

# Graph Theory

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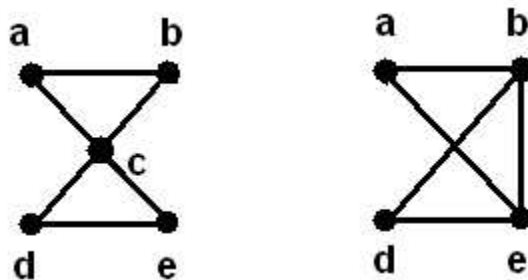
## ■ Plan

1. Euler Cycles
2. Hamiltonian Cycles

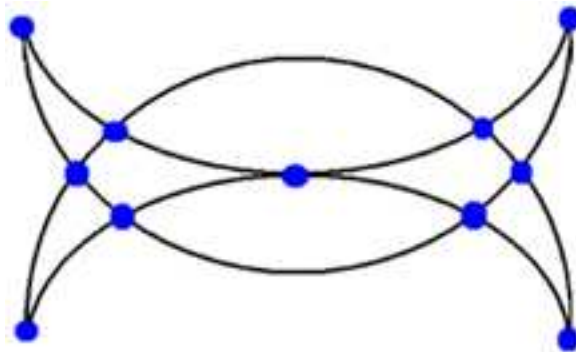
### *Euler Cycles*

#### **Definition.**

An **Euler cycle (or circuit)** is a cycle that traverses every edge of a graph exactly once. If there is an open path that traverse each edge only once, it is called an **Euler path**.



The left graph has an Euler cycle:  $a, c, d, e, c, b, a$  and the right graph has an Euler path  $b, a, e, d, b, e$



**Theorem**

*A connected undirected graph has an Euler cycle  $\iff$  each vertex is of even degree.*

*Proof.*

$\implies$  Each time the path passes through a vertex it contributes two to the vertex's degree, except the starting and ending vertices. If the path terminates where it started, it will contribute two to that degree as well.

$\impliedby$  We start with any vertex and choose a next vertex from all adjacent vertices in random order. We continue in this manner. After finite number of steps, we will end in the vertex we started with. If we traverse all edges, we build an Euler cycle. Otherwise, consider a subgraph  $H$  obtained from the original graph by deleting that cycle. Each vertex of  $H$  has even degree. In the above left picture graph  $H$  might consist of vertices  $c$ ,  $d$ ,  $e$ . The original graph and  $H$  must have one vertex in common. We start with this vertex and repeat the procedure. We repeat all these until all edges are traversed.

QED.

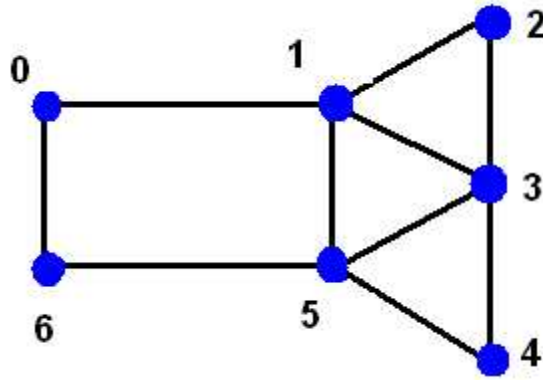
The proof to the theorem produces an algorithm.

**Algorithm** (constructing Euler cycles)

$G$  is a connected graph with even edges

We start at a proper vertex and construct a cycle. We decrease the vertex degree each time we visit it. If this cycle contains all edges of the graph, stop. Otherwise, select a vertex of degree greater than 0 (that belongs to the graph as well as to the cycle!) and construct another cycle. Next, splice these two cycles into one. If a join cycle contains all edges of the graph, stop. If not, select a common vertex of degree greater than 0 and construct a cycle, and so on.

Consider the following example. On each step we maintain an array of degree

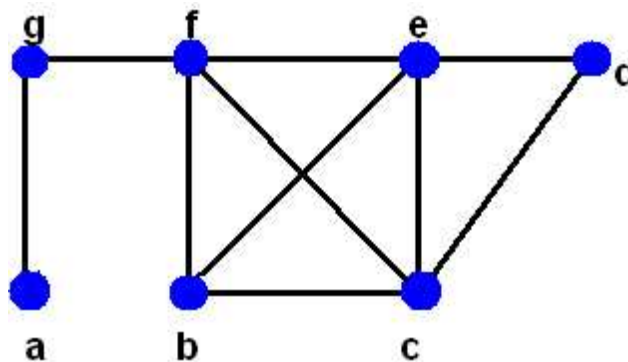


1. Start at 3 and discover cycle (3,5,4,3).
2. Start at 3 and discover cycle (3,2,1,3)
3. splice them in one (3, 5, 4, 3, 1, 2, 3)
4. start at 1 and discover (1,0,6,5,1)
5. splice, to get a final cycle (3, 5, 4, 3, 1, 0, 6, 5, 1, 2, 3)

**Theorem**

*A connected undirected graph has an Euler path (not a cycle)  $\iff$  if it has exactly two vertices of odd degree.*

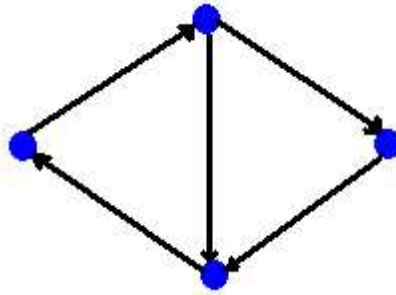
Consider the graph. It has two vertices of odd degrees, since the graph has an Euler path.



**Theorem**

*A digraph has an Euler cycle  $\iff$  if it strongly connected and  $indeg(v_k) = outdeg(v_k)$  for all vertices*

A graph below is not Eulerian.



## *Hamiltonian Cycles*

### **Definition.**

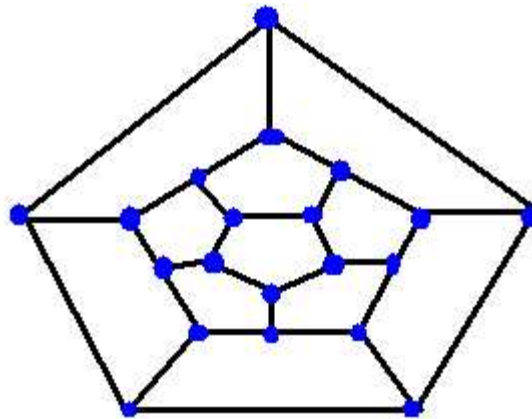
A **Hamiltonian cycle** is a cycle that traverses every vertex of a graph exactly once.

If there is an open path that traverses each vertex only once, it is called a **Hamiltonian path**.

The terminology came from the Icosian puzzle, invented by Hamilton in 1857.

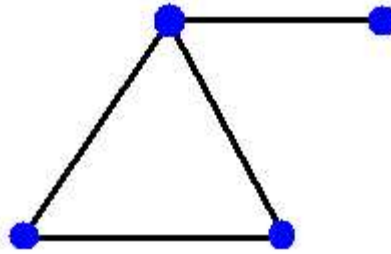
See <http://www.puzzlemuseum.com/month/picm02/200207icosian.htm>

There was a wooden "schematic" dodecahedron with a peg at each vertex labeled with different cities. The goal was to start at a city and travel along edges, visiting each of the other 19 cities exactly once.



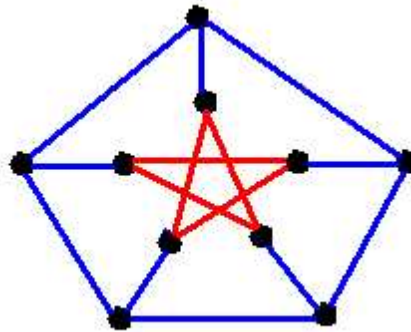
There are no known conditions which are both necessary and sufficient. There are a few sufficient conditions.

For a graph to have a Hamiltonian cycle the degree of each vertex must be two or more.



In general, the more edges a graph has, the more likely it is to have a Hamiltonian cycle.

The Petersen graph does not have a Hamiltonian cycle.



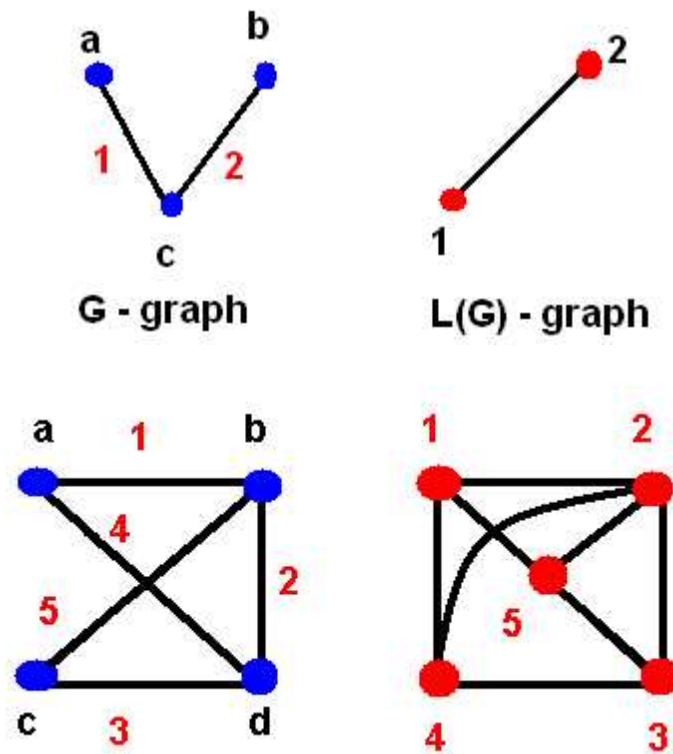
Theorem (Dirac)

Let  $G$  be a simple graph with  $n \geq 3$  vertices. If every vertex has degree at least  $n/2$ , then  $G$  has a Hamiltonian cycle.

The best known algorithm for finding a Hamiltonian cycle has an exponential worst-case complexity.

**Definition.**

The **line graph**  $L(G)$  of graph  $G$  has a vertex for each edge of  $G$ , and two of these vertices are adjacent iff the corresponding edges in  $G$  have a common vertex.



Eulerian cycles of a graph  $G$  translate into Hamiltonian cycles of  $L(G)$ .

**Exercise 1.**

If an edge has a vertex of degree  $d_1$  at one end and a vertex of degree  $d_2$  at the other, what is the degree of its corresponding vertex in the line graph?

*Answer.*

$$d_1 + d_2 - 2$$

**Exercise 2.**

Suppose a graph  $G$  is regular of degree  $d$ . Is  $L(G)$  necessarily a regular graph? and if so of what degree?

*Answer.*

$L(G)$  is a regular graph of degree  $2(d - 1)$

**Exercise 3.**

Why the line graph of a regular graph, if connected, always has an Euler cycle?

*Answer.*

$L(G)$  is a regular graph of even degree.

**Exercise 4.**

Given  $G = \{V, E\}$  with degree sequence  $d_1, d_2, \dots, d_V$ . Count the number of edges in  $L(G) = \{E, X\}$ .

*Answer.*

Each edge in  $L(G)$  corresponds to a pair of adjacent edges in  $G$ . Therefore, we have to count pair of edges associated with each vertex in  $G$

$$X = \sum_{k=0}^V \binom{d_k}{2}$$