!} \left(1 - \frac{\lambda}{n}\right)^{n-m} \\
&\approx^{(1)} \frac{\lambda^m}{m!} \left(1 - \frac{\lambda}{n}\right)^{n-m} \approx \frac{\lambda^m}{m!} \left(1 - \frac{\lambda}{n}\right)^{n} \approx^{(2)} \frac{\lambda^m}{m!} e^{-\lambda}.
\end{aligned}$$

For (1) we used $m \ll n$; for (2) we used $(1 - a/n) \approx e^{-a/n}$ for $a/n \ll 1$.

Poisson Distribution: Definition and Mean

Definition Poisson Distribution with parameter $\lambda > 0$

$$X = P(\lambda) \Leftrightarrow Pr[X = m] = \frac{\lambda^m}{m!} e^{-\lambda}, m \geq 0.$$

Fact: $E[X] = \lambda$.

Proof:

$$\begin{aligned} E[X] &= \sum_{m=1}^{\infty} m \times \frac{\lambda^m}{m!} e^{-\lambda} = e^{-\lambda} \sum_{m=1}^{\infty} \frac{\lambda^m}{(m-1)!} \\ &= e^{-\lambda} \sum_{m=0}^{\infty} \frac{\lambda^{m+1}}{m!} = e^{-\lambda} \lambda \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} \\ &= e^{-\lambda} \lambda e^{\lambda} = \lambda. \end{aligned}$$

□

Summary.

Distributions

- $U[1,\ldots,n] : Pr[X=m] = \frac{1}{n}, m = 1,\ldots,n;$
  $E[X] = \frac{n+1}{2};$
- $B(n,p) : Pr[X=m] = \binom{n}{m} p^m (1-p)^{n-m}, m = 0,\ldots,n;$
  $E[X] = np;$
- $G(p) : Pr[X=n] = (1-p)^{n-1} p, n = 1,2,\ldots;$
  $E[X] = \frac{1}{p};$
- $P(\lambda) : Pr[X=n] = \frac{\lambda^n}{n!} e^{-\lambda}, n \geq 0;$
  $E[X] = \lambda.$