Mathematical Induction

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Lecture 2 (out of three)

Plan

1. Strong Induction
2. Faulty Inductions
3. Induction and the Least Element Principal

Strong Induction

Fibonacci Numbers

Fibonacci number F_n is defined as the sum of two previous Fibonacci numbers

$$F_n = F_{n-1} + F_{n-2}$$

$$F_1 = 1, \ F_0 = 0$$

Claim. Fibonacci numbers are growing exponentially

$$F_n \geq \phi^{n-2}, \ \forall \, n \geq 2$$

where ϕ is the golden ratio

$$\phi = \frac{1+\sqrt{5}}{2}$$

Proof:

Base case: $n = 2$

$$F_2 \geq \phi^{2-2} = 1$$

Inductive hypothesis: assume the following

$$F_n \geq \phi^{n-2}$$

$$F_{n-1} \geq \phi^{n-3}$$

Note, we need two assumptions. Prove that

$$F_{n+1} \geq \phi^{n-1}$$

We start with the definition

$$F_{n+1} = F_n + F_{n-1}$$

Next we use the inductive hypothesis to obtain

$$F_{n+1} = F_n + F_{n-1} \geq \phi^{n-2} + \phi^{n-3} = \phi^{n-3}(\phi + 1)$$

Now we use the property of the golden ratio (prove this!)

$$\phi^2 = \phi + 1$$

Substituting this into the previous formula, we get

$$F_{n+1} \geq \phi^{n-3}(\phi + 1) = \phi^{n-1}$$

Question. Where would be the proof failed if you attempted to prove $F_n = \phi^{n-2}$.

Formal Definition

The weak form induction is stated as

$$P(n_0) \wedge \forall\, n \geq n_0 \,(\, P(n) \Longrightarrow P(n+1)\,)$$

Here $P(n_0)$ is a base case and $P(n)$ is the inductive hypothesis. To prove that $P(n)$ is true for $\forall\, n \geq n_0$, we have to

1. show that $P(n_0)$ is true

2. show that $P(n) \Longrightarrow P(n+1)$ is true for $\forall\, n \geq n_0$

A proof by strong induction only differs from the above in the Inductive Hypothesis step

Prove that $P(n+1)$ *is true whenever* $P(k)$ *is true for all* k *such that* $0 \le k \le n$.

Here is the formal definition

$$P(0) \wedge (P(1) \wedge \ldots P(n)) \Longrightarrow P(n+1)$$

This is called strong induction because you might need some or all previous case to prove the $n+1$ case.

Breaking Chocolate Bar

A chocolate bar consists of a number of squares (say, $n > 0$) arranged in a rectangular pattern. You split the bar into small squares always breaking along the lines between the squares. What is the minimum number of breaks?

Claim: It takes $n - 1$ breaks.

Proof.

Let $P(n)$ denote the number of breaks needed to split a bar with n squares.

Base step: $P(1) = 0$ is true

Induction step: Assume that $P(k)$ is true for $2 \le k \le n$

Prove that $P(n+1) = n$ under the above assumption.

Break a bar into two pieces of sizes n_1 and n_2, so that $n_1 + n_2 = n + 1$. By inductive hypothesis

$$P(n_1) = n_1 - 1$$

$$P(n_2) = n_2 - 1$$

Hence, the total number of breaks is

$$1 + (n_1 - 1) + (n_2 - 1) = n$$

How to justify the proof by strong induction?

The proof is by contradiction.

Suppose that some statements in the list $P(1), \ldots, P(n)$ were actually false. We choose the *first* false statement, say $P(m)$, where $m > 0$. Now we know that $P(0), P(1), \ldots, P(m-1)$ are true. Then by inductive hypothesis, $P(m)$ logically follows from $P(0), P(1), \ldots, P(m-1)$. Therefore, $P(m)$ is true. Contradiction.

Binary Search

The number of comparisons used during binary search in a table of size n in the worst case described by the recurrence

$$a_n = a_{\frac{n}{2}} + 1, \; a_1 = 1$$

with the solution

$$a_n = \log_2 n + 1$$

Proof. Base case: $a_1 = \log_2 1 + 1 = 0 + 1 = 1$

Inductive hypothesis: Assume $a_k = \log_2 k + 1$ for $k = 2, \ldots, n-1$.

Inductive step: prove for $k = n$:

$$a_n = \log_2 n + 1$$

We start with

$$a_n = a_{\frac{n}{2}} + 1$$

and make a use of inductive hypothesis

$$a_{\frac{n}{2}} = \log_2 \frac{n}{2} + 1$$

to obtain

$$a_n = \log_2 \frac{n}{2} + 2$$

Let n is even $n = 2\,p$, then

$$a_{2\,p} = \log_2 p + 2 = \log_2(2\,p) - \log_2 2 + 2 = \log_2(2\,p) + 1$$

Let n is odd $n = 2\,p + 1$, then

$$a_{2\,p+1} = \log_2 \frac{2\,p+1}{2} + 2 = \log_2(2\,p + 1) + 1$$

■ Faulty Inductions

Example 1

Claim: Every positive integer $n \geq 2$ has a unique prime factorization

Proof. Base step: $P(2)$ is true

Induction step: Assume that $P(k)$ is true for $2 \leq k \leq n$

Prove that $P(n+1)$ is true

There are two possibilities:

Case 1: $n+1$ is prime. Then we are done.

Case 2: $n+1$ is composite.

Let $n+1 = p * q$ where $1 < p, q < n+1$. By inductive hypothesis, p and q have unique factorizations. Since the product of two unique factorizations is again unique, we conclude the proof. QED

Explanation. $n+1 = p * q$ is NOT unique.

Example 2

Claim. $6\,n = 0$ for all $n \geq 0$.

Base step: Clearly $6*0 = 0$.

Induction step: Assume that $6\,k = 0$ for all $0 \leq k \leq n$.

We need to show that $6\,(n+1)$ is 0.

Write $n+1 = a+b$, where $a > 0$ and $b > 0$ are natural numbers less that $n+1$. By IH, we have

$$6\,a = 0 \text{ and } 6\,b = 0$$

Therefore,

$$6\,(n+1) = 6\,a + 6\,b = 0 + 0 = 0.$$

Explanation. We cannot write 1 as the sum of two natural numbers.

Example 3

Claim: All Fibonacci numbers are even

Proof by strong induction.

Base step: Clearly $F_0 = 0$ which is even

IH: Assume that F_k are even for all $0 \leq k \leq n$.

IS: We need to show that F_{n+1} is even

It is easy. By definition

$$F_{n+1} = F_n + F_{n-1}$$

F_n and F_{n-1} are even - by inductive hypothesis. Thus, F_{n+1} is even. QED.

Explanation. $F_{n+1} = F_n + F_{n-1}$ is not valid for $n = 0$.

■ Induction and the Least Element Principal

Weak vs. Strong

These two forms of induction are equivalent. They only differ from each other from the point of view of writing a proof. It is always possible to convert a proof using one form of induction into the other.

The conversion from weak to strong form is trivial, because a weak form is already a strong form.

The conversion from a strong form into a weak form is more interesting. Here are two forms

$$P(n_0) \wedge \forall\, n \geq n_0\, (\, P(n) \Longrightarrow P(n+1)\,)$$

$$P(n_0) \wedge \forall\, n \geq n_0\, (\, P(n_0) \wedge P(n_0+1) \wedge \ldots P(n) \Longrightarrow P(n+1)\,)$$

We introduce a new hypothesis $Q(n)$ defined by

$$Q(n) := P(n_0) \wedge P(n_0+1) \wedge \ldots P(n)$$

The base step is identical in both cases, namely $Q(n_0)$.

The inductive step is

$$Q(n) \Longrightarrow P(n+1)$$

Since $Q(n)$ implies itself, we rewrite the above statement as

$$Q(n) \Longrightarrow Q(n) \wedge P(n+1)$$

which is equivalent (by definition of Q)

$$Q(n) \Longrightarrow Q(n+1)$$

Therefore, the strong induction in P can be written as a weak induction in Q

$$Q(n_0) \wedge \forall\, n \geq n_0\, (\, Q(n) \Longrightarrow Q(n+1)\,)$$

Inductions vs. the Least Principal Element

Least Element Principal or Least Number Principal or Well-Ordering Principle:

Each non-empty subset of $\mathbb{N}$ has a least element.

In this section we prove the following

Theorem. *Induction and the Least Element Principal are logically equivalent.*

Proof. We prove that strong induction implies the least element principal. The proof is by contrapositive.

Negate

$$|S| \neq 0 \Longrightarrow \exists \text{min. element}$$

to get

$$\nexists \text{min. element} \Longrightarrow |S| = 0$$

Suppose S is a subset of $\mathbb{N}$ with no minimal element. Define proposition P by

$$P(n) := n \notin S$$

We will show that P satisfies all conditions of strong induction.
Clearly $P(0)$ is true. Assume that $P(0),\ P(1),\ \ldots,\ P(n)$ are all true. This means that none of $1, 2, \ldots, n$ is in S. What about $P(n+1)$? It is true as well, because othewise, we would have that $n+1 \in S$ and, therefore, S would have a minimal element. So, now all conditions of strong induction are satisfied. It follows then that $P(n)$ is true for all $n \geq 0$, hence S is empty.

Exercise 1.

Prove that every positive integer has a unique representation of the form

$$n = F_p + F_{p-1} + \ldots + F_{q-1} + F_q$$

where F_k are Fibonacci numbers.

For example,

$$1000000 = F_{30} + F_{26} + F_{24} + F_{12} + F_{10}$$

Exercise 2.

Show that for any fixed integer $p \geq 1$ the sequence:

$$2,\ 2^2,\ 2^{2^2},\ \ldots \pmod{p}$$

converges to an integer. (Hint: think of the Euler-Fermat theorem and $\phi(n)$)