Recursions

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Plan

1. Convergence of sequences
2. Fractals
3. Counting binary trees

Convergence of Sequences

In the previous lecture we considered a continued fraction for $\sqrt{2}$:

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \dots}}}}}$$

This fraction can be written in a recursive form

$$x_{n+1} = \frac{1}{x_n + 2}$$

$$x_0 = 0$$

Here are a few first values of the above sequence (coded in *Mathematica*)

```
x[0] = 0; x[n_] := 1/(x[n - 1] + 2);
Table[x[n], {n, 0, 10}] // N
```

{0., 1., 0.333333, 0.428571, 0.411765, 0.414634,
0.414141, 0.414226, 0.414211, 0.414214, 0.414213}

This numeric experiment suggests that

$$| x_{n+1} - x_n | \to 0$$

Therefore, we say that a given sequence converges if the limit exists:

$$\lim_{n\to\infty} x_n = a \neq \infty$$

The value to which a sequence converges is called a fixed point. For the sequence x_n a fixed point is $\sqrt{2} - 1$

$$\lim_{n\to\infty} x_n = \sqrt{2} - 1$$

We could prove this by the following argument. Let the sequence converges to number z, namely $x_n \to z$. Clearly, that $x_{n+1} \to z$. We find the value of z from the sequence definition

$$z = \frac{1}{z+2}$$

Solving the equation, we get two roots $z_1 = -\sqrt{2} - 1$, $z_2 = +\sqrt{2} - 1$. The positive root is the limit.

Towers of Hanoi

Consider the Towers of Hanoi recursion

$$x_{n+1} = 2\,x_n + 1$$

$$x_1 = 1$$

Here are the first few values of the sequence

```
Clear[x];
x[1] = 1; x[n_] := 2 x[n - 1] + 1
Table[x[n], {n, 1, 7}] // N
```

{1., 3., 7., 15., 31., 63., 127.}

We say that

$$\lim_{n\to\infty} x_n = \infty$$

Therefore, the sequence diverges.

More on the Golden Ratio

Consider the following recursion

$$x_{n+1} = \frac{1}{x_n + 1}$$

$$x_0 = 0$$

Does it converge? What does it converge to? Let us assume that $x_n \to z$, then $x_{n+1} \to z$ and

$$z = \frac{1}{z+1} \implies z^2 + z - 1 = 0$$

Solving this, yeilds

$$\lim_{n\to\infty} x_n = \phi - 1$$

where ϕ is the golden ratio. This immediately leads to a continued fraction for ϕ

$$\phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ldots}}}}}$$

Recall the Euclidean algorithm

$$\begin{array}{ccll}
a & = & b * q_1 + r_1, & 0 \le r_1 < b \\
b & = & r_1 * q_2 + r_2, & 0 \le r_2 < r_1 \\
r_1 & = & r_2 * q_3 + r_3, & 0 \le r_3 < r_2 \\
 & \ldots & \ldots & \ldots \\
r_{k-2} & = & r_{k-1} * q_k + r_k, & 0 \le r_k < r_{k-1} \\
r_{k-1} & = & r_k * q_{k+1} + 0 &
\end{array}$$

The continued fraction above implies that all quotients in the Euclidean algorithm applied to GCD(ϕ, 1) are ones. At the same time, we know that such property holds for GCD(F_{n+1}, F_n). Therefore, we can conject a relation between F_n and ϕ

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \phi$$

Exercise. Given a sequence

$$x_{n+1} = \sqrt{1 + x_n}$$

$$x_0 = 0$$

that represents a nested radical

$$\sqrt{1 + \sqrt{1 + \sqrt{1 + \ldots}}}$$

What does it converge to?

Exercise. Find the fixed point of the folllowing sequence

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{4}{x_n}\right)$$

$$x_0 = 1$$

Consider a general form recursion

$$x_{n+1} = f(x_n)$$

If x_n converges to a number z^*, than z^* is a fixed point

$$z^* = f(z^*)$$

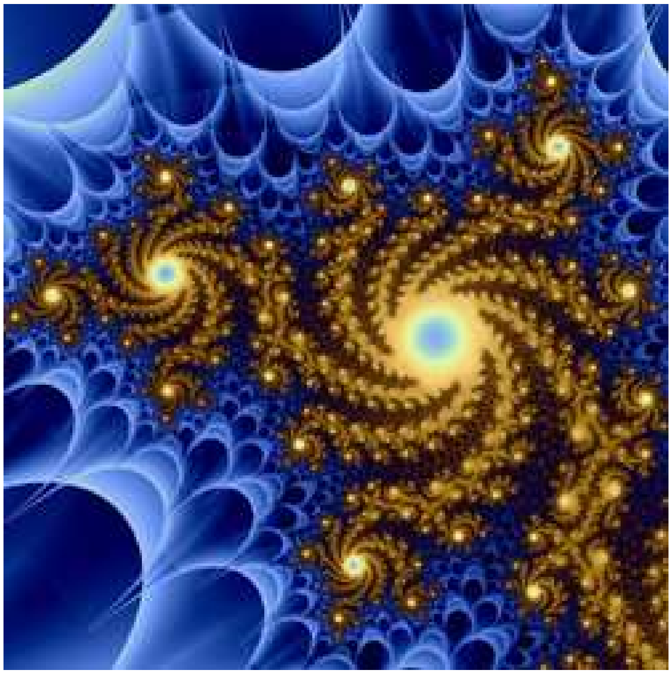
Solving this equation, we find z^*. The functional analysis question: for what classes of function f the sequence x_n converges?

Fractals

Fractals are geometric objects that are self-similar, i.e. composed of infinitely many pieces, all of which look the same.

Some fractals are mundane

But some fractals are extremely complicated



Since producing fractals requires repeating a certain step over an over again on smaller and smaller scales, it can be easily drawn by a computer.

Mandelbrot Set

The Mandelbrot set is a set of complex numbers z for which the following recurrence converges

$$a_{n+1} = a_n^2 + z$$

$$a_0 = z$$

Let $z = 1$, we get

$$a_{n+1} = a_n^2 + 1$$

$$a_0 = 1$$

Here are the first few value of the sequence

```
Clear[x];
x[0] = 1; x[n_] := x[n - 1]^2 + 1
Table[x[n], {n, 1, 5}] // N
```

{2., 5., 26., 677., 458330.}

The sequence does not converge. However, if $z = \frac{1}{5}$ then

$$a_0 = \frac{1}{5} = 0.2$$

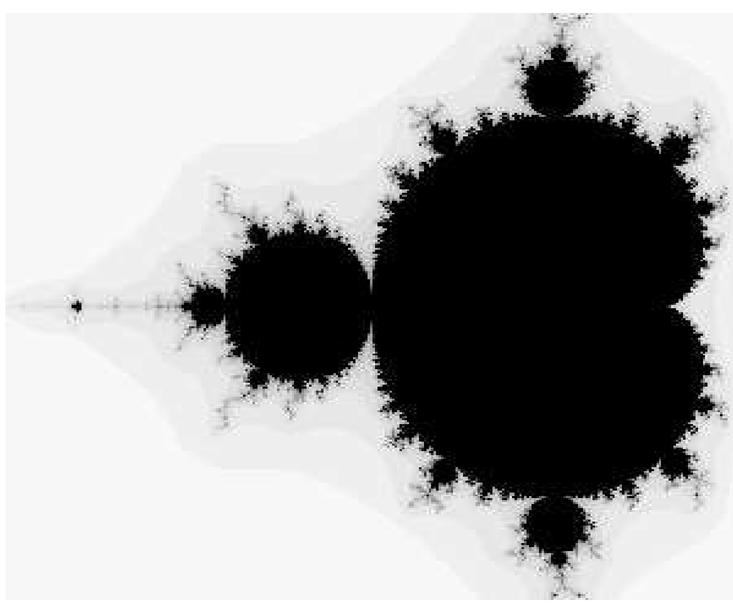
$$a_1 = \left(\frac{1}{5}\right)^2 + \frac{1}{5} = \frac{6}{25} = 0.24$$

$$a_2 = \left(\frac{6}{25}\right)^2 + \frac{1}{5} = \frac{161}{625} = 0.2576$$

$$a_3 = \left(\frac{161}{625}\right)^2 + \frac{1}{5} = \frac{104046}{390625} = 0.26636$$

the sequence does converge to 0.276393.

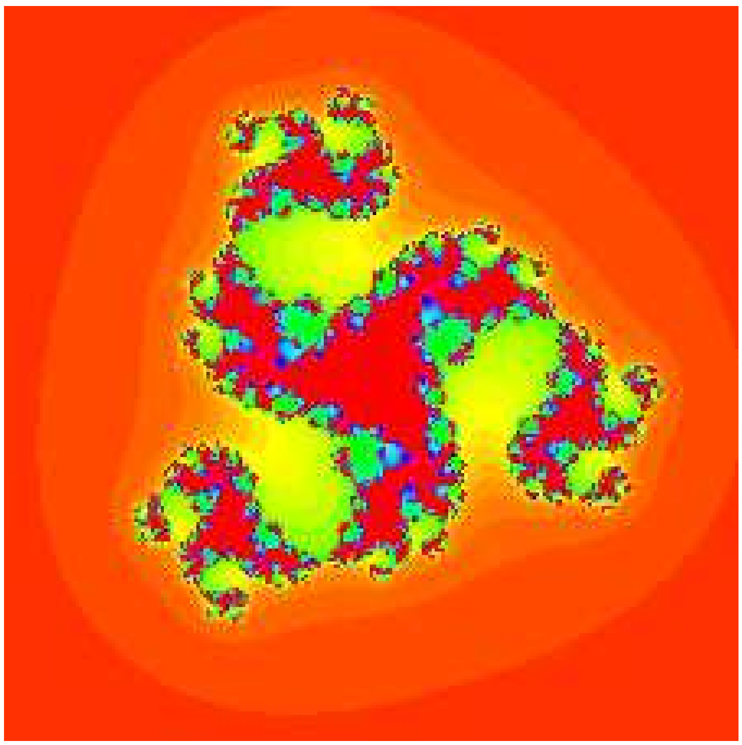
Here is a picture of the number of iterations that takes until a fixed point has been reached. Different shadows of gray correspondent to a different number of iterations.



■ **Julia Set**

Julia sets are defined by iterating a function starting at the arbitarary point in the plane. If after some number of iterations the result does not drift to infinity, but instead tends to a fixed point, then that starting point belongs to the Julia set. Here is a picture of the Julis set for

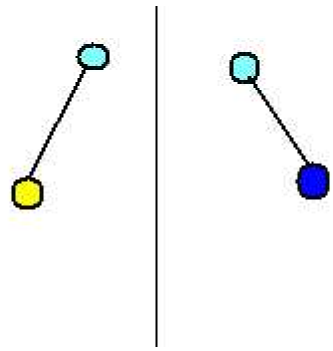
$$x_{n+1} = \text{Conjugate}(x_n)^3 - 0.53 - 0.4 * i$$



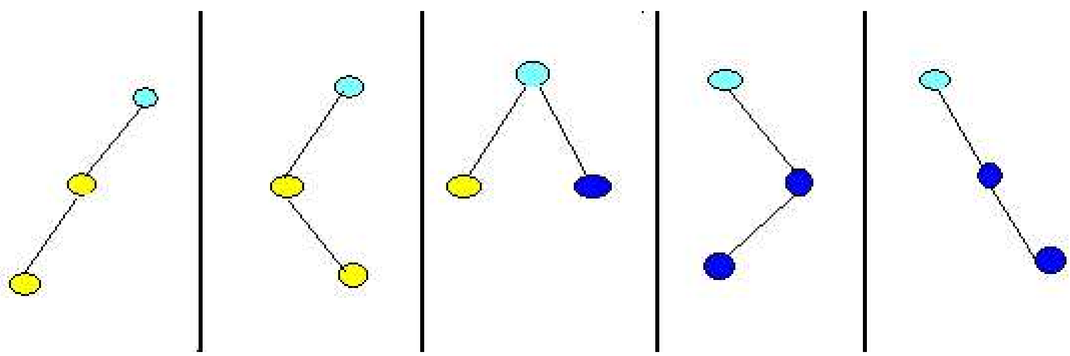
Counting Binary Trees

A binary tree is made of nodes, where each node contains a "left" reference, a "right" reference, and a data element.The left and right references recursively point to smaller "subtrees" on either side. The topmost node in the tree is called the "root". A recursive definition: a binary tree is either empty or consists of a root, a left subtree and a right subtree.

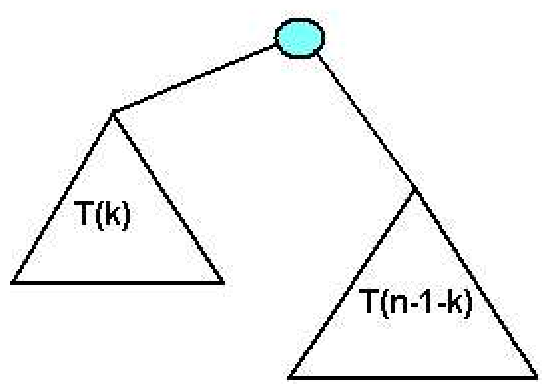
In this section we will count the number of binary trees with n nodes. Conisder a few particluar cases. If $n = 1$, there is only one binary tree. If $n = 2$, there two trees



If $n = 3$, there five trees



In general case we will derive a recursive formula for the number of trees based on the recursive definition of a binary tree. Let $T(n)$ denote the number of binary trees with n nodes. Suppose the left subtree (LT) has k nodes, then the right one (RT) has $n-1-k$ nodes.



Thus altogether we can create $T(k) * T(n-1-k)$ binary trees with k nodes in the left subtree. Since the left subtree can have any number of nodes in the interval $0 \le k \le n-1$, we have to sum up over all such k

$$T(n) = \sum_{k=0}^{n-1} T(k) * T(n-1-k)$$

The solution to this recurrence is known as the Catalan numbers after the Belgian Eugene Charles Catalan:

$$T(n) = \frac{1}{n+1} \binom{2n}{n}$$

where $\binom{2n}{n}$ stands for binomial coefficients.

Exercise. Give a recurrence relation for the number of ways to climb n stairs if the climber can take either one or two stairs at a time.