Photo by Arthur Gretton, CMU Machine Learning Protestors at G20

AIN311

Fundamentals of Machine Learning

Lecture 15: Support Vector Machines

HACETTEPE UNIVERSITY COMPUTER VISION LAB

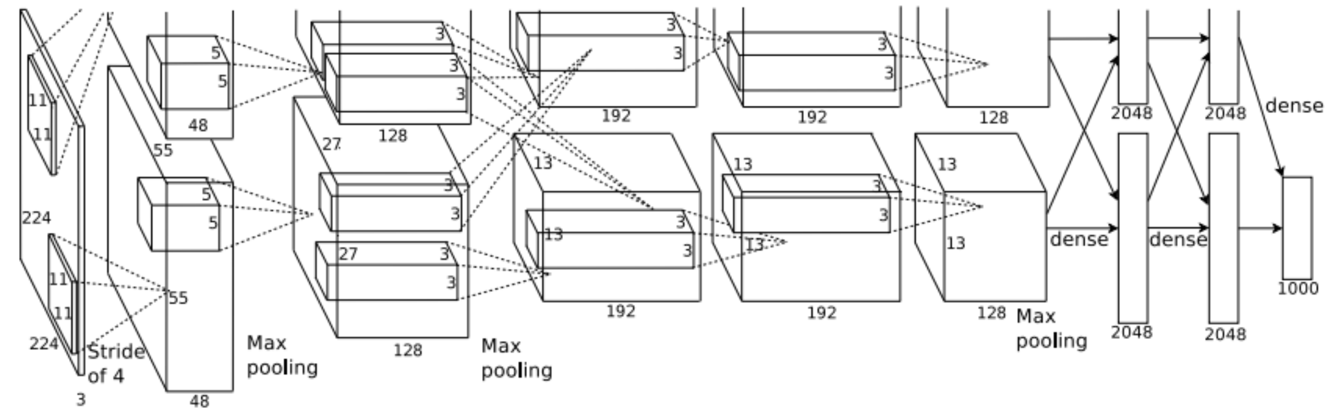
Erkut Erdem // Hacettepe University // Fall 2023

Announcement

- Midterm exam will be held on Nov 29, 2023 at 09:40 (D1).
- No class next Monday, will use it as an office hour.

Last time…

AlexNet [Krizhevsky et al. 2012]

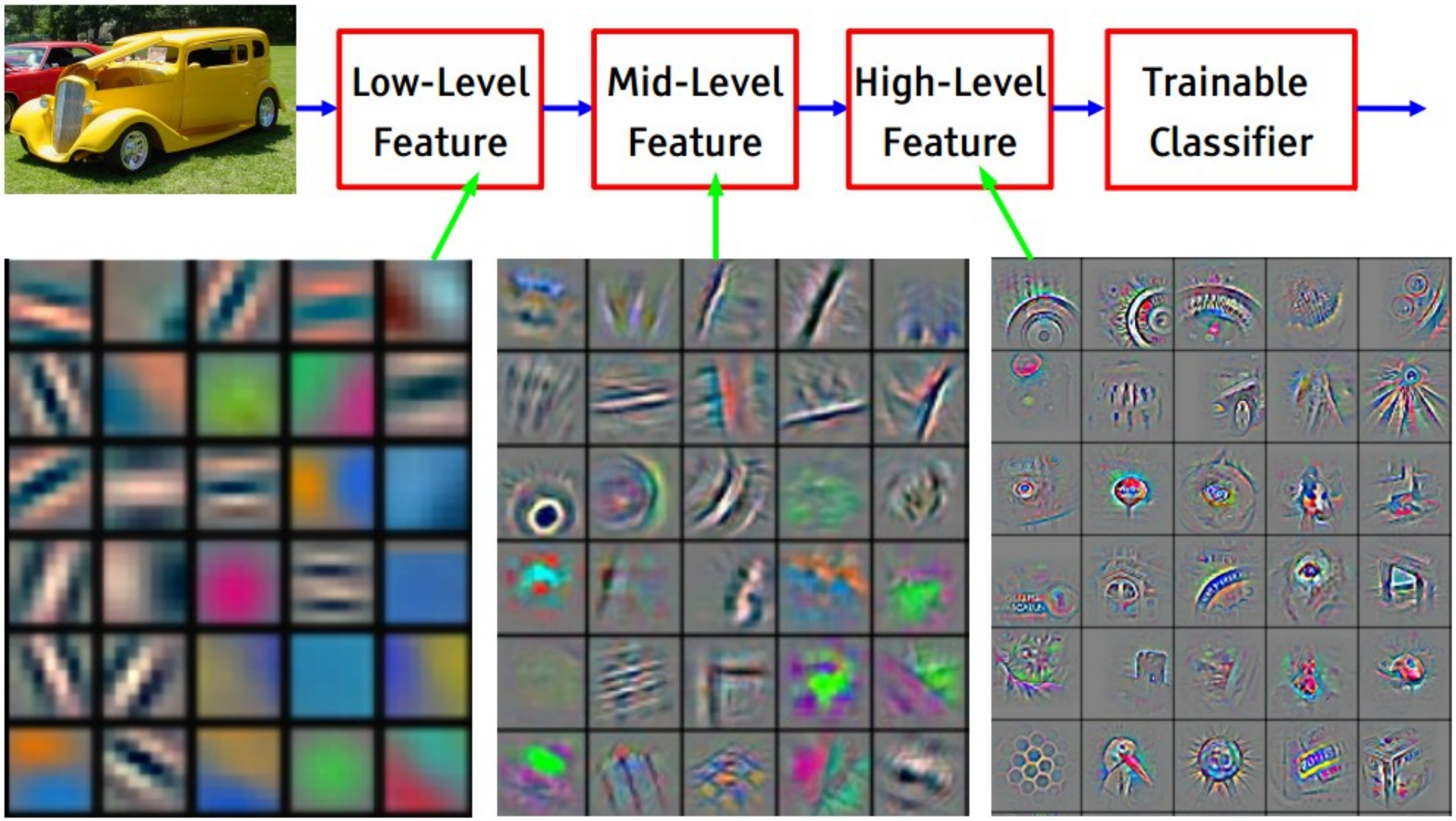


Full (simplified) AlexNet architecture:
[227x227x3] INPUT
[55x55x96] CONV1: 96 11x11 filters at stride 4, pad 0
[27x27x96] MAX POOL1: 3x3 filters at stride 2
[27x27x96] NORM1: Normalization layer
[27x27x256] CONV2: 256 5x5 filters at stride 1, pad 2
[13x13x256] MAX POOL2: 3x3 filters at stride 2
[13x13x256] NORM2: Normalization layer
[13x13x384] CONV3: 384 3x3 filters at stride 1, pad 1
[13x13x384] CONV4: 384 3x3 filters at stride 1, pad 1
[13x13x256] CONV5: 256 3x3 filters at stride 1, pad 1
[6x6x256] MAX POOL3: 3x3 filters at stride 2
[4096] FC6: 4096 neurons
[4096] FC7: 4096 neurons
[1000] FC8: 1000 neurons (class scores)

Details/Retrospectives:
- first use of ReLU
- used Norm layers (not common anymore)
- heavy data augmentation
- dropout 0.5
- batch size 128
- SGD Momentum 0.9
- Learning rate 1e-2, reduced by 10 manually when val accuracy plateaus
- L2 weight decay 5e-4
- 7 CNN ensemble: 18.2% -> 15.4%

slide by Fei-Fei Li, Andrej Karpathy & Justin Johnson

Last time.. Understanding ConvNets



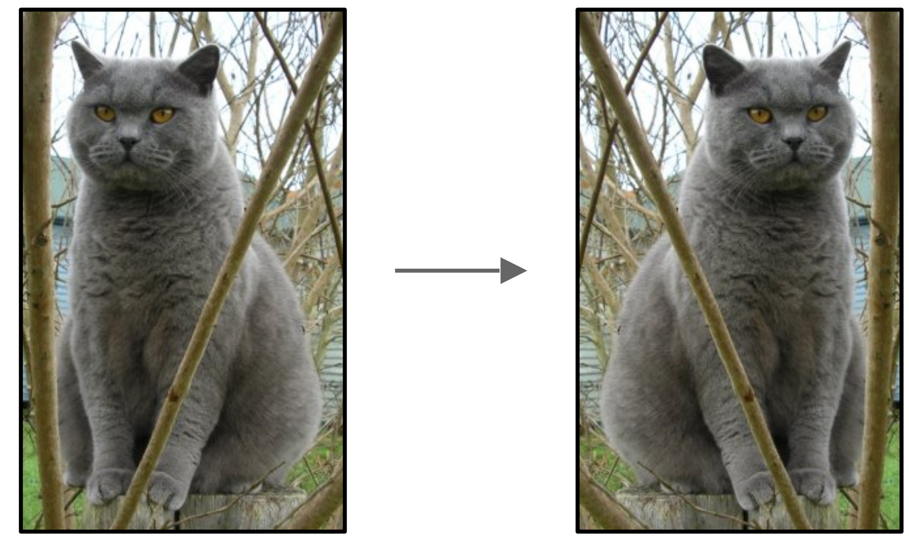
Feature visualization of convolutional net trained on ImageNet from [Zeiler & Fergus 2013]

http://cs.nyu.edu/~fergus/papers/zeilerECCV2014.pdf
http://cs.nyu.edu/~fergus/presentations/nips2013_final.pdf

Last time… Data Augmentation

Random mix/combinations of:

- translation
- rotation
- stretching
- shearing,
- lens distortions, …



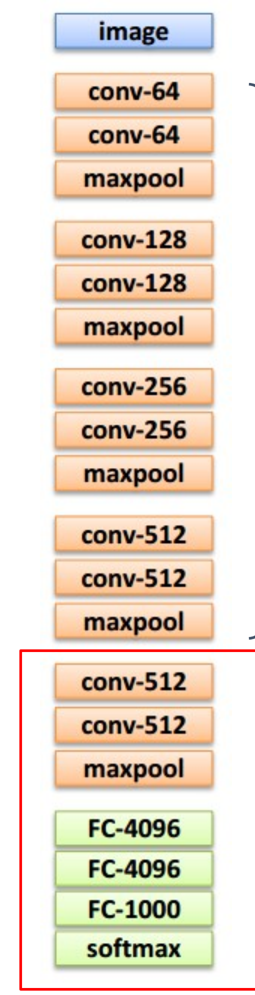
Last time… Transfer Learning with Convolutional Networks

1. Train on Imagenet

2. Small dataset: **feature extractor**

Freeze these

Train this



3. Medium dataset: **finetuning**

more data = retrain more of the network (or all of it)

Freeze these

tip: use only ~1/10th of the original learning rate in finetuning top layer, and ~1/100th on intermediate layers

Train this

slide by Fei-Fei Li, Andrej Karpathy & Justin Johnson

Today

- Support Vector Machines
  - Large Margin Separation
  - Optimization Problem
  - Support Vectors

Recap: Binary Classification Problem

- Training data: sample drawn i.i.d. from set $X \subseteq \mathbb{R}^N$ according to some distribution D,

$$S = ((x_1, y_1), \ldots, (x_m, y_m)) \in X \times \{-1, +1\}.$$

- Problem: find hypothesis $h : X \mapsto \{-1, +1\}$ in H (classifier) with small generalization error $R_D(h)$.

- Linear classification:
  - Hypotheses based on hyperplanes.
  - Linear separation in high-dimensional space.

slide by Mehryar Mohri

Example: Spam

- Imagine 3 features (spam is "positive" class):
  1. free (number of occurrences of "free")
  2. money (occurrences of "money")
  3. BIAS (intercept, always has value 1)

x	$f(x)$	w	$w \cdot f(x)$ $\sum_i w_i \cdot f_i(x)$
"free money"	BIAS : 1 free : 1 money : 1 ...	BIAS : -3 free : 4 money : 2 ...	$(1)(-3)\ +$ $(1)(4)\ +$ $(1)(2)\ +$ $\dots$ $= 3$

$w \cdot f(x) > 0$ → **SPAM!!!**

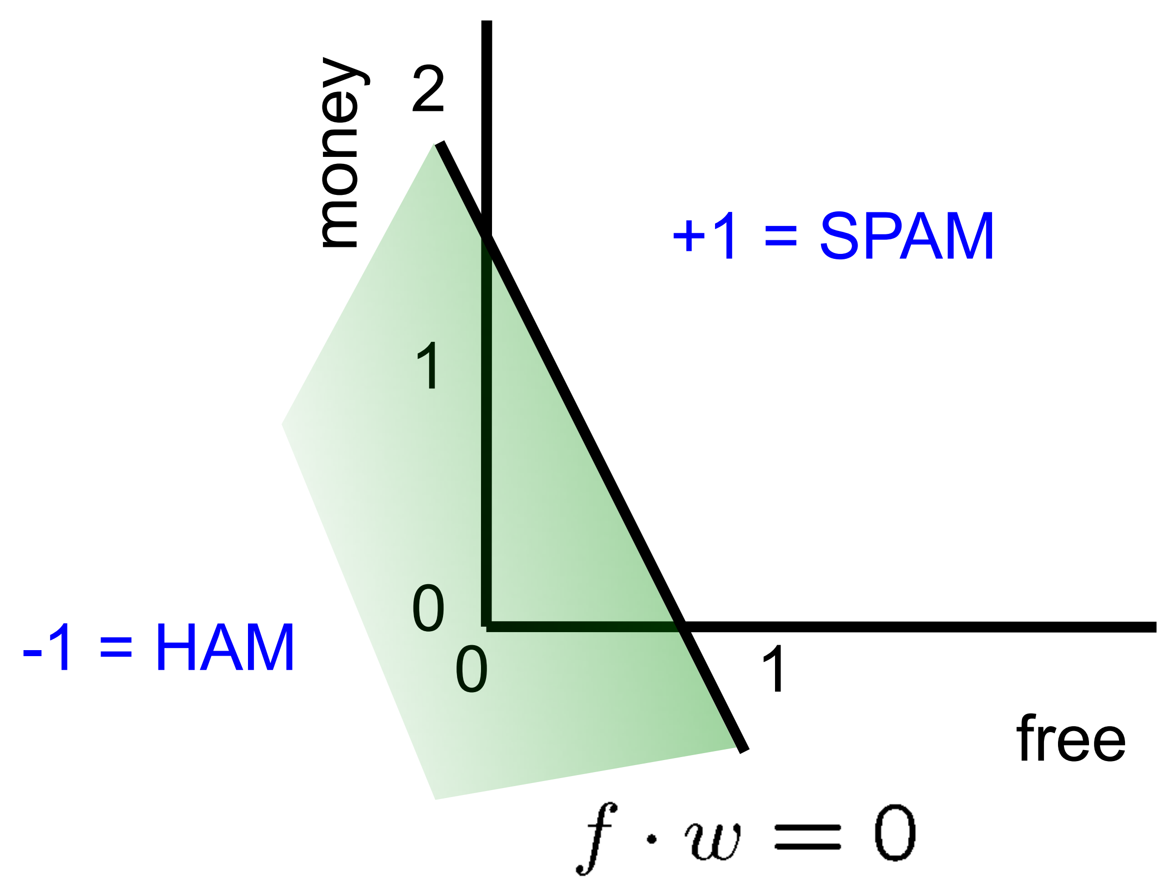
slide by David Sontag

Binary Decision Rule

- In the space of feature vectors
  - Examples are points
  - Any weight vector is a hyperplane
  - One side corresponds to $\mathrm{Y} = +1$
  - Other corresponds to $\mathrm{Y} = -1$

w

```
BIAS  : -3
free  :  4
money :  2
...
```



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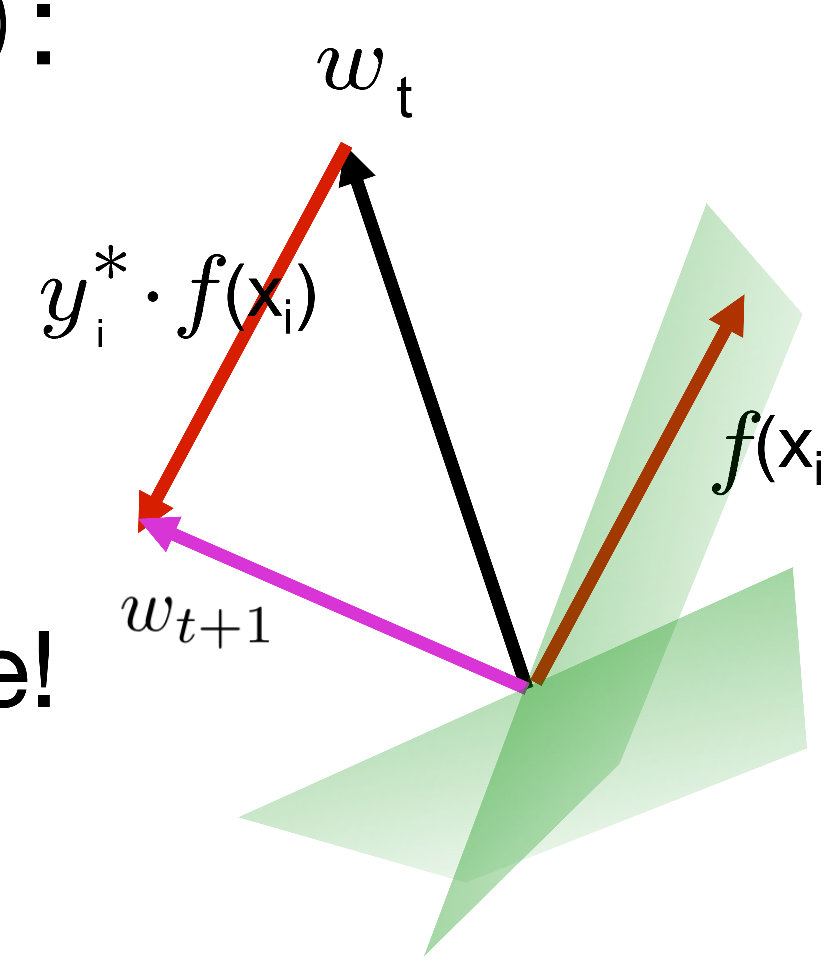
The perceptron algorithm

- Start with weight vector = $\vec{0}$
- For each training instance (x_i, y_i^*):
  - Classify with current weights

$$y_i = \begin{cases} +1 & \text{if } \; w \cdot f(x_i) \geq 0 \\ -1 & \text{if } \; w \cdot f(x_i) < 0 \end{cases}$$

  - If correct (i.e. $y = y_i^*$), no change!
  - If wrong: update

$$w = w + y_i^* f(x_i)$$



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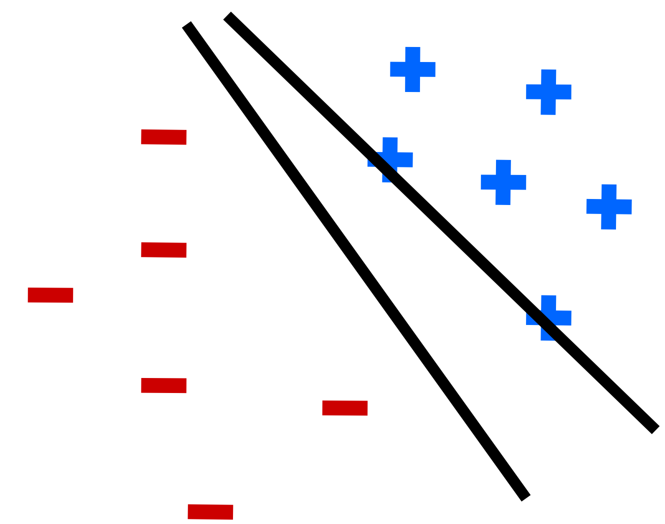
Properties of the perceptron algorithm

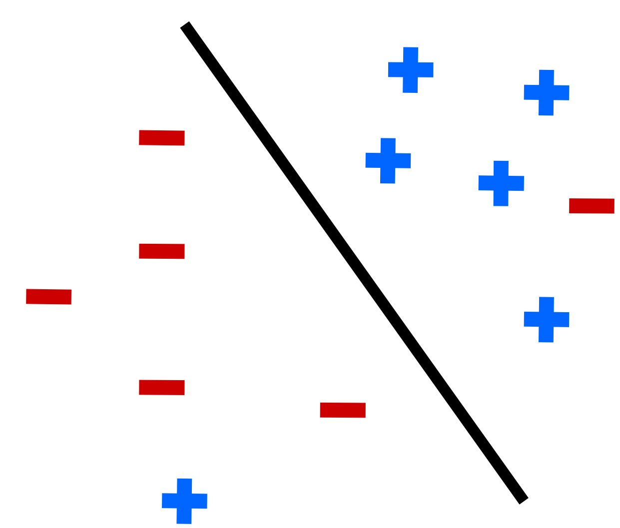
- Separability: some parameters get the training set perfectly correct

- Convergence: if the training is linearly separable, perceptron will eventually converge

Separable



Non-Separable



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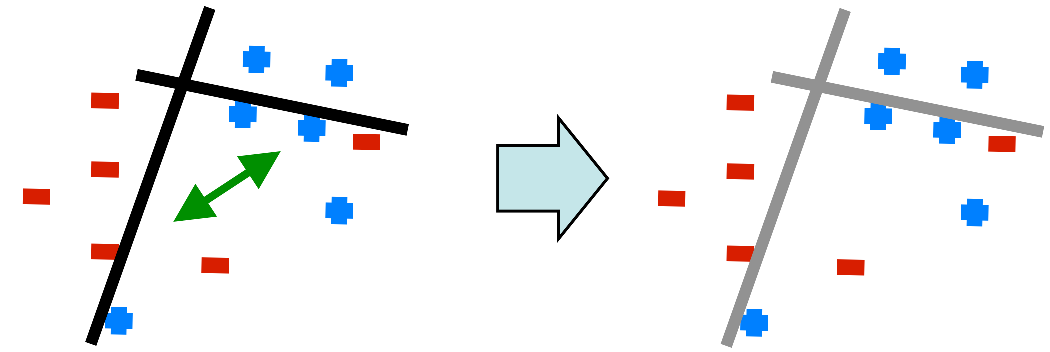
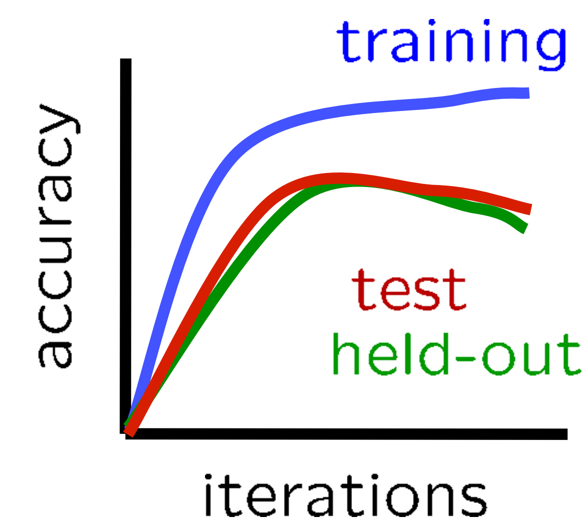
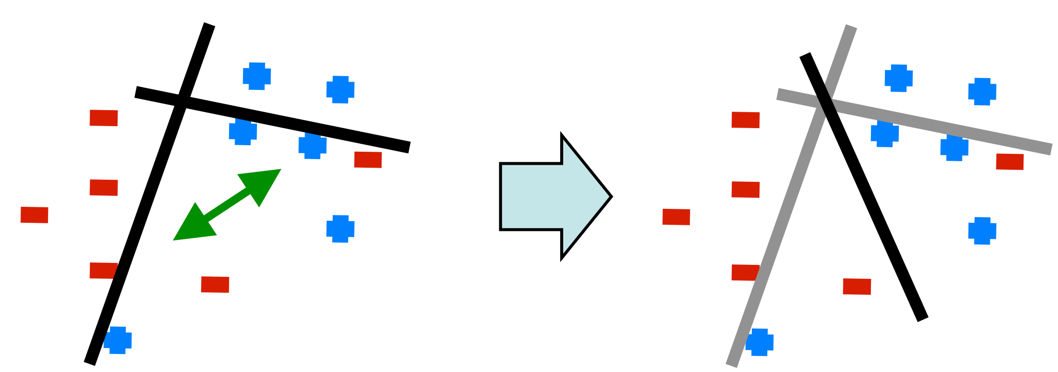
Problems with the perceptron algorithm



- **Noise:** if the data isn't linearly separable, no guarantees of convergence or accuracy
- Frequently the training data is linearly separable! **Why?**
  - When the number of features is much larger than the number of data points, there is lots of flexibility
  - As a result, Perceptron can significantly **overfit** the data



- **Averaged** perceptron is an algorithmic modification that helps with both issues
  - Averages the weight vectors across all iterations



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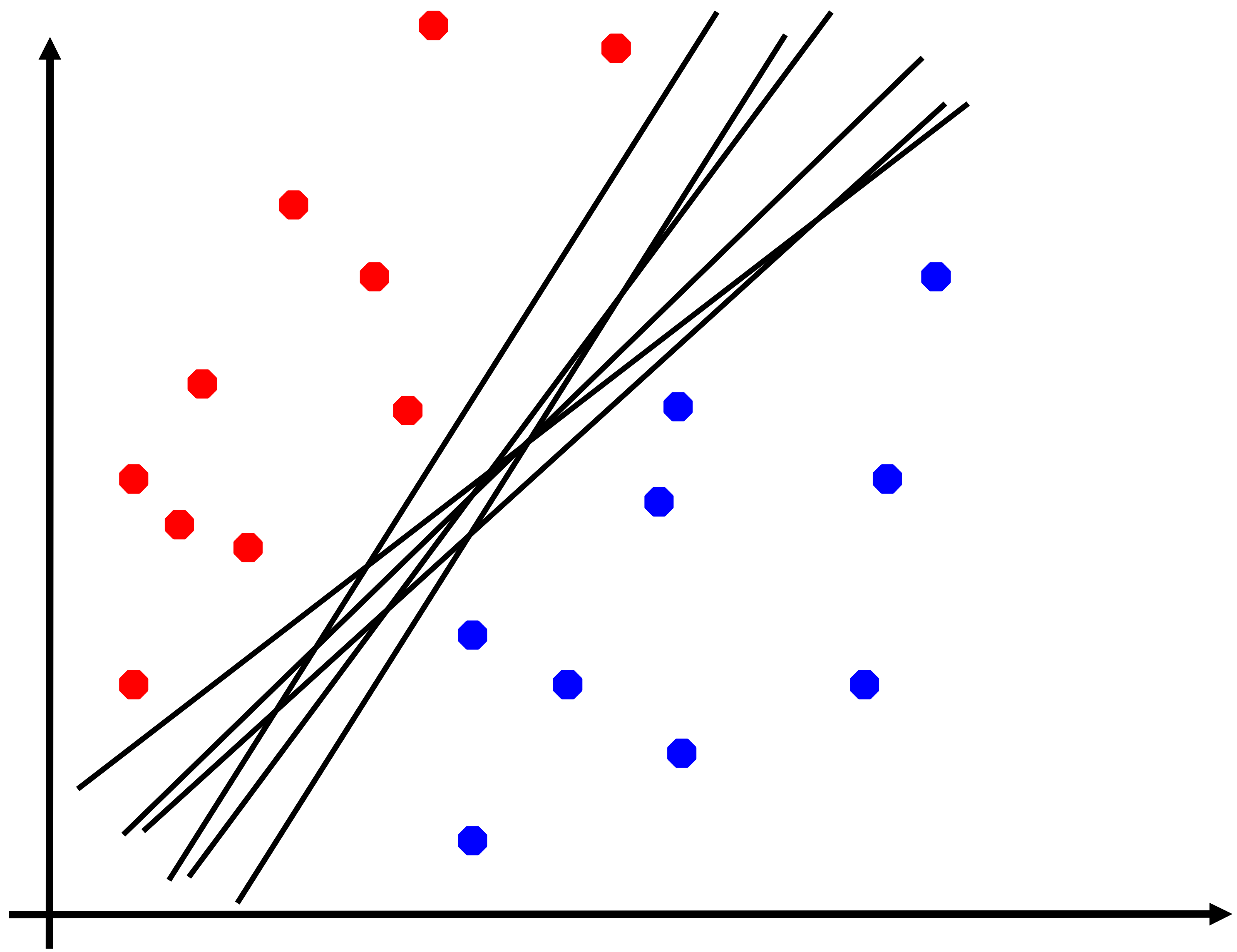
Linear Separators

- Which of these linear separators is optimal?



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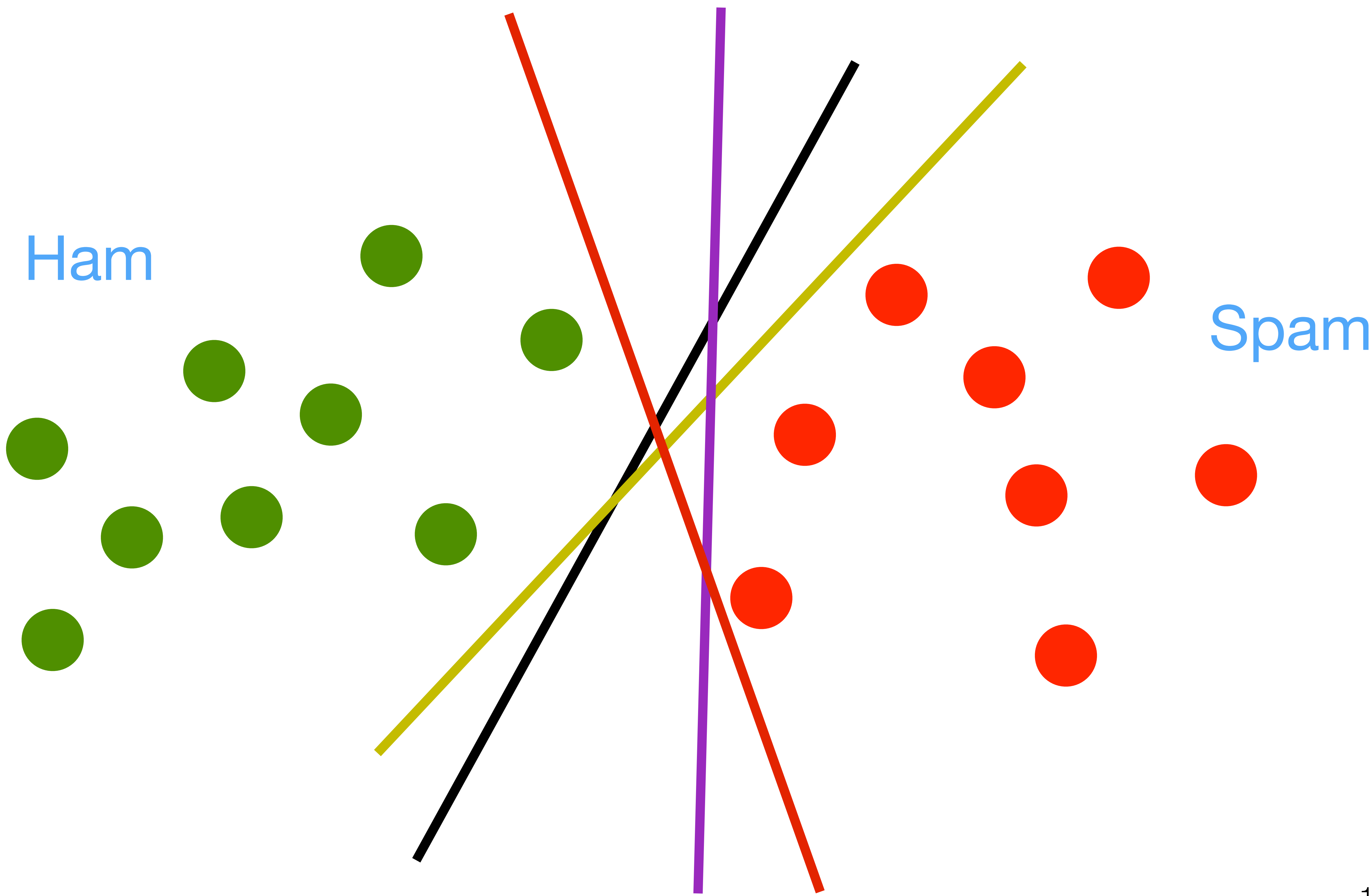
Support Vector Machines

Linear Separator



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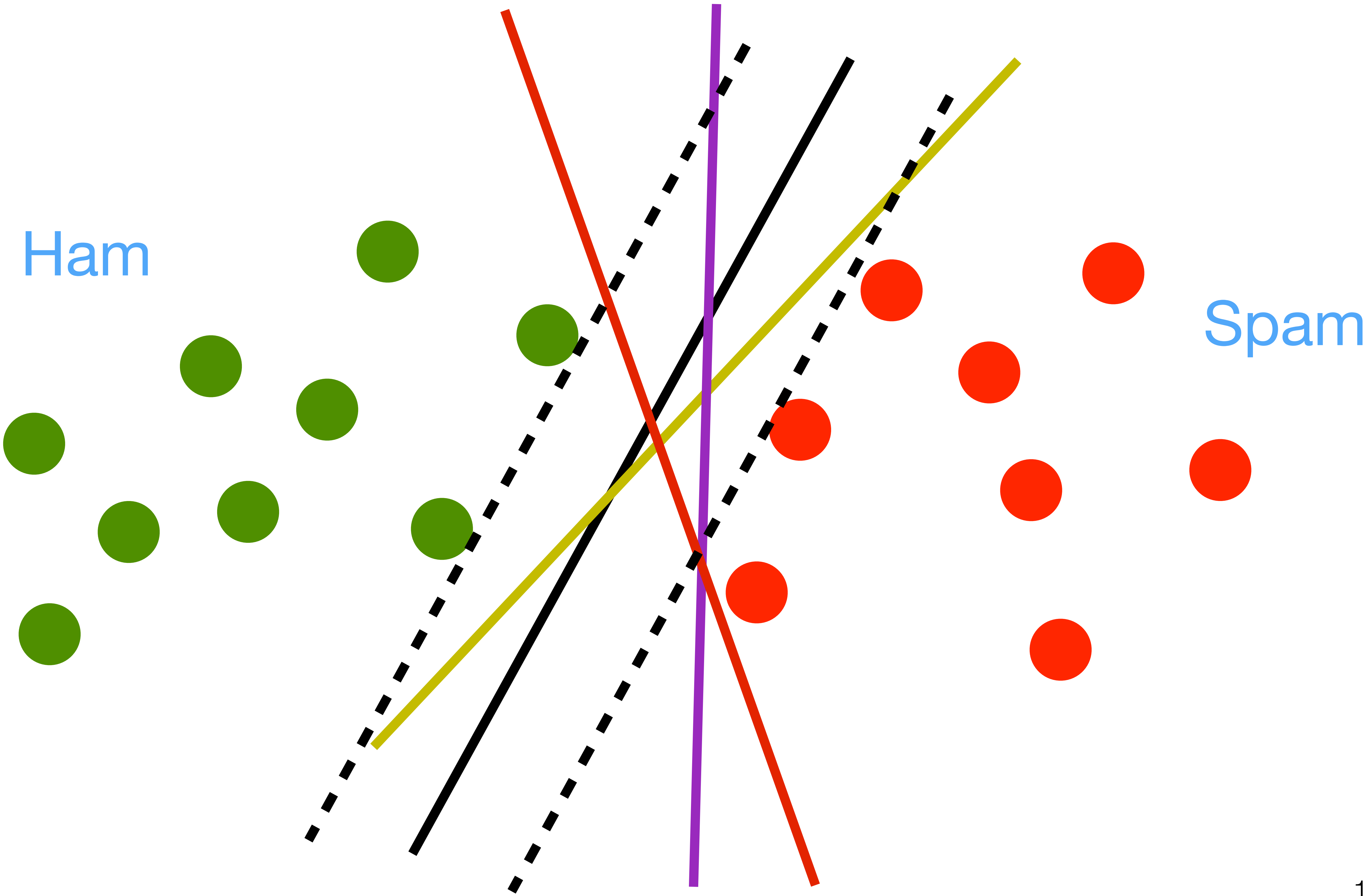
Large Margin Classifier



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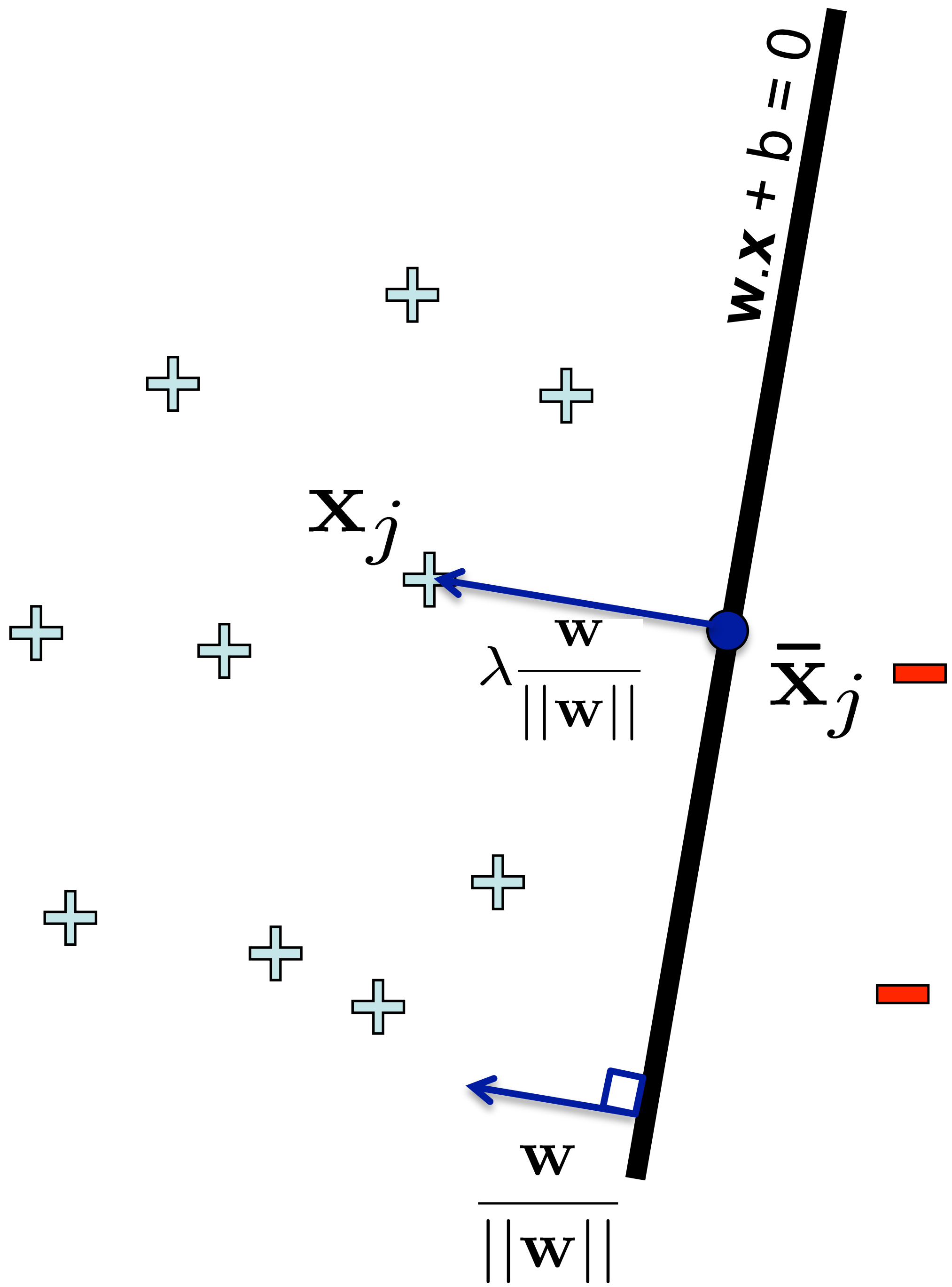
Review: Normal to a plane



$\frac{\mathbf{w}}{\|\mathbf{w}\|}$ -- unit vector normal to w

$\bar{\mathbf{x}}_j$ -- projection of x_j onto the plane

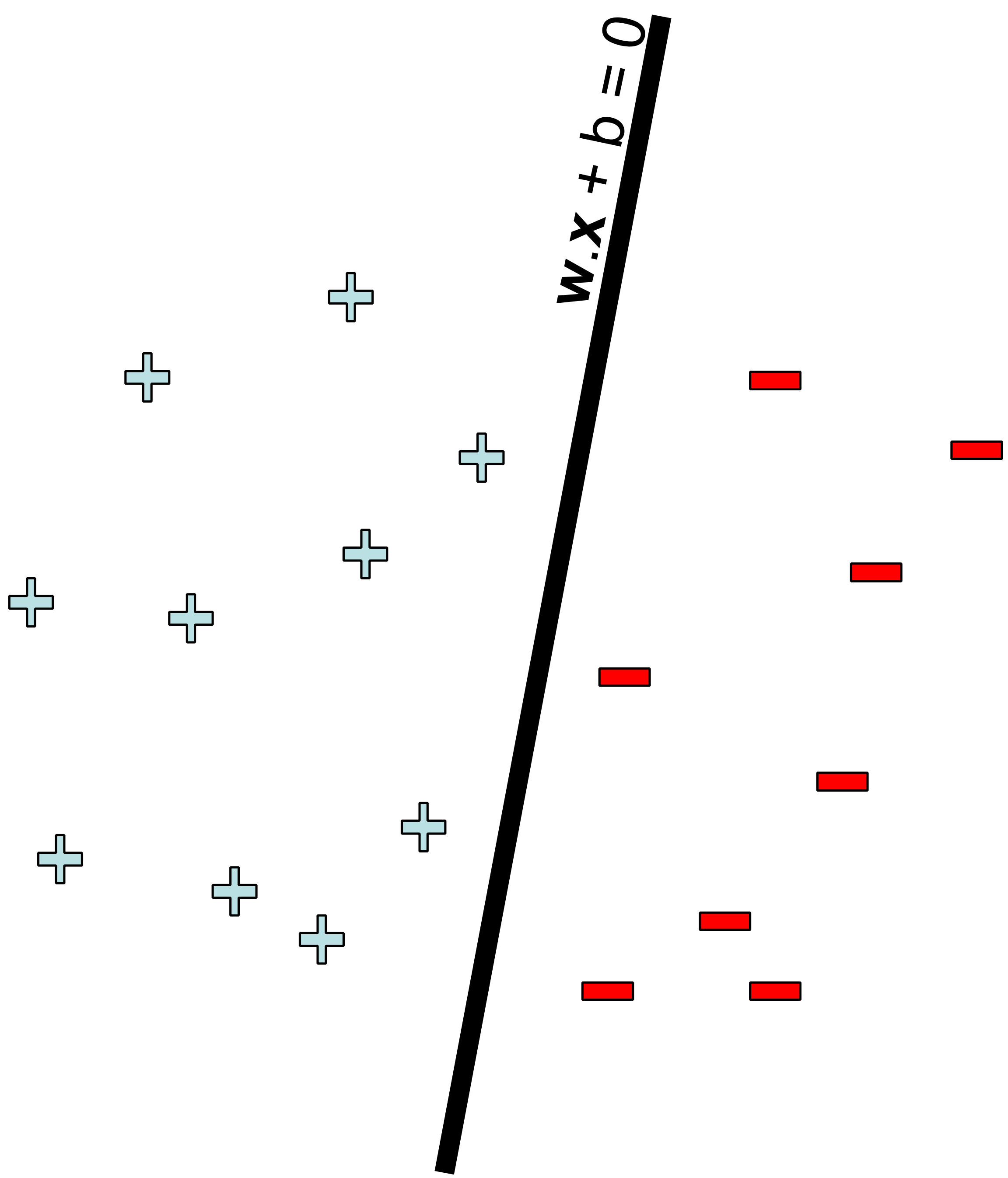
$$x_j - \bar{x}_j = \lambda \frac{w}{\|w\|}$$

λ Is the length of the vector, i.e.

$$x_j - \bar{x}_j = \frac{\lambda}{\|w\|} \|w\| = \lambda$$

slide by David Sontag

Scale invariance

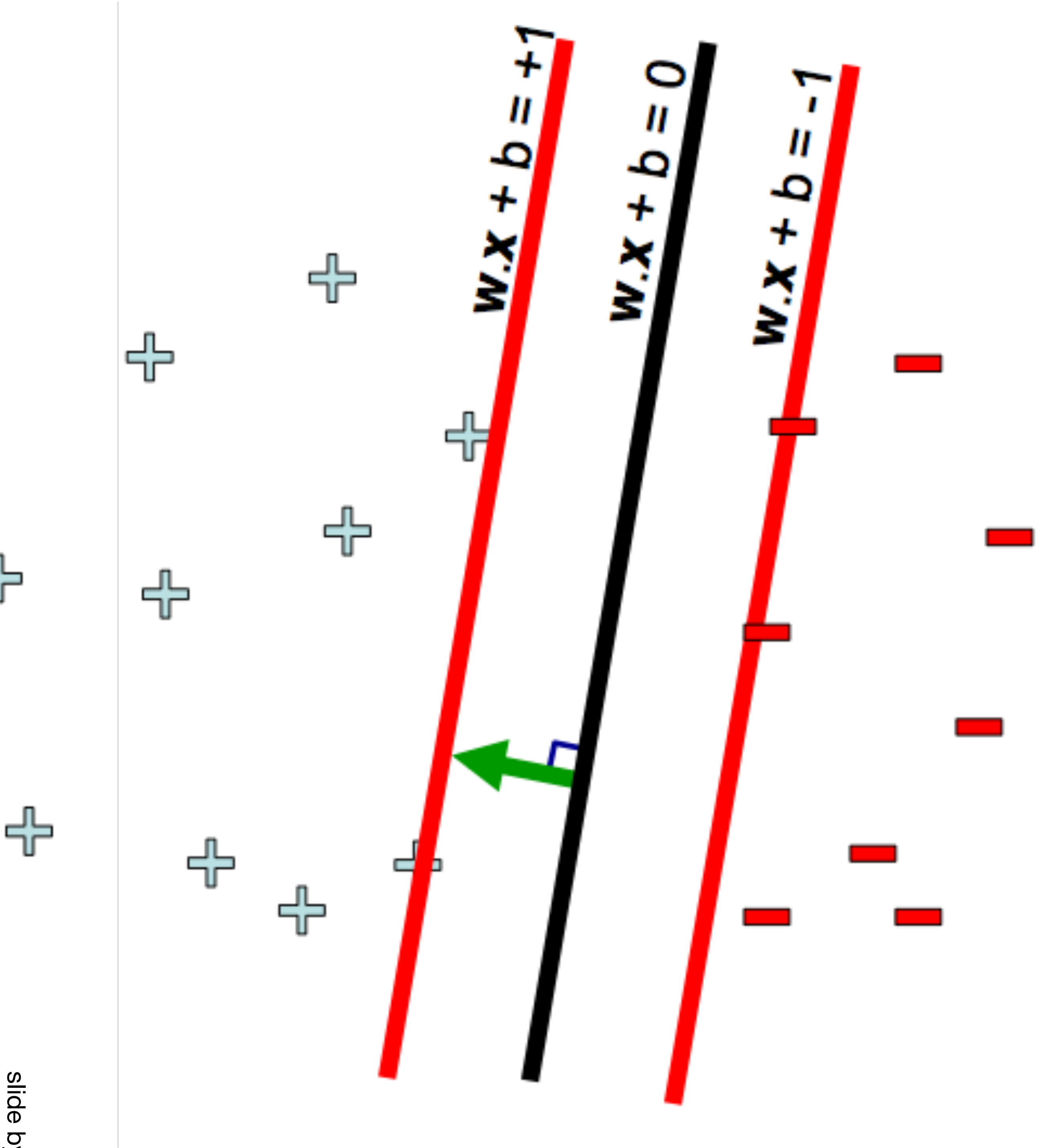


Any other ways of writing the same dividing line?

- **w**.**x**+b=0
- 2**w**.**x**+2b=0
- 1000**w**.**x** + 1000b = 0
- ....

slide by David Sontag

Scale invariance



During learning, we set the scale by asking that, for all *t*,

for $y_t = +1$, $w \cdot x_t + b \geq 1$

and for $y_t = -1$, $w \cdot x_t + b \leq -1$

That is, we want to satisfy all of the **linear** constraints

$$y_t(w \cdot x_t + b) \geq 1 \quad \forall t$$

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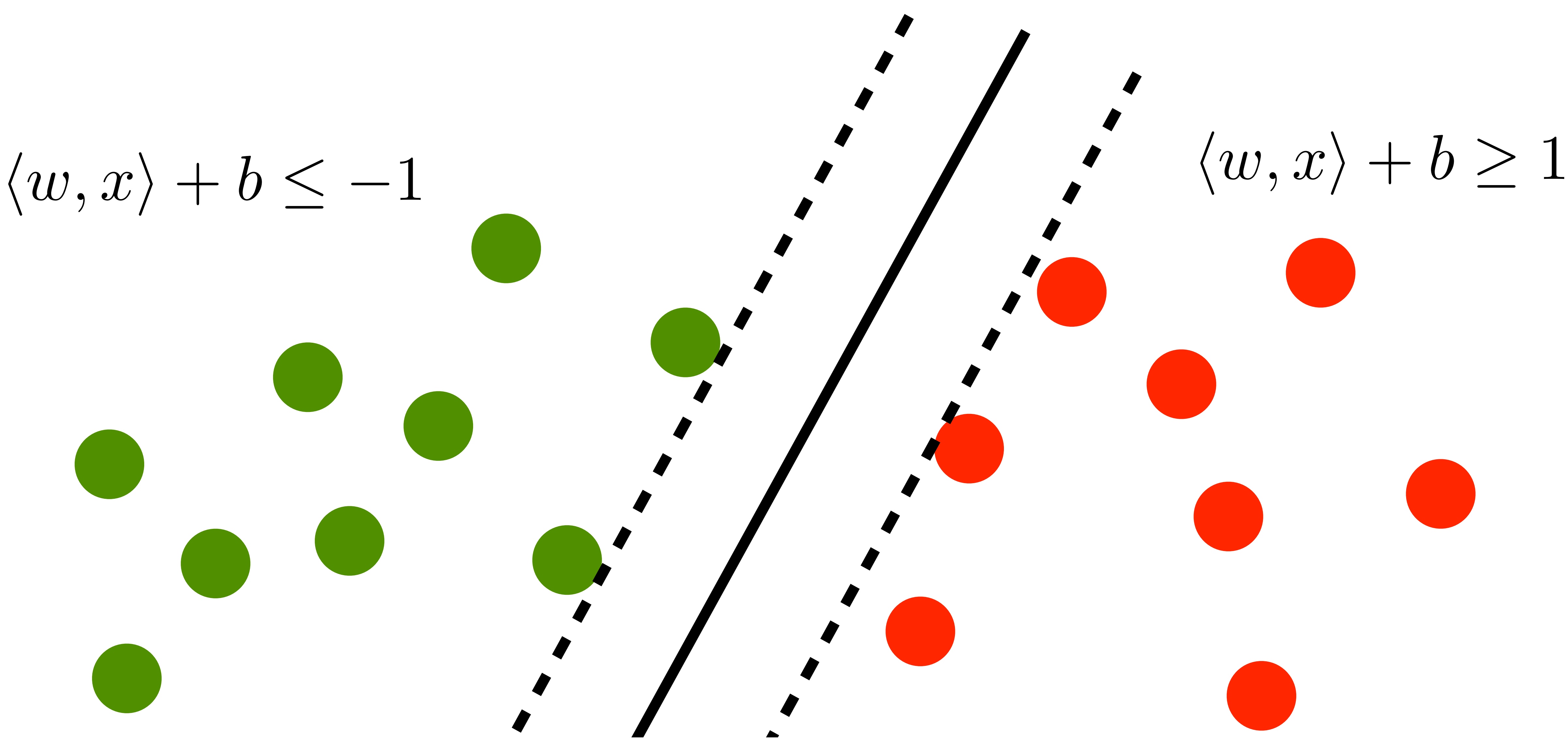
Large Margin Classifier



linear function

$$f(x) = \langle w, x \rangle + b$$

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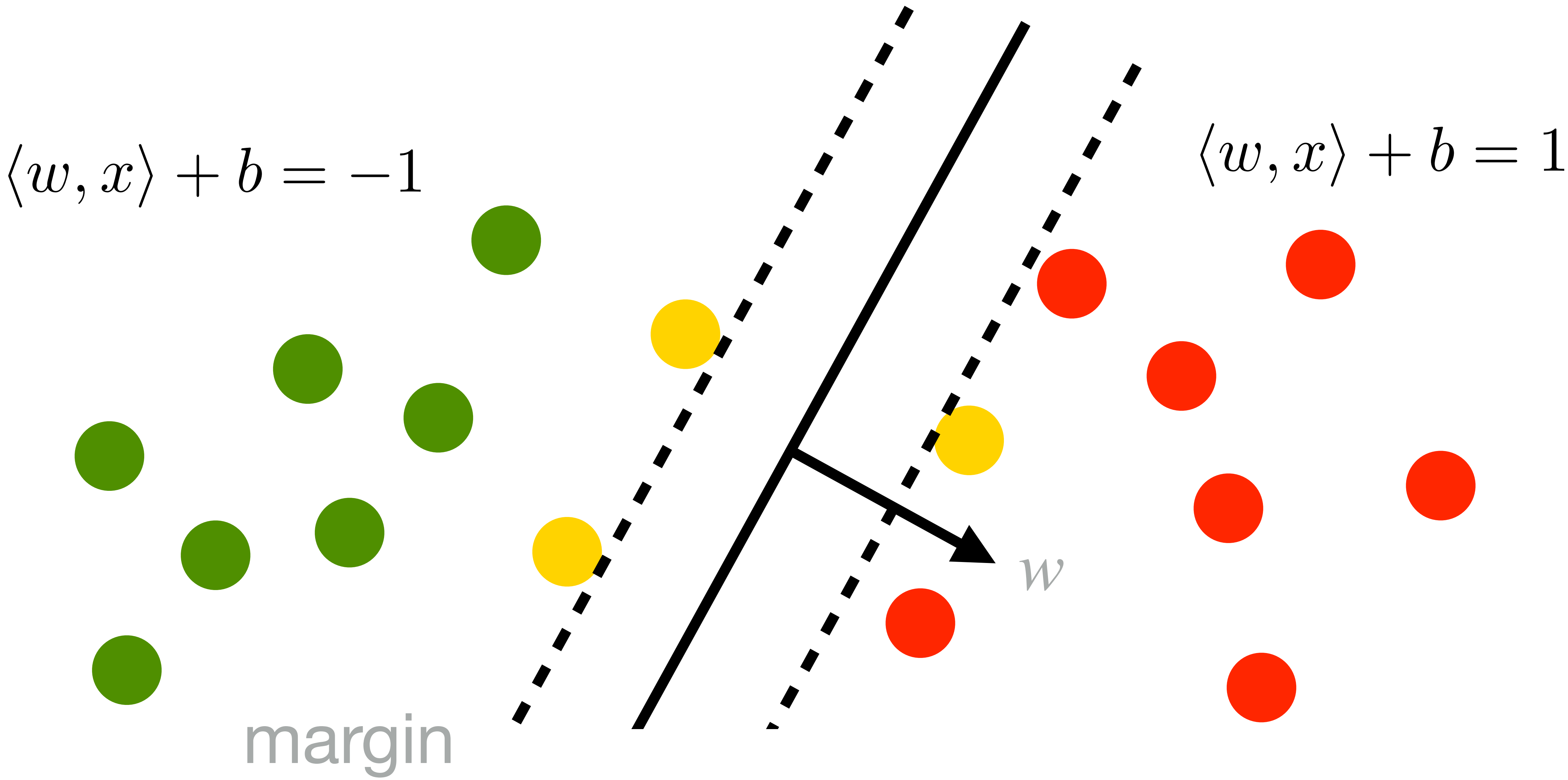
Large Margin Classifier



$$\frac{\langle x_+ - x_-, w \rangle}{2\,\|w\|} = \frac{1}{2\,\|w\|}\left[\left[\langle x_+, w \rangle + b\right] - \left[\langle x_-, w \rangle + b\right]\right] = \frac{1}{\|w\|}$$

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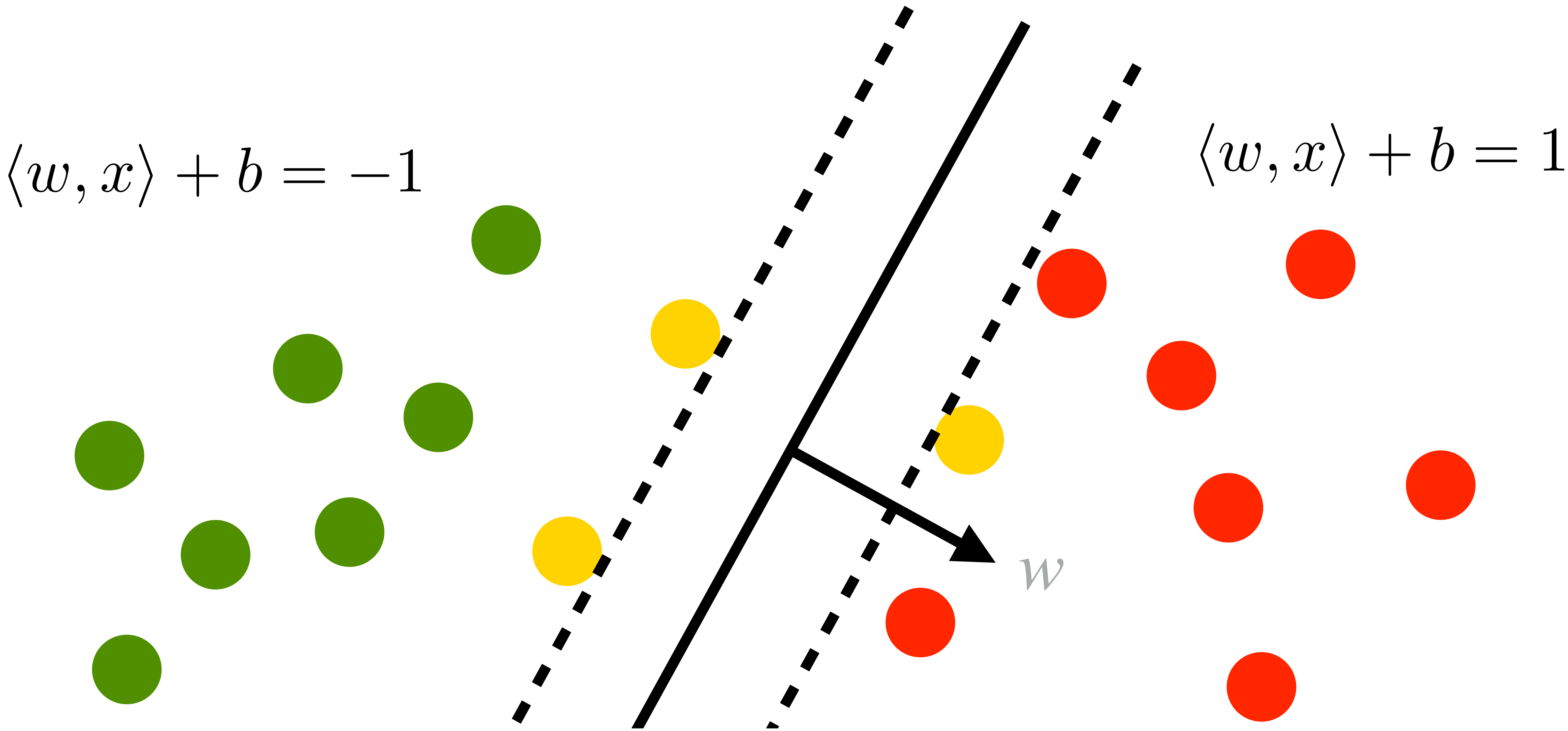
Large Margin Classifier



optimization problem

$$\underset{w,b}{\text{maximize}} \frac{1}{\|w\|} \text{ subject to } y_i \left[\langle x_i, w \rangle + b \right] \geq 1$$

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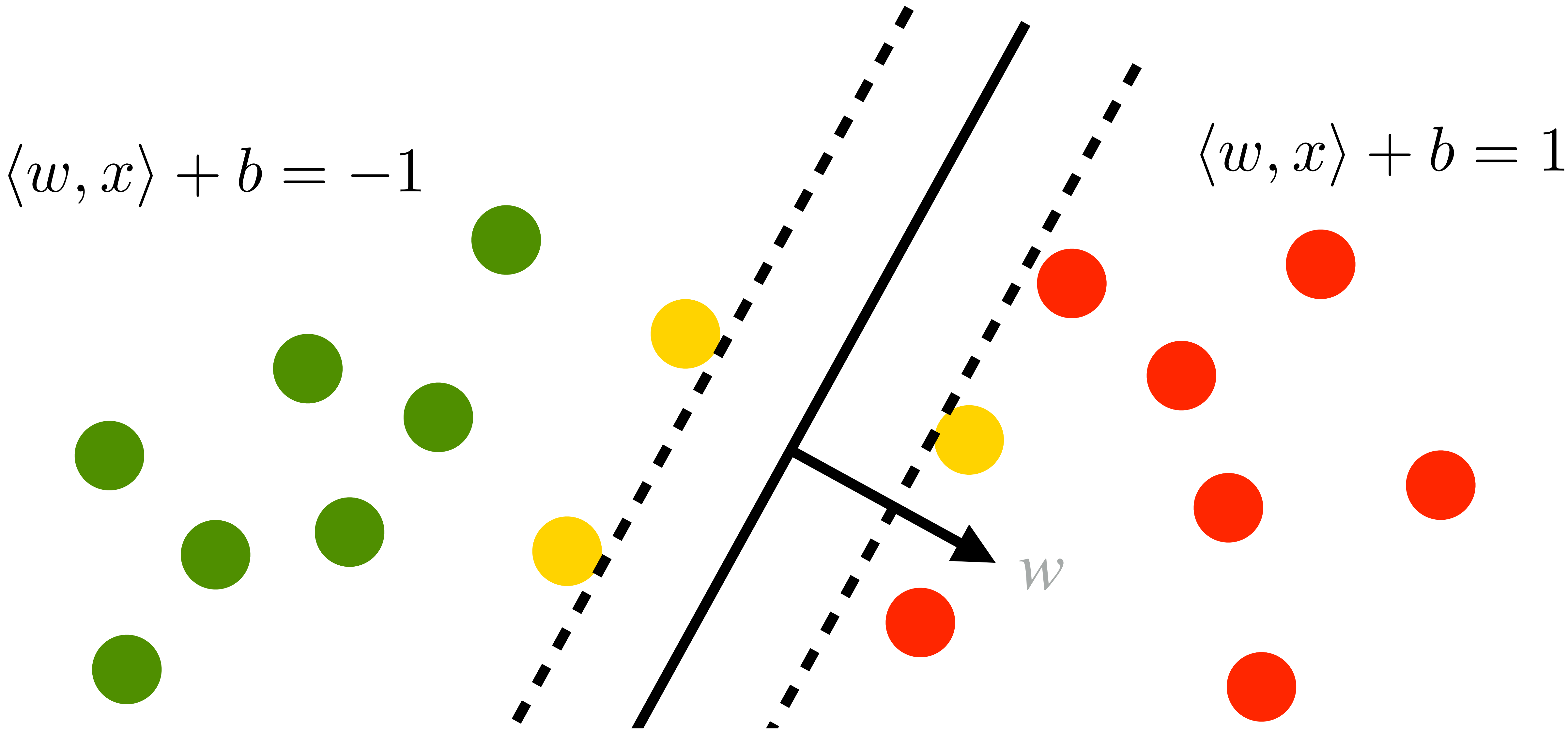
Large Margin Classifier



optimization problem

$$\underset{w,b}{\text{minimize}} \ \frac{1}{2}\left\|w\right\|^2 \ \text{ subject to } y_i\left[\langle x_i, w\rangle + b\right] \geq 1$$

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Convex Programs for Dummies

- Primal optimization problem

$$\underset{x}{\text{minimize}}\, f(x) \text{ subject to } c_i(x) \leq 0$$

- Lagrange function

$$L(x, \alpha) = f(x) + \sum_i \alpha_i c_i(x)$$

- First order optimality conditions in x

$$\partial_x L(x, \alpha) = \partial_x f(x) + \sum_i \alpha_i \partial_x c_i(x) = 0$$

- Solve for x and plug it back into L

$$\underset{\alpha}{\text{maximize}}\, L(x(\alpha), \alpha)$$

(keep explicit constraints)

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Dual Problem

- Primal optimization problem

$$\underset{w,b}{\text{minimize}} \frac{1}{2} \|w\|^2 \text{ subject to } y_i \left[\langle x_i, w \rangle + b\right] \geq 1$$

- Lagrange function

constraint

$$L(w, b, \alpha) = \frac{1}{2} \|w\|^2 - \sum_i \alpha_i \left[y_i \left[\langle x_i, w \rangle + b\right] - 1\right]$$

Optimality in w, b is at saddle point with α

- Derivatives in w, b need to vanish

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Dual Problem

- Lagrange function

$$L(w, b, \alpha) = \frac{1}{2} \|w\|^2 - \sum_i \alpha_i \left[y_i \left[\langle x_i, w \rangle + b \right] - 1 \right]$$

- Derivatives in w, b need to vanish

$$\partial_w L(w, b, a) = w - \sum_i \alpha_i y_i x_i = 0$$

$$\partial_b L(w, b, a) = \sum_i \alpha_i y_i = 0$$

- Plugging terms back into L yields

$$\underset{\alpha}{\text{maximize}} - \frac{1}{2} \sum_{i,j} \alpha_i \alpha_j y_i y_j \langle x_i, x_j \rangle + \sum_i \alpha_i$$

$$\text{subject to } \sum_i \alpha_i y_i = 0 \text{ and } \alpha_i \geq 0$$

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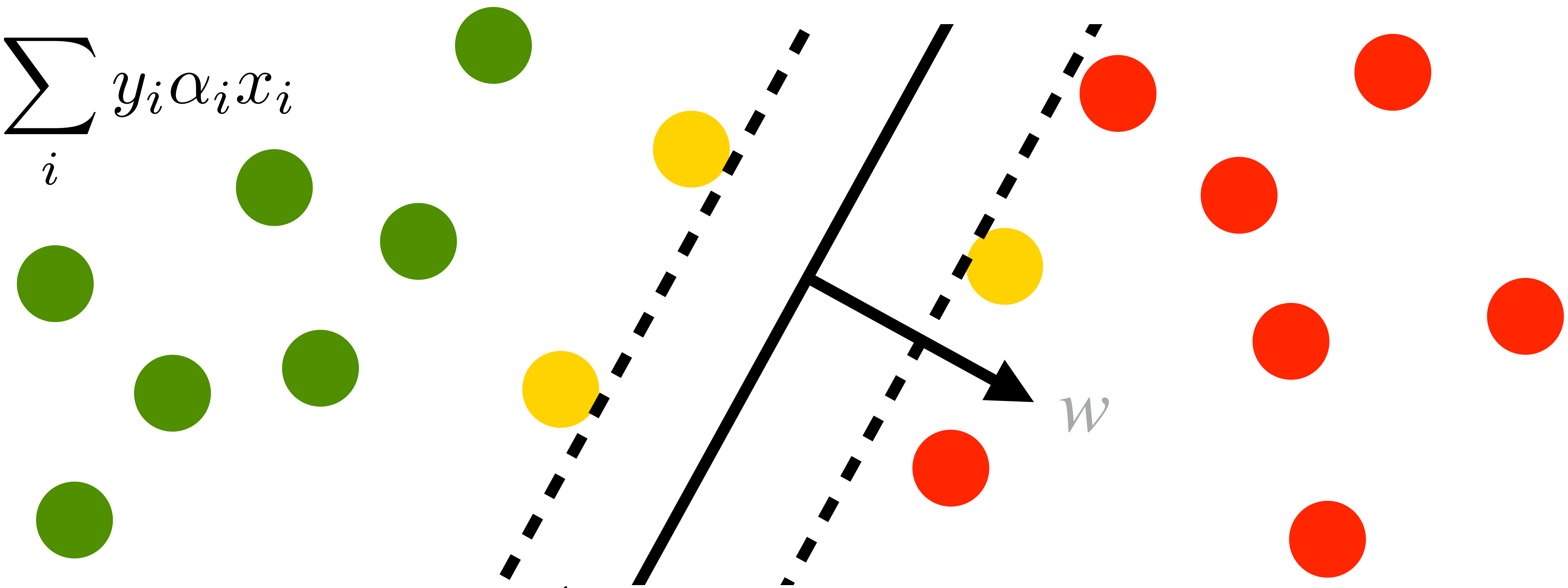
Support Vector Machines

$$\underset{w,b}{\text{minimize}}\ \frac{1}{2}\|w\|^2 \ \text{ subject to } y_i\left[\langle x_i, w\rangle + b\right] \geq 1$$

$$w = \sum_i y_i \alpha_i x_i$$



$$\underset{\alpha}{\text{maximize}} - \frac{1}{2}\sum_{i,j} \alpha_i \alpha_j y_i y_j \langle x_i, x_j\rangle + \sum_i \alpha_i$$

$$\text{subject to } \sum \alpha_i y_i = 0 \text{ and } \alpha_i \geq 0$$

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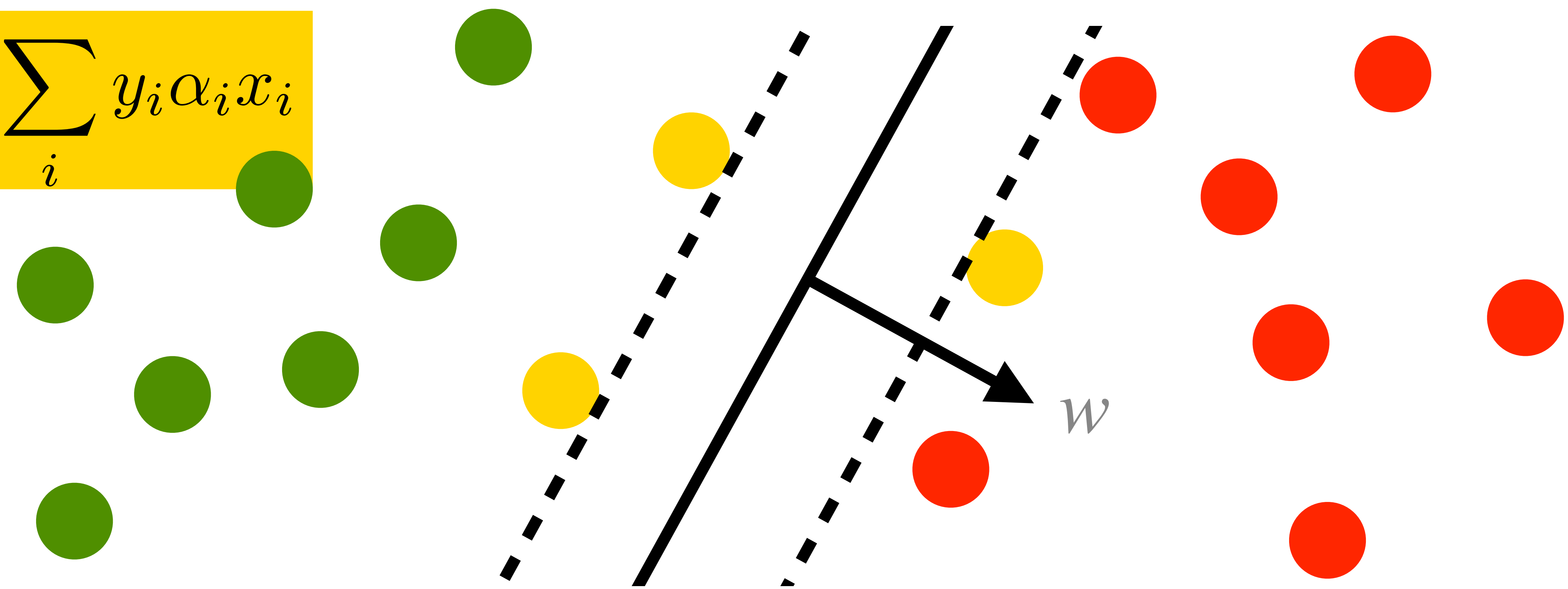
Support Vectors

$$\underset{w,b}{\text{minimize}} \ \frac{1}{2}\|w\|^2 \ \text{subject to} \ y_i\left[\langle x_i, w\rangle + b\right] \geq 1$$

$$\boxed{w = \sum_i y_i \alpha_i x_i}$$



Karush Kuhn Tucker
Optimality condition

$$\alpha_i\left[y_i\left[\langle w, x_i\rangle + b\right] - 1\right] = 0$$

$$\alpha_i = 0$$

$$\boxed{\alpha_i > 0 \Longrightarrow y_i\left[\langle w, x_i\rangle + b\right] = 1}$$

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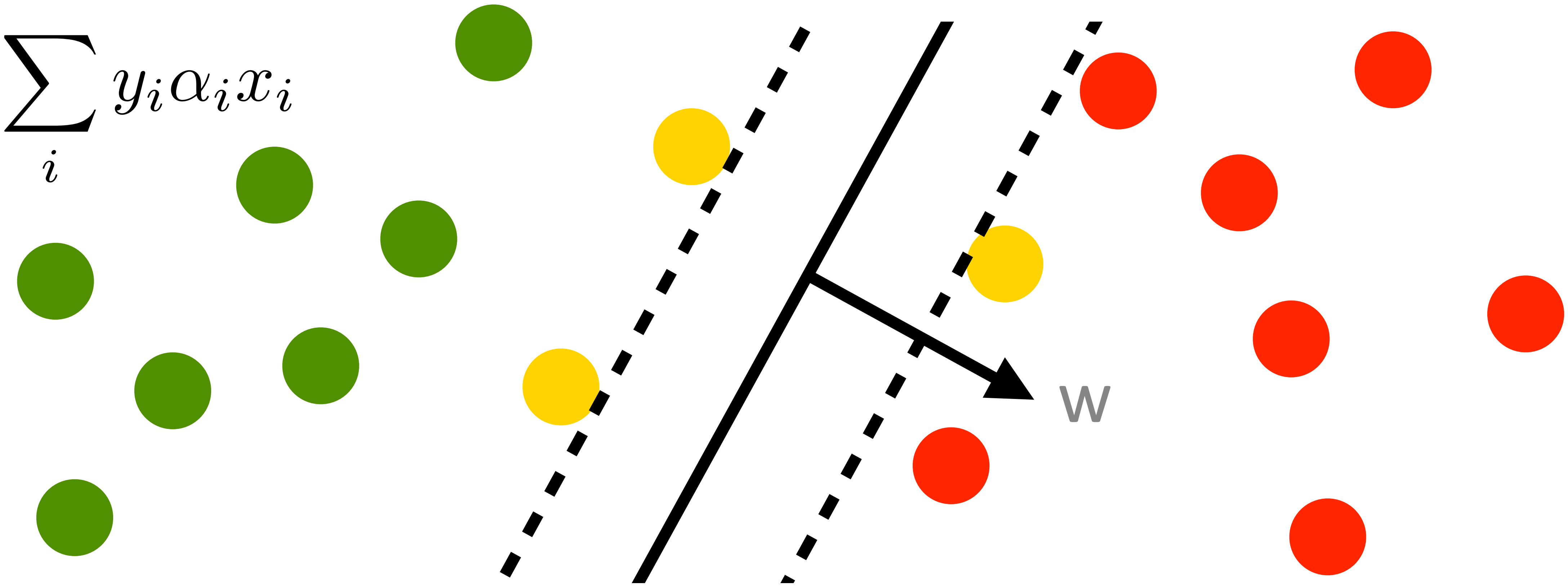
Properties

$$w = \sum_i y_i \alpha_i x_i$$



- Weight vector w as weighted linear combination of instances
- Only points on margin matter (ignore the rest and get same solution)
- Only inner products matter
  - Quadratic program
  - We can replace the inner product by a kernel
- Keeps instances away from the margin

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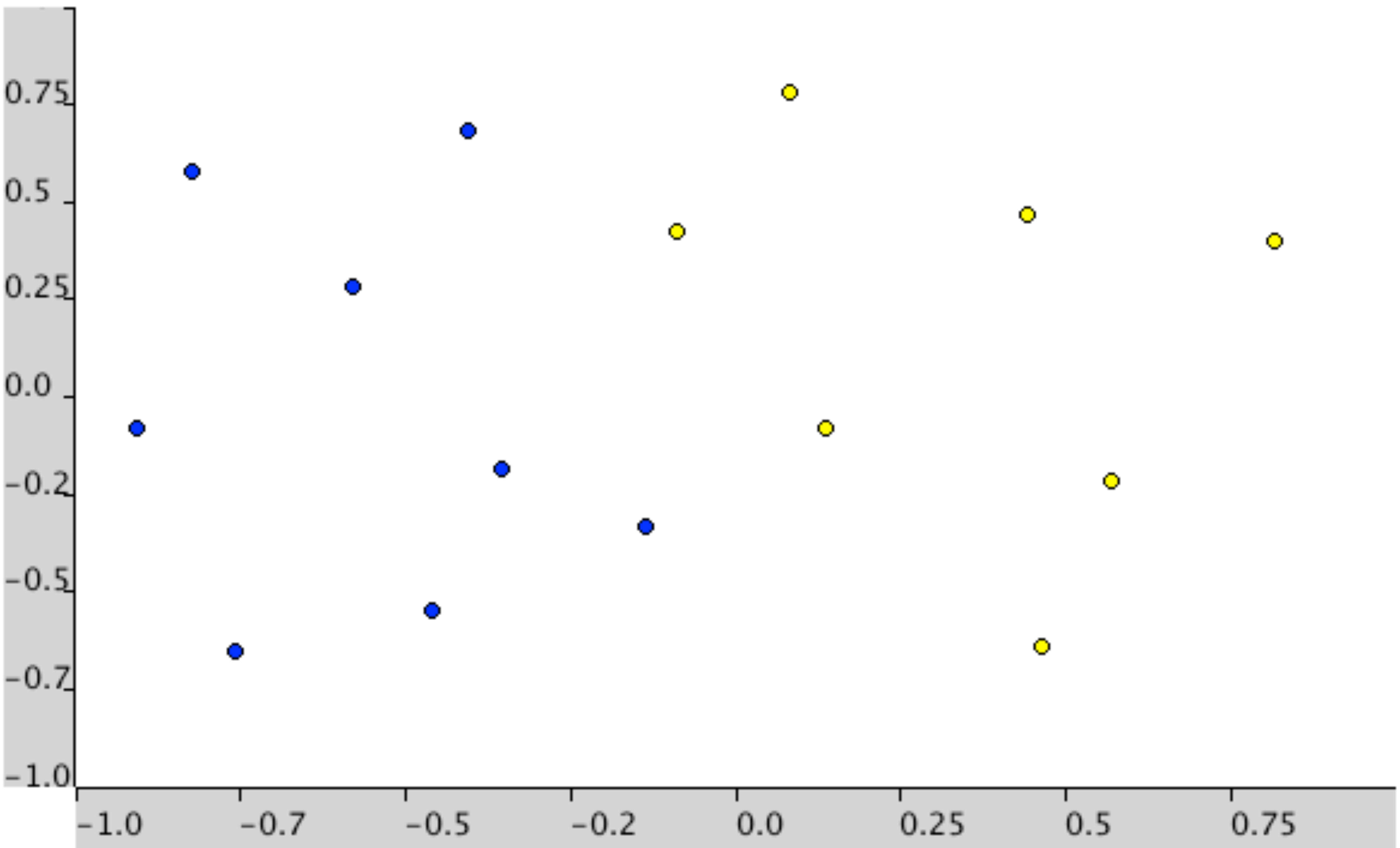
Example



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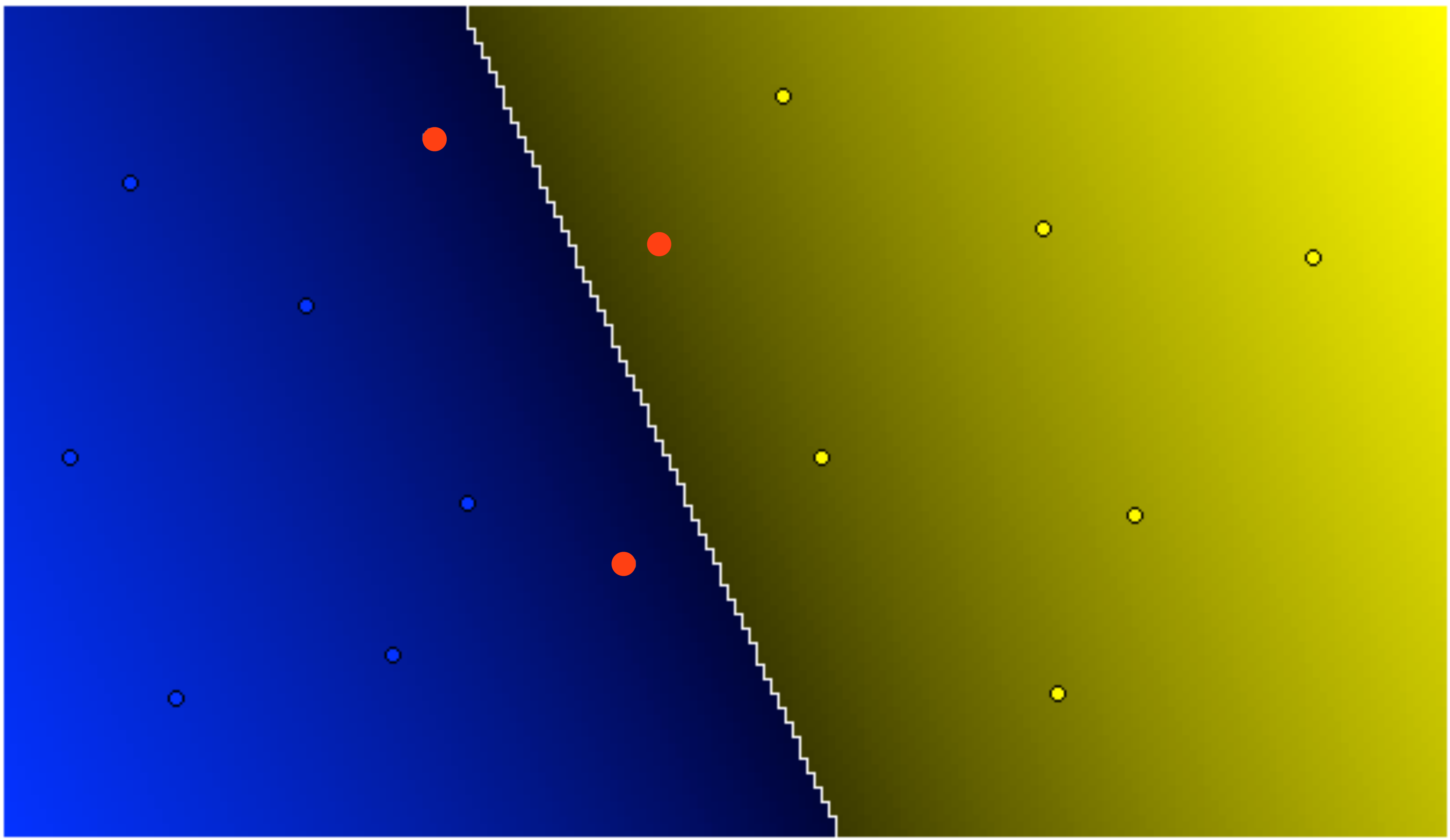
Example

Number of Support Vectors: **3** (-ve: 2, +ve: 1) Total number of points: 15



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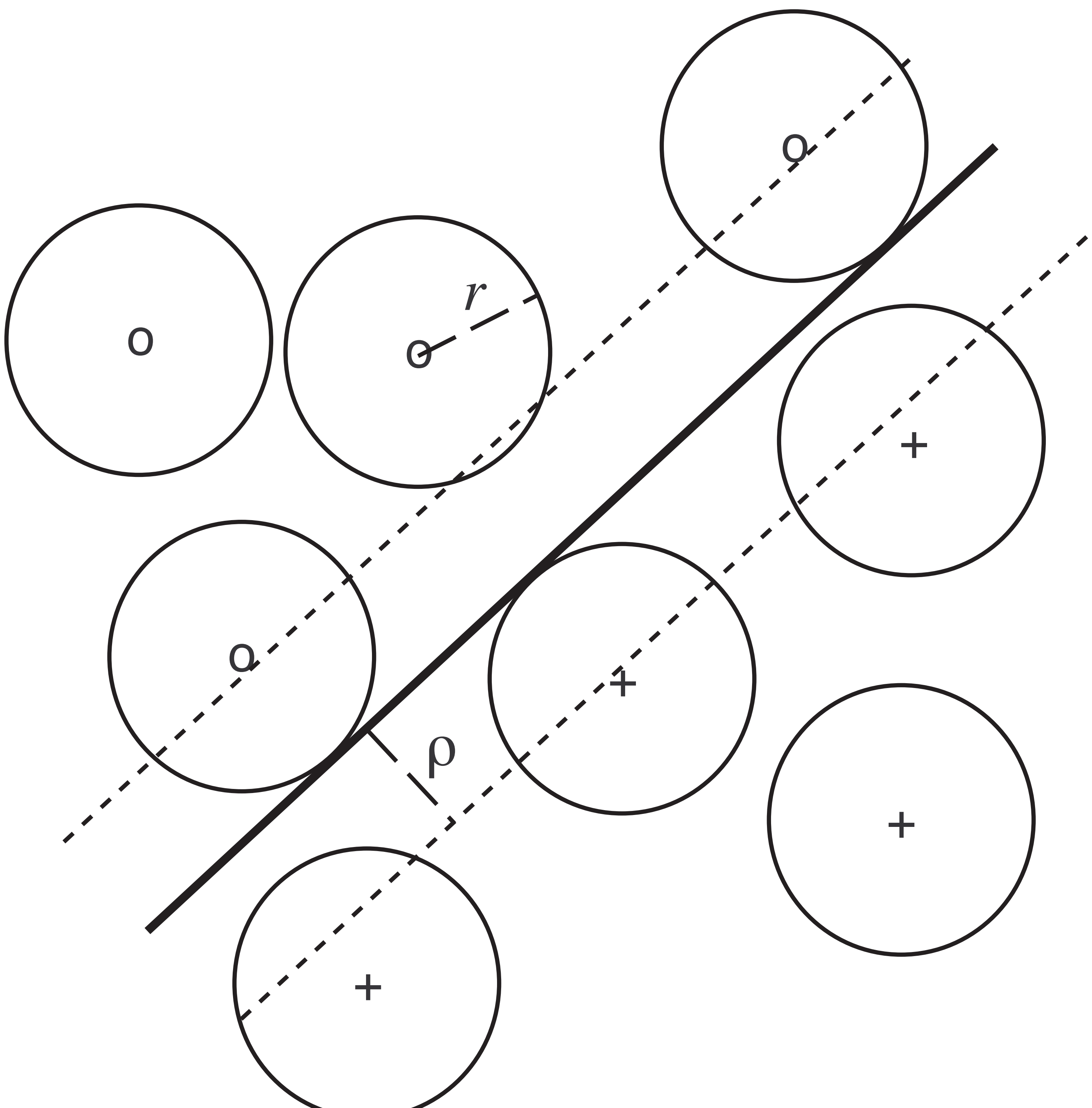
Why Large Margins?



- Maximum robustness relative to uncertainty
- Symmetry breaking
- Independent of correctly classified instances
- Easy to find for easy problems

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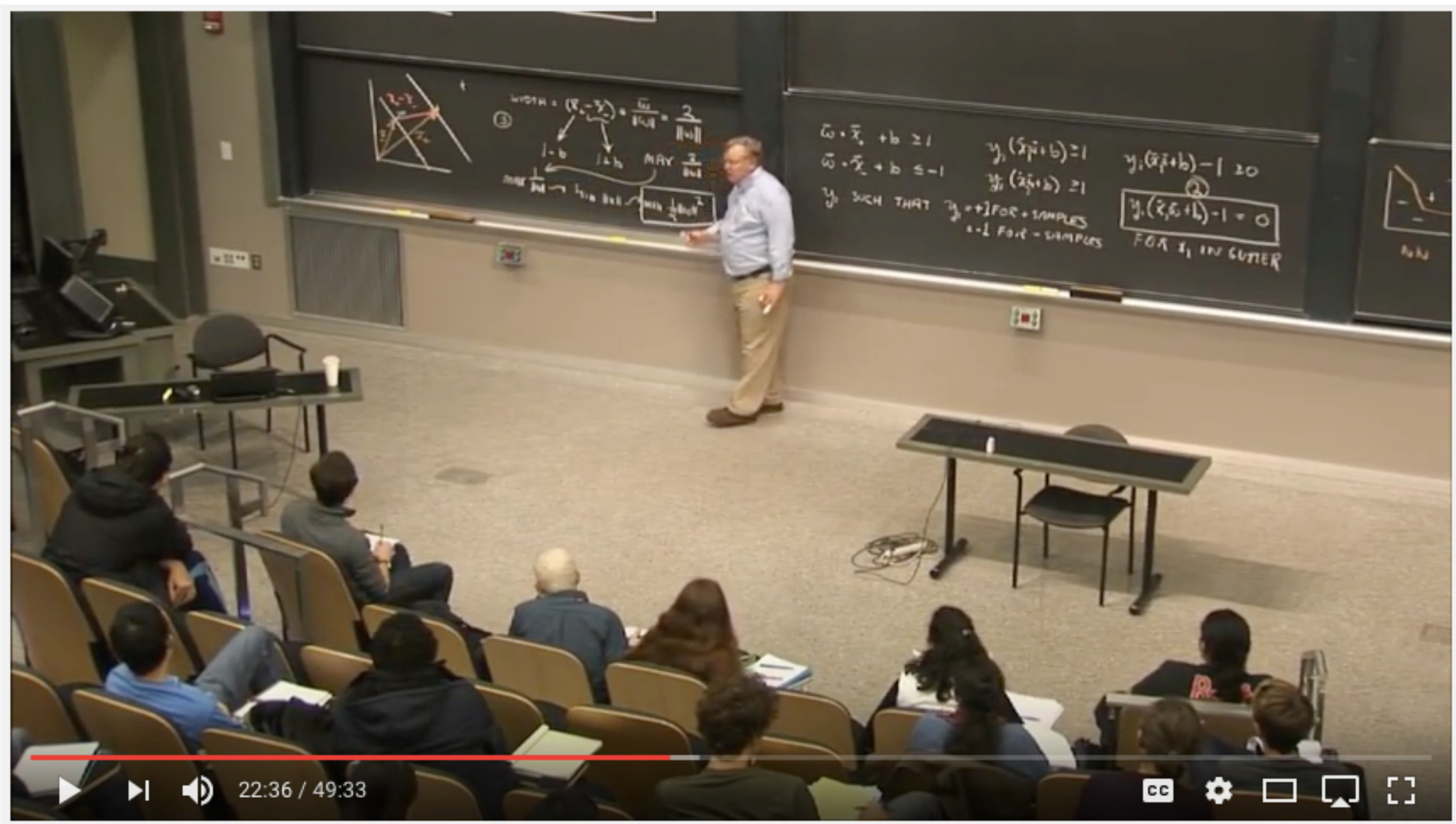
Watch: Patrick Winston, Support Vector Machines



https://www.youtube.com/watch?v=_PwhiWxHK8o

Next Lecture:

Soft Margin Classification,
Multi-class SVMs