## Week 5 Convex Hulls in 2D <br> <br> C <br> <br> C Me大n

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## Week 5 Convex Hulls in 2D



Applications
－Collision avoidance
－Fitting ranges with a line
－Smallest box
－Shape Analysis
Applications
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－Fitting ranges with a line
－Smallest box
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Applications
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－Smallest box
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Applications


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Applications
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& \text { Applications } \\
& \text { - Collision avoidance } \\
& \text { - Fitting ranges with a line }
\end{aligned}
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$\qquad$ Formal Segment Definition - The segment $x y$ is the set of all points of the

- $\alpha x+\beta y=\alpha x+(1-\alpha) y=\alpha(x-y)+y=x+\beta(y-x)$


## form $\alpha x+\beta y$ with $\alpha \geq 0, \beta \geq 0$ and $\alpha+\beta=1$

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\begin{aligned}
& \frac{|x v|}{|x y|}=\beta \\
& \frac{|v y|}{|x y|}=\alpha
\end{aligned}
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 The segment xy is the set of all points of the

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\alpha x+\beta y=\alpha x+(1-\alpha) y=\alpha(x-y)+y=x+\beta(y-x)
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## Convex Combination <br> $\qquad$

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\begin{align*}
& \text { Convex Comb } \\
& \text { - A convex combination of poi }  \tag{eㅡㄹ}\\
& \text {. a sum of the form } \alpha_{1} x_{1}+\alpha_{2} x_{2} \\
& \text { - } a_{i} \geq 0 \text { for all I } \\
& \text { - } a_{1}+\alpha_{2}+\ldots+\alpha_{k}=1
\end{align*}
$$

- A convex combination of points $x_{1}, x_{2}, \ldots, x_{k}$ is
- a sum of the form $\alpha_{1} x_{1}+\alpha_{2} x_{2}+\ldots+\alpha_{k} x_{k}$

$$
x_{2} \underbrace{a_{1} x_{1}+\alpha_{2} x_{2}+\alpha_{3} x_{3}}_{x_{3}}
$$

$0^{\alpha_{1} x_{1}+\alpha_{2} x_{2}+\alpha_{3} x_{3}}$
$0^{\alpha_{1} x_{1}+\alpha_{2} x_{2}+\alpha_{3} x_{3}}$
$0^{\alpha_{1} x_{1}+\alpha_{2} x_{2}+\alpha_{3} x_{3}}$
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－The convex hull of a set of points $S$
－in d dime set of all convex combinations of d
points of $S$
For d＝2，convex hull is the combing
triangles
Definition ll
The convex hull of a set of points S
：in die set of all convex combinations of d－
points of S
For d＝2，convex hull is the combing
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Convex Hull

- Definition IV
- The convex hull of a set of points $S$ is
- the intersection of all halfspaces that contain $S$

Definition IV
Definition IV

- Definition V
- Definition IV -
- Definition IV
- Definition IV -
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Definition IV

- the intersection of all halfspaces that contain

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Definition IV

- The convex hull of a set of points $S$ is

- 

intersection of all halfspaces that contain S =
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he convex hull of a set of points $S$ is
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- Definition V
- The convex hull of a set of points S is
. the smallest convex polygon that encloses S

Definition V
Definition V

- the smallest convex polygon that encloses S


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Definition V

- the smallest convex polygon that encloses S

- Definition V
- The convex hull of a set of points S is
- the smallest convex polygon that encloses S


Definition V

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Definition V

- the smallest convex polygon that encloses S
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## Convex Hull

- Definition VI
- The convex hull of a set of points $S$ is
- the enclosing convex polygon with smallest area
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## Output

- A convex hull algorithm may have the following outputs:
- all the points on the hull, in arbitrary order
- extreme points, in arbitrary order
- all the points on the hull, in BTO*
- extreme points, in BTO
* BTO: Boundary Traversal Order


## Extreme Points

- Extreme points
- are the vertices of the convex hull at which the
interior angle is strictly convex
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interior angle is strictly convex
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## Naive Algorithms

- Nonextreme Points
- A point is nonextreme iff it is inside some triangle whose vertices are points of the set and is not itself a corner of that triangle
a comer ot that triangle



## Naive Algorithms

- Algorithm: INTERIOR POINTS
- for each ido
- for each $\mathrm{j} \neq \mathrm{i}$ do
- for each $\mathrm{k} \neq \mathrm{j} \neq \mathrm{i}$ do
- if $p_{I}$ in triangle $p_{i} p_{j} p_{k}$ then $p_{I}$ is nonextreme

$$
\text { - for each } I \neq k \neq j \neq i \text { do }
$$




Naive Algorithms .


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p_{1} \text { is nonextreme }
$$

$$
\text { - } \mathrm{O}\left(\mathrm{n}^{4}\right)!!!
$$

## Extreme Edges

- for each i do
- for each $\mathrm{j} \neq \mathrm{i}$ do
- for each $k \neq j \neq i$ do


## - Algorithm: EXTREME EDGES



## Gift Wrapping 

- All edges of the convex hull are connected
- Find one, then search for the next
- Use the lowest vertex to start with
- Works much faster: O(nh)
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- Use the lowest vertex to start with
- Works much faster: $O(n h)$
18
18 All edges of the convex hull are conned
Gift Wrapping
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- Find one, then search for the next
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- Works much faster: O(nh) ,
18
- All edges of the convex hull are conne
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. Works much faster: $O(n h)$
18 All edges of the convex hull are conned $\nabla$ $\square$ $\square$
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- All edges of the convex hull are conn
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- All edges of the convex hull are conn
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- Use the lowest vertex to start with
- Works much faster: O(nh)

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- All edges of the convex hull are conn
- Find one, then search for the next
- Use the lowest vertex to start with
- Works much faster: $\mathrm{O}(\mathrm{nh})$
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\begin{align*}
& \text { Gift Wrapping } \\
& \text { - For each edge } \\
& \text { - Compute theta with } \mathrm{O}(\mathrm{n}) \text { vertices } \\
& \text { - Choose the vertex with the smallest theta }  \tag{2}\\
& \text { - There are only } \mathrm{h} \text { edges on the boundary } \\
& \text { - O(n) time for each edge } \\
& \text { - Overall running time } \mathrm{O}(\mathrm{nh})
\end{align*}
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\section*{Quickhull}

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- $\mathrm{c} \leftarrow$ index of point with max distance from ab
index of point with max distance from ab









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- $B \leftarrow$ points strictly right of $(c, b)$
return Quick hull (a, c, A) + (c) +QURKHULL(, b, B)
- 
- return QUICKHULL(a, c, A) + ( c ) + QUICKHULL(c, b, B)
- A $\leftarrow$ points strictly right of ( $\mathrm{a}, \mathrm{c}$ )

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$c$ index of point with max dis from ab $+$ 0

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\begin{array}{r}
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\text { - Algorith } \\
\text { - If } S= \\
=\text { else } \\
=\text { c } \leftarrow \\
=A \leftarrow \\
=B \\
~-~ r e t u ~ \tag{0}
\end{array}
$$

- If $S=\varnothing$ then return

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## - else <br> <br> ,

 <br> <br> ,} <br> <br> ,}- $\varnothing$ then return

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## Quickhull <br> <br> $2-2 \rightarrow+\square$

 <br> <br> $2-2 \rightarrow+\square$}Start with the leftmost
and rightmost vertices

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Start with the leftmost $\begin{aligned} & \text { and rightmost vertices }\end{aligned}$
Start with the leftmost
and rightmost vertices
Start with the leftmost
and rightmost vertices

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Start with the leftmost
and rightmost vertices



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Start with the leftmost $\begin{aligned} & \text { and rightmost vertices }\end{aligned}$



Start with the leftmost
and rightmost vertices

## Quickhull

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Points in the triangle act
are interior to the hull
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are interior to the hull
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are interior to the hull
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are interior to the hull
Points in the triangle acb
are interior to the hull



Points in the triangle acb
are interior to the hull
Points in the triangle act
are interior to the hull


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Points in the triangle act
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\begin{aligned}
& \text { Points in the triangle acb } \\
& \text { are interior to the hull }
\end{aligned}
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Points in the triangle acb
are interior to the hull

## Quickhull <br> ?

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## Quickhull <br> (2)

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Quickhull
－Running Time：
Running Time：
－Initial extremes abb and separating $S$ ：$O(n)$
．Finding extreme point $c$ and eliminating points in
$\quad$ ．Recursive steps：$T(n)=O(n)+T(\alpha)+T(\beta)$
$\quad$ Best case：$T(n)=2 T(n / 2)+O(n)=O(n \log n)$
$\quad$ Worst case：$T(n)=O(n)+T(n-1)=O\left(n^{2}\right)$
Running Time：
－Initial extremes a，b and separating $S: O(n)$
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Running Time：
－Initial extremes a，b and separating $S$ ：$O(n)$
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$\quad$ Recursive steps：$T(n)=O(n)+T(\alpha)+T(\beta)$
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$\quad$ Worst case：$T(n)=O(n)+T(n-1)=O\left(n^{2}\right)$
－Running Time：
－Initial extremes a，b and separating $S: O(r$
－Finding extreme point c and eliminating $p$
triangle acb：$O(n)$
－Recursive steps：$T(n)=O(n)+T(\alpha)+T(\beta)$
－Best case：$T(n)=2 T(n / 2)+O(n)=O(n \log n)$
－Worst case：$T(n)=O(n)+T(n-1)=O\left(n^{2}\right)$

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\text { Worst case: } \mathrm{T}(\mathrm{n})=\mathrm{O}(\mathrm{n})+\mathrm{T}(\mathrm{n}-1)=\mathrm{O}\left(\mathrm{n}^{2}\right)
$$

Running Time：
－Initial extremes abb and separating $S$ ：$O(n)$
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 Running Time：
－Initial extremes a，b and separating $S$ ：$O(n)$
－Finding extreme point c and eliminating points in
triangle acb：$O(n)$
$\quad$ Recursive steps：$T(n)=O(n)+T(\alpha)+T(\beta)$
• Best case：$T(n)=2 T(n / 2)+O(n)=O(n \log n)$

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  Running Time：
－Initial extremes $a, b$ and separating $S: O(n)$
$\quad$ Finding extreme point $c$ and eliminating points in
$\quad$ Recursive steps：$T(n)=O(n)+T(\alpha)+T(\beta)$
$\quad$ Best case：$T(n)=2 T(n / 2)+O(n)=O(n \log n)$ Running Time：
－Initial extremes $a, b$ and separating $S: O(n)$
－Finding extreme point $c$ and eliminating points in
triangle acb：$O(n)$
－Recursive steps：$T(n)=O(n)+T(\alpha)+T(\beta)$
－case：$T(n)=2 T(n / 2)+O(n)=O(n \log n)$ Running Time：
－Initial extremes $a, b$ and separating $S: O(n)$
－Finding extreme point $c$ and eliminating points in
triangle acb：$O(n)$
－Recursive steps：$T(n)=O(n)+T(\alpha)+T(\beta)$
－case：$T(n)=2 T(n / 2)+O(n)=O(n \log n)$

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Running Time：
－Initial extremes abb and separating $S$ ：$O(n)$
$\quad$ Finding extreme point $c$ and eliminating points in acb：$O(n)$
$\quad$ ．Recursive steps：$T(n)=O(n)+T(\alpha)+T(\beta)$
$\quad$ ．West case：$T(n)=2 T(n / 2)+O(n)=O(n \log n)$


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 Running Time：
－Initial extremes $a, b$ and separating $S: O(n)$
－Finding extreme point $c$ and eliminating points in
triangle acb：$O(n)$
－Recursive steps：$T(n)=O(n)+T(\alpha)+T(\beta)$
－case：$T(n)=2 T(n / 2)+O(n)=O(n \log n)$




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Running Time：
－Initial extremes a，b and separating $S: O(n)$
．Finding extreme point $c$ and eliminating points in
－Recursive steps：$T(n)=O(n)+T(\alpha)+T(\beta)$
－Worst case：$T(n)=O(n)+T(n-1)=O\left(n^{2}\right)$

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of Computational Geometry (1972)

- Bell laboratories required the hull of $\sim=10,000$ vertices
- The $\mathrm{O}(\mathrm{n} 2)$ algorithm took too much time
- n2 = 100,000,000
- Graham invented this $O(n \log n)$ algorithm
- $n \log n=133,000$

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## Outline

- Choose a vertex v inside the hull
- Sort the rest of the vertices in counter clockwise angular order around $v$
- All turns on the convex hull are left turns during a boundary traversal
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boundary traversal



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## ！ing

Running Time
－Sorting the vertices：On log n）
－Each vertex is added to the stack once
－May be removed only once
－Overall running time：$O(n \log n)$
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## Some Problems

- The algorithm so far is not perfect
- What happens at the end if the bottom vertex is not a hull vertex?
- What happens at the start if the second vertex is not a hull vertex?
- Which vertex to use as origin for sorting?
- How about collinear vertices?


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not a hull vertex?


震 Solutions
－Always a hull vertex
－After the sort，the first vertex is also a hull vertex <br> －Can be found in $O(n)$ time <br> －Can be found in $O(n)$ lime}



## －Origin of sorting：The rightmost vertex with smallest y－coordinate smallest y－coordinate <br> －

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- Always a hull vertex

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## Solutions

 －Hull collinearities－require a strict left turn
－Arbitrary solutions cause hull overlaps
－or bugs in the implementation
－Delete the vertices closer to the origin
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Consider the following pseudocode on a set \(S\) of \(n\) points:
\[
\begin{array}{l}\mathrm{i}^{\mathrm{i}} \leftarrow 1 \\ \mathrm{~S}_{1} \leftarrow \mathrm{~S} \\ n_{1} \leftarrow \mathrm{n} \\ \text { while }\left(n_{i}>3\right) \\ \text { compute the convex hull } H\left(\mathrm{~S}_{\mathrm{i}}\right) \text { of } \mathrm{S}_{\mathrm{i}} \\ \mathrm{S}_{\mathrm{i}+1} \leftarrow \mathrm{~S}_{\mathrm{i}}-\mathrm{H}\left(\mathrm{S}_{\mathrm{i}}\right) \\ n_{i+1} \leftarrow \mathrm{n}_{\mathrm{i}}-\left|\mathrm{H}\left(\mathrm{S}_{\mathrm{i}}\right)\right| \\ \mathrm{i} \leftarrow \mathrm{i}+1\end{array}
\]
where \(\left|\mathrm{H}\left(\mathrm{S}_{\mathrm{i}}\right)\right|\) is the number of points on \(\mathrm{H}\left(\mathrm{S}_{\mathrm{i}}\right)\) What is the maximum value i can obtain at the end?


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