Lecture 5

Solving Recursions using Generating Functions

Generating Functions

What is a generating function?

A *generating function* is a "function" that stores the numbers in a sequence in its coefficients.

For example, the sequence

$$A = 1, 1, 1, 1, 1, 1, \ldots$$

can be stored as coefficients of the function

$$f(x) = 1 + 1x + 1x^2 + 1x^3 + \ldots$$

So, a_n the n^{th} term in sequence A, is the coefficient of x^n in $f(x)$.

What is a generating function?

$f(x) = 1 + 1x + 1x^2 + 1x^3 + \ldots$

$f(x) = 1 + x + x^2 + x^3 + \ldots$

$xf(x) = x + x^2 + x^3 + x^4 \ldots$

$f(x) - xf(x) = 1$

$f(x) = \frac{1}{1-x}$

Why use generating functions?

- Generating functions are easier to use and remember
- You don't have to remember all numbers in a sequence
- **More importantly** you can use certain operators on them to obtain other generating functions for other sequences
- **Even more importantly** They can be used to solve recurrences

The most important generating function

$S = 1, 1, 1, 1, 1, 1, \ldots$

$f(x) = 1 + x + x^2 + x^3 + x^4 + \ldots$

$f(x) = \dfrac{1}{1 - x}$

Derive another one

Consider,

$$S = 2, 2, 2, 2, 2, 2, 2, \ldots$$

Derive another one

Consider,

$$S = 2, 2, 2, 2, 2, 2, 2, \ldots$$

This can be represented by,

$$f(x) = 2 + 2x + 2x^2 + 2x^3 + \ldots$$

then, a similar trick,

$$xf(x) = 2x + 2x^2 + 2x^3 + 2x^4 + \ldots$$

$$f(x) - xf(x) = 2$$

therefore:

$$f(x) = \frac{2}{1 - x}$$

Generalize

Consider,

$$S = k, k, k, k, k, \ldots$$

This can be represented by,

$$f(x) = k + kx + kx^2 + kx^3 + \ldots$$

then, a similar trick,

$$xf(x) = kx + kx^2 + kx^3 + kx^4 + \ldots$$

$$f(x) - xf(x) = k$$

therefore:

$$f(x) = \frac{k}{1-x}$$

Derive another one

Consider,

$S = 1, 2, 4, 8, 16, \ldots$

Derive another one

Consider,

$S = 1, 2, 4, 8, 16, \dots$

Then,

$f(x) = 1 + 2x + 4x^2 + 8x^3 + \dots$

then, a similar trick,

$xf(x) = x + 2x^2 + 4x^3 + 8x^4 + \dots$

$2xf(x) = 2x + 4x^2 + 8x^3 + 16x^4 + \dots$

$f(x) - 2xf(x) = 1$

therefore:

$$f(x) = \frac{1}{1 - 2x}$$

Derive another one

Consider,

$S = 1, 3, 9, 27, 81, \ldots$

Derive another one

Consider,

$$S = 1, 3, 9, 27, 81, \ldots$$

Then,

$$f(x) = 1 + 3x + 9x^2 + 27x^3 + 81x^4 \ldots$$

then, a similar trick,

$$xf(x) = x + 3x^2 + 9x^3 + 27x^4 + \ldots$$

$$3xf(x) = 3x + 9x^2 + 27x^3 + 81x^4 + \ldots$$

$$f(x) - 3xf(x) = 1$$

therefore:

$$f(x) = \frac{1}{1 - 3x}$$

Generalize

$$S = 1, k, k^2, k^3, k^4, \ldots$$

$$f(X) = \frac{1}{1 - kx}$$

Derive another

$S = 1, -1, 1, -1, 1, -1, \ldots$

Simply use $k = -1$

$$f(x) = \frac{1}{1+x}$$

Derive another

$S = k, k^2, k^3, k^4, \ldots$

Start with

$S = 1, k, k^2, k^3, k^4, \ldots$

Summations

What about the sequence

$2, 4, 10, 28, 82, \ldots$

Notice that each term is 1 more than a power of 3.

$3^0 + 1, 3^1 + 1, 3^2 + 1, 3^3 + 1, \ldots$

Summations

So, the sequence is actually sum of two sequences:

$$S = (1, 1, 1, 1, 1, \dots) + (1, 3, 9, 27, 81, \dots)$$

$$S = \frac{1}{1-x} + \frac{1}{1-3x}$$

Alternating sequences

What if we replace x by x^2 in $\dfrac{1}{1-x}$?

Alternating sequences

$$f(x) = \frac{1}{1 - x^2}$$

results in

$1 + x^2 + x^4 + x^6 + \ldots$ which is the sequence $1, 0, 1, 0, 1, 0 \ldots$

Alternating sequences

$f(x) = \dfrac{2}{1 - x^2}$?

Alternating sequences

$f(x) = \frac{2}{1-x^2}$?

$2 + 2x^2 + 2x^4 + 2x^6 + \ldots$

$2, 0, 2, 0, 2, 0, \ldots$

Alternating sequences

What about $0, 1, 0, 1, 0, 1, \ldots$?

How can we get it?

We know that $f(x) = \dfrac{1}{1-x^2}$ produces $1, 0, 1, 0, 1, 0, \ldots$

Alternating sequences

$S = 1, 0, 1, 0, 1, 0, \ldots$ is $f(x) = \dfrac{1}{1 - x^2}$

$f(x) = 1 + x^2 + x^4 + x^6 + \ldots$

$xf(x) = x + x^3 + x^5 + x^7 + \ldots$ is $\dfrac{x}{1 - x^2}$

and this gives $S = 0, 1, 0, 1, 0, 1, \ldots$

Multiplying with x

Note that multiplying with x is like shifting right.

$1, 1, 1, 1, 1, \ldots$ is $\dfrac{1}{1-x}$

$0, 1, 1, 1, 1, \ldots$ is $\dfrac{x}{1-x}$

$0, 0, 1, 1, 1, \ldots$ is $\dfrac{x^2}{1-x}$

Multiplying with x

Find the generating function for

$$0, 0, 1, 2, 4, 8, 16, \ldots$$

Multiplying with x

$0, 0, 1, 2, 4, 8, 16, \ldots$

We know $1, 2, 4, 8, 16, \ldots$ is $\frac{1}{1-2x}$.

Now shift right twice by multiplying by x twice:

$$x * x * \frac{1}{1-2x} = \frac{x^2}{1-2x}$$

What if?

What happens if we add $1, 0, 1, 0, 1, 0, \ldots$ and $0, 1, 0, 1, 0, 1, \ldots$?

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What happens if we add $1, 0, 1, 0, 1, 0, \ldots$ and $0, 1, 0, 1, 0, 1, \ldots$?

$$\frac{1}{1-x^2} + \frac{x}{1-x^2}$$

$$= \frac{1+x}{1-x^2}$$

$$= \frac{1+x}{(1-x)(1+x)}$$

$$= \frac{1}{1-x}$$

That is, $1, 1, 1, 1, 1, \ldots$

Derivatives

What if we take the derivative of $\dfrac{1}{1-x}$?

Derivatives

Derivative of $\frac{1}{1-x}$ is $\frac{1}{(1-x)^2}$. If we take the derivative of the corresponding generating function:

$$f(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + \dots$$

$$f'(x) = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots$$

So, the corresponding sequence is $1, 2, 3, 4, 5, \dots$!

Second derivative

What about the second derivative?

$f''(x) = 2 + 6x + 12x^2 + 20x^3 + \ldots$ which is for $\dfrac{2}{(1-x)^3}$

so, $\dfrac{1}{(1-x)^3} = 1 + 3x + 6x^2 + 10x^3 + \ldots$

Note that, the sequence $1, 3, 6, 10, \ldots$ is the partial sum sequence (a.k.a. triangular numbers).

Differencing

Consider the sequence $1, 3, 5, 7, 9, \ldots$

What is the corresponding generating function?

Differencing

Consider the differences between consecutive items in $1, 3, 5, 7, 9, \ldots$: $2, 2, 2, 2, 2, \ldots$

So, right shift the original sequence and compute the difference:

$$f(x) = 1 + 3x + 5x^2 + 7x^3 + \ldots$$

$$xf(x) = x + 3x^2 + 5x^3 + \ldots$$

$$f(x) - xf(x) = 1 + 2x + 2x^2 + 2x^3 + \ldots$$

That is equivalent to $\dfrac{2}{1-x} - 1 = \dfrac{1+x}{1-x}$

So, $f(x) = \dfrac{1+x}{(1-x)^2}$

Multiplication and partial sums

What happens if you multiply two sequences?

Consider multiplying $1, 1, 1, 1, \ldots$ and $1, 1, 1, 1, \ldots$

Multiplication

$(1 + x + x^2 + x^3 + x^4 + \dots) \times (1 + x + x^2 + x^3 + x^4 + \dots)$

$= 1 \times 1, 1 \times x + x \times 1, 1 \times x^2 + x \times x + x^2 \times 1, \dots$

$= 1 + 2x + 3x^2 + 4x^3 + \dots$

That is $\dfrac{1}{(1-x)^2}$, which is expected because $1, 1, 1, 1, \dots$ is $\dfrac{1}{1-x}$.

Multiplication

Multiplying a sequence with $1, 1, 1, 1, \ldots$ is like obtaining a sequence of partial sums.

Multiply $1, 2, 4, 8, 16, 32, \ldots$ with $1, 1, 1, 1, \ldots$

Multiplication

Multiply $1, 2, 4, 8, 16, 32, \ldots$ with $1, 1, 1, 1, \ldots$

$$\frac{1}{1-2x} \times \frac{1}{1-x}$$

gives us the sequence

$$1, 3, 7, 15, 31, \ldots$$

That is the same as subtracting $1, 1, 1, 1, \ldots$ from $2, 4, 8, 16, 32, \ldots$.

$$\frac{2}{1-2x} - \frac{1}{1-x} = \frac{1}{(1-2x)(1-x)}$$

Solving Recurrences

Example

Solve the recurrence $a_n = 3a_{n-1} - 2a_{n-2}$ with initial conditions $a_0 = 1$ and $a_1 = 3$.

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Solve the recurrence $a_n = 3a_{n-1} - 2a_{n-2}$ with initial conditions $a_0 = 1$ and $a_1 = 3$.

Start with writing the very first terms of the sequence A: $1, 3, 7, 15, 31, \ldots$

Let's turn this into a generating function,

$A = 1 + 3x + 7x^2 + 15x^3 + 31x^4 + \ldots$

Example

Given the recurrence $a_n = 3a_{n-1} - 2a_{n-2}$, we know that $a_n - 3a_{n-1} + 2a_{n-2} = 0$, except the initial conditions. Consider the following:

$$A = 1 + 3x + 7x^2 + 15x^3 + 31x^4 + \dots$$

$$-3xA = 0 - 3x - 9x^2 - 21x^3 - 45x^4 + \dots$$

$$+2x^2A = 0 + 0x + 2x^2 + 6x^3 + 14x^4 + \dots$$

Note that, each column after the initial 2 items, cancels out.

$A(1 - 3x + 2x^2) = 1$

Therefore,

$A = \frac{1}{(1-2x)(1-x)}$

Example

Now that we have obtained a generating function, we need to solve for *partial fraction decomposition*:

$$\frac{1}{(1-2x)(1-x)} = \frac{a}{1-2x} + \frac{b}{1-x}$$

$$a - ax + b - 2bx = 1$$

$a + b = 1$ and $a + 2b = 0$, Solve for a and b to get $a = 2$ and $b = -1$

That is:

$$\frac{2}{1-2x} + \frac{-1}{1-x}$$

The first is 2^{n+1} and the second is -1, so the solution of the recurrence is $2^{n+1} - 1$.

Another example

Solve the recurrence $a_n = 2a_{n-1} - 3a_{n-2}$ with $a_0 = 1$ and $a_1 = 0$.

Another example

Solve the recurrence $a_n = 2a_{n-1} - a_{n-2}$ with $a_0 = 1$ and $a_1 = 0$.

$A = 1, 0, -1, -2, -3, -4, \dots$

We know $a_n - 2a_{n-1} + a_{n-2} = 0$

Hence,

$$\begin{aligned} A &= 1 + 0x - 1x^2 - 2x^3 - 3x^4 - 4x^5 + \dots \\ -2xA &= 0 - 2x + 0x^2 + 2x^3 + 4x^4 + 6x^5 + \dots \\ +x^2A &= 0 + 0x + 1x^2 + 0x^3 - 1x^4 - 2x^5 + \dots \end{aligned}$$

$A(1 - 2x + x^2) = 1 - 2x$

$$A = \frac{1 - 2x}{1 - 2x + x^2}$$

Another example

$$A = \frac{1-2x}{1-2x+x^2}$$

$$\frac{1-2x}{1-2x+x^2} = \frac{1-x}{(1-x)^2} - \frac{x}{(1-x)^2}$$

$$\frac{1}{1-x} - \frac{x}{(1-x)^2}$$

where, the first term is $1, 1, 1, 1, \ldots$ and the second term is $0, 1, 2, 3, 4, 5, \ldots$.
Hence, the n^{th} term is $a_n = 1 - n$.