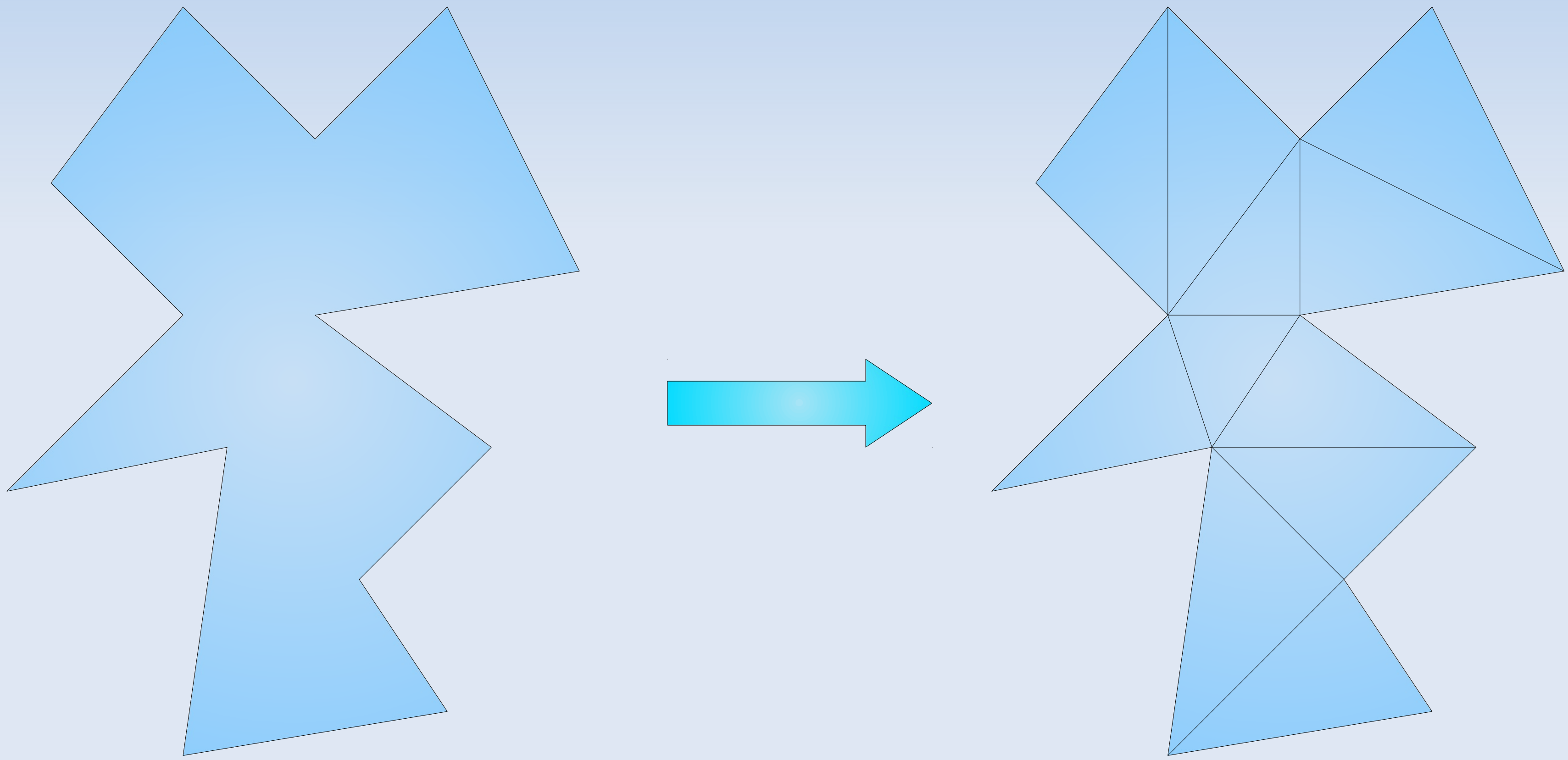
Week 2 Polygon Triangulation

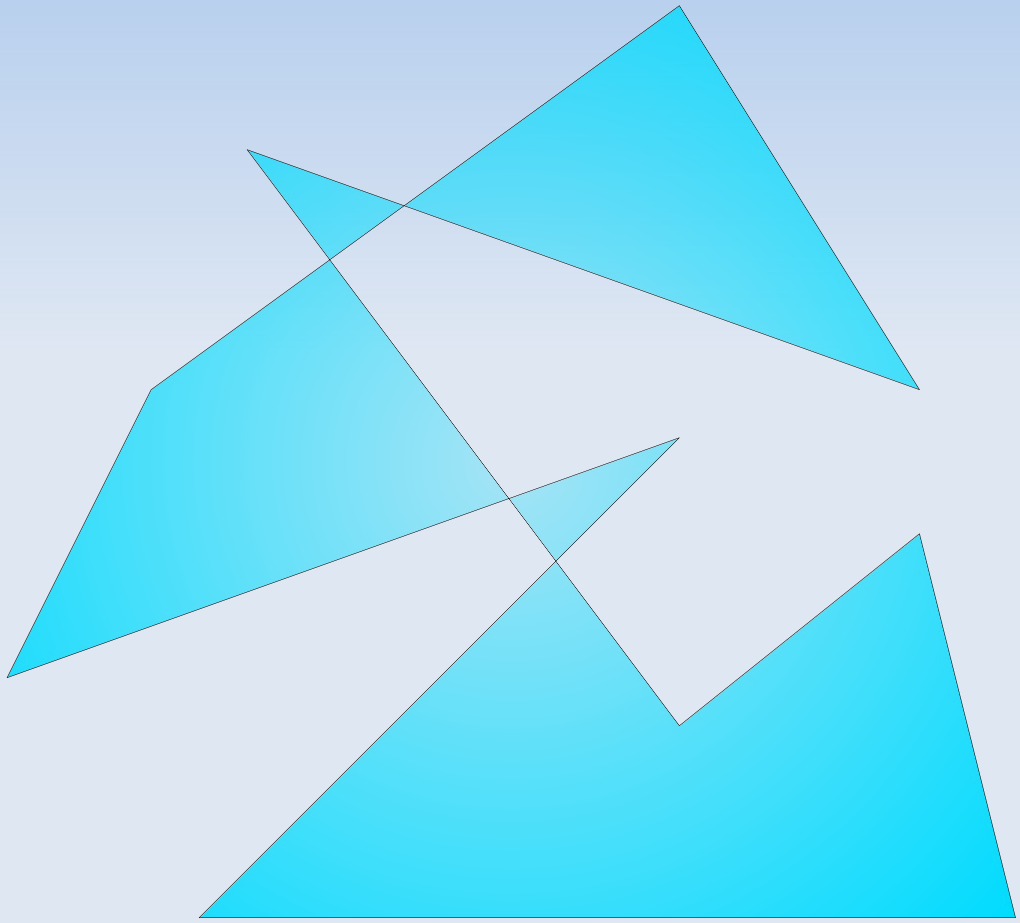


What is a polygon?

Last week

- A polygonal chain is a connected series of line segments
- A closed polygonal chain is a polygonal chain, such that there is also a line segment between the first and the last vertices
- A polygon is a 2D region bounded by a closed polygonal chain.

Is this a polygon?



Simple polygons

This is a polygon, but not a simple polygon!

Most people omit non-simple polygons

- We will do the same

When we say polygon, we mean a simple polygon

Formal definition

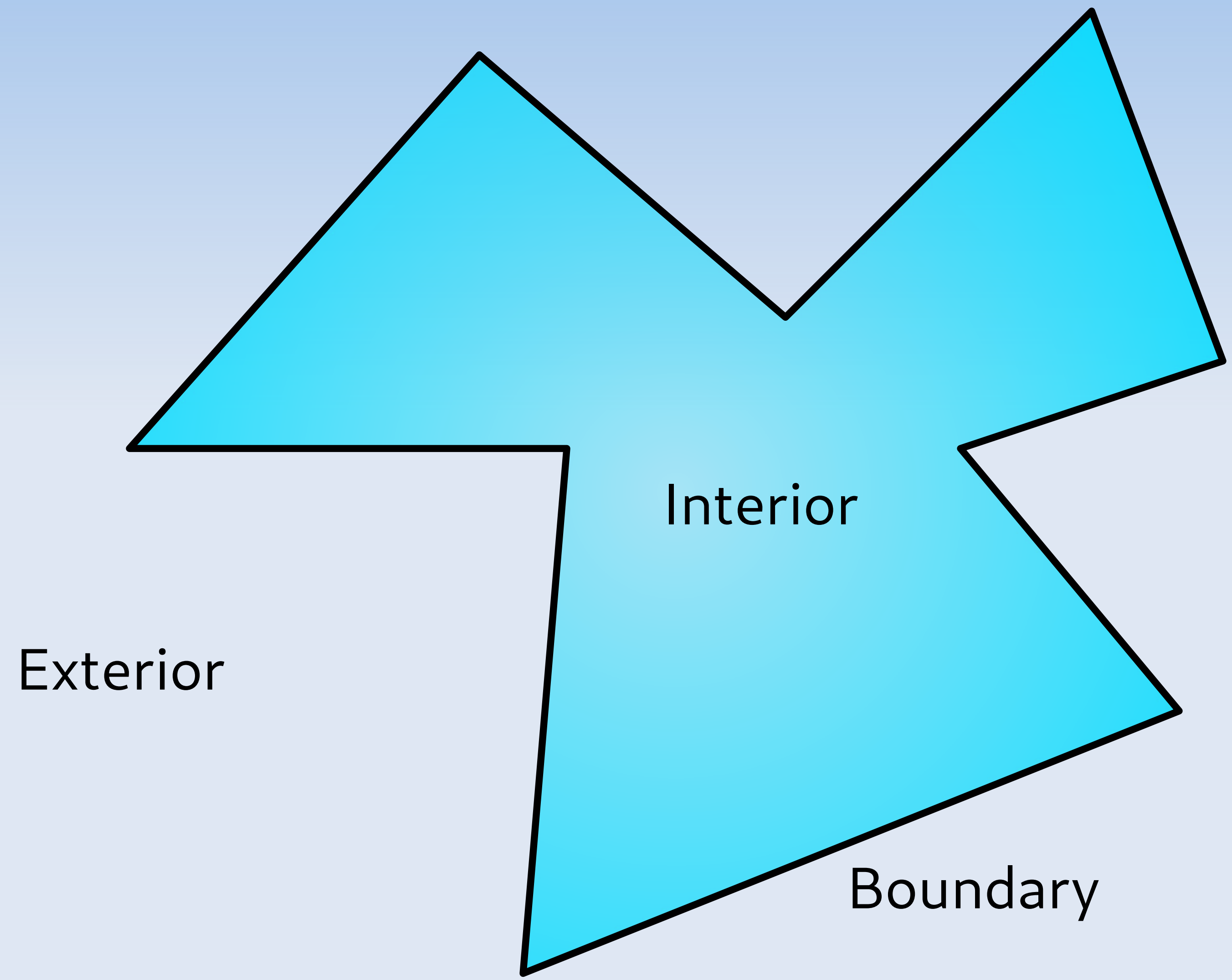
- A polygon is the region of a plane bounded by a finite collection of line segments forming a simple closed curve.
  - The intersection of adjacent segments is the shared end point
  - Nonadjacent segments do not intersect

Jordan Curve Theorem

Every simple closed plane curve divides the plane into two components

- A closed polygonal chain P is a simple closed plane curve!
  - it divides the plane into two components
  - the chain is the boundary ∂P
  - the components are the interior and exterior of P

Boundary, interior, exterior



Polygon = Boundary + Interior

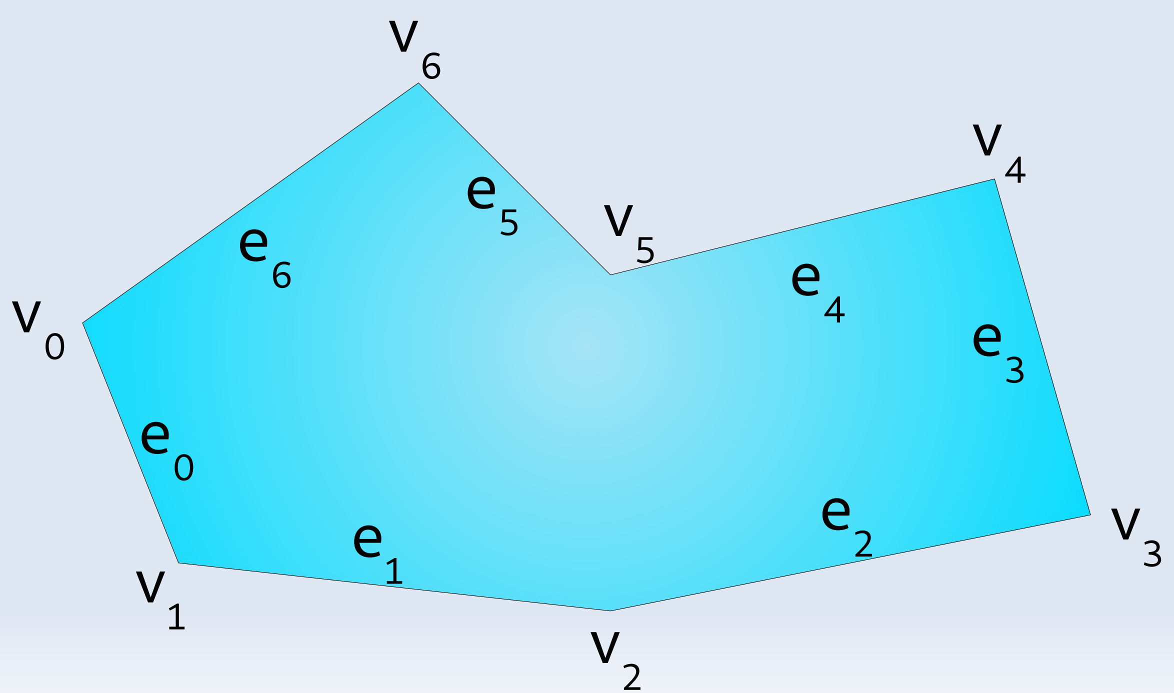
Cyclic ordering of vertices and edges

A polygon defines a cycling order of its vertices

$v_0, v_1, v_2, \ldots, v_{n-1}$

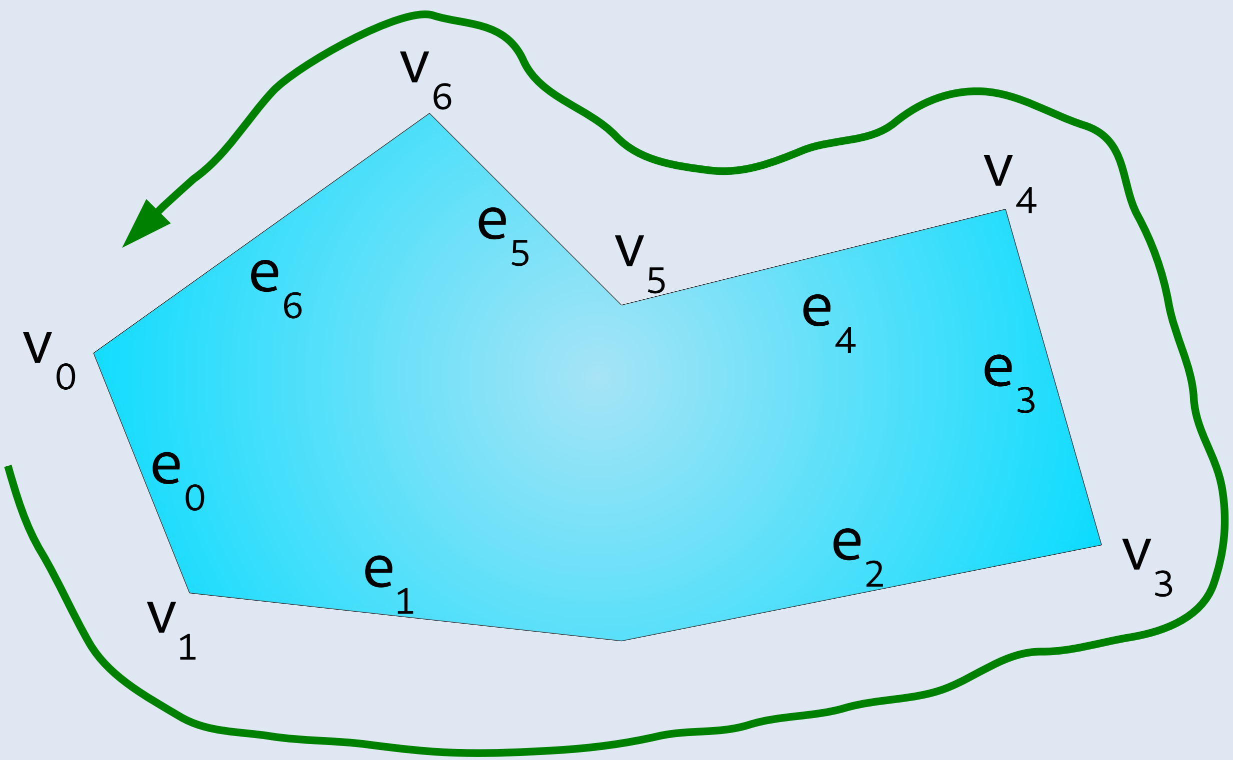
...and edges

$e_0 = v_0v_1, e_1 = v_1v_2, e_2 = v_2v_3, \ldots, e_{n-1} = v_{n-1}v_0$



Boundary Traversal

Visiting every vertex/edge in the given cyclic order is called a boundary traversal

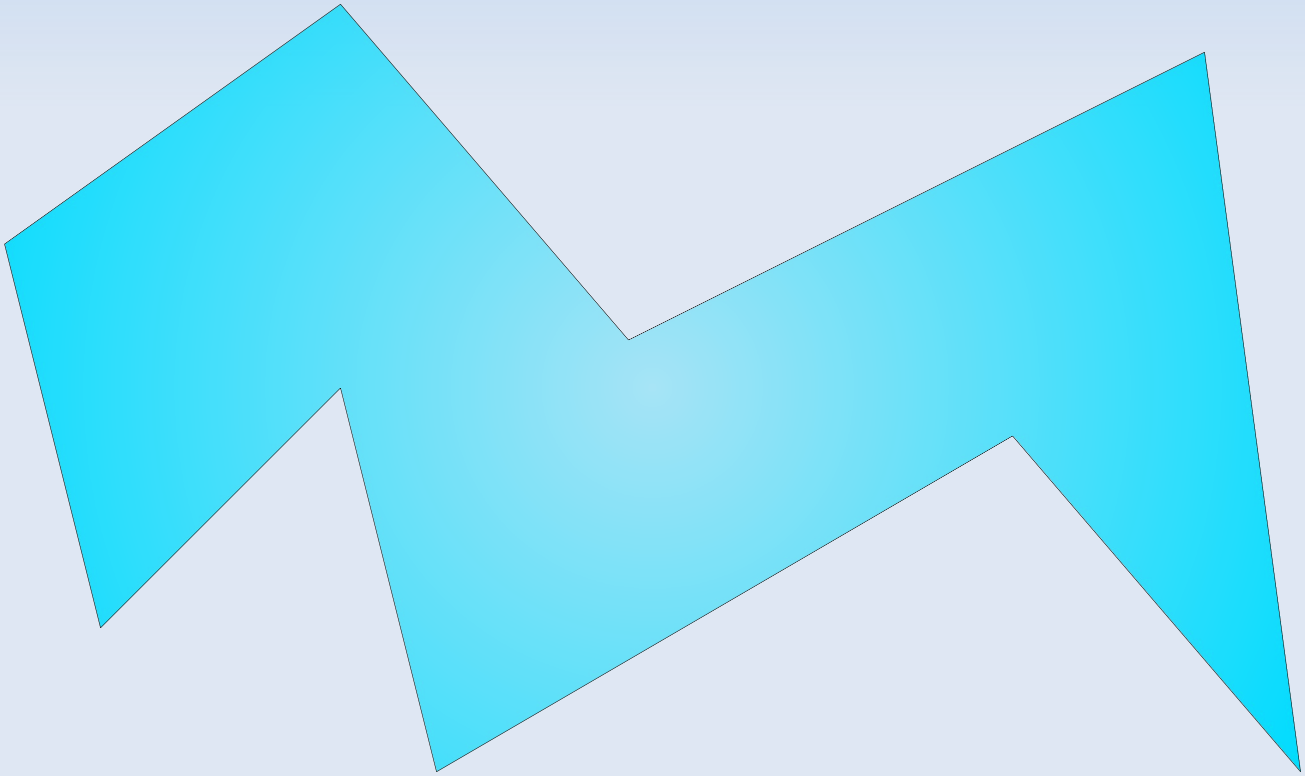


The Art Gallery Problem

- Given an art gallery room whose floor plan can be modelled as a polygon with n vertices:
  - How many stationary guards do you need to secure the gallery?
- Guards can see in 360°
- But not through walls!

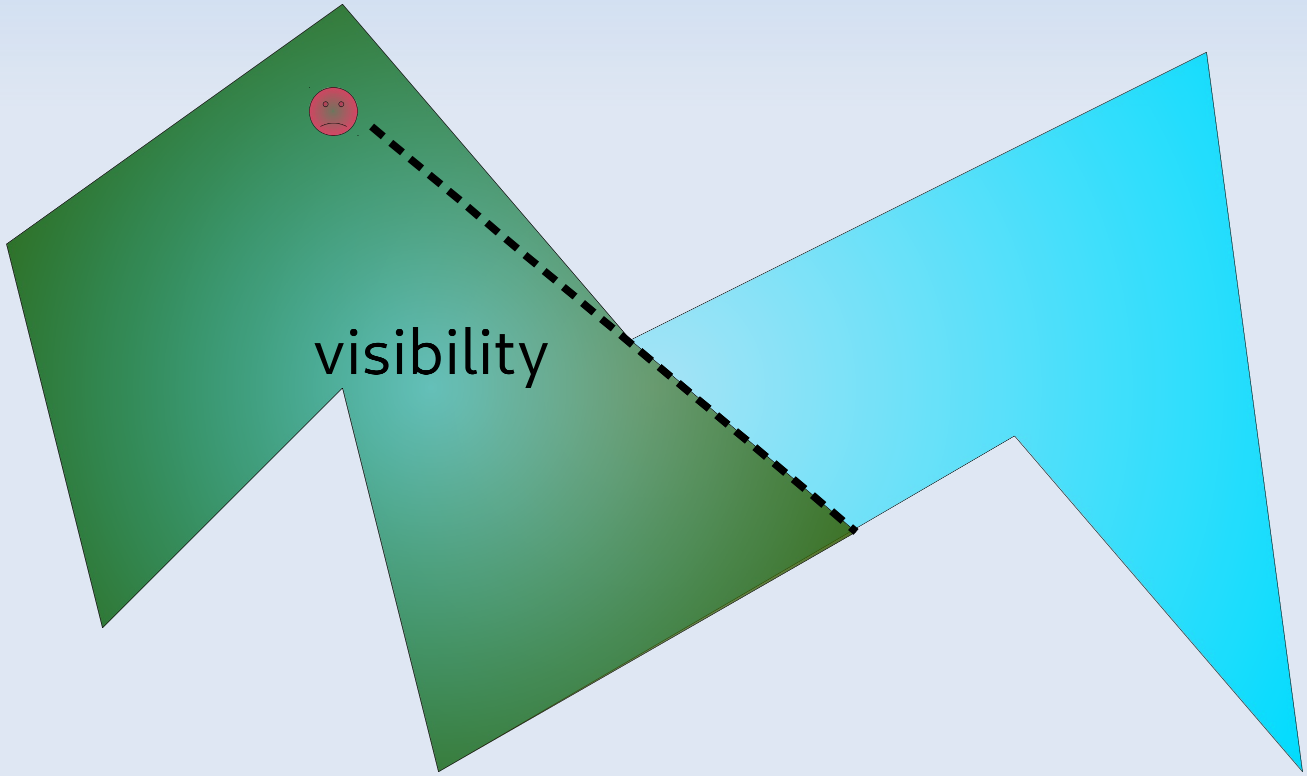
Example

How many guards do you need?



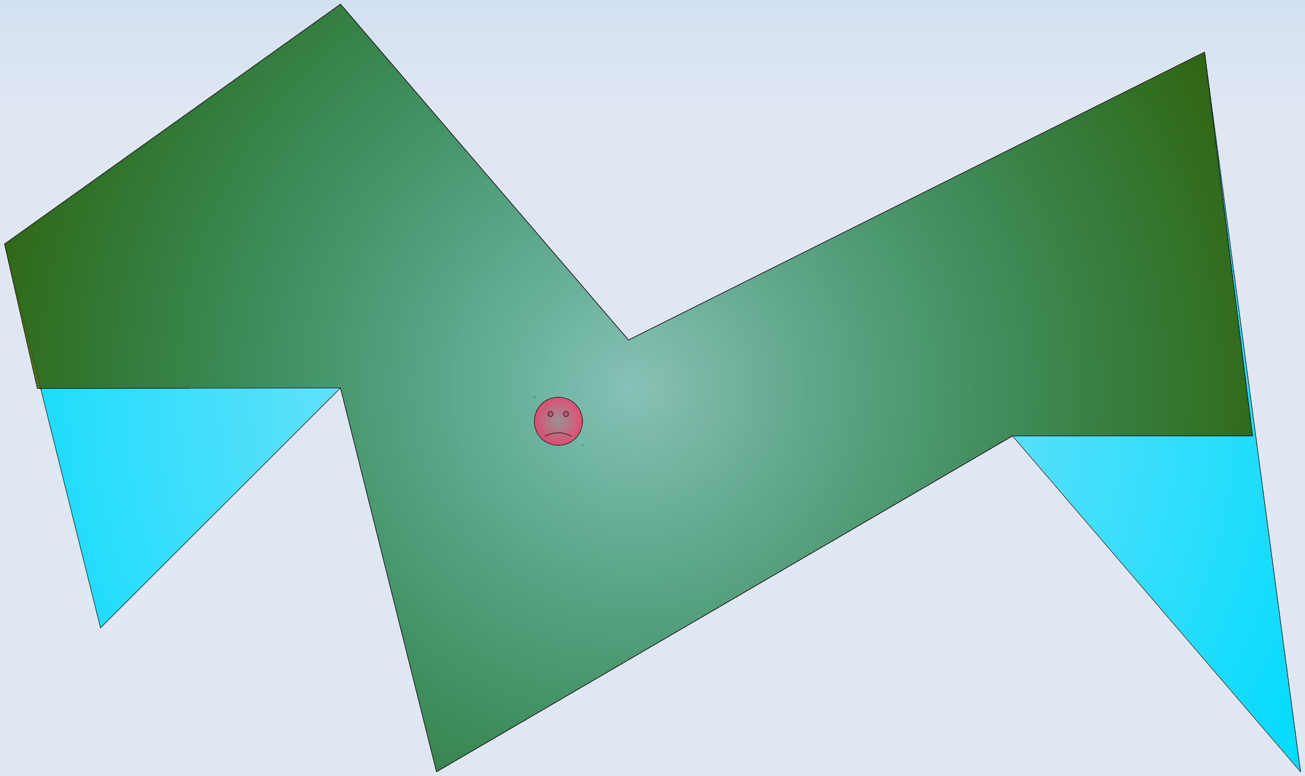
Example

How many guards do you need?



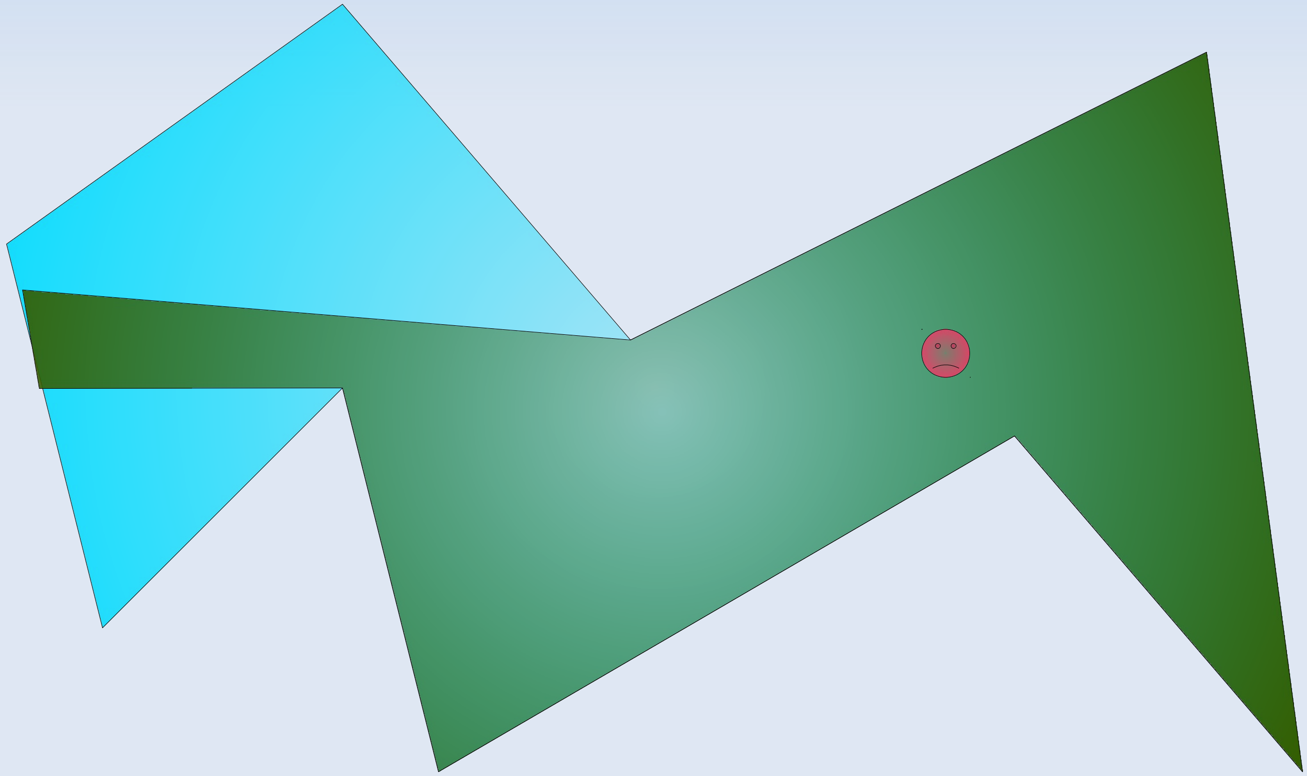
Example

How many guards do you need?



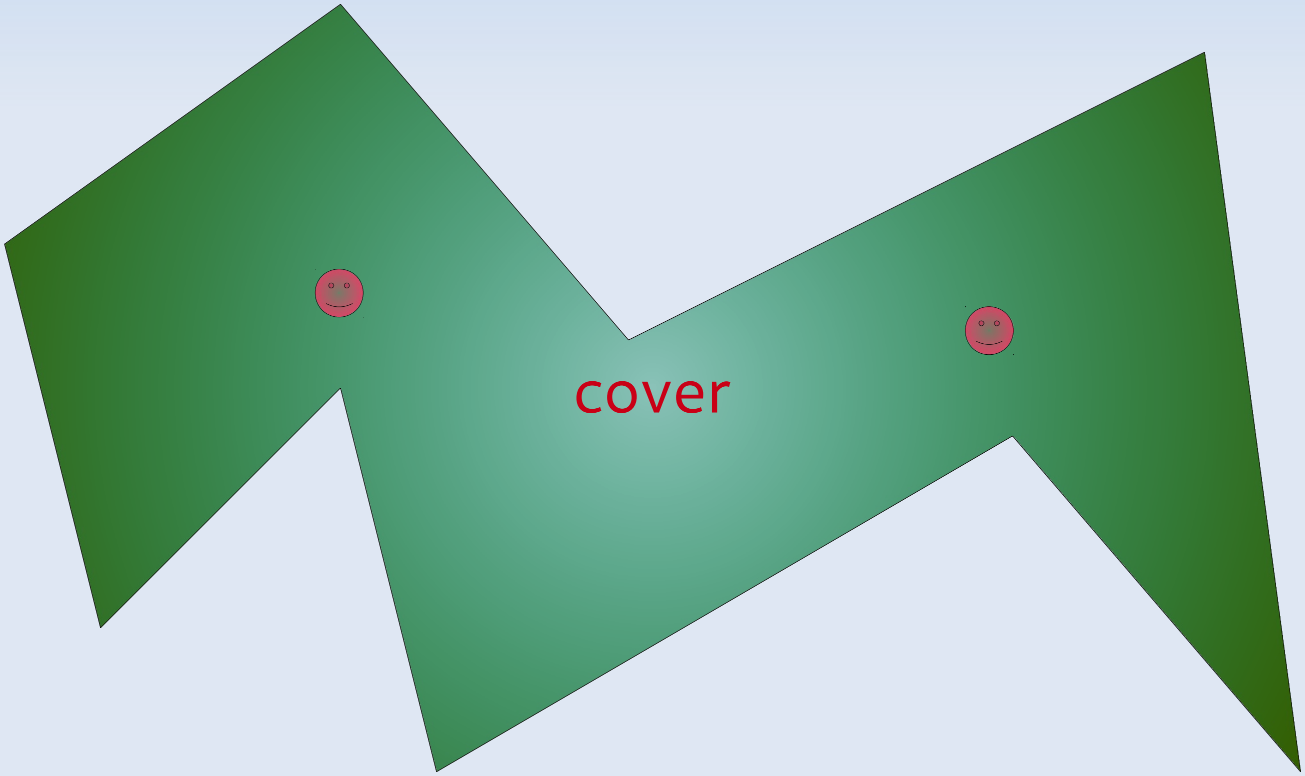
Example

How many guards do you need?

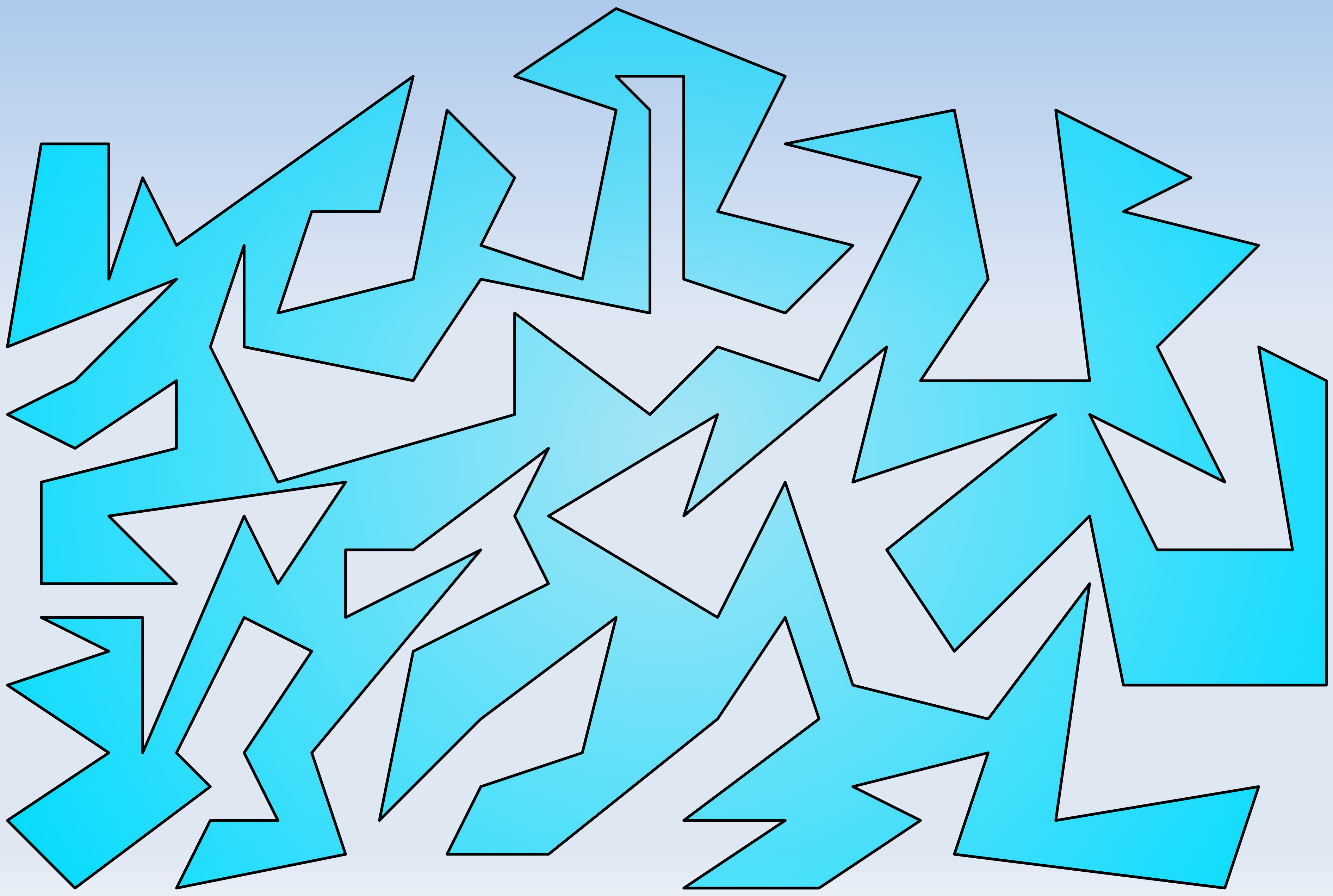


Example

Two guards do the job!



What now?!



Formal definition

How many points do you need to cover a given polygon with n vertices?

How many points are sufficient to cover any polygon with n vertices?

Max over min formulation

- This is called a max over min formulation problem
- Over all polygons of n vertices, find the maximum of minimum number of points needed to cover the polygon
  - Consider all polygons of n vertices
  - Compute the minimum number of points to cover each polygon
  - Calculate the maximum of these minimums

Empirical Exploration

How many points do you need for $n = 3$?

…for $n = 4$?

…for $n = 5$?

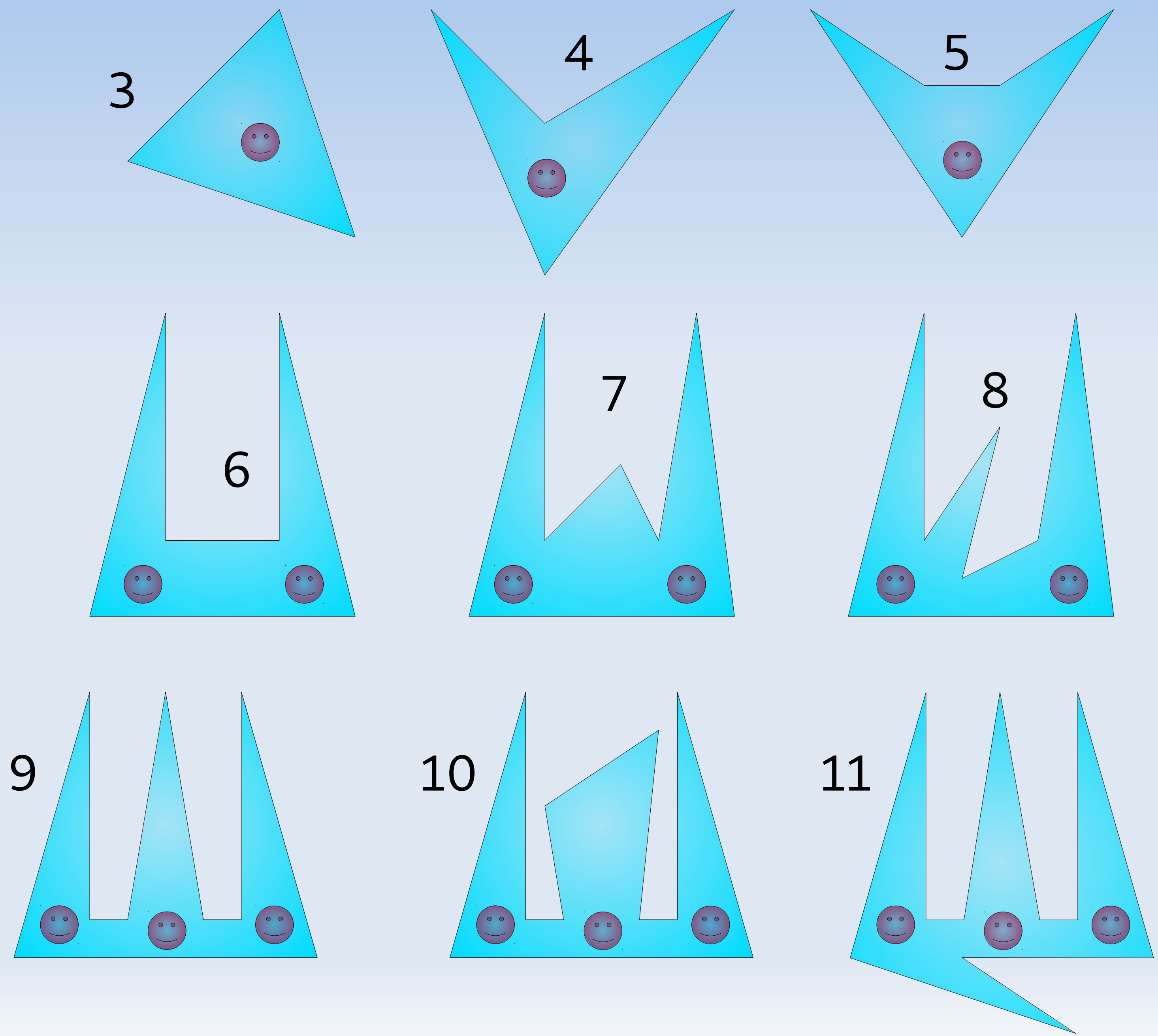
…for $n = 6$?

…for $n = 7$?

…for $n = 8$?

…for $n = 9$?

Empirical Exploration



Empirical Exploration

$n = 3 \rightarrow p = 1$

$n = 4 \rightarrow p = 1$

$n = 5 \rightarrow p = 1$

$n = 6 \rightarrow p = 2$

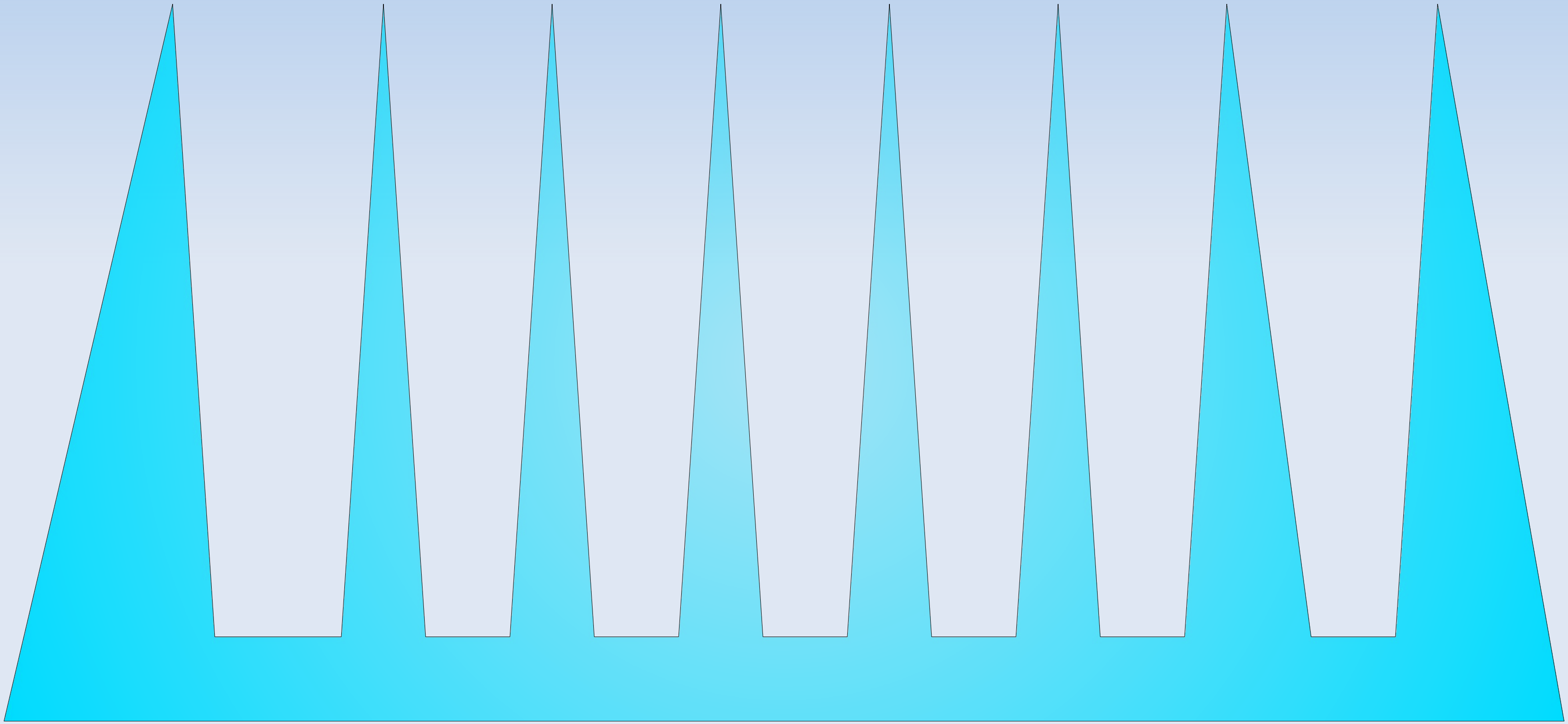
$n = 7 \rightarrow p = 2$

$n = 8 \rightarrow p = 2$

$n = 9 \rightarrow p = 3$

$n \rightarrow p = \text{floor}(n/3)$

Necessity of floor(n/3)

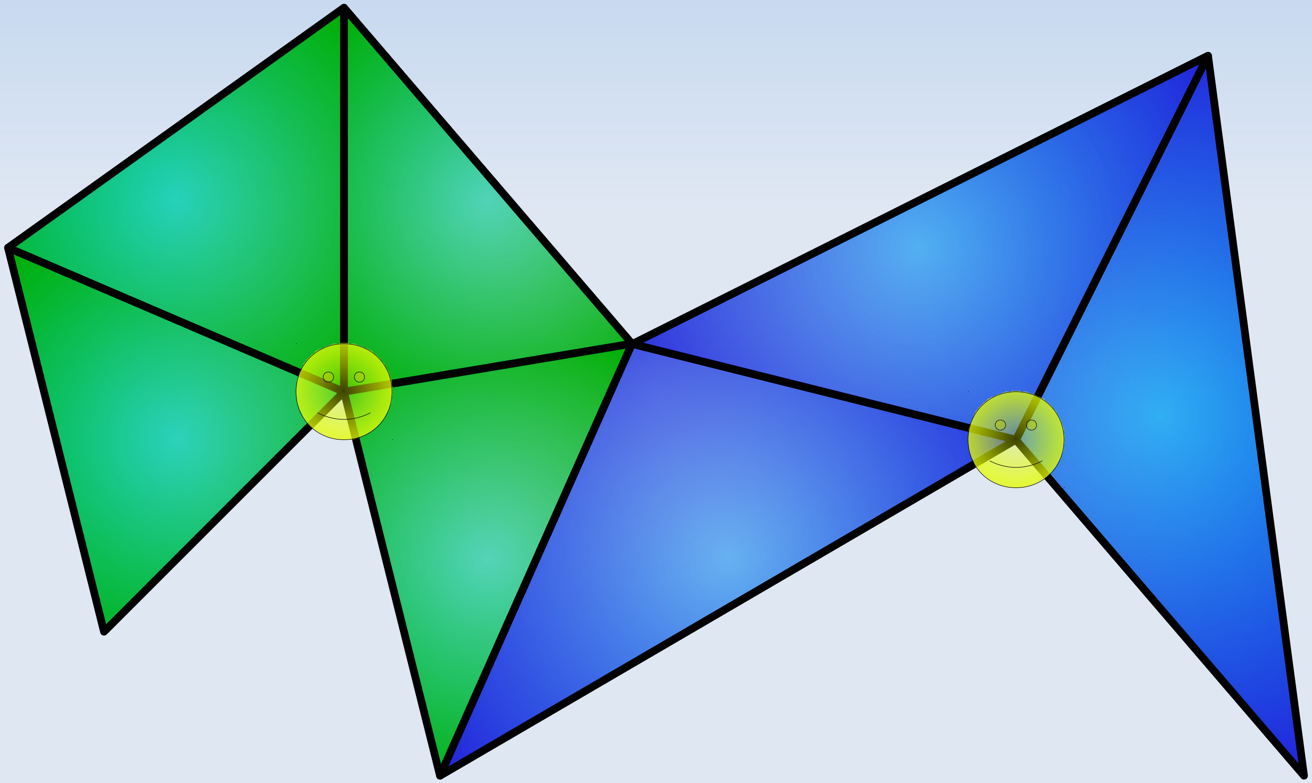


each prong needs 1 guard

Sufficiency

- Fisk proved sufficiency via partitioning the polygon into triangles using diagonals
  - Each triangle can be covered by one guard
  - If k triangles share a vertex, a guard at this vertex covers all k triangles!
- A diagonal of a polygon P is a line segment between two of its vertices which are clearly visible to each other.

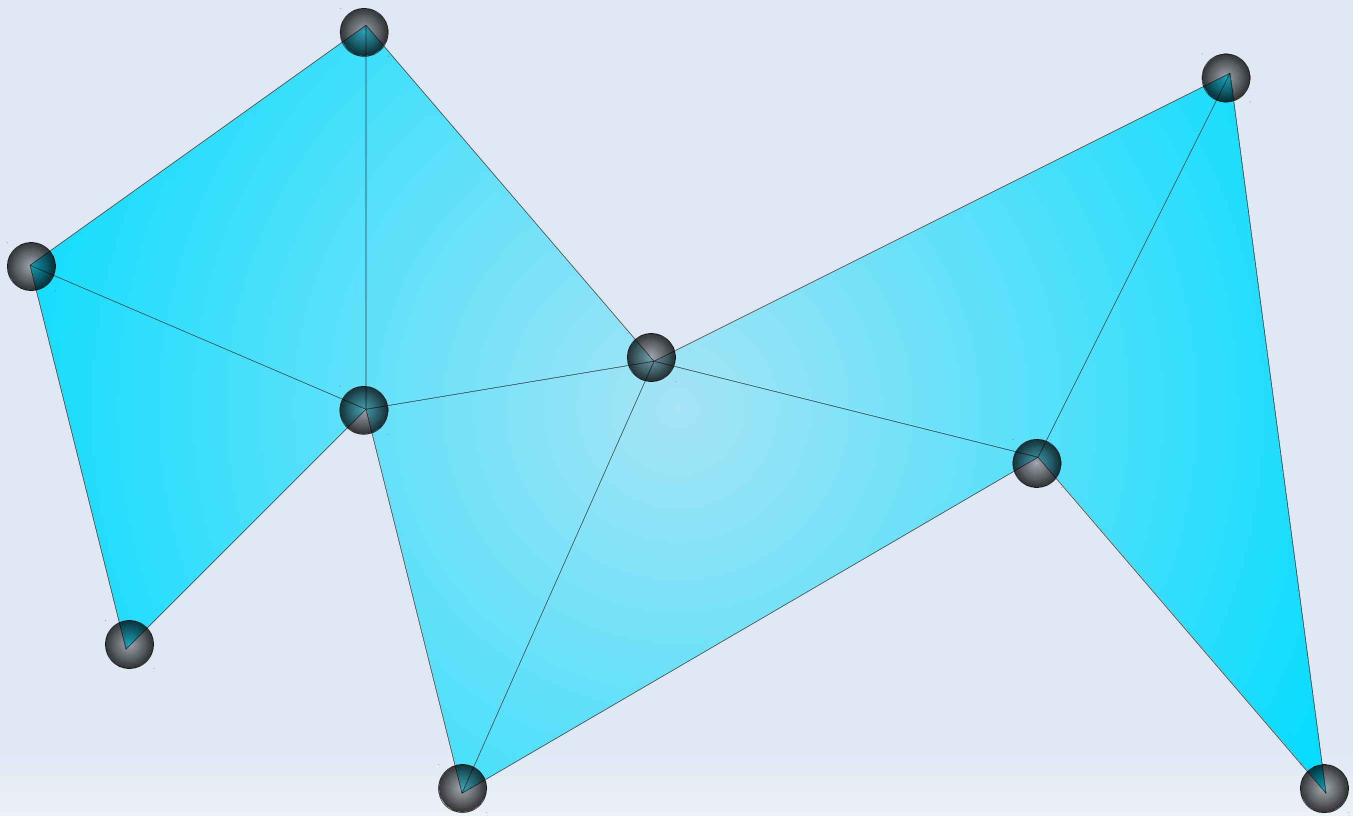
Example



Graph of Triangulation

Define a graph

- nodes are the vertices of the polygon
- arcs are the edges and the diagonals



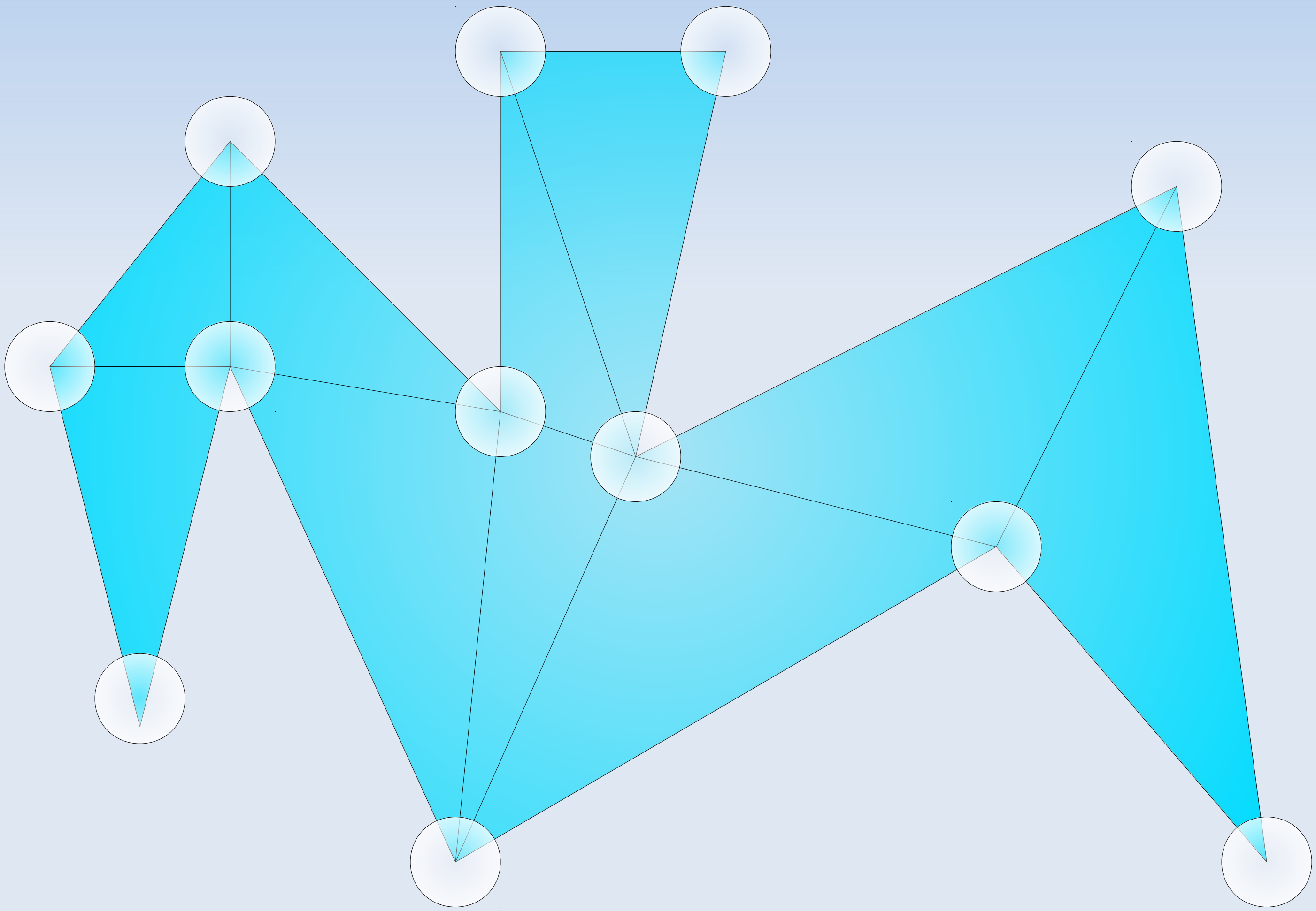
Graph Coloring

Assign a color to each node, such that the end nodes of each edge is colored differently

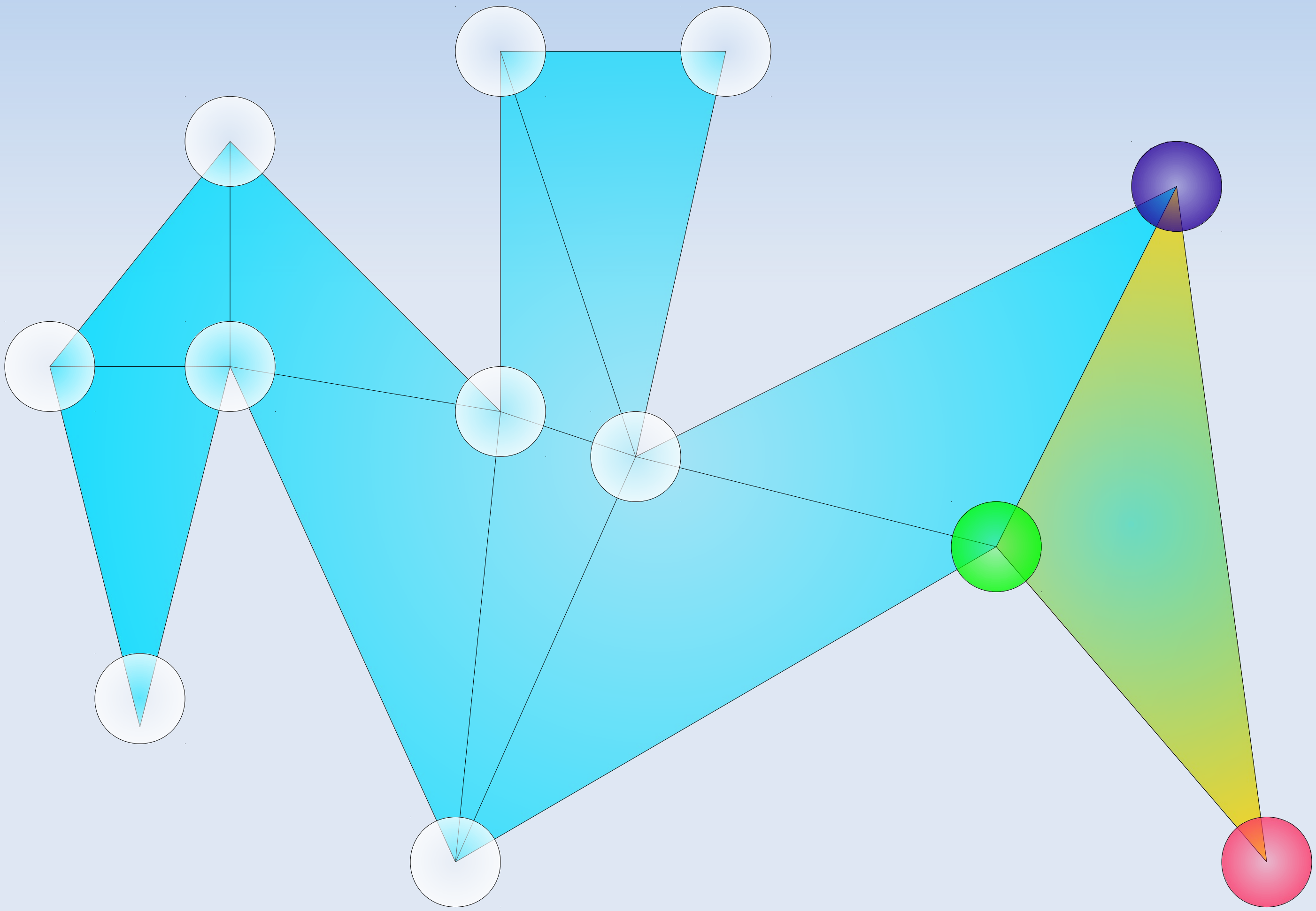
What is the minimum number of colors you need for a given graph G?

Fisk showed that every polygon graph can be 3-colored!

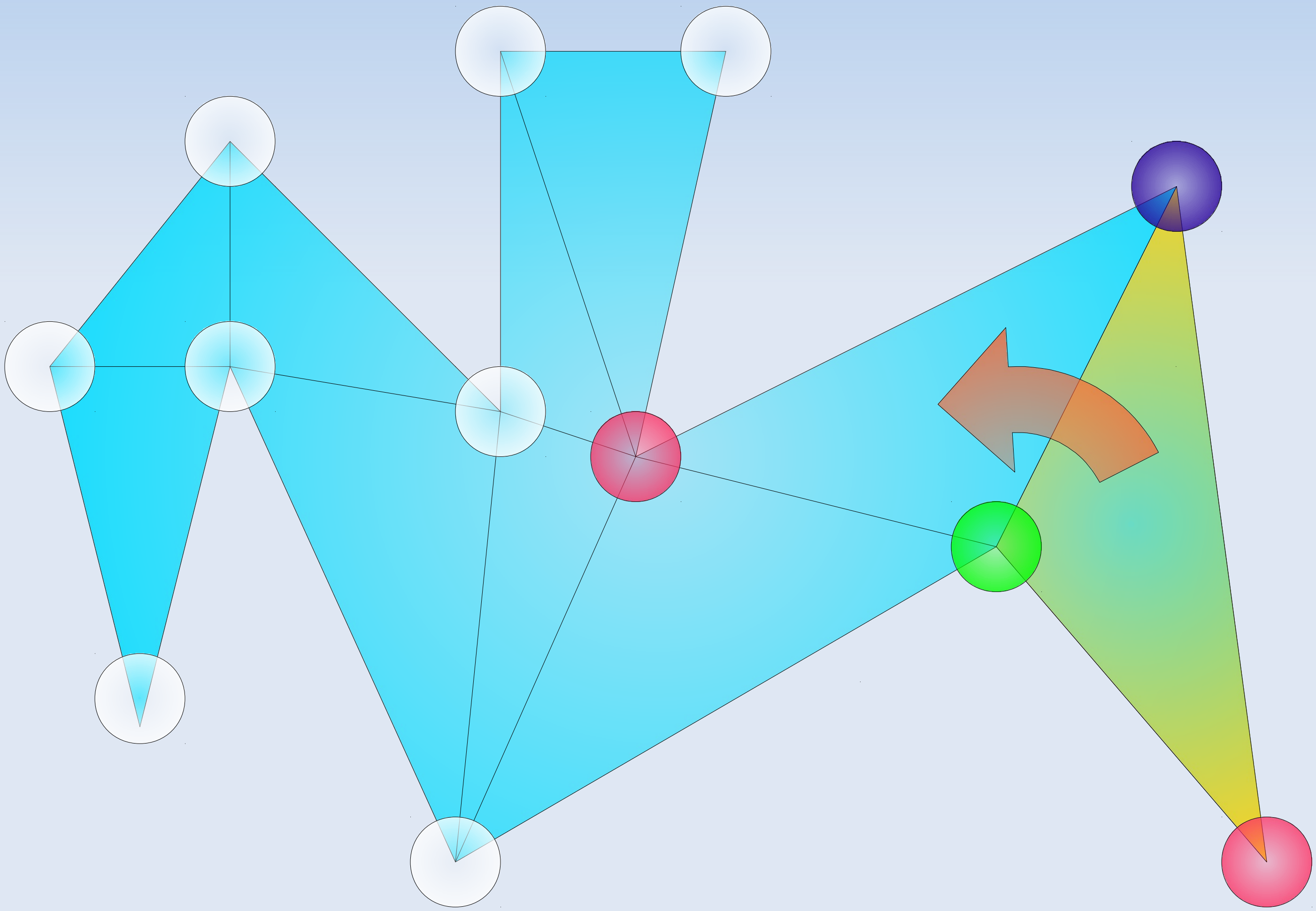
Example



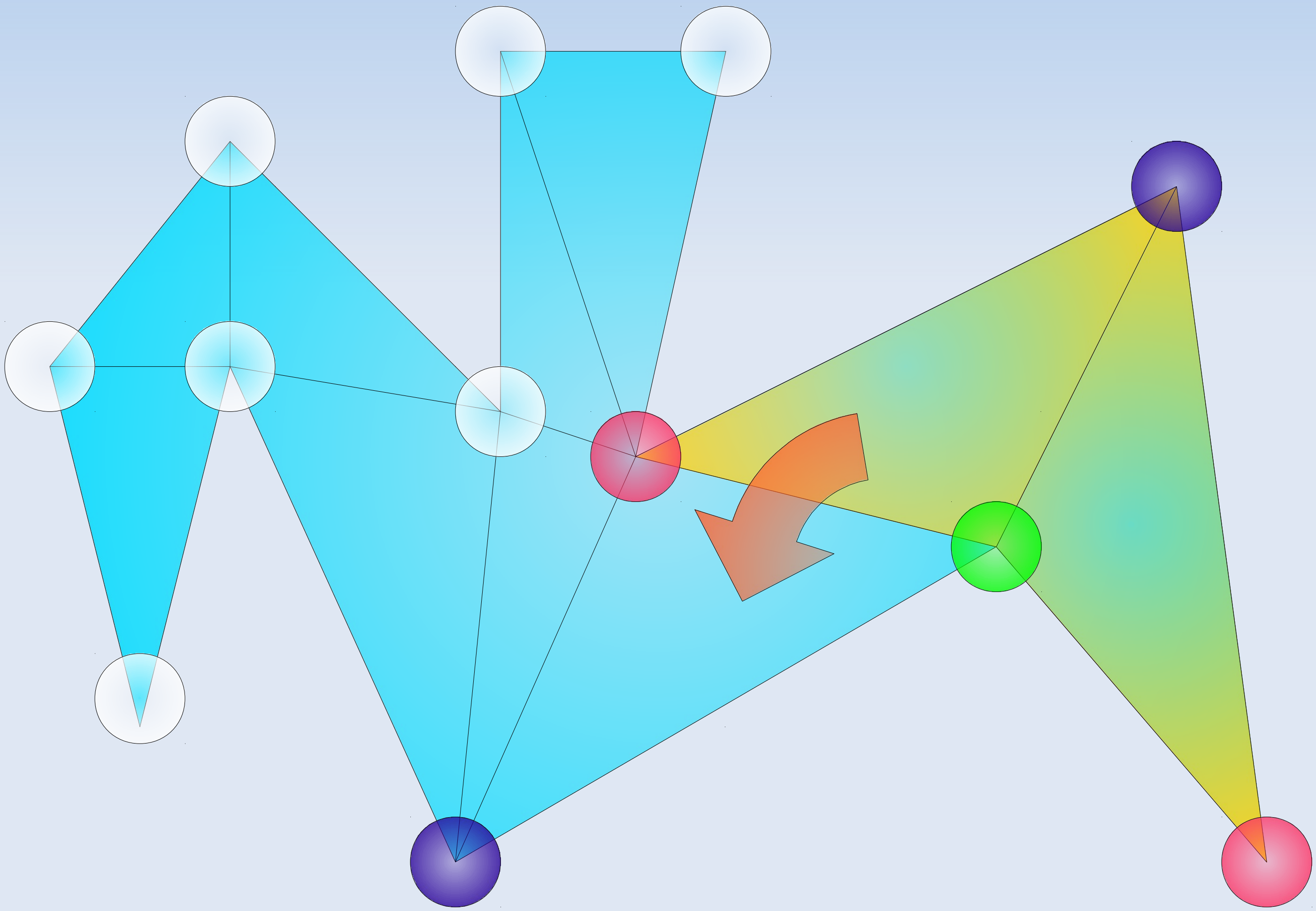
Example



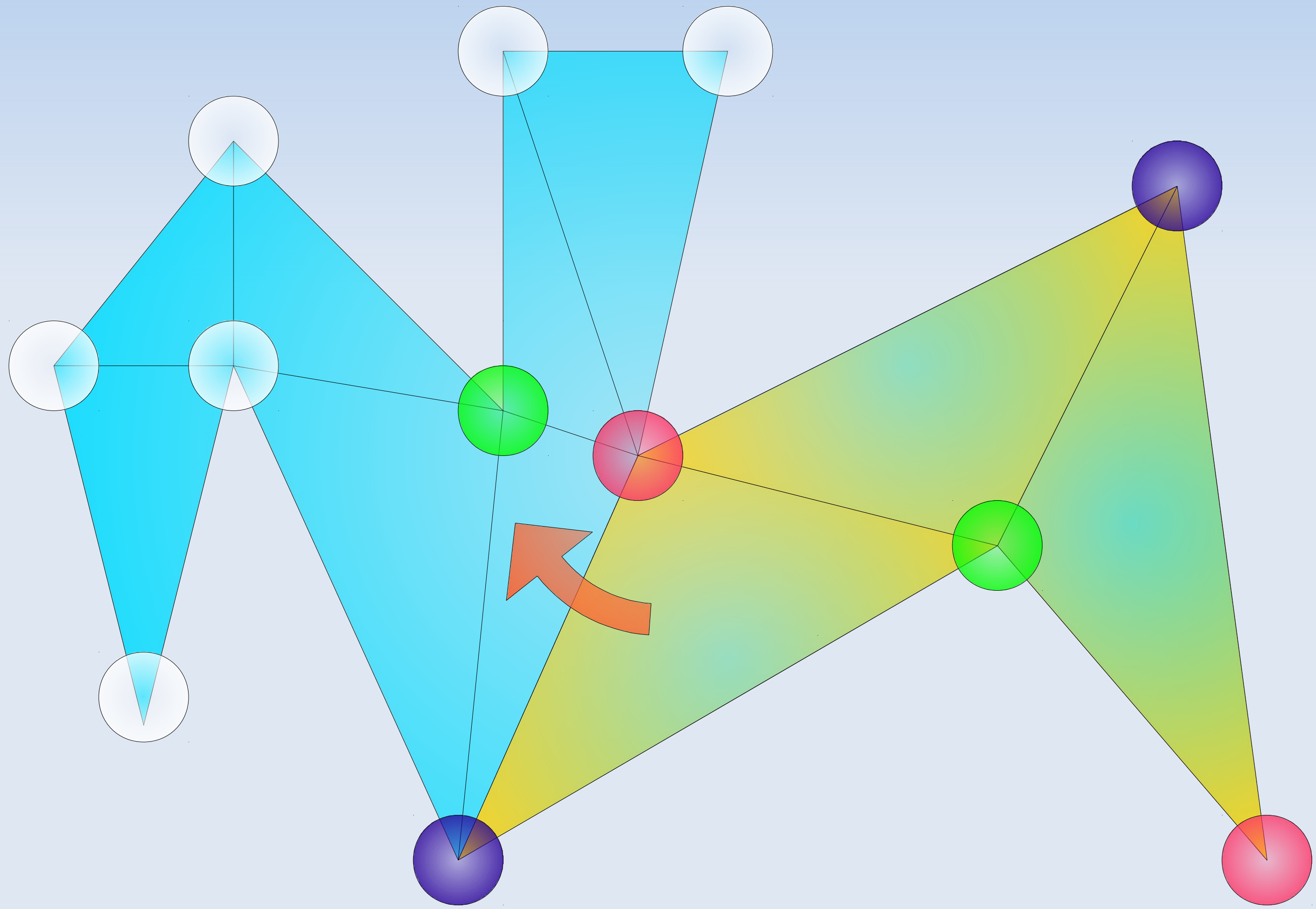
Example



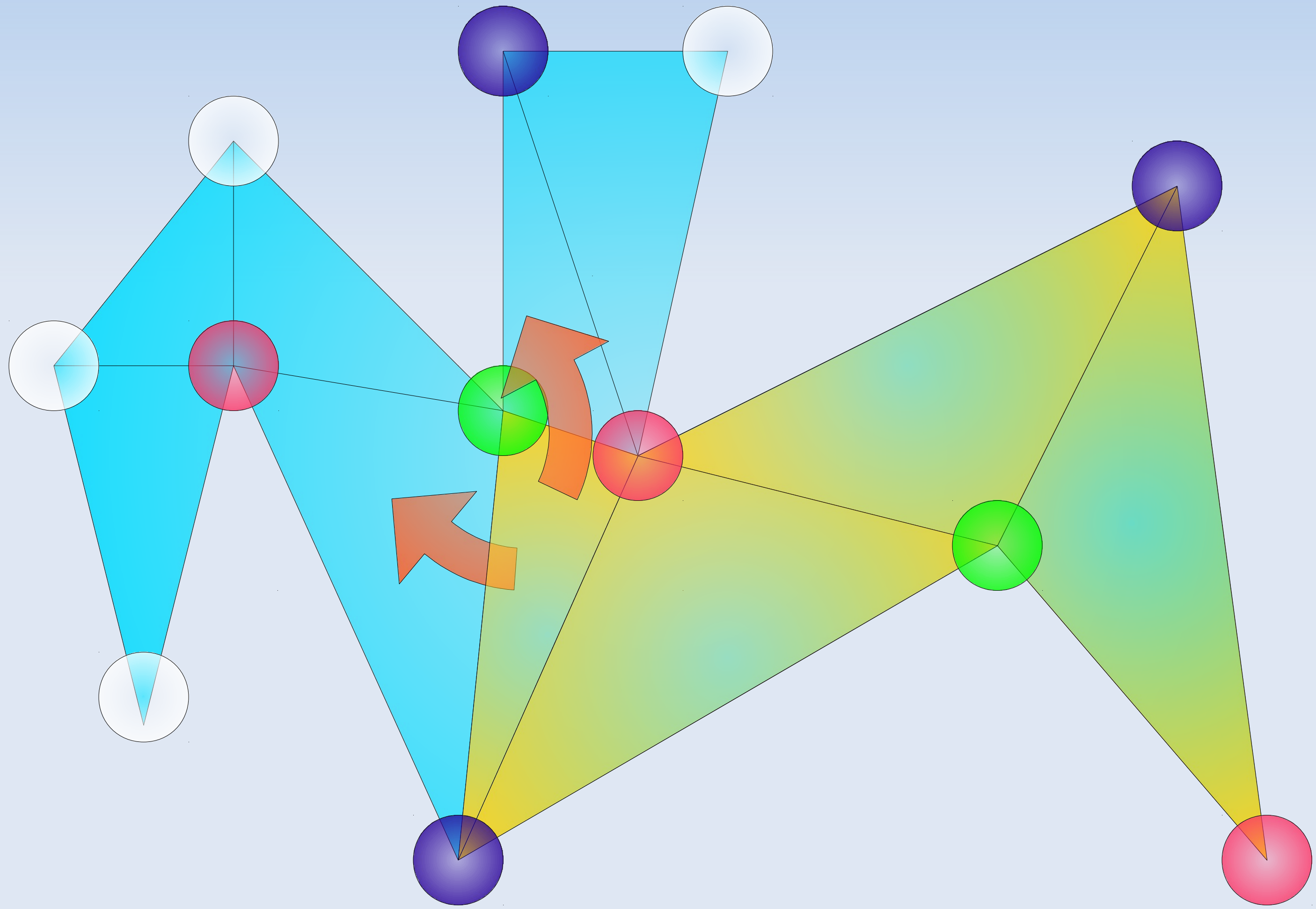
Example



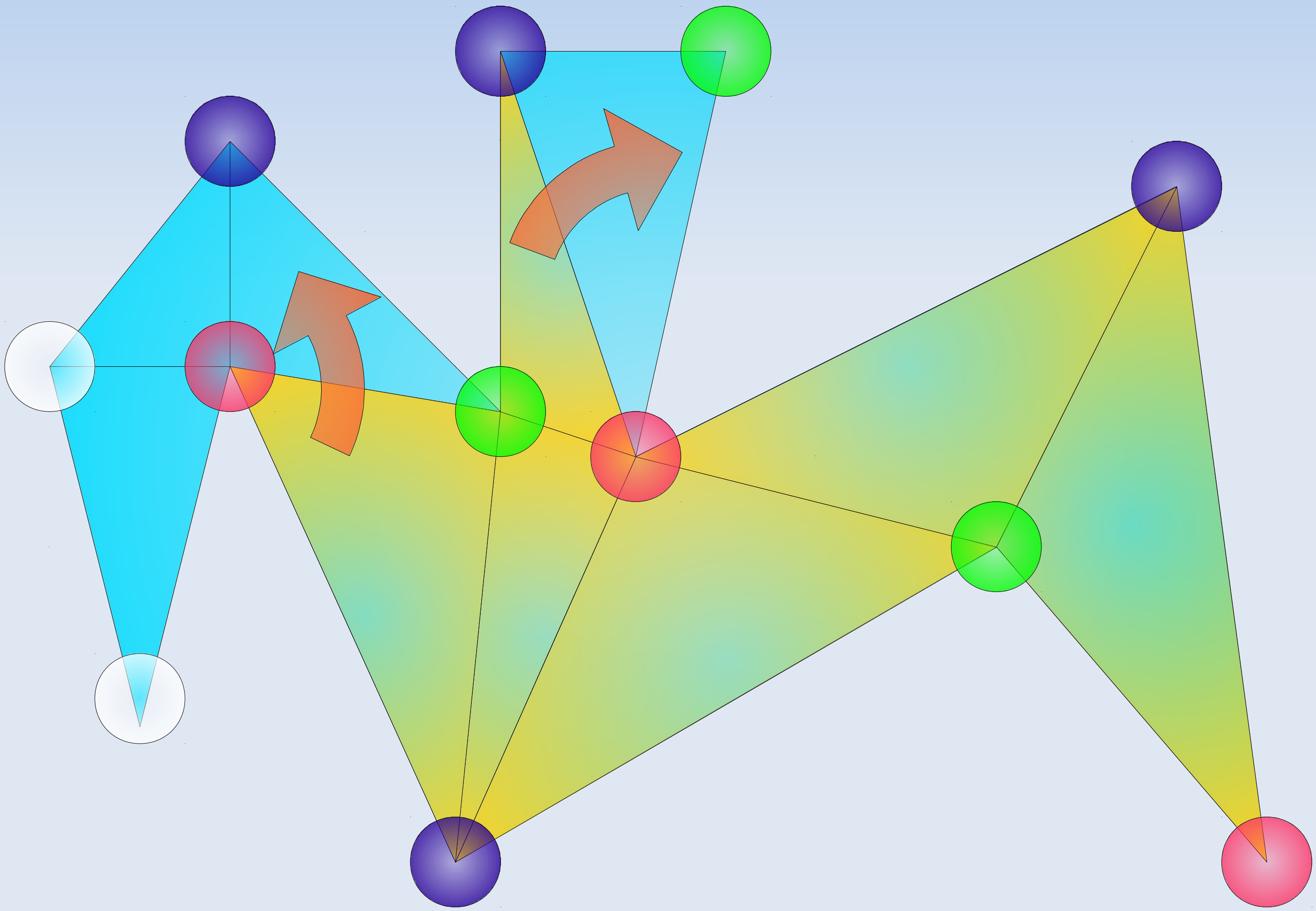
Example



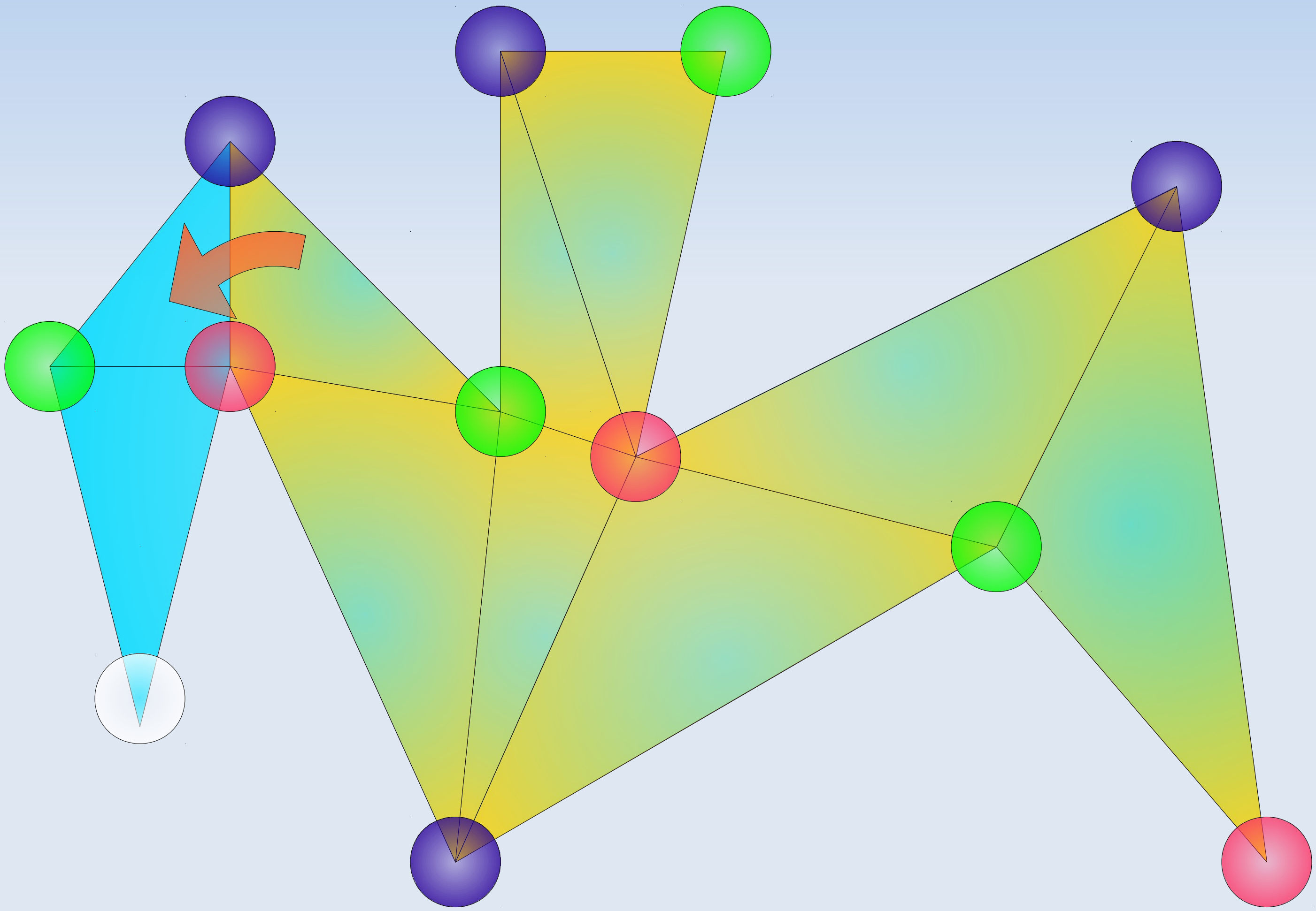
Example



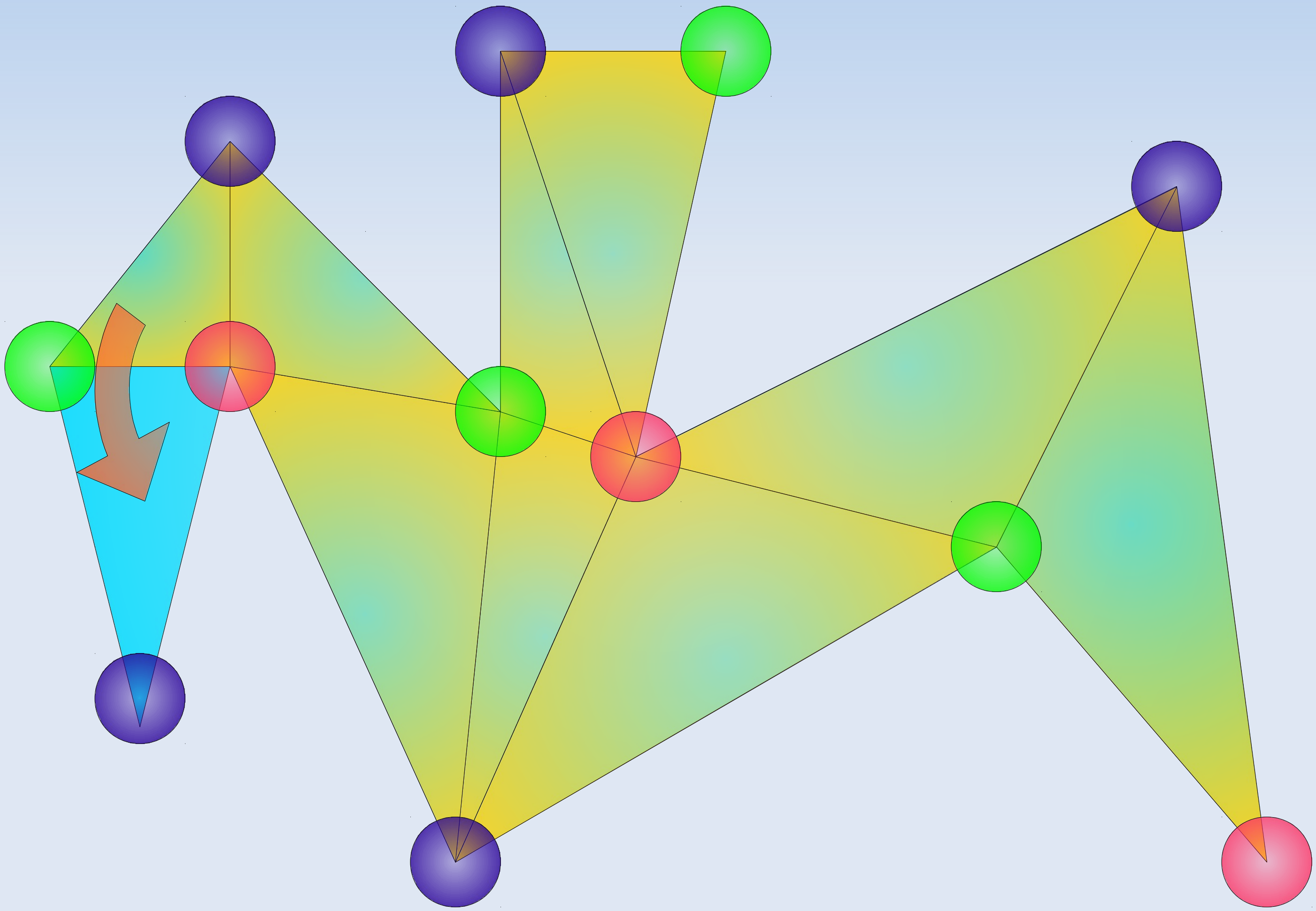
Example



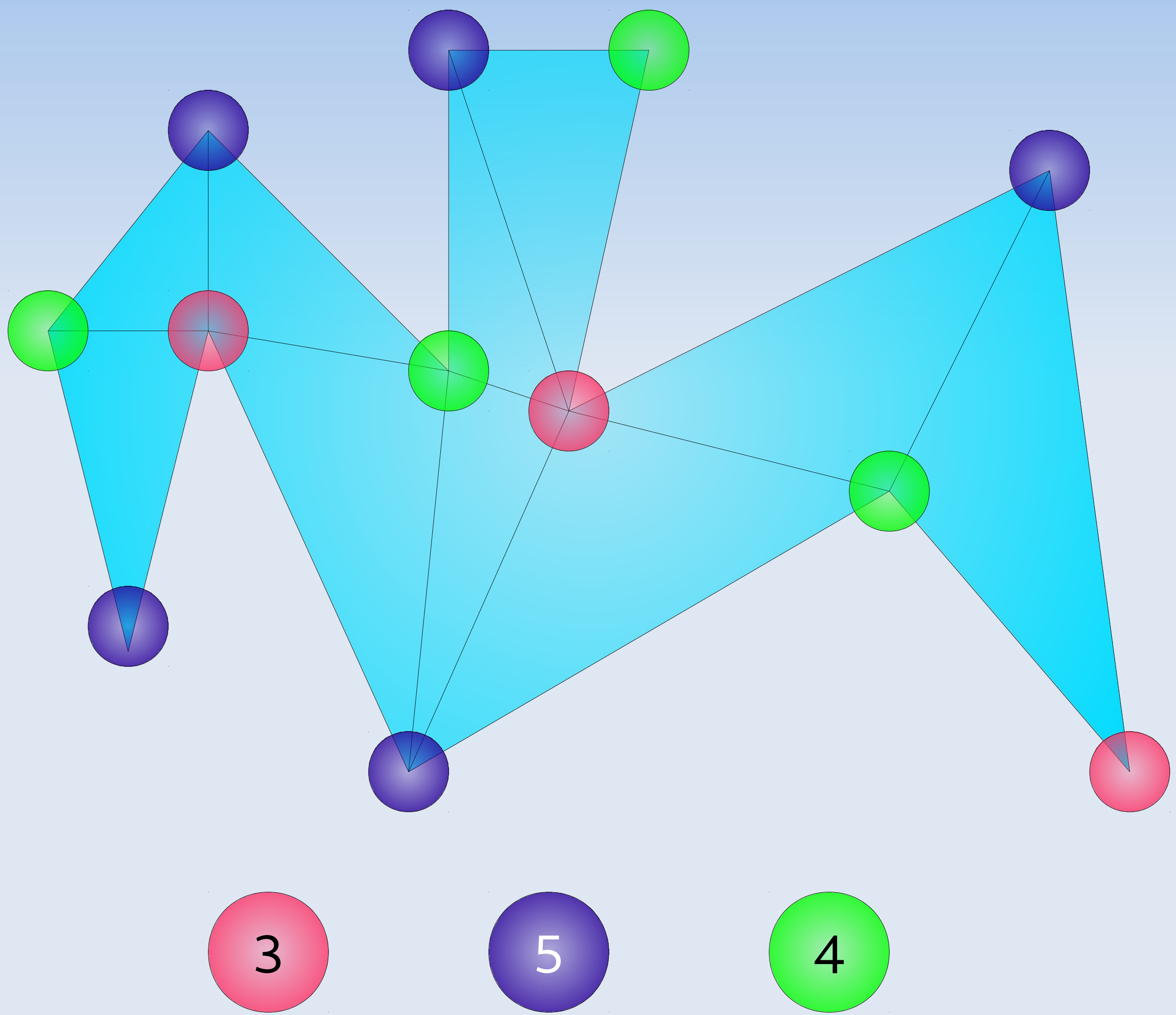
Example



Example

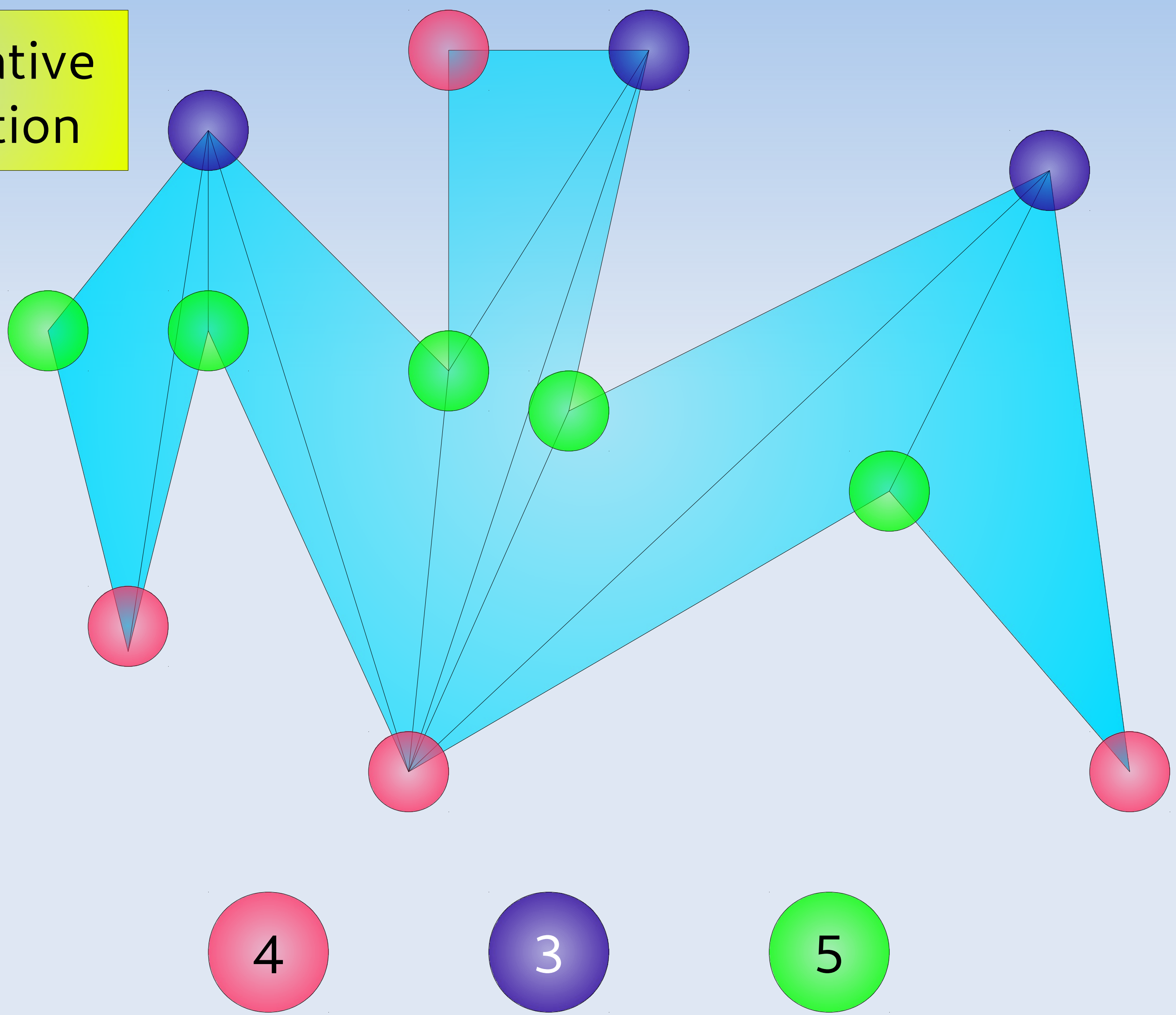


Example



Example

An alternative triangulation



Pigeon Hole

- $r + g + b = 12$
- Pigeon-hole theorem
  - one of r, g, b must be ≤ 4 [floor(12/3)]
- Place a guard at the vertices with that color!!!
  - Number of guards ≤ 4
  - Each triangle has its own guard!

Triangulation Theory

Every polygon has a triangulation

Establish basic properties of triangulations

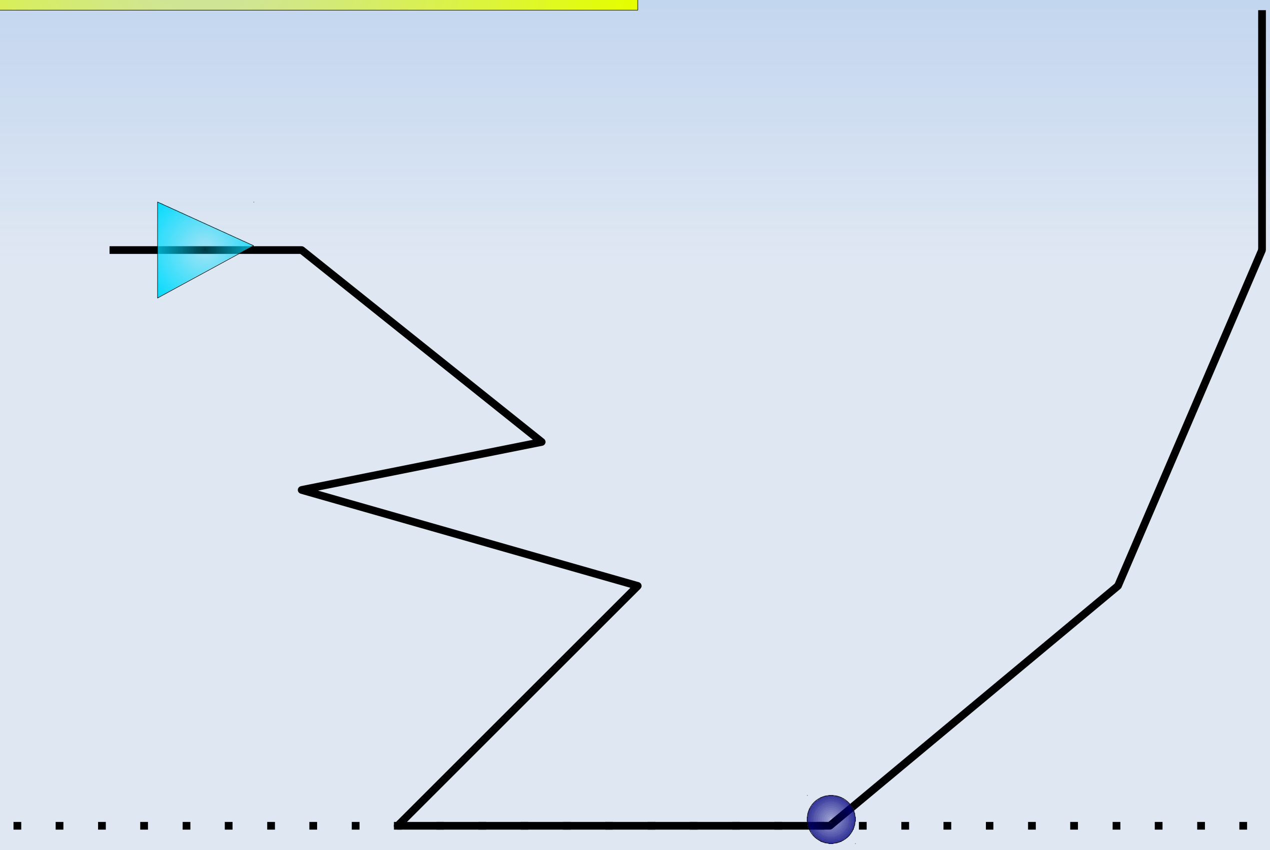
Algorithms for constructing triangulations

Existence of a Diagonal

- First show that
  - Every polygon must have at least one strictly convex vertex
    - A vertex with angle < 180

Existence of a Convex Vertex

a convex vertex is a left turn

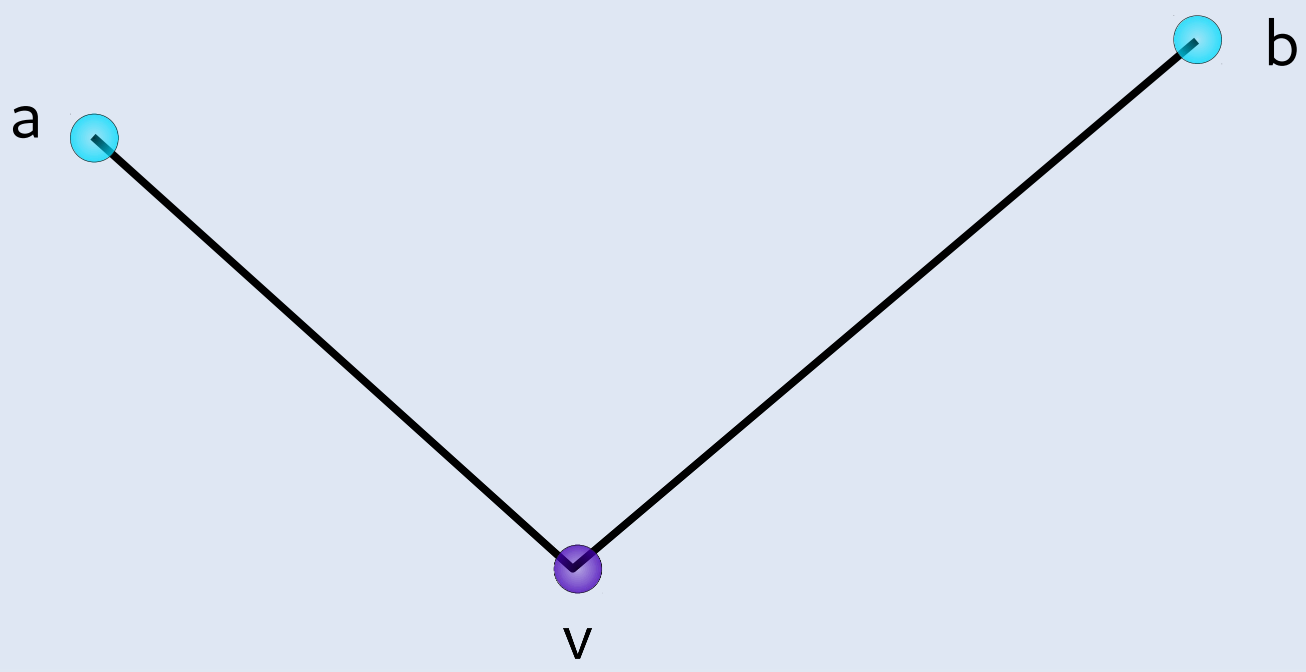


v = vertex with
minimum y coordinate

Existence of a Diagonal

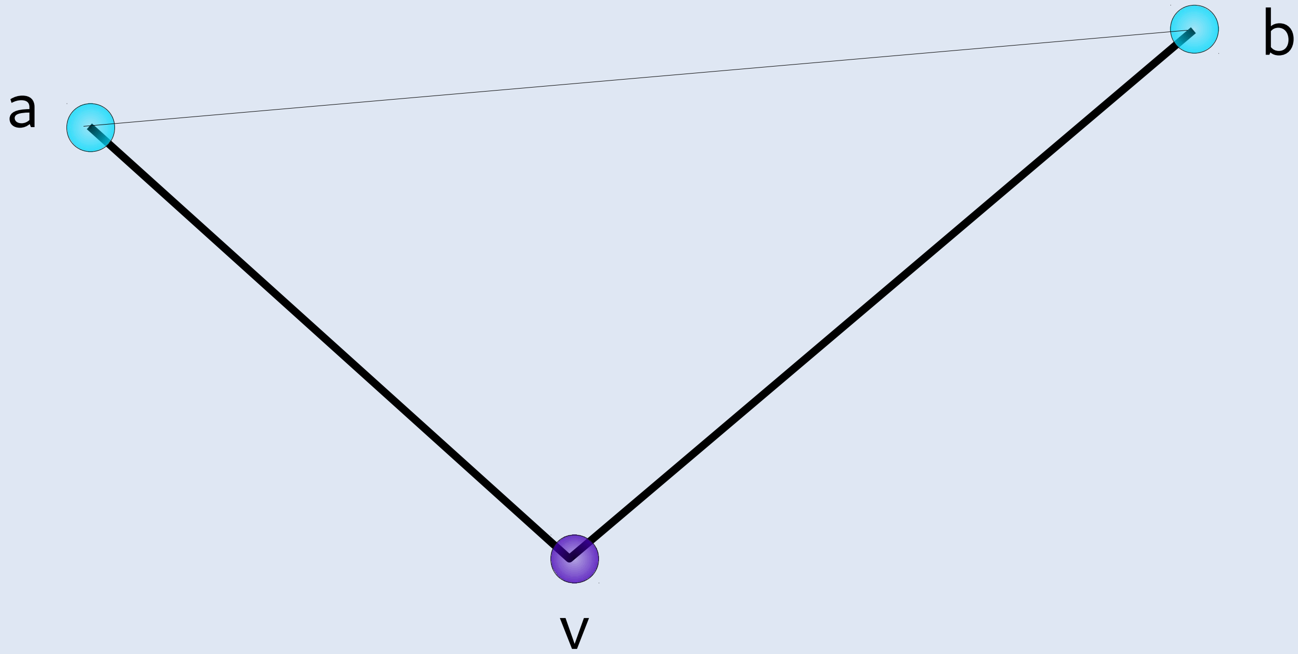
Every polygon of $n \geq 4$ vertices has a diagonal

- Let v be a convex vertex
- Let a and b be the adjacent vertices



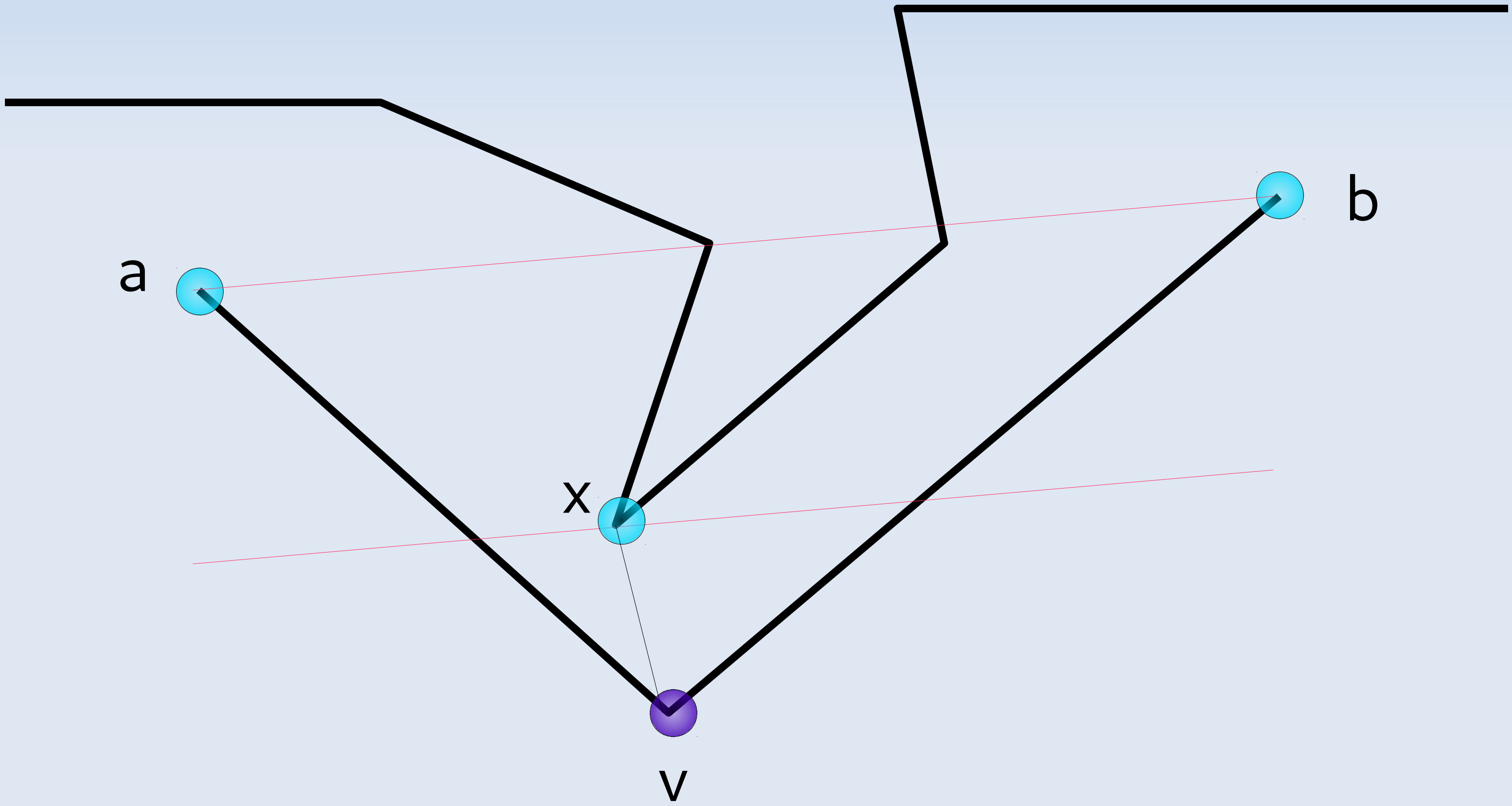
Looking for a Diagonal

If ab is already a diagonal, we are done



Looking for a Diagonal

If ab is not a diagonal, then xv is a diagonal



Triangulation

Theorem

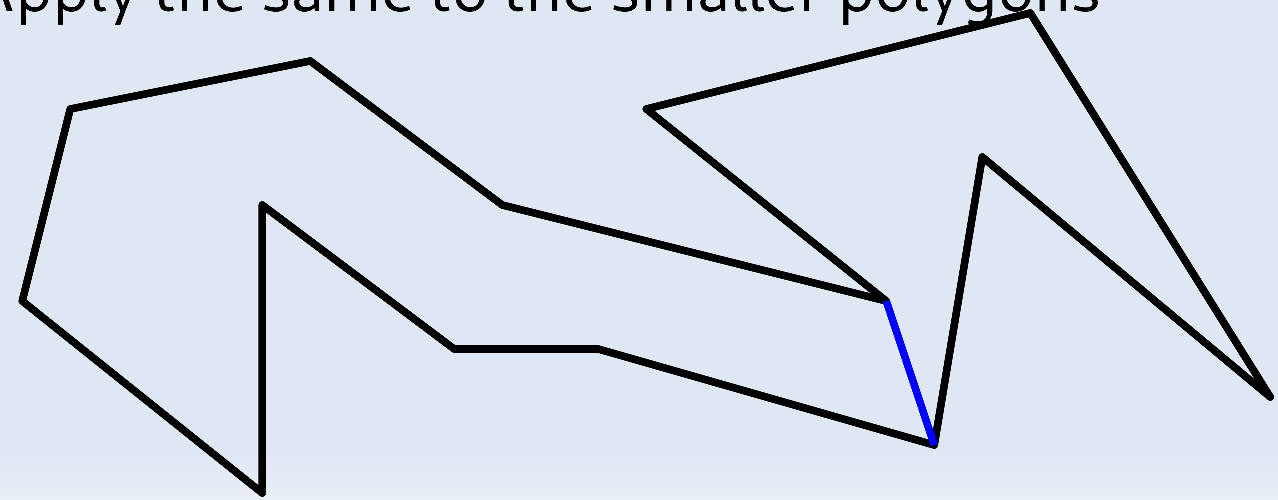
Every polygon with n vertices may be partitioned into triangles by the addition of (zero or more) diagonals

Proof

If n = 3, we are done

For n ≥ 4,

- Let d = ab be a diagonal
  - We showed that d exists
- d partitions the polygon into two smaller polygons
  - Apply the same to the smaller polygons



Properties of Triangulations

Number of Diagonals

Given a triangulation of a polygon with n vertices

how many diagonals do exist?

how many triangles do exist?

Sum of Angles

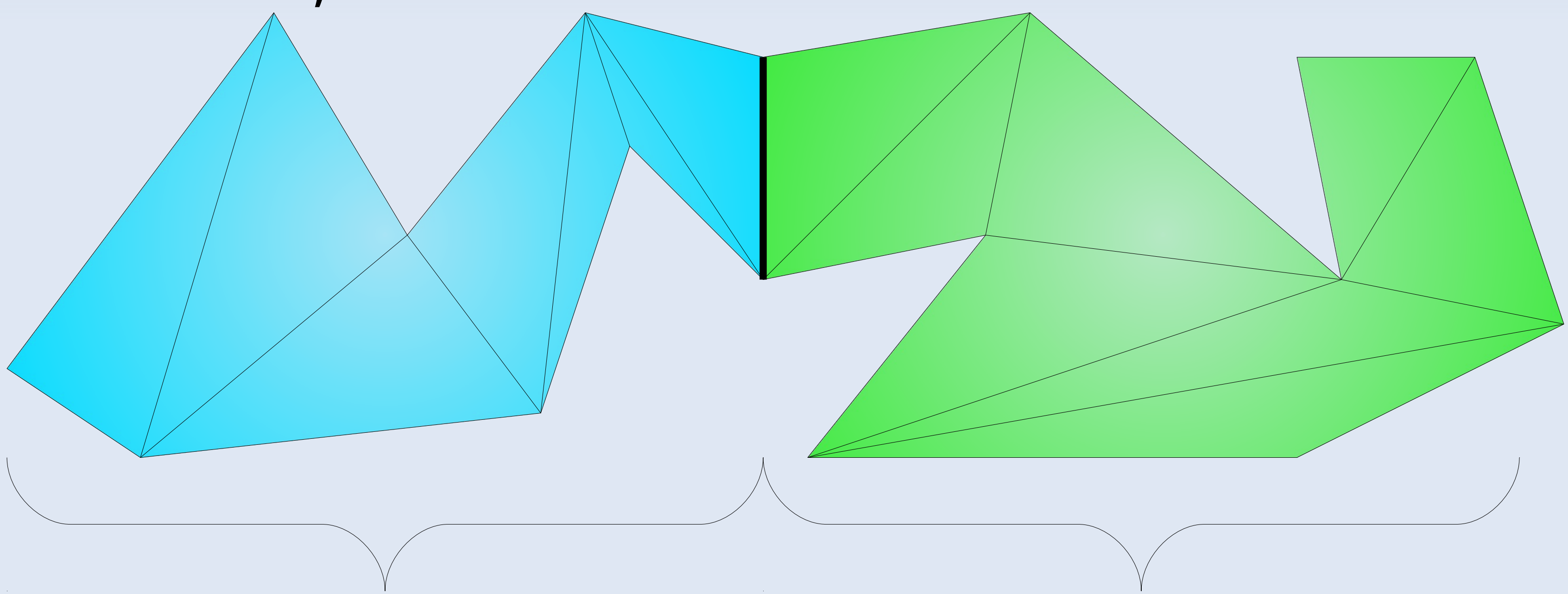
Given a polygon with n vertices, what is the sum of internal angles?

Number of Diagonals & Triangles

Every triangulation of a polygon of n vertices has n-3 diagonals and n-2 triangles

Holds for $n = 3$ (0 diagonals, 1 triangle)

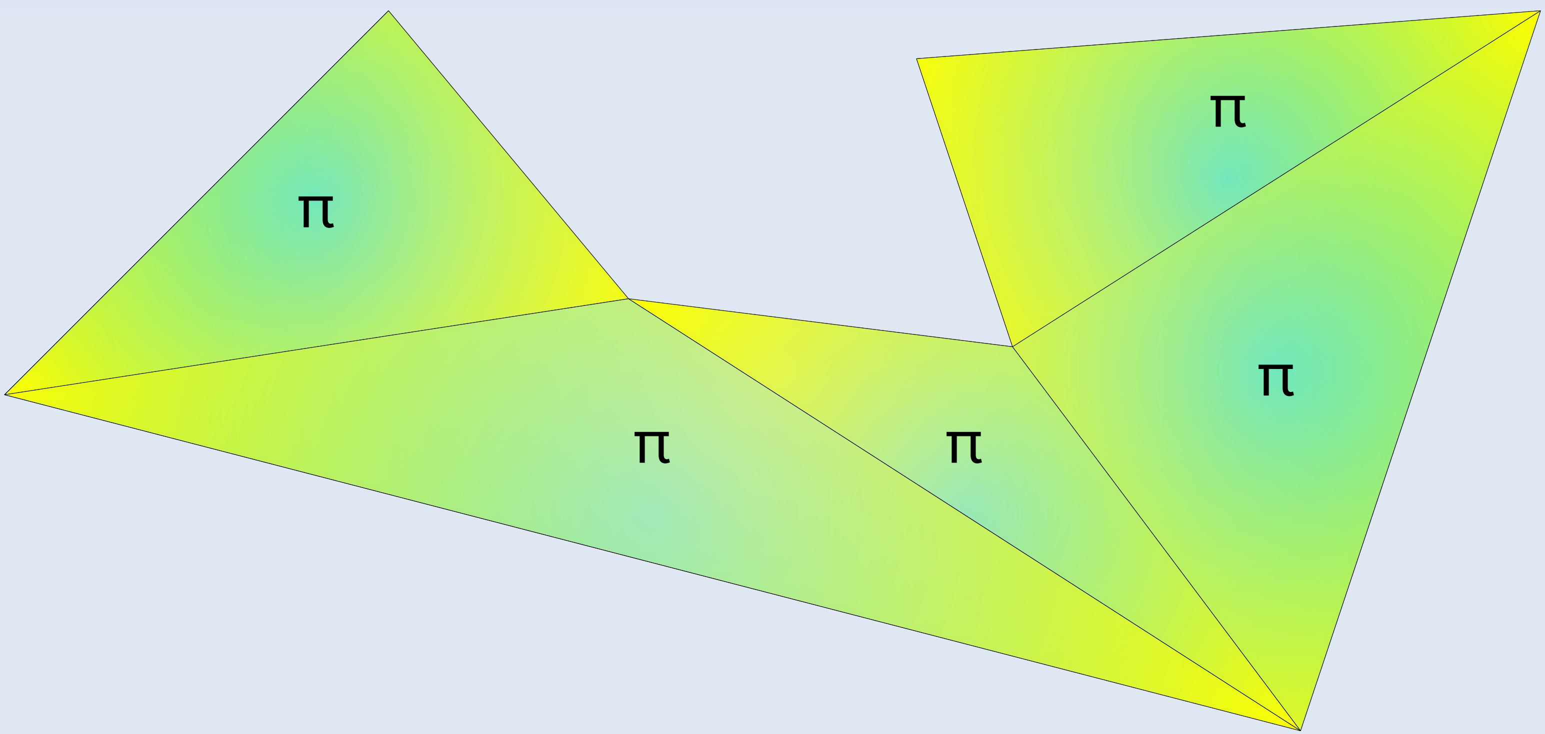
For $n \geq 4$,



Vertices	n_1	+	n_2	$= n + 2$
Diagonals	$n_1 - 3$	+ 1 +	$n_2 - 3$	$= n - 3$
Triangles	$n_1 - 2$	+	$n_2 - 2$	$= n - 2$

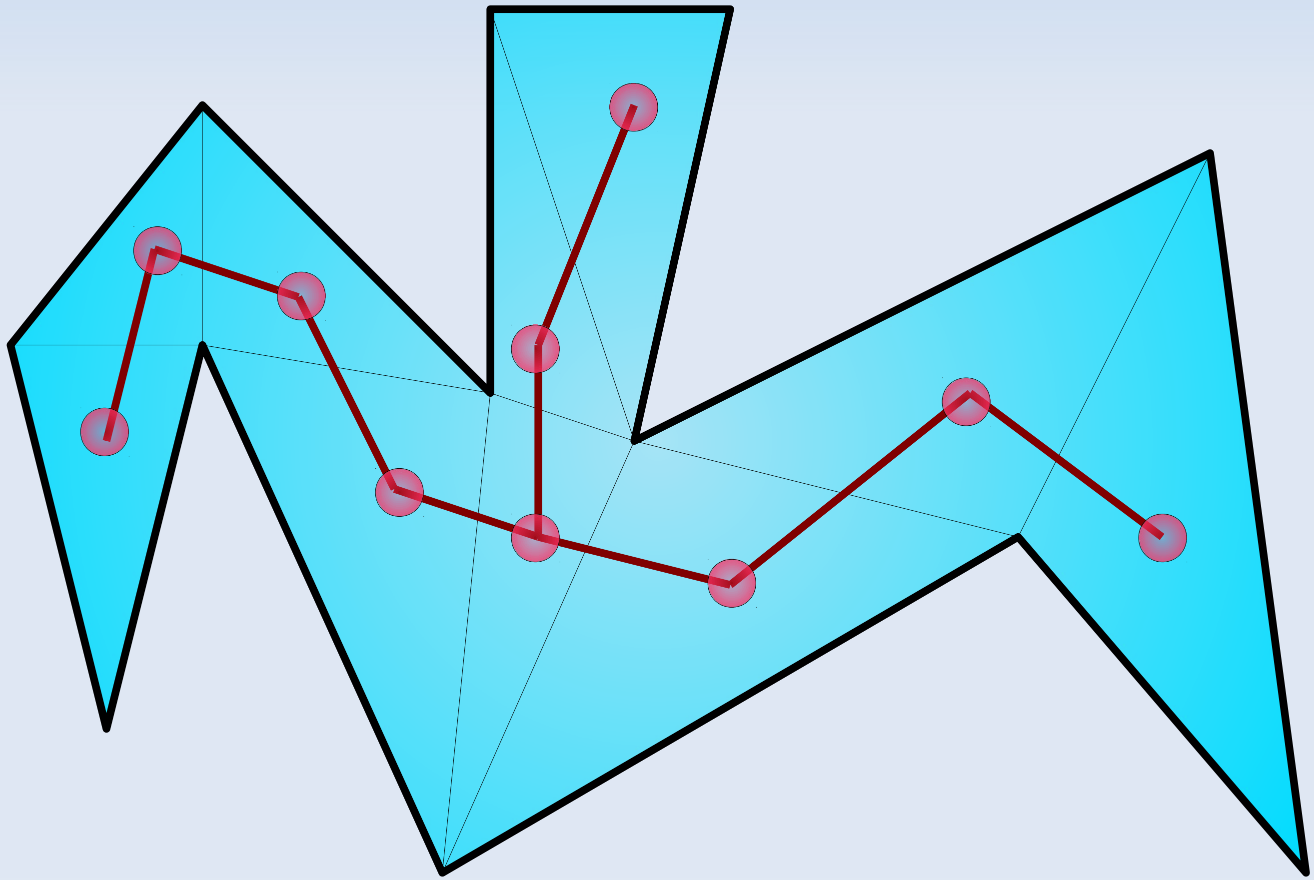
Sum of Internal Angles

The sum of the internal angles of a polygon of n vertices is $(n-2)\pi$



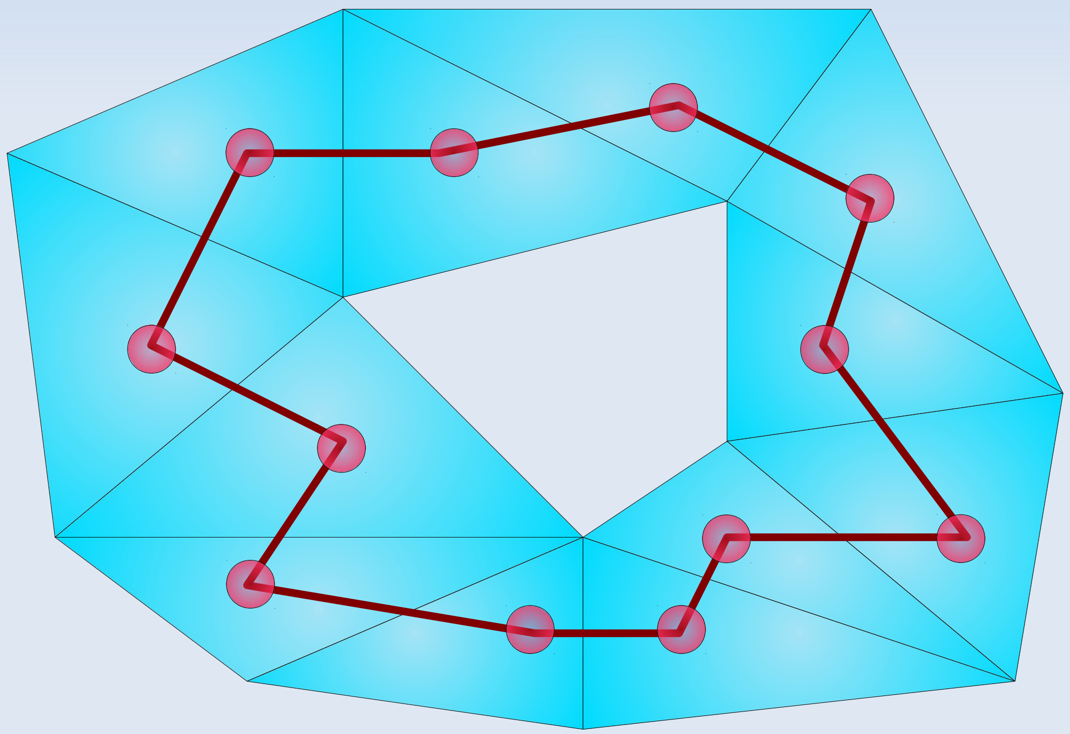
Triangulation Dual

The dual T of a triangulation is a tree, with each node of degree at most 3



Triangulation Dual

If T is not a tree, then a cycle must exist



But then the boundary is disconnected!

Two Ears Theorem

Three consecutive vertices a,b,c of a polygon form an ear if ac is a diagonal

Theorem: Every polygon of $n \geq 4$ vertices has at least two nonoverlapping ears.

- Proof: The triangulation dual has at least two leaves!
  - Each leaf is an ear

3-coloring a Triangulation Graph

- The triangulation graph of a polygon P of n vertices can be 3-colored
  - Proof by induction
    - A triangle is 3-colorable
    - Assume every triangulation graph of n-1 vertices is 3-colorable
    - By Two Ears Theorem, there is an ear abc
      - Remove b
      - Rest is 3-colorable
      - Color b according to a and c

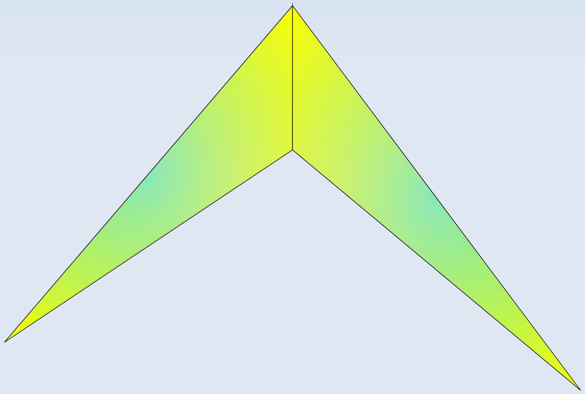
End of Lecture

QUIZ TIME!!!

QUIZ

Construct a polygon P with at least 8 vertices

- such that there is only one possible triangulation
- triangulate P



An example with 4 vertices

Make sure that your drawing is clear

- Edges must be straight
- Vertices must be emphasised