

Binomial Coefficients

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■ Plan

1. Binomial Expansion
2. Combinatorics
3. Binomial Coefficients mod 2

■ Binomial Expansion

There are several ways to introduce binomial coefficients. We choose one that explains the name. A **binomial** is an algebraic expression that contains two terms, for example, $x + y$. We are going to multiply binomials

$$(x + y)^2 = (x + y)(x + y) = 1x^2 + 2xy + 1y^2$$

$$(x + y)^3 = (x + y)^2(x + y) = 1x^3 + 3x^2y + 3xy^2 + 1y^3$$

$$(x + y)^4 = (x + y)^3(x + y) = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$$

The numbers that appear as the coefficients of the terms in a binomial expansion, called **binomial coefficients**. The binomial coefficient of n and k is written either

$$C(n, k) \text{ or } \binom{n}{k}$$

and read as " n choose k ". To explain the latter name let us consider the quadratic form.

$$(x + y)^2 = (x + y)(x + y) = \binom{2}{0}x^2y^0 + \binom{2}{1}xy + \binom{2}{0}y^2x^0$$

Think of each binomial term in $(x + y)(x + y)$ as a box containing two items, x and y . To form x^2 in the expansion, we need to pick two x 's from both boxes and no y 's are needed. There is

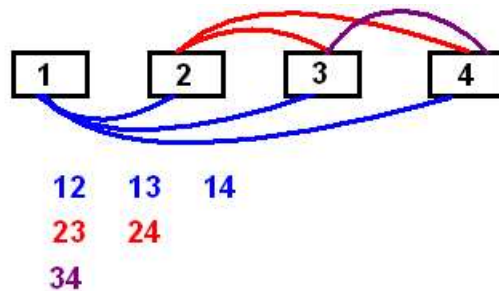
only one choice to do this. To form term xy we have two choices: either x from the first box and y from the second or vice versa.

$$(x + y)^3 = (x + y)(x + y)(x + y) = \binom{3}{0}x^3 + \binom{3}{1}x^2y + \binom{3}{1}xy^2 + \binom{3}{0}y^3$$

In case of cubic form $(x + y)^3$ we have three boxes. The coefficient by x^2y means the number of choices to pick two x 's and one y . All we do is decide which box we are going to pick the y from and then pick the x 's from all the other boxes. Thus, the coefficient by x^2y is the number of ways we can pick one y from a set of 3 boxes.

$$(x + y)^4 = \binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{1}xy^3 + \binom{4}{0}y^4$$

In case of $(x + y)^4$ we have four boxes. The coefficient by x^3y means the number of choices to pick one y from four boxes. There are three choices to do so. The coefficient by x^2y^2 means the number of choices two pick two x 's from 4 boxes. This number is 6 as it follows from the picture below



■ Combinatorics

Binomial coefficients are important in combinatorics where they provide formulas for certain counting problems. In this section we obtain a formula to calculate $C(n, k)$.

Let us start with [ordered sets](#). In an ordered set, there is a first element, a second element and so on. Here are ordered sets of two elements x and y

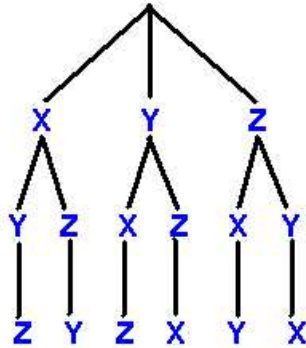
$$\{x, y\}, \{y, x\}$$

Consider ordered sets with x, y and z . There are six sets:

$$\{x, y, z\}, \{x, z, y\}, \{y, x, z\}, \{z, x, y\}, \{y, z, x\}, \{z, y, x\}$$

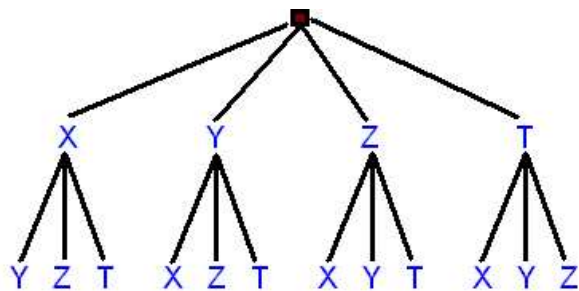
What is the number of ordered sets with n elements?

Generally, the number of ordered sets with n elements is then given by $n!$ We demonstrate this in the following way: There are n choices to put an element into the first position. Once the first position is filled, there are $n - 1$ choices to pick an element from the rest for the second position, and so on. This number is called the number of **arrangements**. The following decision tree makes it easy to visualize the proof

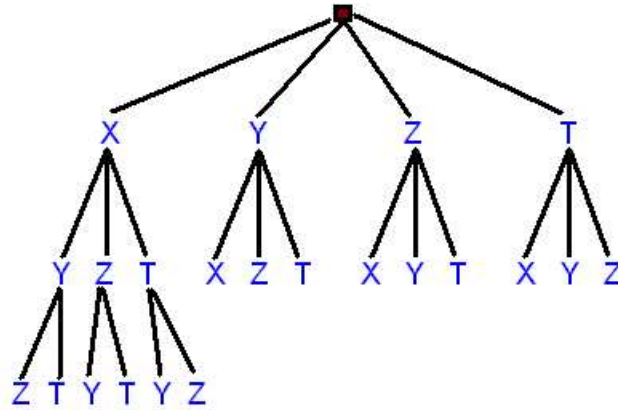


Each path from the root to a leaf corresponds to an ordered set.

Next we count the number of ways of obtaining an ordered subset of $k \leq n$ elements from a set of n elements. This is the number of **permutations**, denoted by $P(n, k)$. It reads the number of permutations of n taken k at a time. We visualize this number by drawing the decision tree. Here is the number of subsets of size 2 that can be made of a set of size 4



Each path from the root to a leaf corresponds to an ordered set. Thus, $P(4, 2) = 12$. If we wanted the number of ordered sets of size 3 made from a set of size 4, we would draw another level of nodes



Thus, $P(4, 3) = 24$. Generally,

$$P(n, k) = n * (n - 1) * \dots * (n - k + 1) = \frac{n!}{(n - k)!}$$

Finally, we consider permutations of unordered sets. **Combinations** are arrangements of elements without regard to their order or position. There is only one unordered set of 2 elements

$$\{x, y\}$$

All subsets of size k of the set of n elements is denoted by $C(n, k)$.

$$C(n, n) = 1$$

Here are unordered subsets of 2 elements out of $\{x, y, z, t\}$:

$$\{x, y\}, \{x, z\}, \{x, t\}, \{y, z\}, \{y, t\}, \{z, t\}$$

$$C(4, 2) = 6$$

IN order to derive a formula for $C(n, k)$, we reconsider the problem of finding the number of ordered sets of size k in a set of size n . We can solve this problem in a different way. Here is the algorithm:

- Create all unordered subsets of size k .
- Order the elements of these subsets.

Consider all unordered subsets of 2 elements out of $\{x, y, z, t\}$:

$$\{x, y\}, \{x, z\}, \{x, t\}, \{y, z\}, \{y, t\}, \{z, t\}$$

How would you order elements in each subset? The ordering is simple, we order elements in direct and then in reverse ways:

$$\{x, y\}, \{x, z\}, \{x, t\}, \{y, z\}, \{y, t\}, \{z, t\}$$

$$\{y, x\}, \{z, x\}, \{t, x\}, \{z, y\}, \{t, y\}, \{t, z\}$$

$$C(4, 2) * 2 = P(4, 2)$$

As another example, consider all subsets of 3 elements out of $\{x, y, z, t\}$:

$$\{x, y, z\}, \{x, y, t\}, \{y, z, t\}, \{x, z, t\}$$

There are six choices to order each subset. Here are choices for the first subset:

$$\{x, y, z\}, \{x, z, y\}, \{y, x, z\}, \{z, x, y\}, \{y, z, x\}, \{z, y, x\}$$

$$C(4, 3) * 6 = P(4, 3)$$

Therefore, we demonstrated that

$$C(n, k) P(k, k) = P(n, k)$$

It follows that

$$C(n, k) = \frac{P(n, k)}{P(k, k)} = \frac{n!}{(n-k)! k!}$$

Example. (The game of poker) How many different full houses are there in poker? There are 13 *denominations*: 2, 3, ..., king and ace. There are 4 *suits*: spades, hearts, diamonds and clubs. A full house consists of 3 denomination + 2 other denomination.

Solution.

Pick a denomination for the 3-set. Then pick a denomination for the 2-set. These can be done in $13 * 12$ ways. Next, we need to choose 3 cards out of 4 and 2 cards out of 4. Here is the total number of different fullhouses

$$13 * 12 * C(4,3) * C(4,2)=3744$$

Theorem.

$$C(n, k) = C(n, n - k)$$

Proof.

$$C(n, k) = \frac{n!}{(n-k)!k!}$$

$$C(n, n-k) = \frac{n!}{(n-(n-k))!(n-k)!} = \frac{n!}{(n-k)!k!}$$

Back to binomial expansions. We have showed, for example, that

$$(x+y)^3 = \binom{3}{0}x^3 + \binom{3}{1}x^2y + \binom{3}{1}xy^2 + \binom{3}{0}y^3$$

In a view of the above theorem,

$$\binom{3}{1} = \binom{3}{2}, \quad \binom{3}{0} = \binom{3}{3}$$

thus

$$(x+y)^3 = \binom{3}{0}x^3 + \binom{3}{1}x^2y + \binom{3}{2}xy^2 + \binom{3}{3}y^3$$

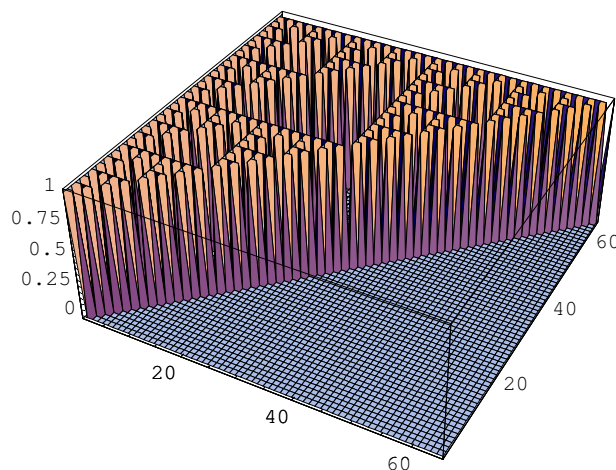
Exercise. Prove combinatorially (without using the above theorem) that

$$C(n, k) = C(n-1, k) + C(n-1, k-1)$$

■ Binomial Coefficients mod 2

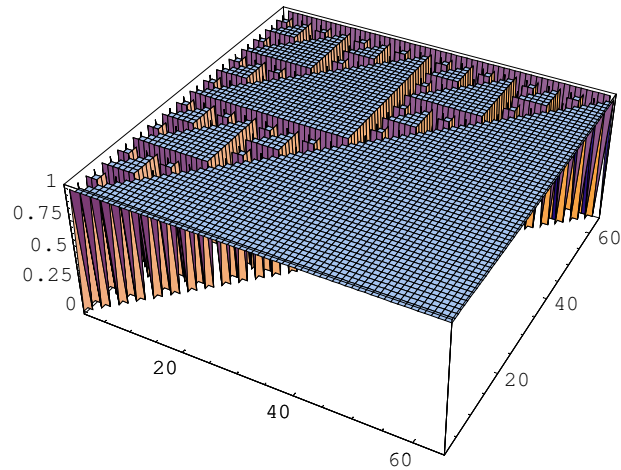
In this section we provide a picture of binomial coefficients modulo 2

```
ListPlot3D[Table[Mod[Binomial[n, k], 2], {n, 0, 2^6}, {k, 0, 2^6}]];
```



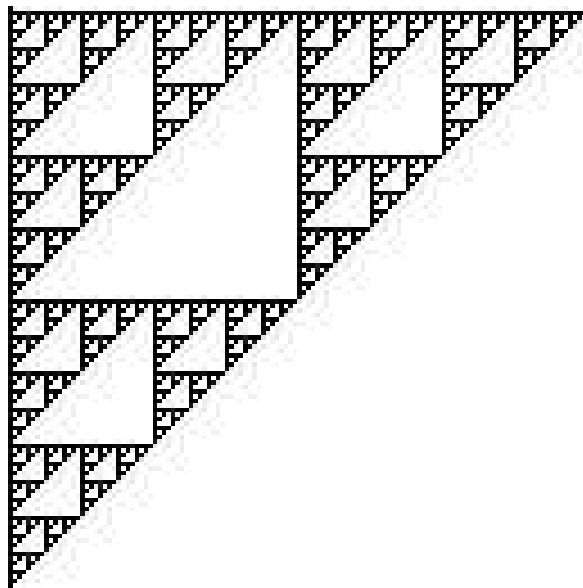
Let us put this picture upside down:

```
ListPlot3D[1 - Table[Mod[Binomial[n, k], 2], {n, 0, 2^6}, {k, 0, 2^6}]];
```



Finally, we flatten it

```
tab = 1 -  
  Table[Mod[Binomial[i, j], 2], {i, 0, 2^7}, {j, 0, 2^7}];  
Show[Graphics[Raster[tab], AspectRatio -> 1, FrameTicks -> None]];
```



The picture reminds the famous Sierpinski triangle. It was first observed by S. Wolfram as an instance of a 1-D example of cellular automata.

■ The Sierpinski Triangle

This was described by Sierpinski in 1915. Let us draw an equilateral triangle. Then we find the middle points of each side and connect them, so that one triangle will generate three new triangles. We remove the middle triangle and divide each remaining triangle in three. Here is the picture after seven iteration

```
spawn[Line[{p1_, p2_, p3_, p1_}]] :=  
  With[{p12 = (p1 + p2) / 2, p23 = (p2 + p3) / 2, p13 = (p1 + p3) / 2}, {Line[  
    {p1, p12, p13, p1}], Line[{p2, p12, p23, p2}], Line[{p3, p13, p23, p3}]}];  
spawn[s_List] := Flatten[spawn /@ s];  
Show[Graphics[Nest[spawn,  
  Line[{{0, 0}, {1/2., Sqrt[3]/2.}, {1., 0}, {0, 0}}, 7]]];
```

