Graph Theory

Victor Adamchik

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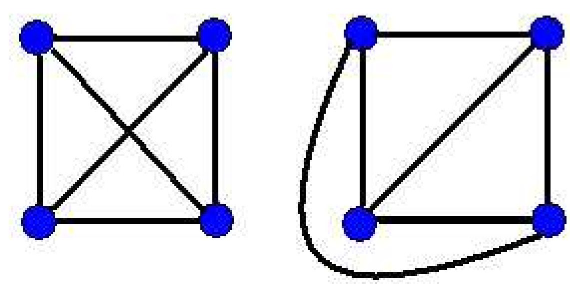
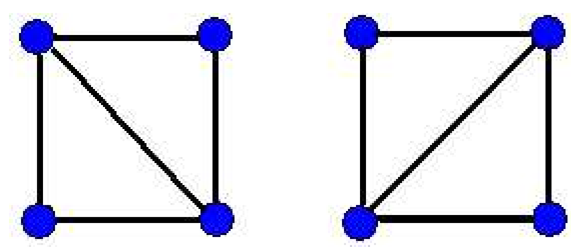
1. Graph Isomorphism
2. Graph Enumeration
3. Planar Graphs

Graphs Isomorphism

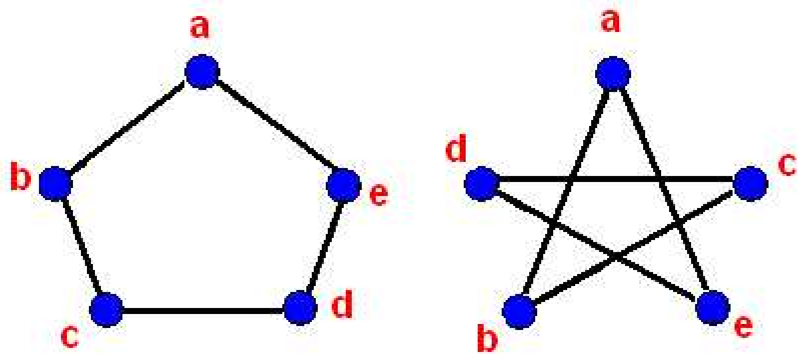
There are different ways to draw the same graph. Consiser the following two graphs.



You probably feel that these graphs do not differ from each other. What about this pair?



Another pair

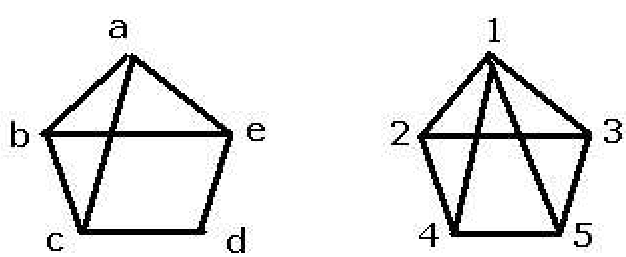


Are they the same? To answer this question, we need to look at the most essential property of a graph. Recall the graph representation. We discuss two models - an adjacency list and an adjacency matrix. So, we can say that a notion about which pair of vertices are adjacent and which are not is the most important property. Base on this, we can say that two graphs above must be identical - check the pair of adjacent vertices.

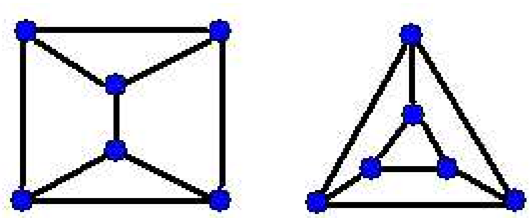
Definition. Graphs $G_1 = \{V_1, E_1\}$ and $G_2 = \{V_2, E_2\}$ are ***isomorphic*** if

1. there is a bijection (one-to-one correspondence) f from V_1 to V_2 and
2. there is a bijection g from E_1 to E_2 that maps each edge (v, u) to $(f(v), f(u))$

Example of non-isomorphic graphs



Exercise. Are these two graphs isomorphic?



In practice, it is not a simple task to prove that two graphs are isomorphic. It is much simpler to show that two graphs are not isomorphic by showing an ***invariant*** property that one has and other does not. An invariant is a property such that if a graph has it all isomorphic graphs have it. Some of such properties are the number of vertices, the number of edges, degree of a vertex and some others. Note, a complete set of such invariants is unknown.

Theorem.
Two graphs are isomorphic $\Longleftrightarrow$ *for some ordering of their vertices their adjacency matrices are equal.*

Question. How many different adjacency matrices do a graph with n vertices have?

Answer. It's equal to the number of permutations of n elements, which is $n!$.

Graph Enumeration

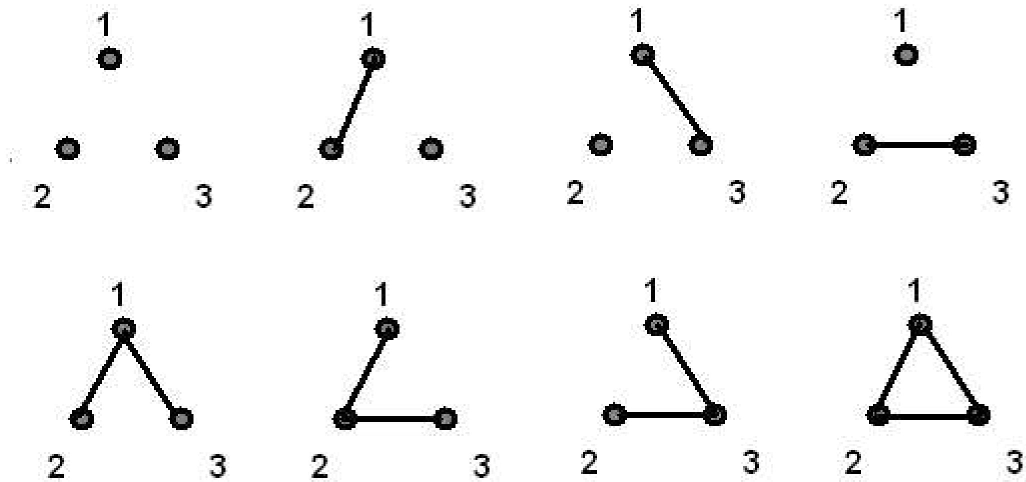
The subject of graph enumeration is concerned with the problem of finding out how many non-isomorphic graphs there are which posses a given property.

Main Question of this section:

How many are there simple undirected non-isomorphic graphs with n vertices?

We will try to answer this question into two steps. First, we count all labelled graphs. How many are there simple undirected labelled graphs with n vertices?

Here is a set of all such graphs for $n = 3$



There are only two choices for each edge, it either exists or it does not. Therefore, since the maximum number of edges is $\binom{n}{2}$, the total number of undirected labelled graphs is $2^{\binom{n}{2}}$

Next, we will count all unlabelled graphs. We observe that in the above picture there are some isomorphic graphs. Excluding them, we obtain only four unlabelled graphs (with 3 vertices)

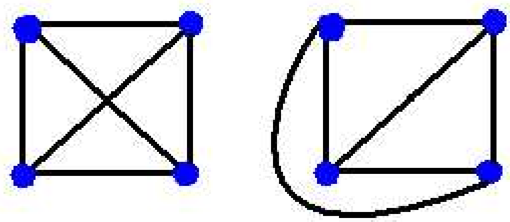
Therefore, to count all unlabelled graphs we need to count the number of equivalent classes. Here are some partciluar cases

vertices	**1**	**2**	**3**	**4**	**5**
num of graphs	**1**	**2**	**4**	**11**	**34**

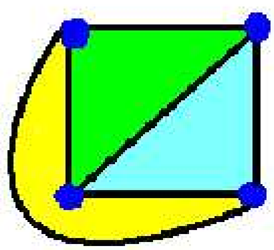
Counting the number of equivalent classes is far out of this course.

Planar graphs

An undirected graph is called a ***planar*** graph if it can be drawn on a paper without having two edges cross.



We say that a graph can be embedded in the plane, if it planar. A planar graph divides the plane into regions (bounded by the edges), called faces. The following planar graph has 4 faces.



Theorem (Euler's formula)

For any connected planar graph $G = (V, E)$, the following formula holds

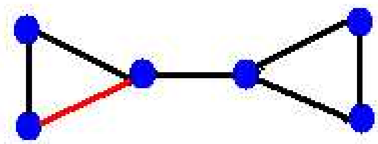
$$V + F - E = 2$$

where F stands for the number of faces.

Proof (by induction on the number of faces)

Base step. G is connected and has only one face. It is a tree, so $E = V - 1$ and therefore $V + 1 - E = 2$.

Suppose the formula holds for a connected graph with n faces. Prove it for $n + 1$ faces.



Choose an edge e connecting two different faces of G, and remove it. The graph remains connected. This removal decreases both the number of faces and edges by one. So, we can use inductive hypothesis.

$$V - E + F = V - (E - 1) + (F - 1) = 2.$$

QED.

Theorem (Euler's formula for a disconnected graph)
For any planar graph $G = (V, E)$ with k components, the following formula holds

$$V + F - E = 1 + k$$

Proof. Choose an edge e within the same face and remove it. The graph becomes disconnected. This removal decreases the number of edges by one and increase the number of componenets by one. Since the formula holds for $G - e$, it also holds for G.
QED