

BBM444

FUNDAMENTALS OF COMPUTATIONAL PHOTOGRAPHY

Lecture #06 – Gradient-Domain Image Processing



HACETTEPE
UNIVERSITY
COMPUTER
VISION LAB

Erkut Erdem // Hacettepe University // Spring 2026

Today's Lecture

- Gradient-domain image processing
- Basics on images and gradients
- Integrable vector fields
- Poisson blending
- Flash/no-flash photography
- Gradient-domain rendering and cameras

Disclaimer: The material and slides for this lecture were borrowed from

—Ioannis Gkioulekas' 15-463/15-663/15-862 "Computational Photography" class

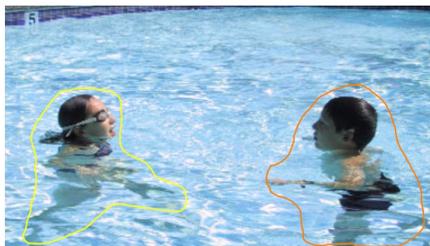
—Amit Agrawal's slides on "Gradient-Domain Based Flash/No-flash Photography"

—Adrien Gruson's slides on "Gradient-Domain Rendering"

—Davide Scaramuzza's tutorial on "Event-based Cameras"

Gradient-domain image processing

Application: Poisson blending



originals



copy-paste

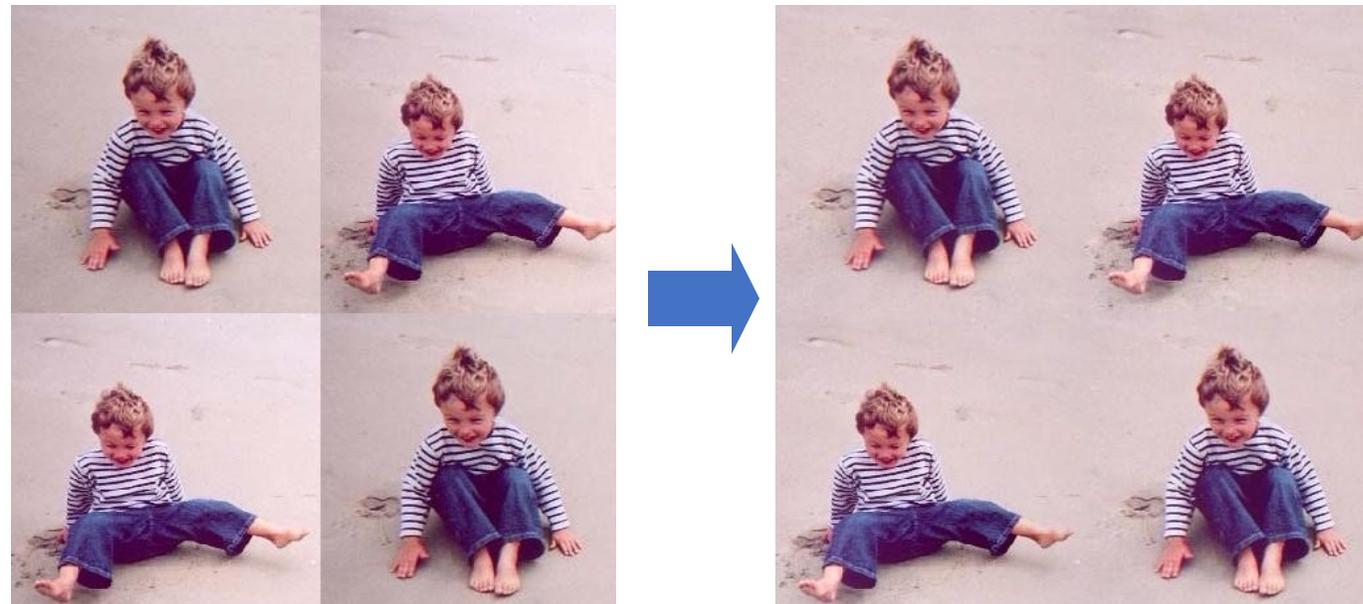


Poisson blending

More applications



Removing Glass Reflections



Seamless Image Stitching

Yet more applications



Fusing day and night photos



Tonemapping

Entire suite of image editing tools

GradientShop: A Gradient-Domain Optimization Framework for Image and Video Filtering

Pravin Bhat¹ C. Lawrence Zitnick² Michael Cohen^{1,2} Brian Curless¹
¹University of Washington ²Microsoft Research



(a) Input image



(b) Saliency-sharpening filter



(c) Pseudo-relighting filter



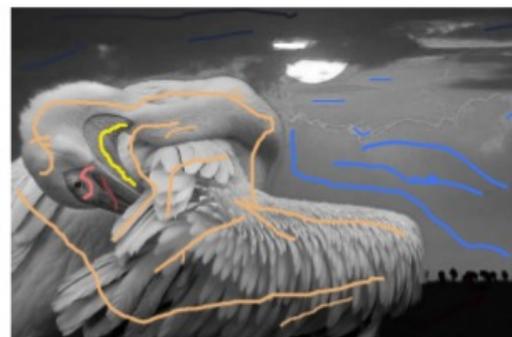
(d) Non-photorealistic rendering filter



(e) Compressed input-image



(f) De-blocking filter

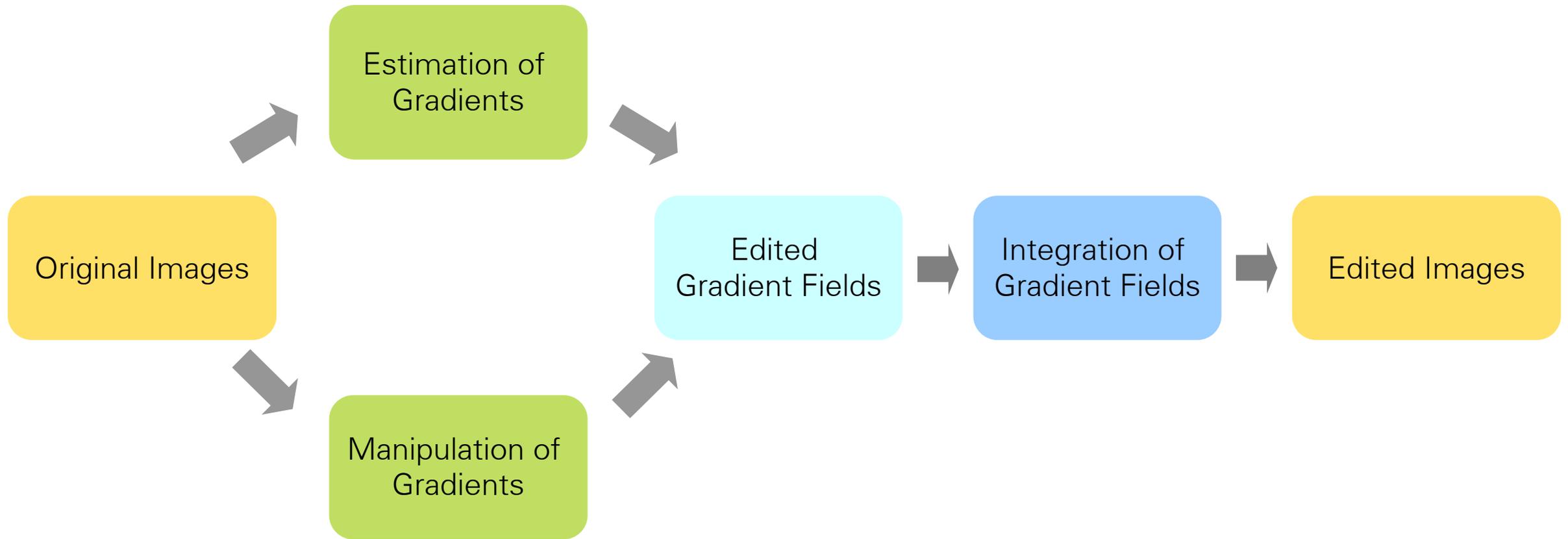


(g) User input for colorization



(h) Colorization filter

Main pipeline



Basics of gradients and fields

Some vector calculus definitions in 2D

Scalar field: a function assigning a scalar to every point in space.

$$I(x, y): \mathbb{R}^2 \rightarrow \mathbb{R}$$

Vector field: a function assigning a vector to every point in space.

$$[u(x, y) \quad v(x, y)]: \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

Can you think of examples of scalar fields and vector fields?

Some vector calculus definitions in 2D

Scalar field: a function assigning a scalar to every point in space.

$$I(x, y): \mathbb{R}^2 \rightarrow \mathbb{R}$$

Vector field: a function assigning a vector to every point in space.

$$[u(x, y) \quad v(x, y)]: \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

Can you think of examples of scalar fields and vector fields?

- A grayscale image is a scalar field.
- A two-channel image is a vector field.
- A three-channel (e.g., RGB) image is also a vector field, but of higher-dimensional range than what we will consider here.

Some vector calculus definitions in 2D

Nabla (or del): vector differential operator.

$$\nabla = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix}$$

Think of this as
a 2D vector.

Some vector calculus definitions in 2D

Nabla (or del): vector differential operator.

$$\nabla = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix}$$

Think of this as
a 2D vector.

Gradient (grad): product of nabla with a scalar field.

$$\nabla I(x, y) = ?$$

Divergence: inner product of nabla with a vector field.

$$\nabla \cdot [u(x, y) \quad v(x, y)] = ?$$

Curl: cross product of nabla with a vector field.

$$\nabla \times [u(x, y) \quad v(x, y)] = ?$$

Some vector calculus definitions in 2D

Nabla (or del): vector differential operator.

$$\nabla = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix}$$

Think of this as
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Gradient (grad): product of nabla with a scalar field.

$$\nabla I(x, y) = \begin{bmatrix} \frac{\partial I}{\partial x}(x, y) & \frac{\partial I}{\partial y}(x, y) \end{bmatrix}$$

What is the
dimension of this?

Divergence: inner product of nabla with a vector field.

$$\nabla \cdot [u(x, y) \quad v(x, y)] = \frac{\partial u}{\partial x}(x, y) + \frac{\partial v}{\partial y}(x, y)$$

What is the
dimension of this?

Curl: cross product of nabla with a vector field.

$$\nabla \times [u(x, y) \quad v(x, y)] = \left(\frac{\partial v}{\partial x}(x, y) - \frac{\partial u}{\partial y}(x, y) \right) \hat{k}$$

What is the
dimension of this?

Some vector calculus definitions in 2D

Nabla (or del): vector differential operator.

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$$\nabla I(x, y) = \begin{bmatrix} \frac{\partial I}{\partial x}(x, y) & \frac{\partial I}{\partial y}(x, y) \end{bmatrix}$$

This is a
vector field.

Divergence: inner product of nabla with a vector field.

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Curl: cross product of nabla with a vector field.

$$\nabla \times [u(x, y) \quad v(x, y)] = \left(\frac{\partial v}{\partial x}(x, y) - \frac{\partial u}{\partial y}(x, y) \right) \hat{k}$$

This is a vector field.
This is a scalar field.

Combinations

Curl of the gradient:

$$\nabla \times \nabla I(x, y) = ?$$

Divergence of the gradient:

$$\nabla \cdot \nabla I(x, y) = ?$$

Combinations

Curl of the gradient:

$$\nabla \times \nabla I(x, y) = \frac{\partial^2}{\partial y \partial x} I(x, y) - \frac{\partial^2}{\partial x \partial y} I(x, y)$$

Divergence of the gradient:

$$\nabla \cdot \nabla I(x, y) = \frac{\partial^2}{\partial x^2} I(x, y) + \frac{\partial^2}{\partial y^2} I(x, y) \equiv \Delta I(x, y)$$

Laplacian: scalar differential operator.

$$\Delta \equiv \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Inner product of
del with itself!

Simplified notation

Nabla (or del): vector differential operator.

$$\nabla = [\quad x \quad y]$$

Think of this as
a 2D vector.

Gradient (grad): product of nabla with a scalar field.

$$\nabla I = [I_x \quad I_y]$$

This is a
vector field.

Divergence: inner product of nabla with a vector field.

$$\nabla \cdot [u \quad v] = u_x + v_y$$

This is a
scalar field.

Curl: cross product of nabla with a vector field.

$$\nabla \times [u \quad v] = (v_x - u_y) \hat{k}$$

This is a vector field.
This is a scalar field.

Simplified notation

Curl of the gradient:

$$\nabla \times \nabla I = I_{yx} - I_{xy}$$

Divergence of the gradient:

$$\nabla \cdot \nabla I = I_{xx} + I_{yy} \equiv \Delta I$$

Laplacian: scalar differential operator.

$$\Delta \equiv \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Inner product of
del with itself!

Image representation

We can treat grayscale images as scalar fields (i.e., two dimensional functions)

$$I(x, y): \mathbb{R}^2 \rightarrow \mathbb{R}$$

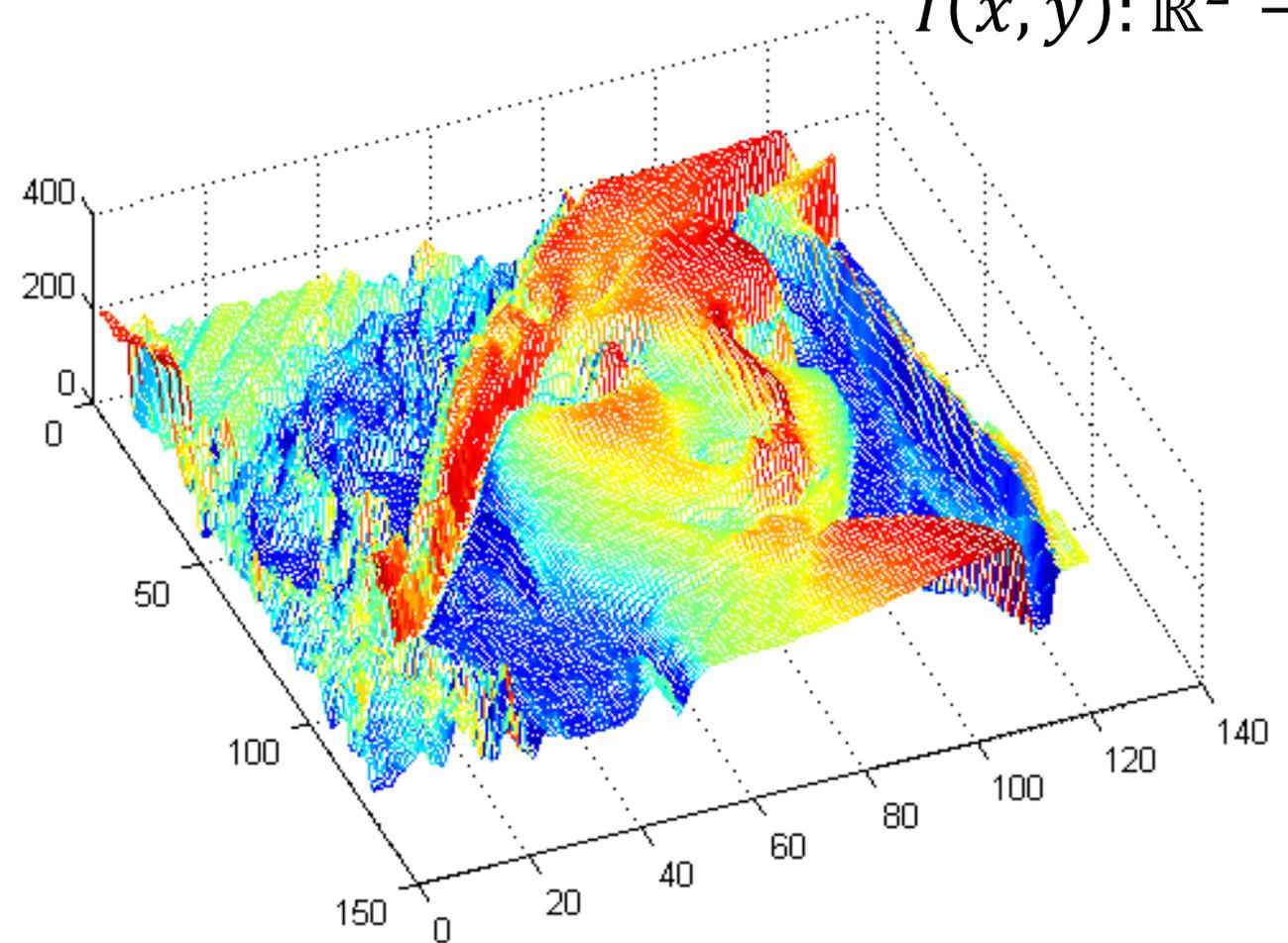


Image gradients

Convert the scalar field into a vector field through differentiation.



scalar field $I(x, y): \mathbb{R}^2 \rightarrow \mathbb{R}$



vector field $\nabla I(x, y) = \begin{bmatrix} \frac{\partial I}{\partial x}(x, y) & \frac{\partial I}{\partial y}(x, y) \end{bmatrix}$

Image gradients

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- How do we do this differentiation in real discrete images?

Finite differences

High-school reminder: definition of a derivative using forward difference.

$$\frac{\partial I}{\partial x}(x, y) = \lim_{h \rightarrow 0} \frac{I(x + h, y) - I(x, y)}{h}$$

For discrete scalar fields: remove limit and set $h = 1$.

$$\frac{\partial I}{\partial x}(x, y) = I(x + 1, y) - I(x, y)$$

What convolution kernel does this correspond to?

Finite differences

High-school reminder: definition of a derivative using forward difference.

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For discrete scalar fields: remove limit and set $h = 1$.

$$\frac{\partial I}{\partial x}(x, y) = I(x + 1, y) - I(x, y)$$

-1	1	?
1	-1	?

Finite differences

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$$\frac{\partial I}{\partial x}(x, y) = \lim_{h \rightarrow 0} \frac{I(x + h, y) - I(x, y)}{h}$$

For discrete scalar fields: remove limit and set $h = 1$.

$$\frac{\partial I}{\partial x}(x, y) = I(x + 1, y) - I(x, y)$$

partial-x derivative filter

1	-1
---	----

Note: common to use central difference, but we will not use it in this lecture.

$$\frac{\partial I}{\partial x}(x, y) = \frac{I(x + 1, y) - I(x - 1, y)}{2}$$

Finite differences

High-school reminder: definition of a derivative using forward difference.

$$\frac{\partial I}{\partial x}(x, y) = \lim_{h \rightarrow 0} \frac{I(x + h, y) - I(x, y)}{h}$$

For discrete scalar fields: remove limit and set $h = 1$.

$$\frac{\partial I}{\partial x}(x, y) = I(x + 1, y) - I(x, y)$$

partial-x derivative filter

1	-1
---	----

Similarly for partial-y derivative.

$$\frac{\partial I}{\partial y}(x, y) = I(x, y + h) - I(x, y)$$

partial-y derivative filter

1
-1

Discrete Laplacian

How do we compute the image Laplacian?

$$\Delta I(x, y) = \frac{\partial^2 I}{\partial x^2}(x, y) + \frac{\partial^2 I}{\partial y^2}(x, y)$$

Discrete Laplacian

How do we compute the image Laplacian?

$$\Delta I(x, y) = \frac{\partial^2 I}{\partial x^2}(x, y) + \frac{\partial^2 I}{\partial y^2}(x, y)$$

Use multiple applications of the discrete derivative filters:

$$\underbrace{\begin{array}{|c|c|} \hline 1 & -1 \\ \hline \end{array} * \begin{array}{|c|c|} \hline 1 & -1 \\ \hline \end{array}} + \underbrace{\begin{array}{|c|c|} \hline 1 & 1 \\ \hline -1 & -1 \\ \hline \end{array} * \begin{array}{|c|c|} \hline 1 & 1 \\ \hline -1 & -1 \\ \hline \end{array}} = ?$$

What is this?

What is this?

Discrete Laplacian

How do we compute the image Laplacian?

$$\Delta I(x, y) = \frac{\partial^2 I}{\partial x^2}(x, y) + \frac{\partial^2 I}{\partial y^2}(x, y)$$

Use multiple applications of the discrete derivative filters:

$$\underbrace{\begin{array}{|c|c|} \hline 1 & -1 \\ \hline \end{array} * \begin{array}{|c|c|} \hline 1 & -1 \\ \hline \end{array}}_{\frac{\partial^2 I}{\partial x^2}(x, y)} + \underbrace{\begin{array}{|c|} \hline 1 \\ \hline -1 \\ \hline \end{array} * \begin{array}{|c|} \hline 1 \\ \hline -1 \\ \hline \end{array}}_{\frac{\partial^2 I}{\partial y^2}(x, y)} =$$

Laplacian filter

0	1	0
1	-4	1
0	1	0

Discrete Laplacian

How do we compute the image Laplacian?

$$\Delta I(x, y) = \frac{\partial^2 I}{\partial x^2}(x, y) + \frac{\partial^2 I}{\partial y^2}(x, y)$$

Very important to:

- use consistent derivative and Laplacian filters.
- account for boundary shifting and padding from convolution.

Use multiple applications of the discrete derivative filters:

$$\underbrace{\begin{array}{|c|c|} \hline 1 & -1 \\ \hline \end{array} * \begin{array}{|c|c|} \hline 1 & -1 \\ \hline \end{array}}_{\frac{\partial^2 I}{\partial x^2}(x, y)} + \underbrace{\begin{array}{|c|} \hline 1 \\ \hline -1 \\ \hline \end{array} * \begin{array}{|c|} \hline 1 \\ \hline -1 \\ \hline \end{array}}_{\frac{\partial^2 I}{\partial y^2}(x, y)} = \begin{array}{|c|c|c|} \hline 0 & 1 & 0 \\ \hline 1 & -4 & 1 \\ \hline 0 & 1 & 0 \\ \hline \end{array}$$

Laplacian filter

Warning!

Very important for the techniques discussed in this lecture to:

- use consistent derivative and Laplacian filters.
- account for boundary shifting and padding from convolution.

A correct implementation of differential operators should pass the following test:

Equality holds at all pixels except boundary
(first and last row, first and last column).

$$\underbrace{\nabla \cdot (\nabla (\text{img}))}_{\text{divergence operator}} = \underbrace{\Delta (\text{img})}_{\text{Laplacian operator}}$$

Typically requires implementing derivatives in various differential operators differently.

Image gradients

Convert the scalar field into a vector field through differentiation.



scalar field $I(x, y): \mathbb{R}^2 \rightarrow \mathbb{R}$



vector field $\nabla I(x, y) = \begin{bmatrix} \frac{\partial I}{\partial x}(x, y) & \frac{\partial I}{\partial y}(x, y) \end{bmatrix}$

Image gradients

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- Image gradients are very informative!

Application - Seam Carving



[Shai & Avidan, SIGGRAPH 2007]

Application - Seam Carving



Content-aware resizing



Traditional resizing

[Shai & Avidan, SIGGRAPH 2007]

Application - Seam Carving



Shai Avidan
Mitsubishi Electric Research Lab
Ariel Shamir
The interdisciplinary Center & MERL

Seam Carving: Main idea

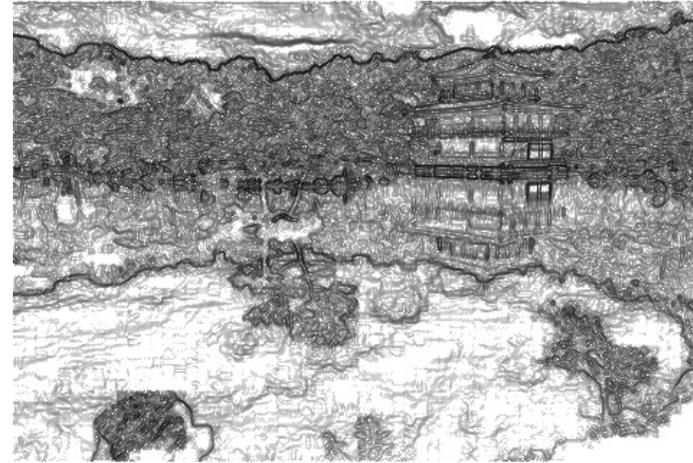


Content-aware resizing

Intuition:

- Preserve the most “interesting/important” content
→ Prefer to remove pixels with low gradient energy
- To reduce or increase size in one dimension, remove irregularly shaped “seams”
→ Optimal solution via dynamic programming.

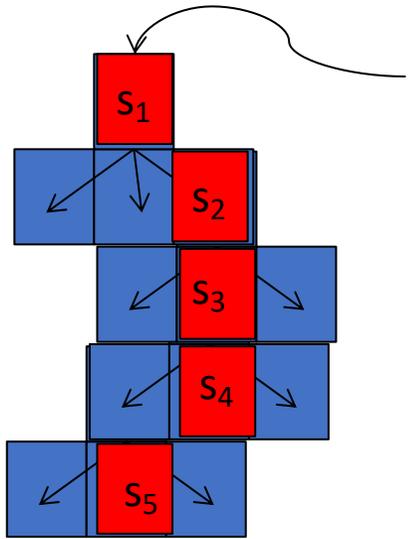
Seam Carving: Main idea



$$Energy(f) = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}$$

- Want to remove seams where they won't be very noticeable:
- Measure "energy" as gradient magnitude
- Choose seam based on **minimum total energy path** across image, subject to 8-connectedness.

Seam Carving: Algorithm



$$Energy(f) = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}$$

- Let a **vertical seam** \mathbf{s} consist of h positions that form an 8-connected path.
- Let the **cost of a seam** be: $Cost(\mathbf{s}) = \sum_{i=1}^h Energy(f(s_i))$
- **Optimal seam** minimizes this cost.
- Compute it efficiently with **dynamic programming**: $\mathbf{s}^* = \min_{\mathbf{s}} Cost(\mathbf{s})$

Image gradients

Convert the scalar field into a vector field through differentiation.



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Image gradients

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- How do we do this differentiation in real discrete images?
- Can we go in the opposite direction, from gradients to images?

Vector field integration

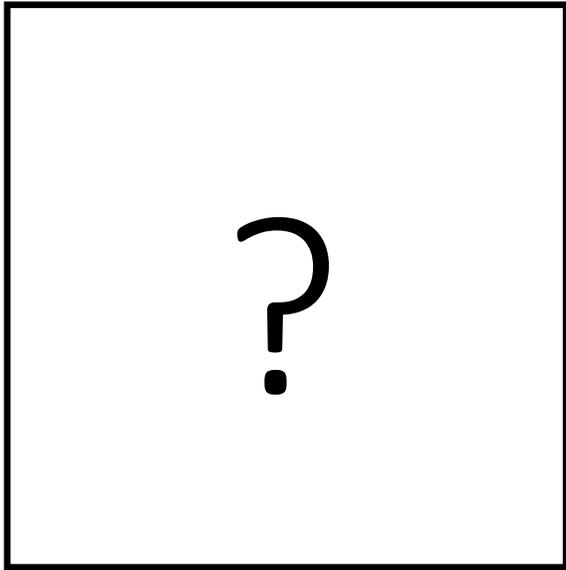
Two fundamental questions:

- When is integration of a vector field possible?
- How can integration of a vector field be performed?

Integrable vector fields

Integrable fields

Given an arbitrary vector field (u, v) , can we always integrate it into a scalar field I ?



$$I(x, y): \mathbb{R}^2 \rightarrow \mathbb{R}$$



$$u(x, y): \mathbb{R}^2 \rightarrow \mathbb{R}$$



$$v(x, y): \mathbb{R}^2 \rightarrow \mathbb{R}$$

such that

$$\frac{\partial I}{\partial x}(x, y) = u(x, y)$$

$$\frac{\partial I}{\partial y}(x, y) = v(x, y)$$

Property of twice-differentiable functions

Curl of the gradient field should be zero:

$$\nabla \times \nabla I = I_{yx} - I_{xy} = 0$$

What does that mean intuitively?

Property of twice-differentiable functions

Curl of the gradient field should be zero:

$$\nabla \times \nabla I = I_{yx} - I_{xy} = 0$$

What does that mean intuitively?

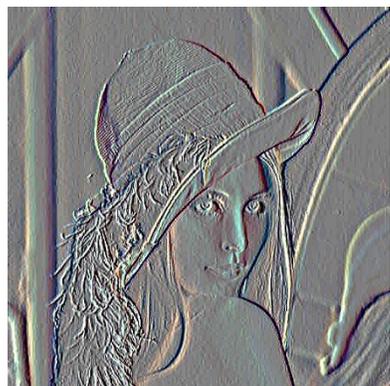
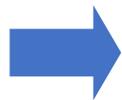
- Same result independent of order of differentiation.

$$I_{yx} = I_{xy}$$

Demonstration



image I



I_x



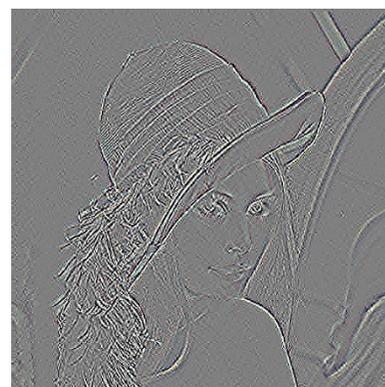
I_y



ΔI

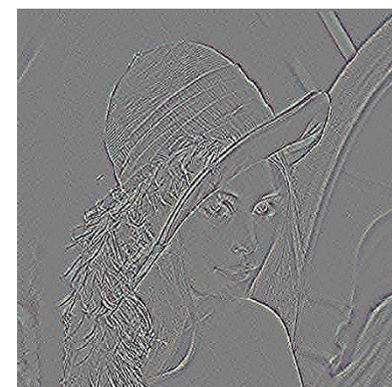


$\nabla \times \nabla I$



I_{xy}

=



I_{yx}

Property of twice-differentiable functions

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What does that mean intuitively?

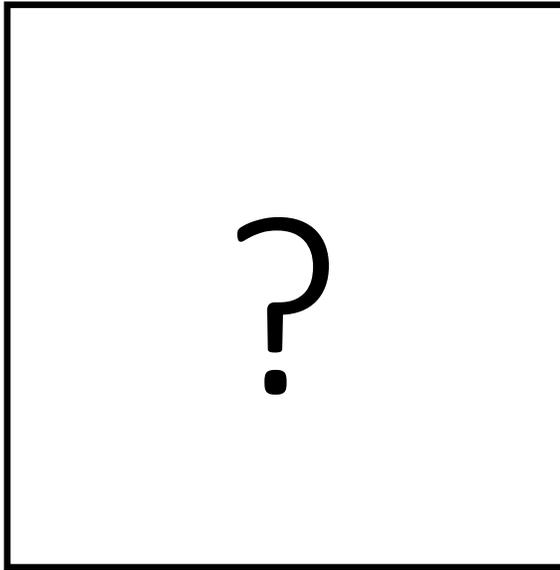
- Same result independent of order of differentiation.

$$I_{yx} = I_{xy}$$

Can you use this property to derive an integrability condition?

Integrable fields

Given an arbitrary vector field (u, v) , can we always integrate it into a scalar field I ?



$$I(x, y): \mathbb{R}^2 \rightarrow \mathbb{R}$$

such that

$$\frac{\partial I}{\partial x}(x, y) = u(x, y)$$
$$\frac{\partial I}{\partial y}(x, y) = v(x, y)$$



$$u(x, y): \mathbb{R}^2 \rightarrow \mathbb{R}$$



$$v(x, y): \mathbb{R}^2 \rightarrow \mathbb{R}$$

Only if:

$$\nabla \times \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix} = 0 \Rightarrow \frac{\partial u}{\partial y}(x, y) = \frac{\partial v}{\partial x}(x, y)$$

Vector field integration

Two fundamental questions:

- When is integration of a vector field possible?
 - Use curl to check for equality of mixed partial second derivatives.
- How can integration of a vector field be performed?

Different types of integration problems

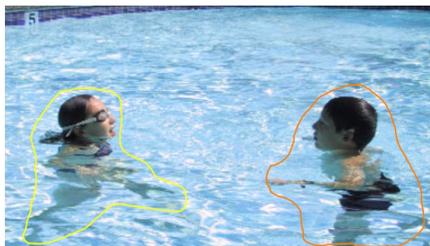
- Reconstructing height fields from gradients
Applications: shape from shading, photometric stereo
- Manipulating image gradients
Applications: tonemapping, image editing, matting, fusion, mosaics
- Manipulation of 3D gradients
Applications: mesh editing, video operations

Key challenge: Most vector fields in applications are not integrable.

- Integration must be done approximately.

A prototypical integration problem: Poisson blending

Application: Poisson blending



originals



copy-paste



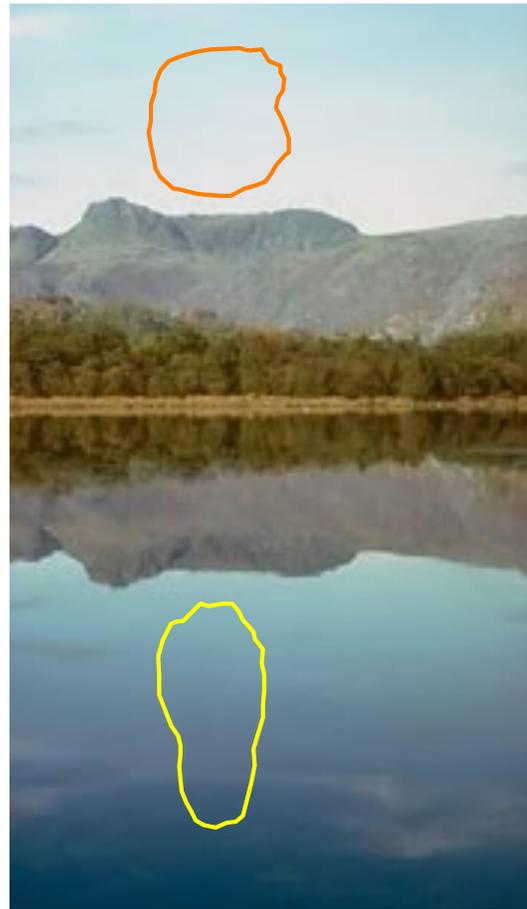
Poisson blending

Key idea

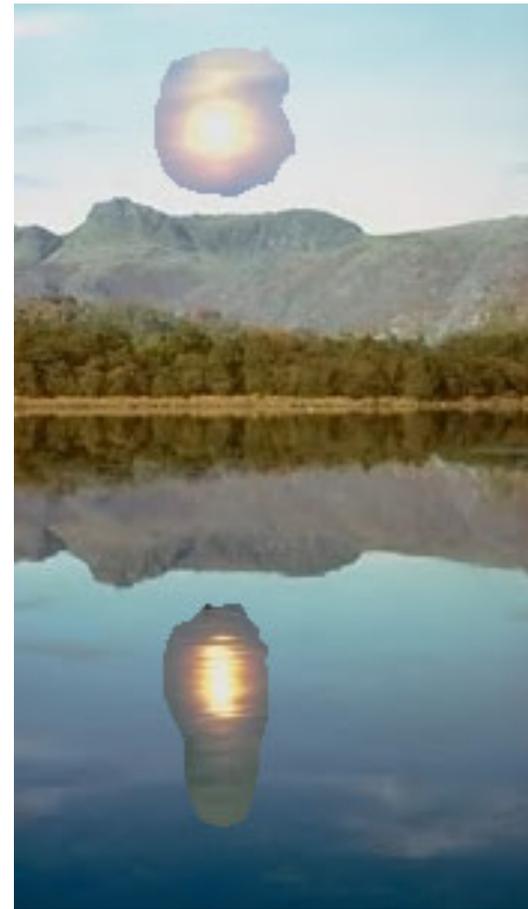
When blending, retain the gradient information as best as possible



source



destination



copy-paste



Poisson blending

Definitions and notation



Notation

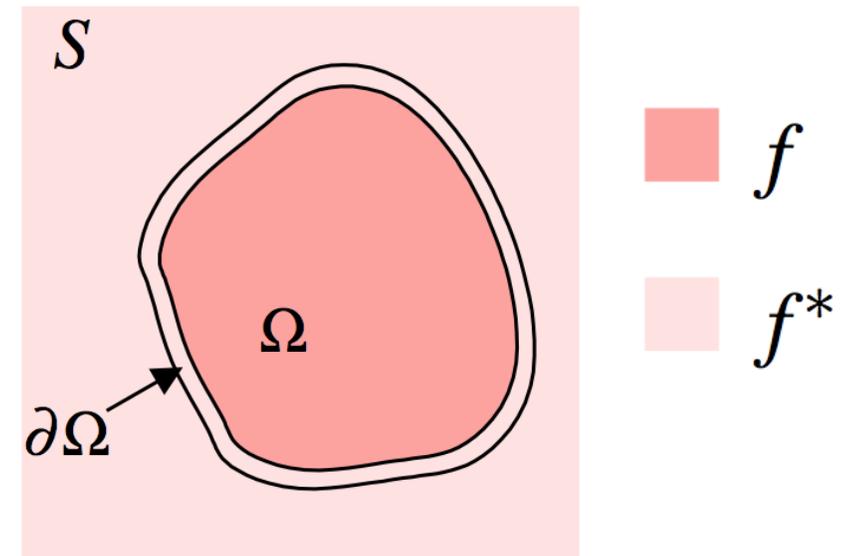
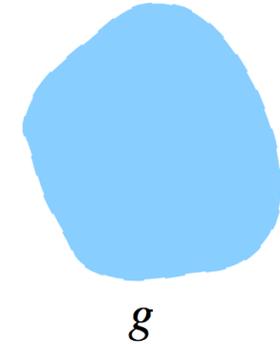
g : source function

S : destination

Ω : destination domain

f : interpolant function

f^* : destination function



Which one is the unknown?

Definitions and notation



Notation

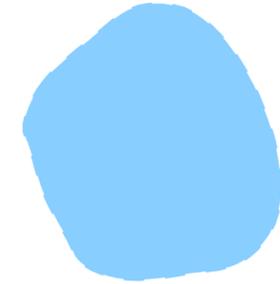
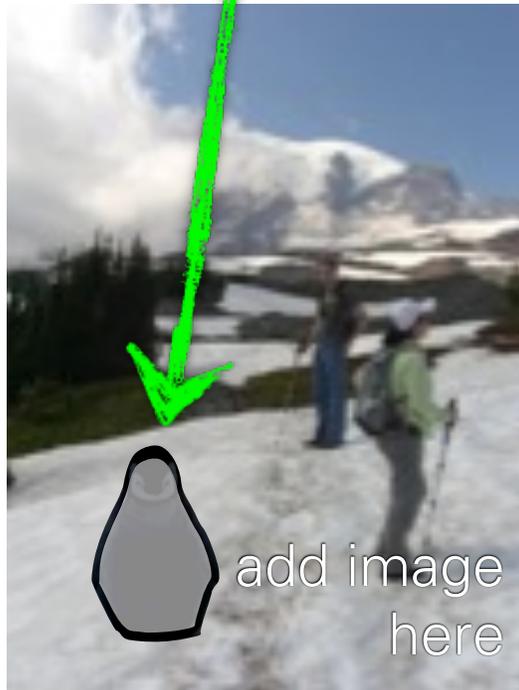
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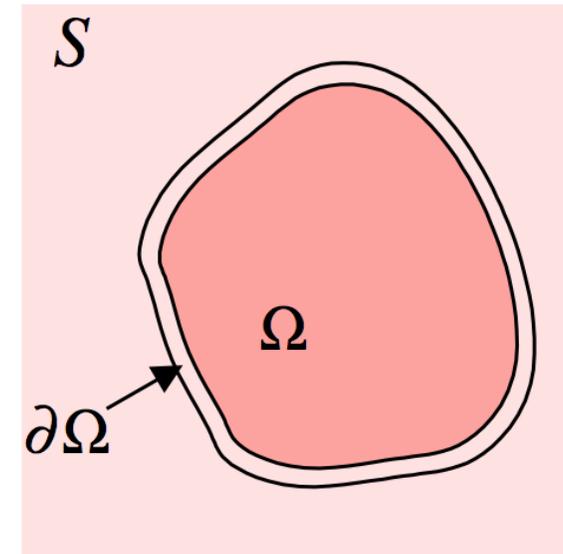
Ω : destination domain

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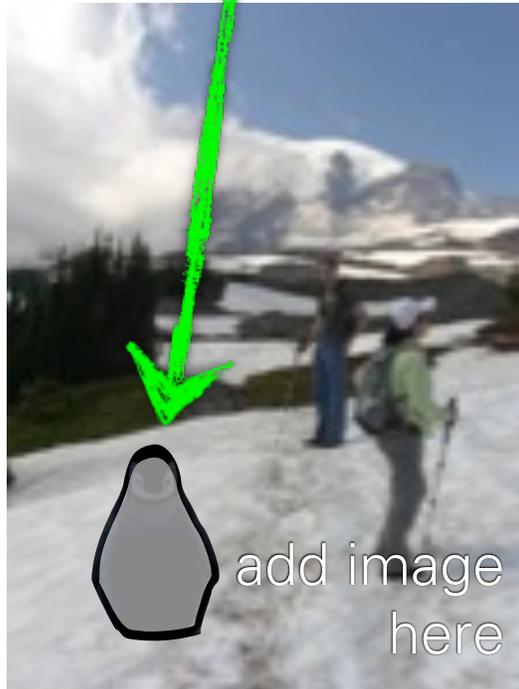
g



How should we determine f ?

- Should it be similar to g ?
- Should it be similar to f^* ?

Definitions and notation



Notation

g : source function

S : destination

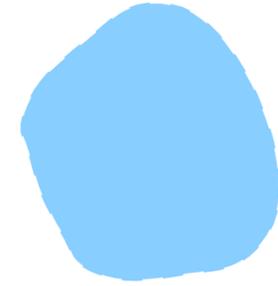
Ω : destination domain

f : interpolant function

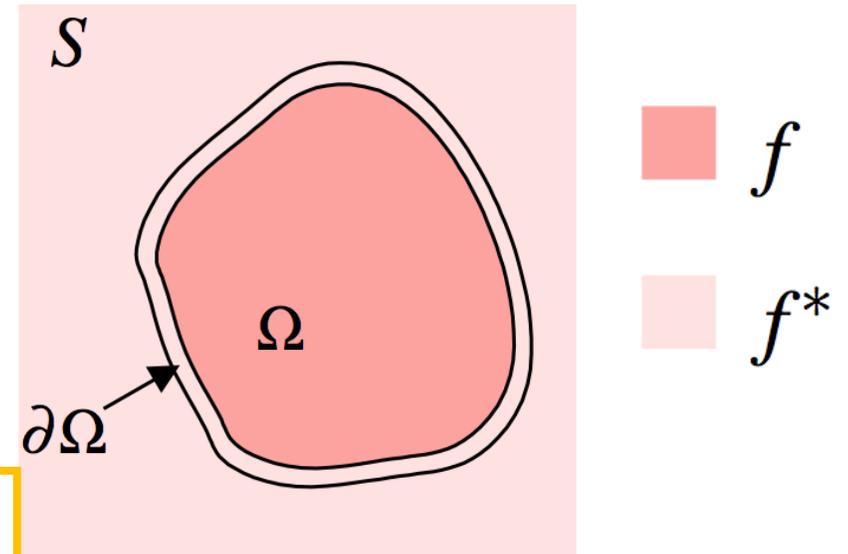
f^* : destination function

Find f such that:

- $\nabla f = \nabla g$ inside Ω .
- $f = f^*$ at the boundary $\partial\Omega$.



g



Poisson blending: integrate vector field ∇g with Dirichlet boundary conditions f^* .

Least-squares integration and the Poisson problem

Least-squares integration

“Variational” means optimization where the unknown is an entire function

Variational problem

$$\min_f \iint_{\Omega} |\nabla f - \mathbf{v}|^2 \quad \text{with} \quad f|_{\partial\Omega} = f^*|_{\partial\Omega}$$

what does this term do?

what does this term do?

Recall ...

Nabla operator definition

$$\nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right]$$

is this known?

$$\mathbf{v} = (u, v)$$

Least-squares integration

“Variational” means optimization where the unknown is an entire function

Variational problem

$$\min_f \iint_{\Omega} |\nabla f - \mathbf{v}|^2 \quad \text{with} \quad f|_{\partial\Omega} = f^*|_{\partial\Omega}$$

gradient of f looks like vector field \mathbf{v}

f is equivalent to f^* at the boundaries

Why do we need boundary conditions for least-squares integration?

Recall ...

Nabla operator definition

$$\nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right]$$

Yes, this is the vector field we are integrating

$$\mathbf{v} = (u, v)$$

Equivalently

The stationary point of the variational loss is the solution to the:

Poisson equation (with Dirichlet boundary conditions)

$$\Delta f = \operatorname{div} \mathbf{v} \quad \text{over } \Omega, \quad \text{with } f|_{\partial\Omega} = f^*|_{\partial\Omega}$$

what does this term do?

This can be derived using the Euler-Lagrange equation.

Recall ...

Laplacian $\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$

Divergence $\operatorname{div} \mathbf{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$

Input vector field:

$$\mathbf{v} = (u, v)$$

Equivalently

The stationary point of the variational loss is the solution to the:

Poisson equation (with Dirichlet boundary conditions)

$$\Delta f = \operatorname{div} \mathbf{v} \quad \text{over } \Omega, \quad \text{with } f|_{\partial\Omega} = f^*|_{\partial\Omega}$$

Laplacian of f same as
divergence of vector field \mathbf{v}

Recall ...

Laplacian $\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$

Divergence $\operatorname{div} \mathbf{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$

Input vector field:

$$\mathbf{v} = (u, v)$$

This can be
derived
using the
Euler-
Lagrange
equation.

In the Poisson blending example...

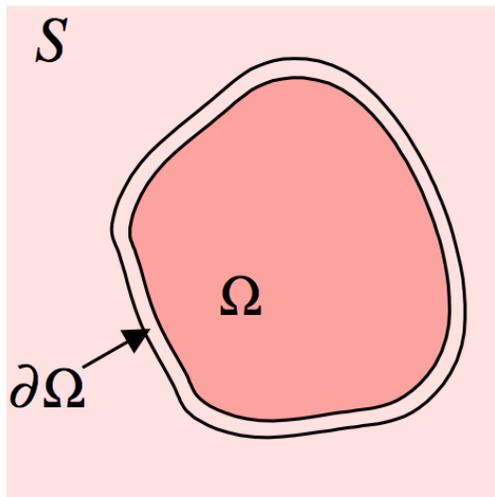
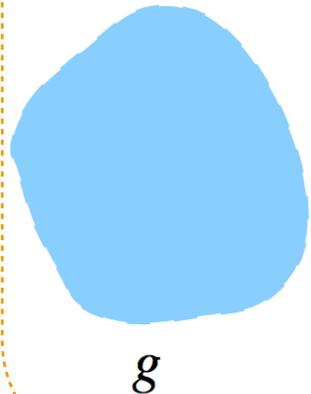
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Poisson equation (with Dirichlet boundary conditions)

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Find f such that:

- $\nabla f = \nabla g$ inside Ω .
- $f = f^*$ at the boundary $\partial\Omega$.



What does the input vector field equal in Poisson blending?

$$\mathbf{v} = (u, v) =$$

In the Poisson blending example...

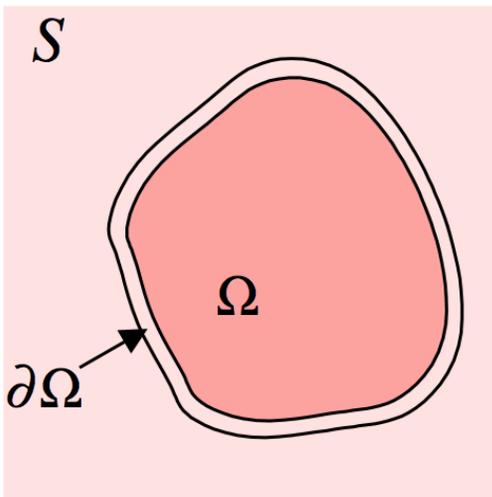
The stationary point of the variational loss is the solution to the:

Poisson equation (with Dirichlet boundary conditions)

$$\Delta f = \operatorname{div} \mathbf{v} \quad \text{over } \Omega, \quad \text{with } f|_{\partial\Omega} = f^*|_{\partial\Omega}$$

Find f such that:

- $\nabla f = \nabla g$ inside Ω .
- $f = f^*$ at the boundary $\partial\Omega$.



What does the input vector field equal in Poisson blending?

$$\mathbf{v} = (u, v) = \nabla g$$

What does the divergence of the input vector field equal in Poisson blending?

$$\operatorname{div} \mathbf{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} =$$

In the Poisson blending example...

The stationary point of the variational loss is the solution to the:

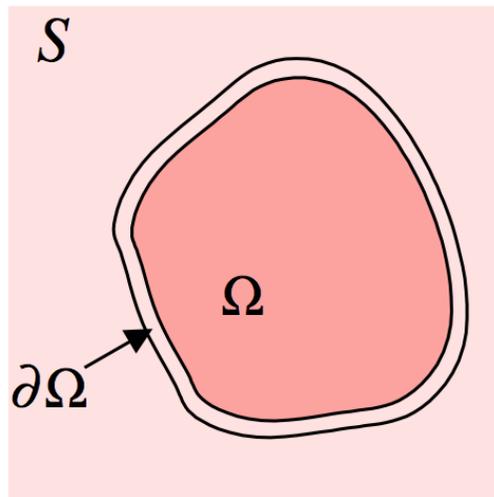
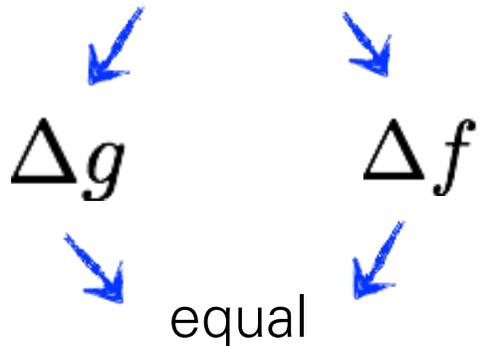
Poisson equation (with Dirichlet boundary conditions)

$$\Delta f = \operatorname{div} \mathbf{v} \quad \text{over } \Omega, \quad \text{with } f|_{\partial\Omega} = f^*|_{\partial\Omega}$$

Find f such that:

- $\nabla f = \nabla g$ inside Ω .
- $f = f^*$ at the boundary $\partial\Omega$.

so make these ...



What does the input vector field equal in Poisson blending?

$$\mathbf{v} = (u, v) = \nabla g$$

What does the divergence of the input vector field equal in Poisson blending?

$$\operatorname{div} \mathbf{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \Delta g$$

Equivalently

The stationary point of the variational loss is the solution to the:

Poisson equation (with Dirichlet boundary conditions)

$$\Delta f = \operatorname{div} \mathbf{v} \quad \text{over } \Omega, \quad \text{with } f|_{\partial\Omega} = f^*|_{\partial\Omega}$$

How do we solve the Poisson equation?

Recall ...

Laplacian $\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$

Divergence $\operatorname{div} \mathbf{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$

Input vector field:

$$\mathbf{v} = (u, v)$$

Discretization of the Poisson equation

Poisson equation (with Dirichlet boundary conditions)

$$\Delta f = \operatorname{div} \mathbf{v} \quad \text{over } \Omega, \quad \text{with } f|_{\partial\Omega} = f^*|_{\partial\Omega}$$

Recall ...

Laplacian
filter

0	1	0
1	-4	1
0	1	0

partial-x
derivative filter

1	-1
---	----

partial-y
derivative filter

1
-1

So for each pixel, do:

$$(\Delta f)(x, y) = (\nabla \cdot \mathbf{v})(x, y)$$

Or for discrete images:

Discretization of the Poisson equation

Poisson equation (with Dirichlet boundary conditions)

$$\Delta f = \operatorname{div} \mathbf{v} \quad \text{over } \Omega, \quad \text{with } f|_{\partial\Omega} = f^*|_{\partial\Omega}$$

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---	----

partial-y
derivative filter

1
-1

So for each pixel, do:

$$(\Delta f)(x, y) = (\nabla \cdot \mathbf{v})(x, y)$$

Or for discrete images:

$$\begin{aligned} & -4f(x, y) + f(x + 1, y) + f(x - 1, y) \\ & \quad + f(x, y + 1) + f(x, y - 1) \\ & = u(x + 1, y) - u(x, y) + v(x, y + 1) \\ & \quad - v(x, y) \end{aligned}$$

Discretization of the Poisson equation

Poisson equation (with Dirichlet boundary conditions)

$$\Delta f = \operatorname{div} \mathbf{v} \quad \text{over } \Omega, \quad \text{with } f|_{\partial\Omega} = f^*|_{\partial\Omega}$$

Recall ...

Laplacian
filter

0	1	0
1	-4	1
0	1	0

partial-x
derivative filter

1	-1
---	----

partial-y
derivative filter

1
-1

So for each pixel, do (more compact notation):

$$(\Delta f)_p = (\nabla \cdot \mathbf{v})_p$$

Or for discrete images (more compact notation):

$$-4f_p + \sum_{q \in N_p} f_q = (u_x)_p + (v_y)_p$$

We can rewrite this as

linear equation
of P variables

$$-4f_p + \sum_{q \in N_p} f_q = (u_x)_p + (v_y)_p$$

one for each
pixel $p = 1, \dots, P$

In vector form:

(each pixel adds another 'sparse' row here)

$$\begin{matrix} \rightarrow & \begin{bmatrix} \vdots \\ 0 & \dots & 1 & \dots & 1 & -4 & 1 & \dots & 1 & \dots & 0 \\ \vdots \end{bmatrix} & \cdot & \underbrace{\begin{bmatrix} f_1 \\ \vdots \\ f_{q_1} \\ \vdots \\ f_{q_2} \\ \vdots \\ f_p \\ \vdots \\ f_{q_3} \\ \vdots \\ f_{q_4} \\ \vdots \\ f_P \end{bmatrix}}_f & = & \underbrace{\begin{bmatrix} (\nabla \cdot \mathbf{v})_1 \\ \vdots \\ (\nabla \cdot \mathbf{v})_{q_1} \\ \vdots \\ (\nabla \cdot \mathbf{v})_{q_2} \\ \vdots \\ (\nabla \cdot \mathbf{v})_p \\ \vdots \\ (\nabla \cdot \mathbf{v})_{q_3} \\ \vdots \\ (\nabla \cdot \mathbf{v})_{q_4} \\ \vdots \\ (\nabla \cdot \mathbf{v})_P \end{bmatrix}}_b \end{matrix}$$

what is
this?

what are the sizes of these?

A

f

b

We can rewrite this as

linear equation
of P variables

$$-4f_p + \sum_{q \in N_p} f_q = (u_x)_p + (v_y)_p$$

one for each
pixel $p = 1, \dots, P$

In vector form:

(each pixel adds another 'sparse' row here)

$$\underbrace{\begin{bmatrix} \vdots & & & & & & & & & & \\ 0 & \dots & 1 & \dots & 1 & -4 & 1 & \dots & 1 & \dots & 0 \\ \vdots & & & & & & & & & & \end{bmatrix}}_A \cdot \underbrace{\begin{bmatrix} f_1 \\ \vdots \\ f_{q_1} \\ \vdots \\ f_{q_2} \\ \vdots \\ f_p \\ \vdots \\ f_{q_3} \\ \vdots \\ f_{q_4} \\ \vdots \\ f_P \end{bmatrix}}_f = \underbrace{\begin{bmatrix} (\nabla \cdot \mathbf{v})_1 \\ \vdots \\ (\nabla \cdot \mathbf{v})_{q_1} \\ \vdots \\ (\nabla \cdot \mathbf{v})_{q_2} \\ \vdots \\ (\nabla \cdot \mathbf{v})_p \\ \vdots \\ (\nabla \cdot \mathbf{v})_{q_3} \\ \vdots \\ (\nabla \cdot \mathbf{v})_{q_4} \\ \vdots \\ (\nabla \cdot \mathbf{v})_P \end{bmatrix}}_b$$

We call this the
Laplacian matrix

Laplacian matrix

For a $m \times n$ image, we can re-organize this matrix into block tridiagonal form as:

$$A_{mn \times mn} = \begin{bmatrix} D & I & 0 & 0 & 0 & \dots & 0 \\ I & D & I & 0 & 0 & \dots & 0 \\ 0 & I & D & I & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & I & D & I & 0 \\ 0 & \dots & \dots & 0 & I & D & I \\ 0 & \dots & \dots & \dots & 0 & I & D \end{bmatrix}$$

This requires ordering pixels in column-major order.

$I_{m \times m}$ is the $m \times m$ identity matrix

$$D_{m \times m} =$$

$$\begin{bmatrix} -4 & 1 & 0 & 0 & 0 & \dots & 0 \\ 1 & -4 & 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & -4 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & -4 & 1 & 0 \\ 0 & \dots & \dots & 0 & 1 & -4 & 1 \\ 0 & \dots & \dots & \dots & 0 & 1 & -4 \end{bmatrix}$$

Discrete Poisson equation

Poisson equation (with Dirichlet boundary conditions)

$$\Delta f = \operatorname{div} \mathbf{v} \quad \text{over } \Omega, \quad \text{with } f|_{\partial\Omega} = f^*|_{\partial\Omega}$$

After discretization, equivalent to:

$$\begin{bmatrix} D & I & 0 & 0 & 0 & \dots & 0 \\ I & D & I & 0 & 0 & \dots & 0 \\ 0 & I & D & I & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & I & D & I & 0 \\ 0 & \dots & \dots & 0 & I & D & I \\ 0 & \dots & \dots & \dots & 0 & I & D \end{bmatrix} \cdot \begin{bmatrix} f_1 \\ \vdots \\ f_{q_1} \\ \vdots \\ f_{q_2} \\ \vdots \\ f_p \\ \vdots \\ f_{q_3} \\ \vdots \\ f_{q_4} \\ \vdots \\ f_p \end{bmatrix} = \begin{bmatrix} (\nabla \cdot \mathbf{v})_1 \\ \vdots \\ (\nabla \cdot \mathbf{v})_{q_1} \\ \vdots \\ (\nabla \cdot \mathbf{v})_{q_2} \\ \vdots \\ (\nabla \cdot \mathbf{v})_p \\ \vdots \\ (\nabla \cdot \mathbf{v})_{q_3} \\ \vdots \\ (\nabla \cdot \mathbf{v})_{q_4} \\ \vdots \\ (\nabla \cdot \mathbf{v})_p \end{bmatrix}$$

Linear system of equations:

$$A f = b$$

How would you solve this?

WARNING: requires special treatment at the borders
(target boundary values are same as source)

Solving the linear system

Convert the system to a linear least-squares problem:

$$E_{LLS} = \|\mathbf{A}f - \mathbf{b}\|^2$$

Expand the error:

$$E_{LLS} = f^\top (\mathbf{A}^\top \mathbf{A})f - 2f^\top (\mathbf{A}^\top \mathbf{b}) + \|\mathbf{b}\|^2$$

Minimize the error:

Set derivative to 0 $(\mathbf{A}^\top \mathbf{A})f = \mathbf{A}^\top \mathbf{b}$

Solve for x $f = (\mathbf{A}^\top \mathbf{A})^{-1} \mathbf{A}^\top \mathbf{b}$ ←

In Matlab:

$$f = A \setminus b$$

Note: You almost never want to compute the inverse of a matrix.

Discrete Poisson equation

Poisson equation (with Dirichlet boundary conditions)

$$\Delta f = \operatorname{div} \mathbf{v} \quad \text{over } \Omega, \quad \text{with } f|_{\partial\Omega} = f^*|_{\partial\Omega}$$

After discretization, equivalent to:

$$\begin{bmatrix} D & I & 0 & 0 & 0 & \dots & 0 \\ I & D & I & 0 & 0 & \dots & 0 \\ 0 & I & D & I & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & I & D & I & 0 \\ 0 & \dots & \dots & 0 & I & D & I \\ 0 & \dots & \dots & \dots & 0 & I & D \end{bmatrix} \cdot \begin{bmatrix} f_1 \\ \vdots \\ f_{q_1} \\ \vdots \\ f_{q_2} \\ \vdots \\ f_p \\ \vdots \\ f_{q_3} \\ \vdots \\ f_{q_4} \\ \vdots \\ f_p \end{bmatrix} = \begin{bmatrix} (\nabla \cdot \mathbf{v})_1 \\ \vdots \\ (\nabla \cdot \mathbf{v})_{q_1} \\ \vdots \\ (\nabla \cdot \mathbf{v})_{q_2} \\ \vdots \\ (\nabla \cdot \mathbf{v})_p \\ \vdots \\ (\nabla \cdot \mathbf{v})_{q_3} \\ \vdots \\ (\nabla \cdot \mathbf{v})_{q_4} \\ \vdots \\ (\nabla \cdot \mathbf{v})_p \end{bmatrix}$$

Linear system of equations:

$$A f = b$$

What is the size of this matrix?

WARNING: requires special treatment at the borders
(target boundary values are same as source)

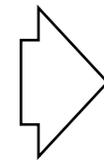
Discrete Poisson equation

Poisson equation (with Dirichlet boundary conditions)

$$\Delta f = \operatorname{div} \mathbf{v} \quad \text{over } \Omega, \quad \text{with } f|_{\partial\Omega} = f^*|_{\partial\Omega}$$

After discretization, equivalent to:

$$\begin{bmatrix} D & I & 0 & 0 & 0 & \dots & 0 \\ I & D & I & 0 & 0 & \dots & 0 \\ 0 & I & D & I & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & I & D & I & 0 \\ 0 & \dots & \dots & 0 & I & D & I \\ 0 & \dots & \dots & \dots & 0 & I & D \end{bmatrix} \cdot \begin{bmatrix} f_1 \\ \vdots \\ f_{q_1} \\ \vdots \\ f_{q_2} \\ \vdots \\ f_p \\ \vdots \\ f_{q_3} \\ \vdots \\ f_{q_4} \\ \vdots \\ f_p \end{bmatrix} = \begin{bmatrix} (\nabla \cdot \mathbf{v})_1 \\ \vdots \\ (\nabla \cdot \mathbf{v})_{q_1} \\ \vdots \\ (\nabla \cdot \mathbf{v})_{q_2} \\ \vdots \\ (\nabla \cdot \mathbf{v})_p \\ \vdots \\ (\nabla \cdot \mathbf{v})_{q_3} \\ \vdots \\ (\nabla \cdot \mathbf{v})_{q_4} \\ \vdots \\ (\nabla \cdot \mathbf{v})_p \end{bmatrix}$$



Linear system of equations:

$$Af = b$$

Matrix is $P \times P \rightarrow$ billions of entries

WARNING: requires special treatment at the borders
(target boundary values are same as source)

Integration procedures

- Poisson solver (i.e., least squares integration)
 - + Generally applicable.
 - Matrices A can become very large.
- Acceleration techniques:
 - + (Conjugate) gradient descent solvers.
 - + Multi-grid approaches.
 - + Pre-conditioning.
 - ...
- Alternative solvers: projection procedures.
 - We will discuss one of these when we cover photometric stereo.

A more efficient Poisson solver

Let's look again at our optimization problem

Variational problem

$$\min_f \iint_{\Omega} |\nabla f - \mathbf{v}|^2 \quad \text{with} \quad f|_{\partial\Omega} = f^*|_{\partial\Omega}$$

gradient of f looks
like vector field \mathbf{v}

f is equivalent to f^*
at the boundaries

Input vector field:

$$\mathbf{v} = (u, v)$$

Recall ...

Nabla operator definition

$$\nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right]$$

Let's look again at our optimization problem

Variational problem

$$\min_f \iint_{\Omega} |\nabla f - \mathbf{v}|^2 \quad \text{with} \quad f|_{\partial\Omega} = f^*|_{\partial\Omega}$$

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at the boundaries

Input vector field:

$$\mathbf{v} = (u, v)$$

Recall ...

Nabla operator definition

$$\nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right]$$

And for discrete images:

partial-x
derivative filter

1	-1
---	----

partial-y
derivative filter

1
-1

Let's look again at our optimization problem

We can use the gradient approximation to discretize the variational problem

Discrete problem

What are G , f , and v ?

$$\min_f \|Gf - v\|^2$$

We will ignore the boundary conditions for now.

Recall ...

Nabla operator definition

$$\nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right]$$

And for discrete images:

partial-x
derivative filter

1	-1
---	----

partial-y
derivative filter

1
-1

Let's look again at our optimization problem

We can use the gradient approximation to discretize the variational problem

Discrete problem

matrix G formed by stacking together discrete gradients

vectorized version of the unknown image

$$\min_f \|Gf - v\|^2$$

vectorized version of the target gradient field

We will ignore the boundary conditions for now.

Recall ...

Image gradient

$$\nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right]$$

And for discrete images:

partial-x derivative filter

1	-1
---	----

partial-y derivative filter

1
-1

Let's look again at our optimization problem

We can use the gradient approximation to discretize the variational problem

Discrete problem

matrix G formed by stacking together discrete gradients

vectorized version of the unknown image

$$\min_f \|Gf - v\|^2$$

vectorized version of the target gradient field

How do we solve this optimization problem?

Recall ...

Image gradient

$$\nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right]$$

And for discrete images:

partial-x derivative filter

1	-1
---	----

partial-y derivative filter

1
-1

Approach 1: Compute stationary points

Given the loss function:

$$E(f) = \|Gf - v\|^2$$

... we compute its derivative:

$$\frac{\partial E}{\partial f} = ?$$

Approach 1: Compute stationary points

Given the loss function:

$$E(f) = \|Gf - v\|^2$$

... we compute its derivative:

$$\frac{\partial E}{\partial f} = G^T Gf - G^T v$$

... and we do what with it?

Approach 1: Compute stationary points

Given the loss function:

$$E(f) = \|Gf - v\|^2$$

... we compute its derivative:

$$\frac{\partial E}{\partial f} = G^T Gf - G^T v$$

... and we set that to zero:

$$\frac{\partial E}{\partial f} = 0 \Rightarrow \underbrace{G^T G}_{} f = \underbrace{G^T}_{} v$$

What is this vector?

What is this matrix?

Approach 1: Compute stationary points

Given the loss function:

$$E(f) = \|Gf - v\|^2$$

... we compute its derivative:

$$\frac{\partial E}{\partial f} = G^T Gf - G^T v$$

... and we set that to zero:

$$\frac{\partial E}{\partial f} = 0 \Rightarrow \underbrace{G^T Gf}_{\text{Laplacian matrix } A} = \underbrace{G^T v}_{\text{vector } b}$$

It is equal to the vector b we derived previously!

It is equal to the Laplacian matrix A we derived previously!

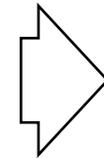
Reminder from variational case

Poisson equation (with Dirichlet boundary conditions)

$$\Delta f = \operatorname{div} \mathbf{v} \quad \text{over } \Omega, \quad \text{with } f|_{\partial\Omega} = f^*|_{\partial\Omega}$$

After discretization, equivalent to:

$$\begin{bmatrix} D & I & 0 & 0 & 0 & \dots & 0 \\ I & D & I & 0 & 0 & \dots & 0 \\ 0 & I & D & I & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & I & D & I & 0 \\ 0 & \dots & \dots & 0 & I & D & I \\ 0 & \dots & \dots & \dots & 0 & I & D \end{bmatrix} \cdot \begin{bmatrix} f_1 \\ \vdots \\ f_{q_1} \\ \vdots \\ f_{q_2} \\ \vdots \\ f_p \\ \vdots \\ f_{q_3} \\ \vdots \\ f_{q_4} \\ \vdots \\ f_p \end{bmatrix} = \begin{bmatrix} (\nabla \cdot \mathbf{v})_1 \\ \vdots \\ (\nabla \cdot \mathbf{v})_{q_1} \\ \vdots \\ (\nabla \cdot \mathbf{v})_{q_2} \\ \vdots \\ (\nabla \cdot \mathbf{v})_p \\ \vdots \\ (\nabla \cdot \mathbf{v})_{q_3} \\ \vdots \\ (\nabla \cdot \mathbf{v})_{q_4} \\ \vdots \\ (\nabla \cdot \mathbf{v})_p \end{bmatrix}$$



Linear system of equations:

$$Af = b$$

Same system as:

$$G^T Gf = G^T v$$

We arrive at the same system, no matter whether we discretize the continuous Poisson equation or the variational optimization problem.

Approach 1: Compute stationary points

Given the loss function:

$$E(f) = \|Gf - v\|^2$$

... we compute its derivative:

$$\frac{\partial E}{\partial f} = G^T Gf - G^T v$$

... and we set that to zero:

$$\frac{\partial E}{\partial f} = 0 \Rightarrow G^T Gf = G^T v$$

Solving this is exactly as expensive as what we had before.

Approach 2: Use gradient descent

Given the loss function:

$$E(f) = \|Gf - v\|^2$$

... we compute its derivative:

$$\frac{\partial E}{\partial f} = G^T Gf - G^T v = Af - b \equiv -r$$

We call this term
the residual

Approach 2: Use gradient descent

Given the loss function:

$$E(f) = \|Gf - v\|^2$$

... we compute its derivative:

$$\frac{\partial E}{\partial f} = G^T Gf - G^T v = Af - b \equiv -r$$

We call this term
the residual

... and then we iteratively compute a solution:

$$f^{i+1} = f^i + \eta^i r^i \quad \text{for } i = 0, 1, \dots, N, \text{ where}$$

η^i are positive step sizes

Selecting optimal step sizes

Make derivative of loss function with respect to η^i equal to zero:

$$E(f) = \|Gf - v\|^2$$

$$E(f^{i+1}) = \|G(f^i + \eta^i r^i) - v\|^2$$

Selecting optimal step sizes

Make derivative of loss function with respect to η^i equal to zero:

$$E(f) = \|Gf - v\|^2$$

$$E(f^{i+1}) = \|G(f^i + \eta^i r^i) - v\|^2$$

$$\frac{\partial E(f^{i+1})}{\partial \eta^i} = [b - A(f^i + \eta^i r^i)]^T r^i = 0 \Rightarrow \eta^i = \frac{(r^i)^T r^i}{(r^i)^T A r^i}$$

Gradient descent

Given the loss function:

$$E(f) = \|Gf - v\|^2$$

Minimize by iteratively computing:

$$r^i = b - Af^i, \quad \eta^i = \frac{(r^i)^T r^i}{(r^i)^T Ar^i}, \quad f^{i+1} = f^i + \eta^i r^i, \quad i = 0, \dots, N$$

Is this cheaper than the pseudo-inverse approach?

Gradient descent

Given the loss function:

$$E(f) = \|Gf - v\|^2$$

Minimize by iteratively computing:

$$r^i = b - Af^i, \quad \eta^i = \frac{(r^i)^T r^i}{(r^i)^T Ar^i}, \quad f^{i+1} = f^i + \eta^i r^i, \quad i = 0, \dots, N$$

Is this cheaper than the pseudo-inverse approach?

- We never need to compute A , only its products with vectors f , r .

Gradient descent

Given the loss function:

$$E(f) = \|Gf - v\|^2$$

Minimize by iteratively computing:

$$r^i = b - Af^i, \quad \eta^i = \frac{(r^i)^T r^i}{(r^i)^T Ar^i}, \quad f^{i+1} = f^i + \eta^i r^i, \quad i = 0, \dots, N$$

Is this cheaper than the pseudo-inverse approach?

- We never need to compute A , only its products with vectors f , r .
- Vectors f , r are images.

Gradient descent

Given the loss function:

$$E(f) = \|Gf - v\|^2$$

Minimize by iteratively computing:

$$r^i = b - Af^i, \quad \eta^i = \frac{(r^i)^T r^i}{(r^i)^T Ar^i}, \quad f^{i+1} = f^i + \eta^i r^i, \quad i = 0, \dots, N$$

Is this cheaper than the pseudo-inverse approach?

- We never need to compute A , only its products with vectors f , r .
- Vectors f , r are images.
- Because A is the Laplacian matrix, these matrix-vector products can be efficiently computed using convolutions with the Laplacian kernel.

In practice: conjugate gradient descent

Given the loss function:

$$E(f) = \|Gf - v\|^2$$

Minimize by iteratively computing:

$$d^i = r^i + \beta^i d^i, \quad \eta^i = \frac{(r^i)^T r^i}{(d^i)^T Ad^i}, \quad f^{i+1} = f^i + \eta^i d^i, \quad i = 0, \dots, N$$

$$r^{i+1} = r^i - \eta^i Ad^i, \quad \beta^i = \frac{(r^{i+1})^T r^{i+1}}{(r^i)^T r^i}$$

- Smarter way for selecting update directions
- Everything can still be done using convolutions
- Only one convolution needed per iteration

Note: initialization

Does the initialization f^0 matter?

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- It doesn't matter in terms of what final f we converge to, because the loss function is convex.

$$E(f) = \|Gf - v\|^2$$

Note: initialization

Does the initialization f^0 matter?

- It doesn't matter in terms of what final f we converge to, because the loss function is convex.

$$E(f) = \|Gf - v\|^2$$

- It does matter in terms of convergence speed.
- We can use a multi-resolution approach:
 - Solve an initial problem for a very low-resolution f (e.g., 2x2).
 - Use the solution to initialize gradient descent for a higher resolution f (e.g., 4x4).
 - Use the solution to initialize gradient descent for a higher resolution f (e.g., 8x8).
 - ...
 - Use the solution to initialize gradient descent for an f with the original resolution $P \times P$.
- Multi-grid algorithms alternative between higher and lower resolutions during the (conjugate) gradient descent iterative procedure.

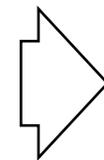
Reminder from variational case

Poisson equation (with Dirichlet boundary conditions)

$$\Delta f = \operatorname{div} \mathbf{v} \quad \text{over } \Omega, \quad \text{with } f|_{\partial\Omega} = f^*|_{\partial\Omega}$$

After discretization, equivalent to:

$$\begin{bmatrix} D & I & 0 & 0 & 0 & \dots & 0 \\ I & D & I & 0 & 0 & \dots & 0 \\ 0 & I & D & I & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & I & D & I & 0 \\ 0 & \dots & \dots & 0 & I & D & I \\ 0 & \dots & \dots & \dots & 0 & I & D \end{bmatrix} \cdot \begin{bmatrix} f_1 \\ \vdots \\ f_{q_1} \\ \vdots \\ f_{q_2} \\ \vdots \\ f_p \\ \vdots \\ f_{q_3} \\ \vdots \\ f_{q_4} \\ \vdots \\ f_p \end{bmatrix} = \begin{bmatrix} (\nabla \cdot \mathbf{v})_1 \\ \vdots \\ (\nabla \cdot \mathbf{v})_{q_1} \\ \vdots \\ (\nabla \cdot \mathbf{v})_{q_2} \\ \vdots \\ (\nabla \cdot \mathbf{v})_p \\ \vdots \\ (\nabla \cdot \mathbf{v})_{q_3} \\ \vdots \\ (\nabla \cdot \mathbf{v})_{q_4} \\ \vdots \\ (\nabla \cdot \mathbf{v})_p \end{bmatrix}$$



Linear system of equations:

$$Af = b$$

Remember that what we are doing is equivalent to solving this linear system.

Note: preconditioning

We are solving this linear system:

$$Af = b$$

For any invertible matrix P , this is equivalent to solving:

$$P^{-1}Af = P^{-1}b$$

When is it preferable to solve this alternative linear system?

Note: preconditioning

We are solving this linear system:

$$Af = b$$

For any invertible matrix P , this is equivalent to solving:

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When is it preferable to solve this alternative linear system?

- Ideally: If A is invertible, and P is the same as A , the linear system becomes trivial! But computing the inverse of A is even more expensive than solving the original linear system.
- In practice: If the matrix $P^{-1}A$ has a better condition number, or its singular values are more uniformly distributed, the linear system becomes more numerically stable.

What preconditioner P should we use?

Note: preconditioning

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What preconditioner P should we use?

- Standard preconditioners like Jacobi.
- More effective preconditioners. Active area of research.

$$P_{\text{Jacobi}} = \text{diag}(A)$$

Note: preconditioning

We are solving this linear system:

$$Af = b$$

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Preconditioning can be incorporated in the conjugate gradient descent algorithm.

When is it preferable to solve this alternative linear system?

- Ideally: If A is invertible, and P is the same as A , the linear system becomes trivial! But computing the inverse of A is even more expensive than solving the original linear system.
- In practice: If the matrix $P^{-1}A$ has a better condition number, or its singular values are more uniformly distributed, the linear system becomes more numerically stable.

What preconditioner P should we use?

- Standard preconditioners like Jacobi.
- More effective preconditioners. Active area of research.

Is this effective for Poisson solvers?

$$P_{\text{Jacobi}} = \text{diag}(A)$$

Discrete Poisson equation

Poisson equation (with Dirichlet boundary conditions)

$$\Delta f = \operatorname{div} \mathbf{v} \quad \text{over } \Omega, \quad \text{with } f|_{\partial\Omega} = f^*|_{\partial\Omega}$$

After discretization, equivalent to:

$$\begin{bmatrix} D & I & 0 & 0 & 0 & \dots & 0 \\ I & D & I & 0 & 0 & \dots & 0 \\ 0 & I & D & I & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & I & D & I & 0 \\ 0 & \dots & \dots & 0 & I & D & I \\ 0 & \dots & \dots & \dots & 0 & I & D \end{bmatrix} \cdot \begin{bmatrix} f_1 \\ \vdots \\ f_{q_1} \\ \vdots \\ f_{q_2} \\ \vdots \\ f_p \\ \vdots \\ f_{q_3} \\ \vdots \\ f_{q_4} \\ \vdots \\ f_p \end{bmatrix} = \begin{bmatrix} (\nabla \cdot \mathbf{v})_1 \\ \vdots \\ (\nabla \cdot \mathbf{v})_{q_1} \\ \vdots \\ (\nabla \cdot \mathbf{v})_{q_2} \\ \vdots \\ (\nabla \cdot \mathbf{v})_p \\ \vdots \\ (\nabla \cdot \mathbf{v})_{q_3} \\ \vdots \\ (\nabla \cdot \mathbf{v})_{q_4} \\ \vdots \\ (\nabla \cdot \mathbf{v})_p \end{bmatrix}$$

Linear system of equations:

$$Af = b$$

Matrix is $P \times P \rightarrow$ billions of entries

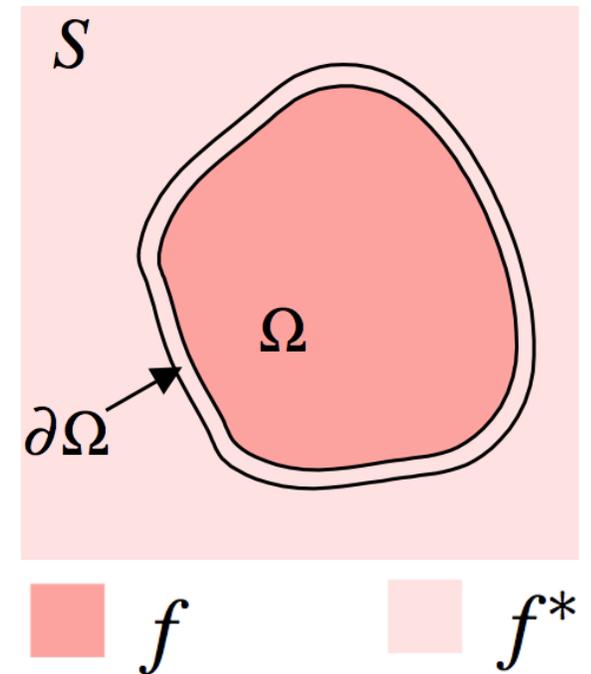
WARNING: requires special treatment at the borders
(target boundary values are same as source)

Note: handling (Dirichlet) boundary conditions

- Form a mask B that is 0 for pixels that should not be updated (pixels on $S-\Omega$ and $\partial\Omega$) and 1 otherwise.
- Use convolution to perform Laplacian filtering over the entire image.
- Use (conjugate) gradient descent rules to only update pixels for which the mask is 1. Equivalently, change the update rules to:

$$f^{i+1} = f^i + B\eta^i r^i \quad (\text{gradient descent})$$

$$f^{i+1} = f^i + B\eta^i d^i \quad (\text{conjugate gradient descent})$$



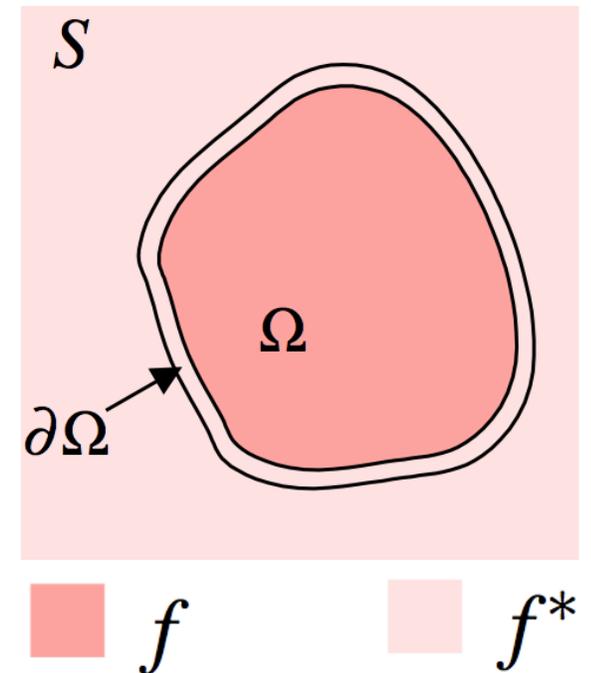
Note: handling (Dirichlet) boundary conditions

- Form a mask \mathbf{B} that is 0 for pixels that should not be updated (pixels on $S-\Omega$ and $\partial\Omega$) and 1 otherwise.
- Use convolution to perform Laplacian filtering over the entire image.
- Use (conjugate) gradient descent rules to only update pixels for which the mask is 1. Equivalently, change the update rules to:

In practice, masking is also required at other steps of (conjugate) gradient descent, to deal with invalid boundaries (e.g., from convolutions).

$$f^{i+1} = f^i + B\eta^i r^i \quad (\text{gradient descent})$$

$$f^{i+1} = f^i + B\eta^i d^i \quad (\text{conjugate gradient descent})$$



Poisson image editing examples

Photoshop's "healing brush"



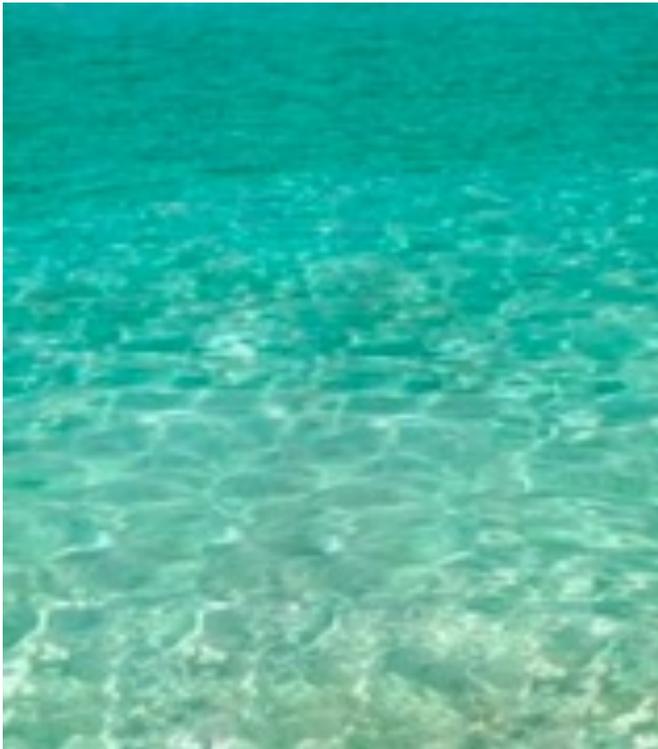
- Slightly more advanced version of what we covered here:
- Uses higher-order derivatives

Contrast problem



Loss of contrast when pasting from dark to bright:

- Contrast is a multiplicative property.
- With Poisson blending we are matching linear differences.



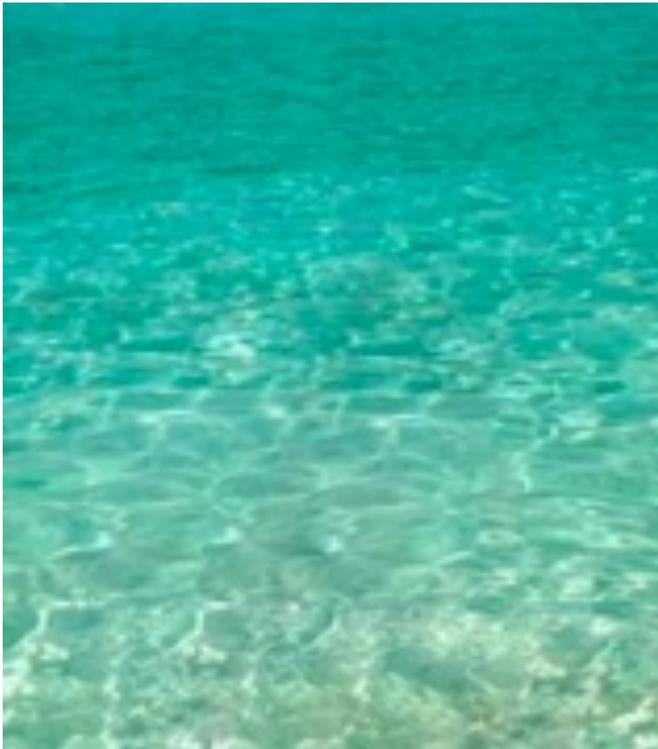
Contrast problem



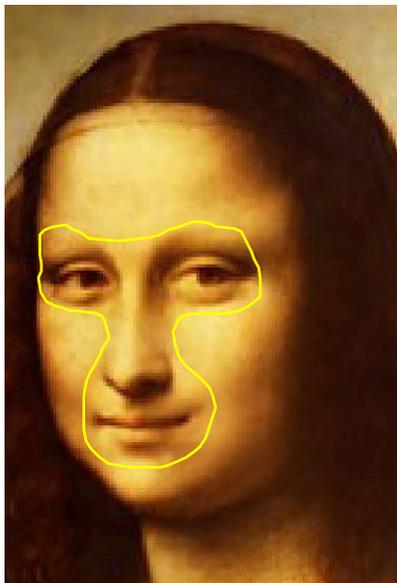
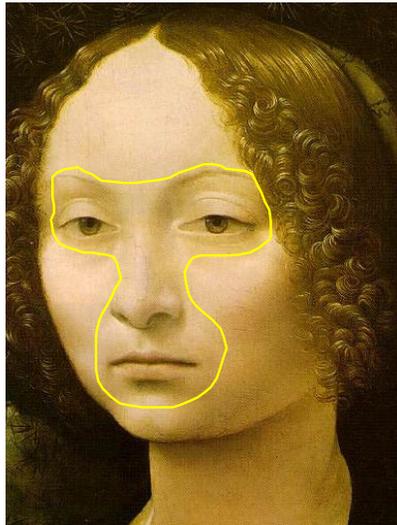
Loss of contrast when pasting from dark to bright:

- Contrast is a multiplicative property.
- With Poisson blending we are matching linear differences.

Solution: Do blending in log-domain.



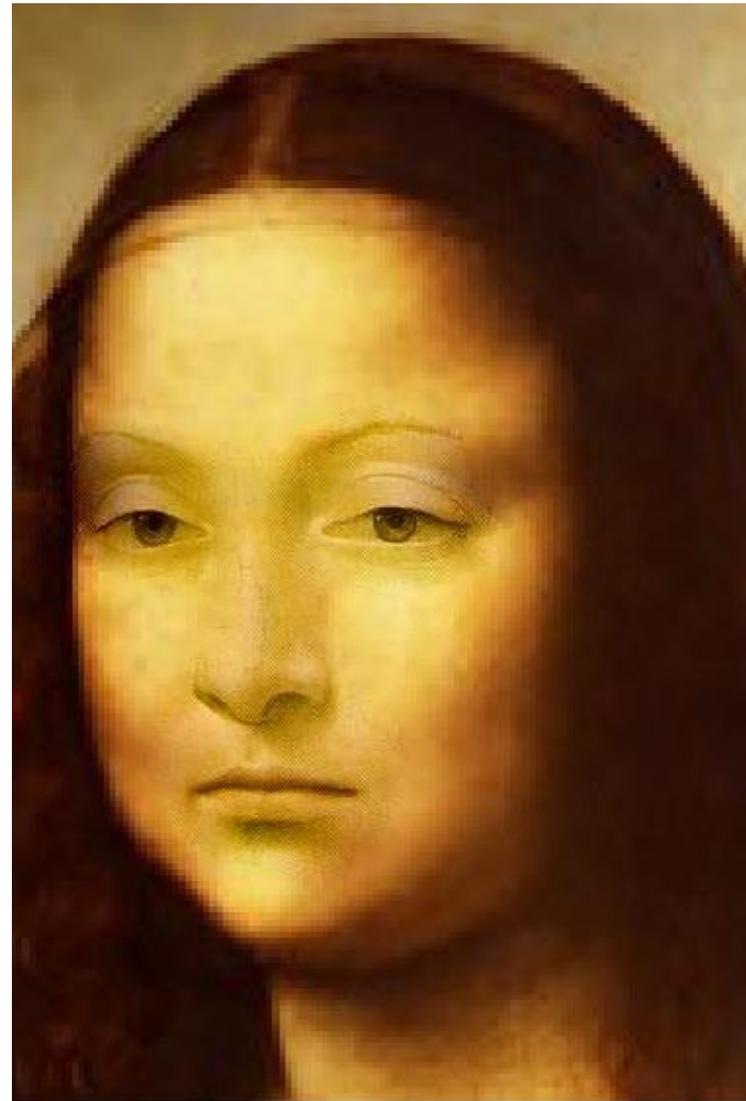
More blending



originals



copy-paste

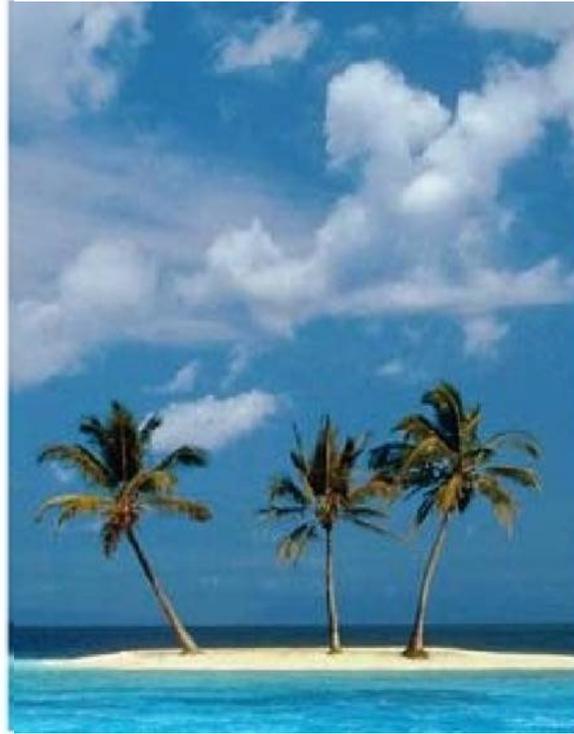


Poisson blending

Blending transparent objects



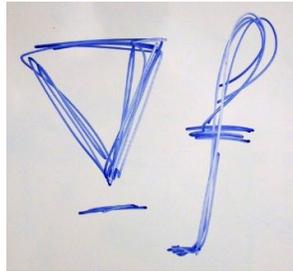
source



destination



Blending objects with holes



color-based cutout and paste



seamless cloning



seamless cloning and destination averaged



mixed seamless cloning

Editing



Concealment



How would you do this with Poisson blending?



Concealment



How would you do this with Poisson blending?

- Insert a copy of the background.

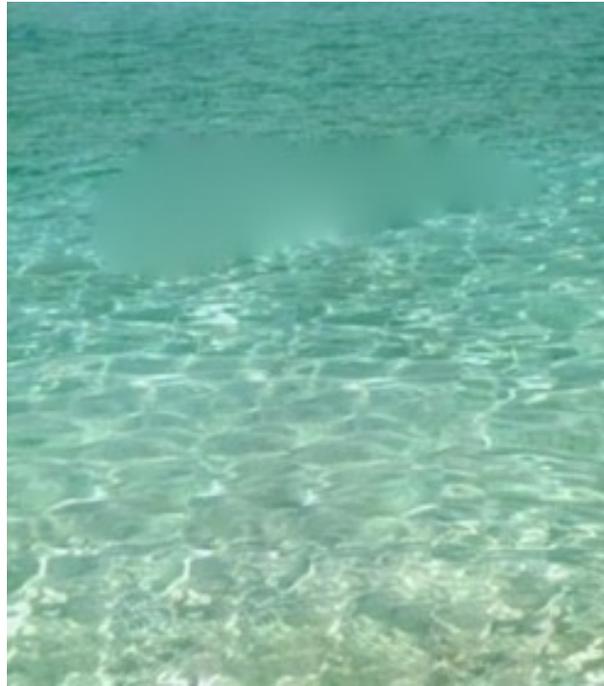


Texture swapping



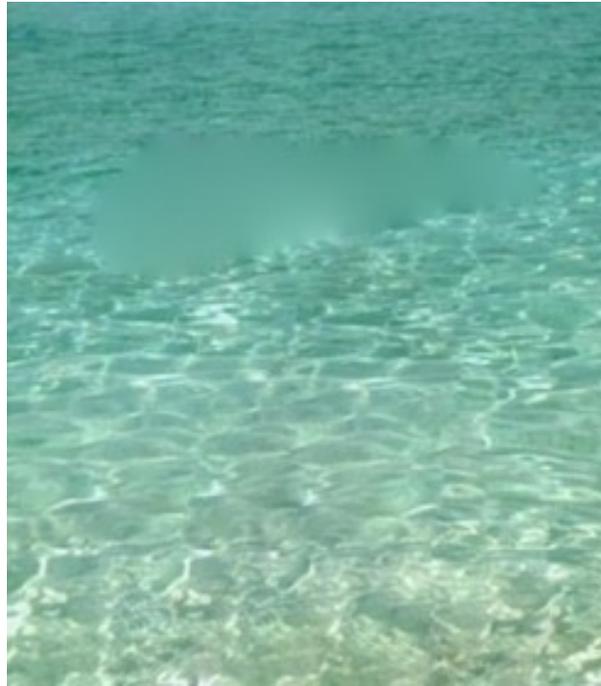
Special case: membrane interpolation

How would you do this?



Special case: membrane interpolation

How would you do this?



Poisson problem

$$\min_f \iint_{\Omega} |\nabla f - \mathbf{v}|^2 \quad \text{with} \quad f|_{\partial\Omega} = f^*|_{\partial\Omega}$$

Laplacian problem

$$\min_f \iint_{\Omega} |\nabla f|^2 \quad \text{with} \quad f|_{\partial\Omega} = f^*|_{\partial\Omega}$$

Entire suite of image editing tools

GradientShop: A Gradient-Domain Optimization Framework for Image and Video Filtering

Pravin Bhat¹ C. Lawrence Zitnick² Michael Cohen^{1,2} Brian Curless¹
¹University of Washington ²Microsoft Research



(a) Input image



(b) Saliency-sharpening filter



(c) Pseudo-relighting filter



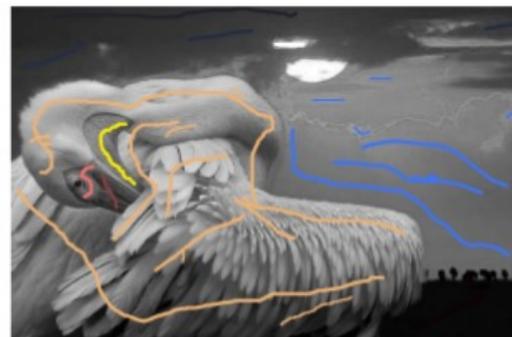
(d) Non-photorealistic rendering filter



(e) Compressed input-image



(f) De-blocking filter



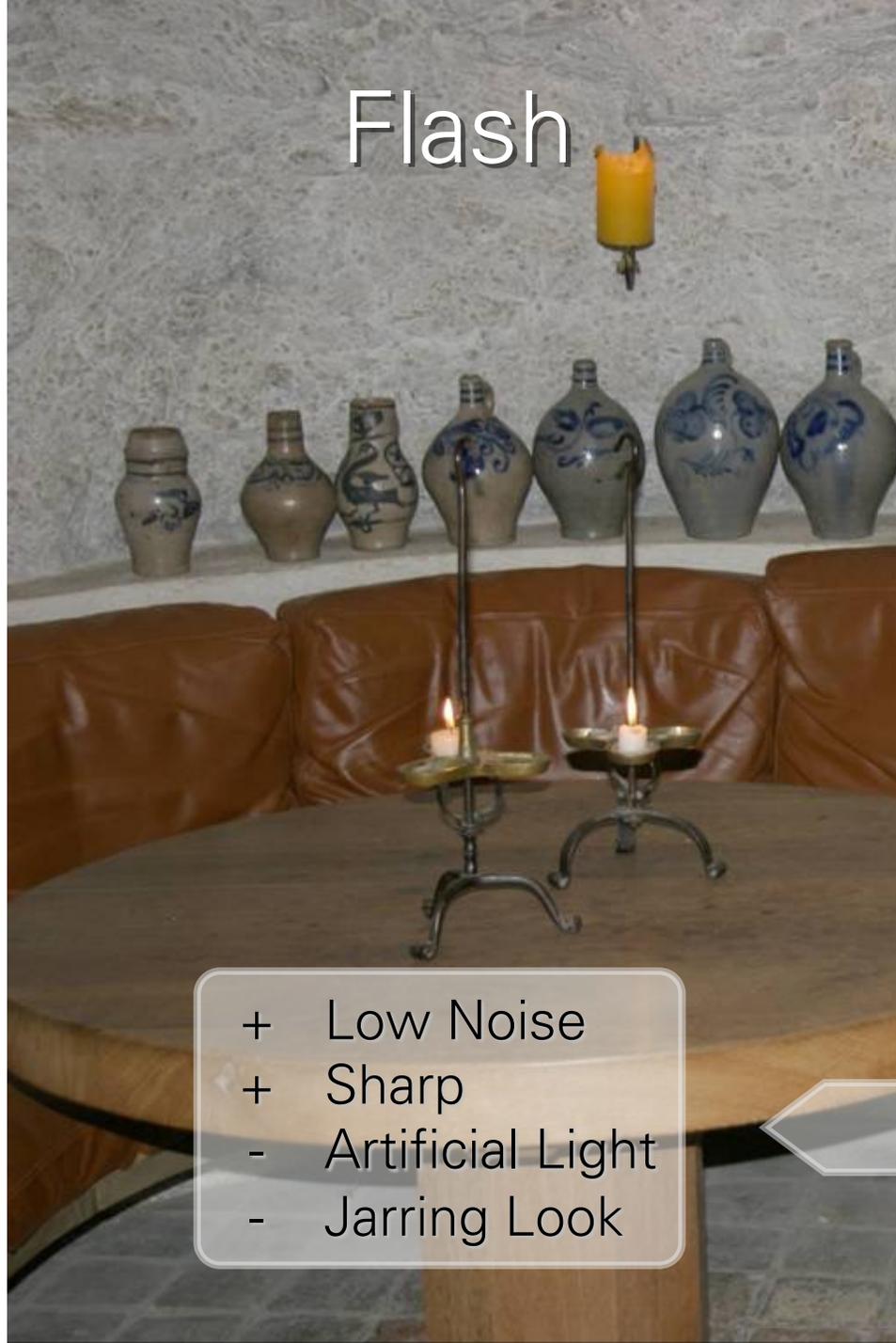
(g) User input for colorization



(h) Colorization filter

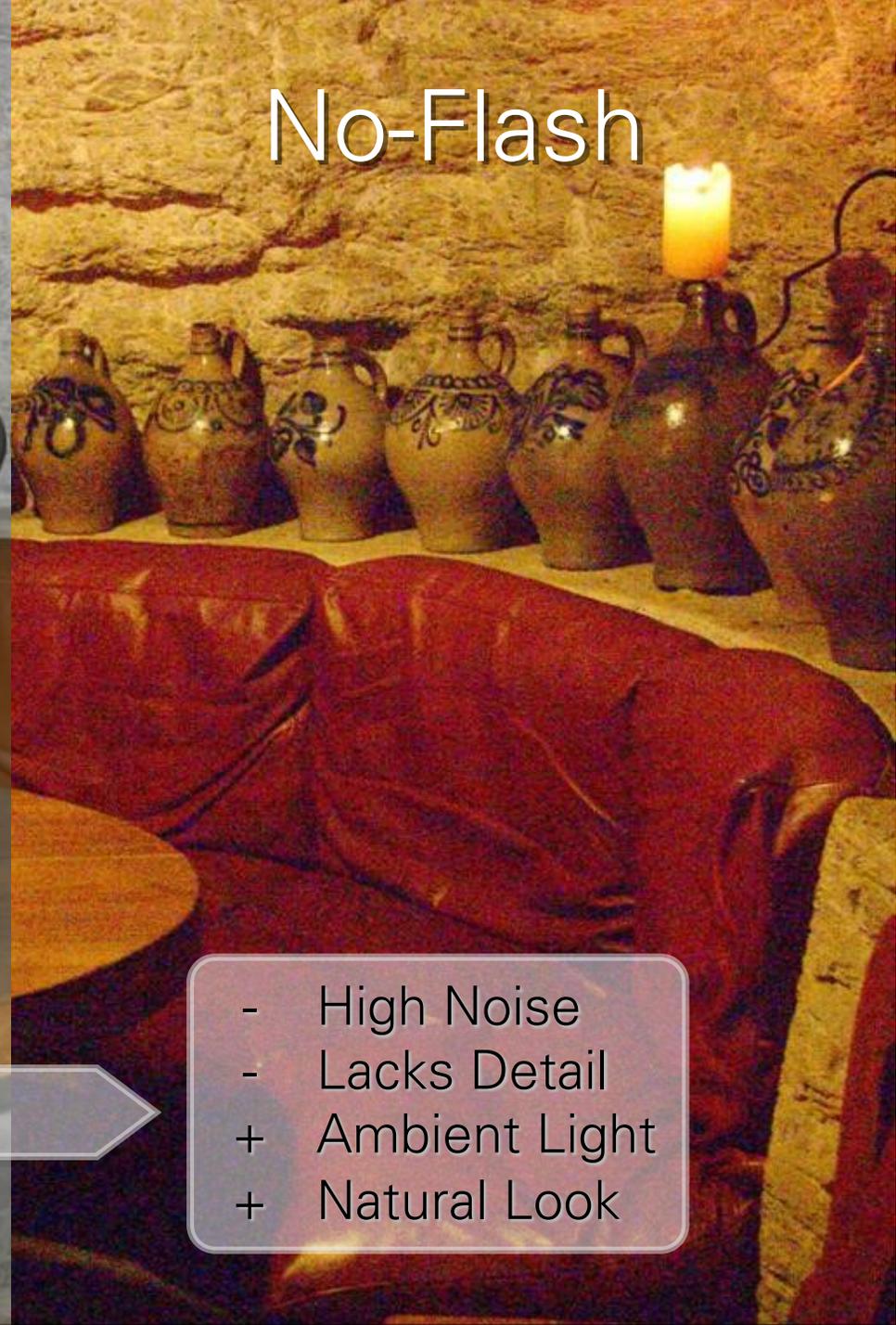
Flash/no-flash photography

Flash



- + Low Noise
- + Sharp
- Artificial Light
- Jarring Look

No-Flash



- High Noise
- Lacks Detail
- + Ambient Light
- + Natural Look



Denoising Result



No-Flash



Denoising Result

Key idea

Denoise the no-flash image while maintaining the edge structure of the flash image.

Can we do similar flash/no-flash fusion tasks with gradient-domain processing?

Photography Artifacts: Flash Hotspot

Ambient



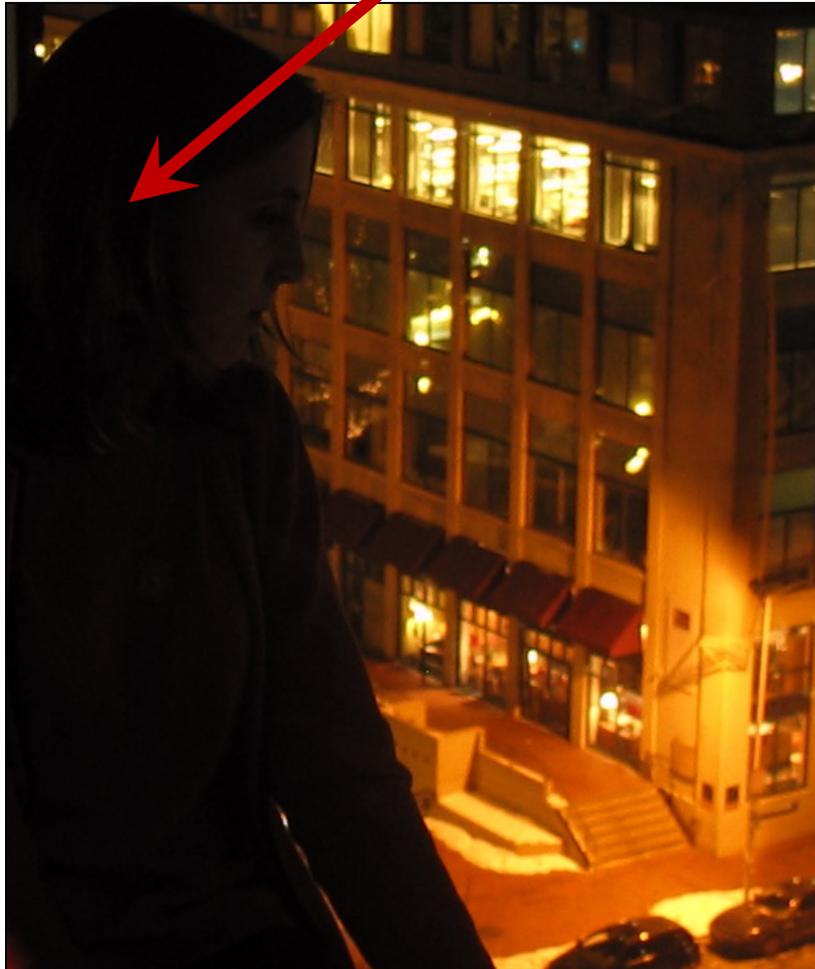
Flash



Flash Hotspot

Reflections due to Flash

Underexposed



Ambient

Reflections



Flash

Distance Dependence

Flash

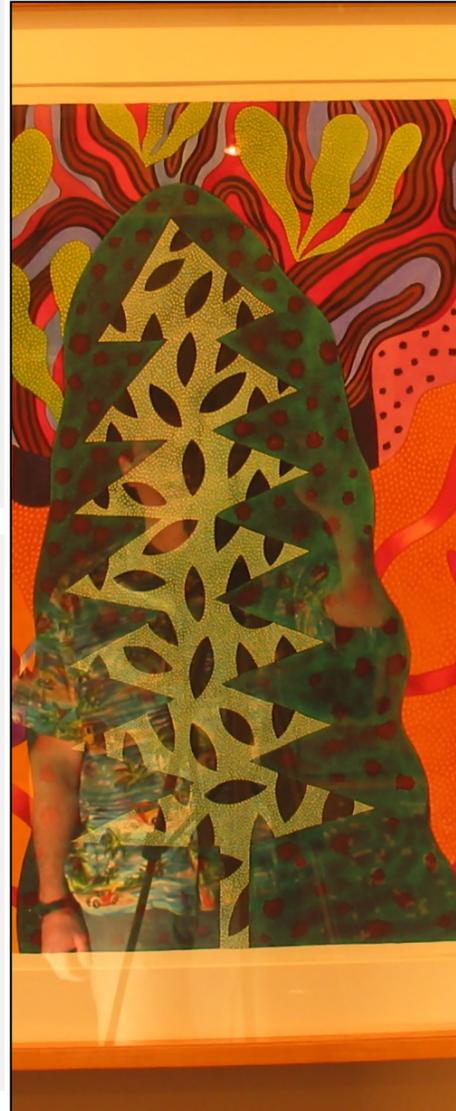
Distant people
underexposed



Removing self-reflections and hot-spots



Ambient



Flash

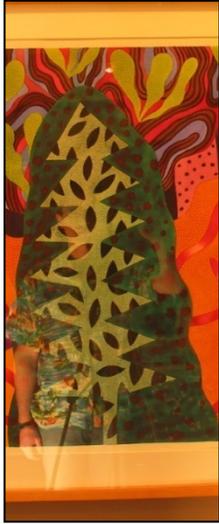


Removing self-reflections and hot-spots



Removing self-reflections and hot-spots

Ambient



Flash



Result

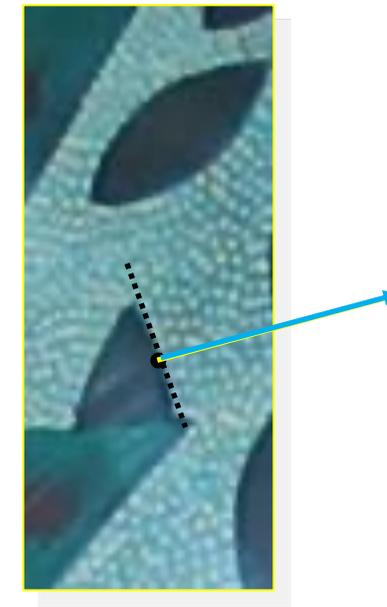
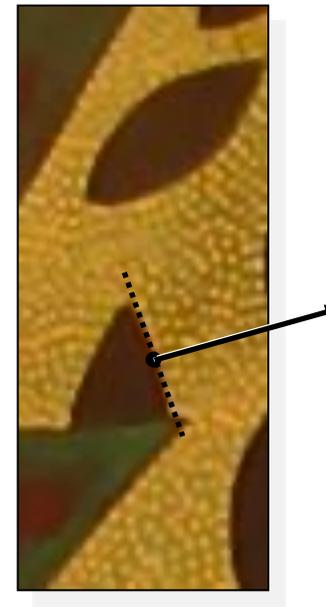
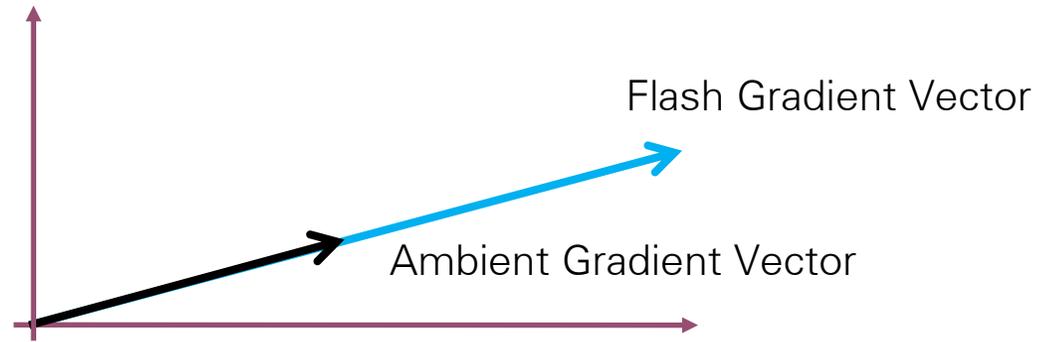


Reflection Layer



Idea: look at how gradients are affected

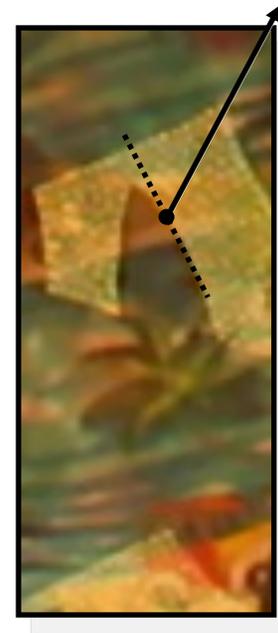
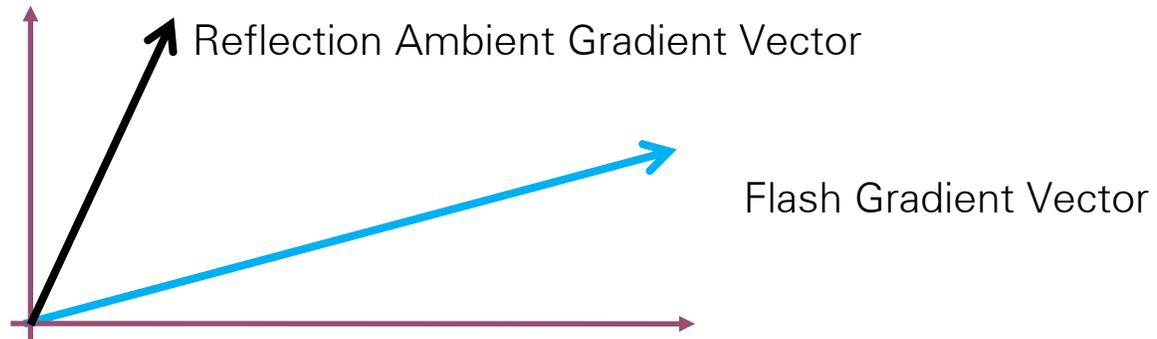
Same gradient vector direction



No reflections

Idea: look at how gradients are affected

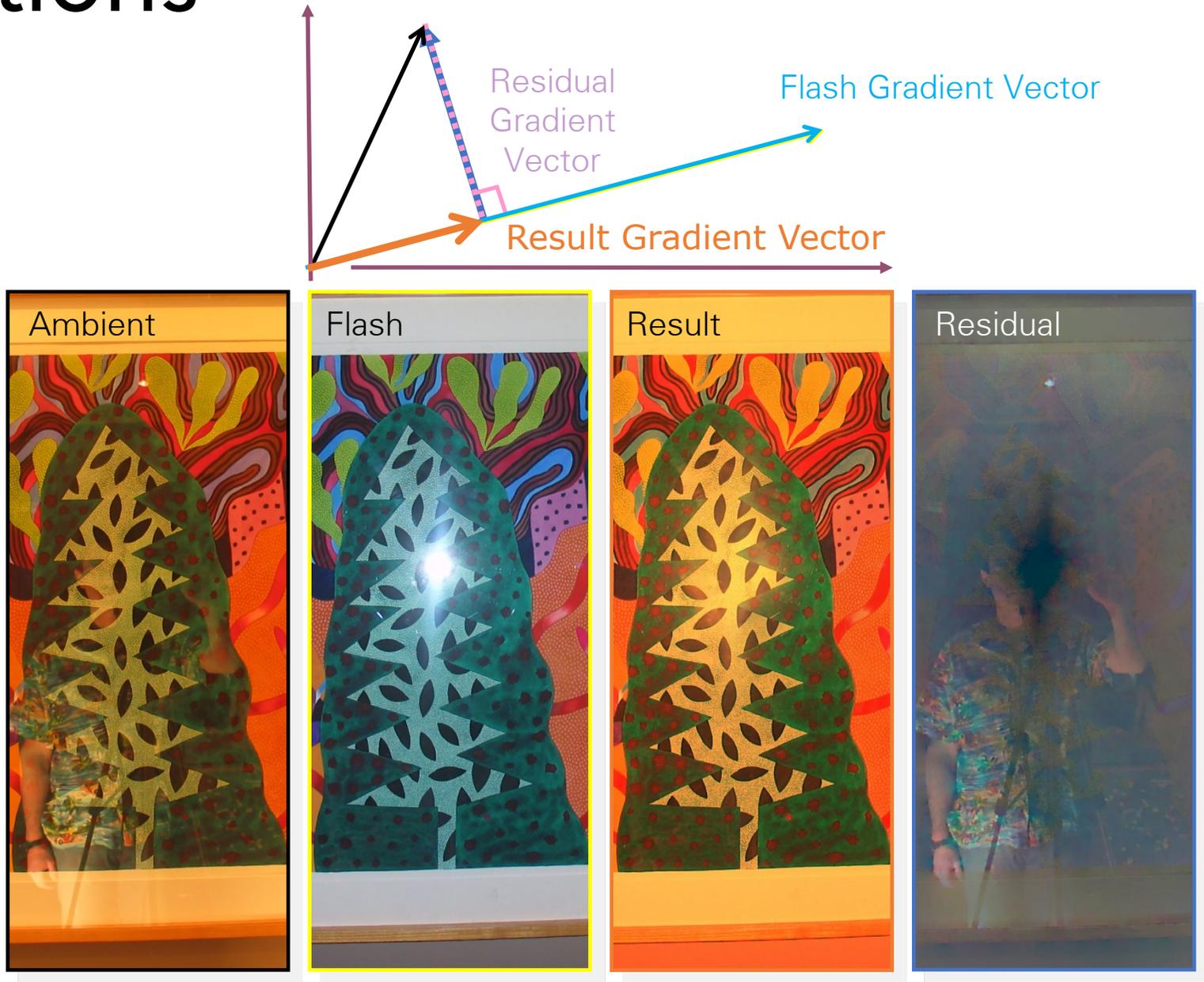
Different gradient vector direction



With reflections

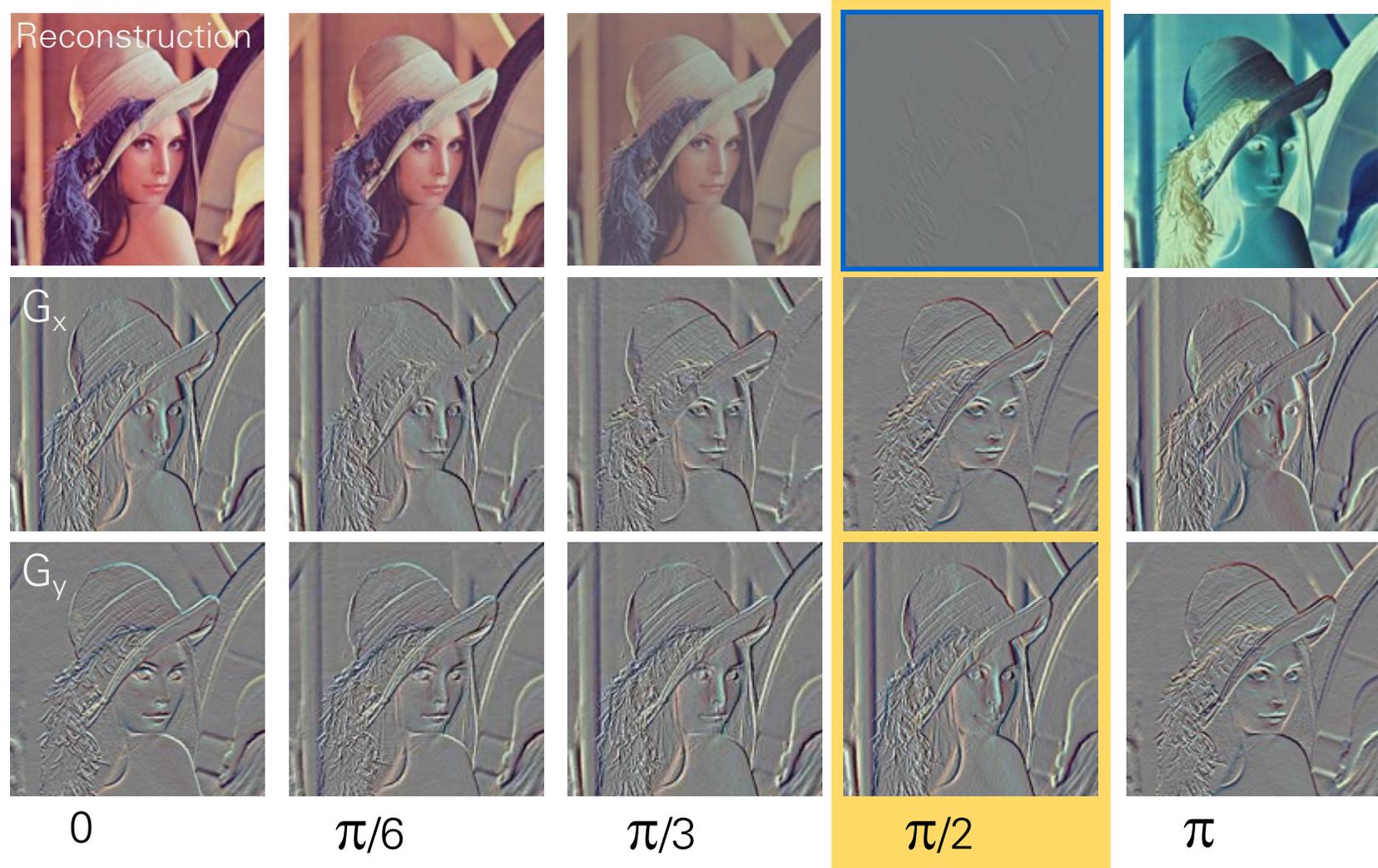
Gradient projections

- Image gradients in flash and ambient images should be aligned.
- Ambient gradient direction is refined by projecting onto the flash gradient.
- "Result" image is formed by 2D integration of the refined gradient.
- Residual gradients after projection create the "reflection layer".
- Gradient projection splits an image into reflection-free and reflection layers.



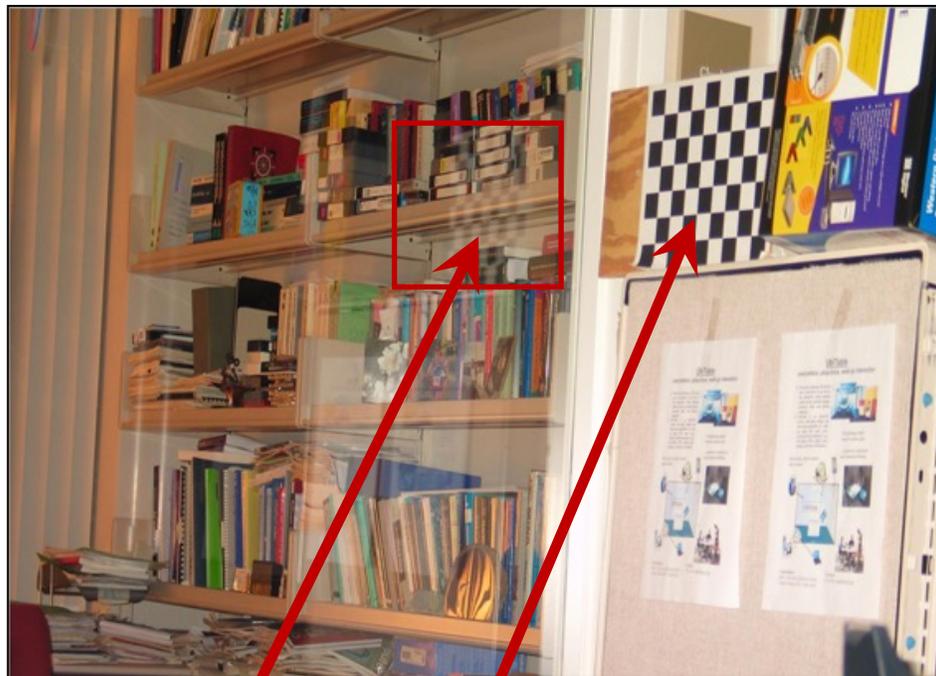
Why projections?

- Projection ensures the gradient direction is preserved, even with a new magnitude.
- Orthogonal gradients holds minimal visual information.
- Rotating gradients by 90° yields zero divergence.
- 90° rotation results in no image detail.
- 180° rotation creates a negative image.

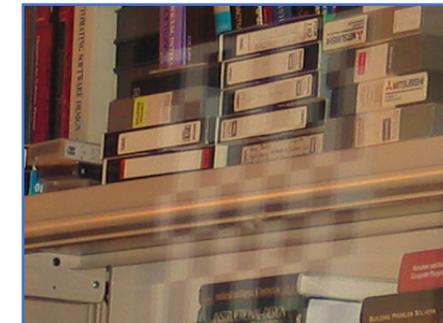
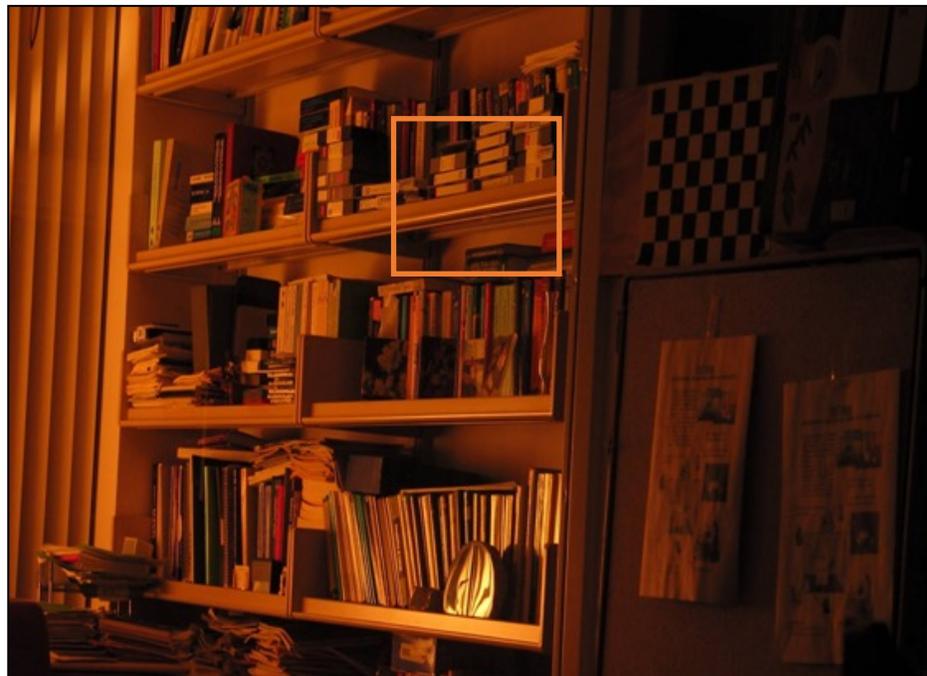


Flash/no-flash with gradient-domain processing

Flash



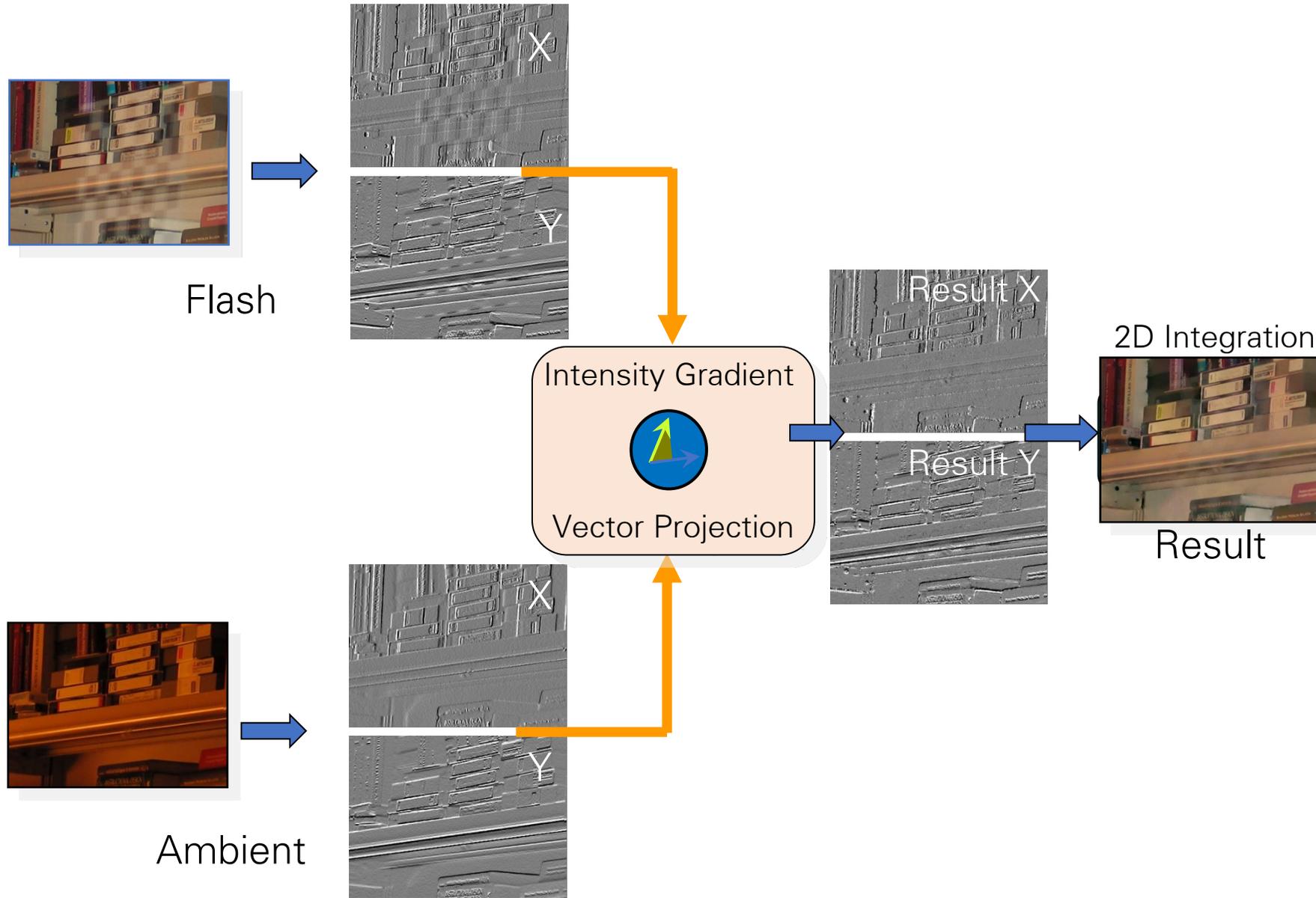
Ambient



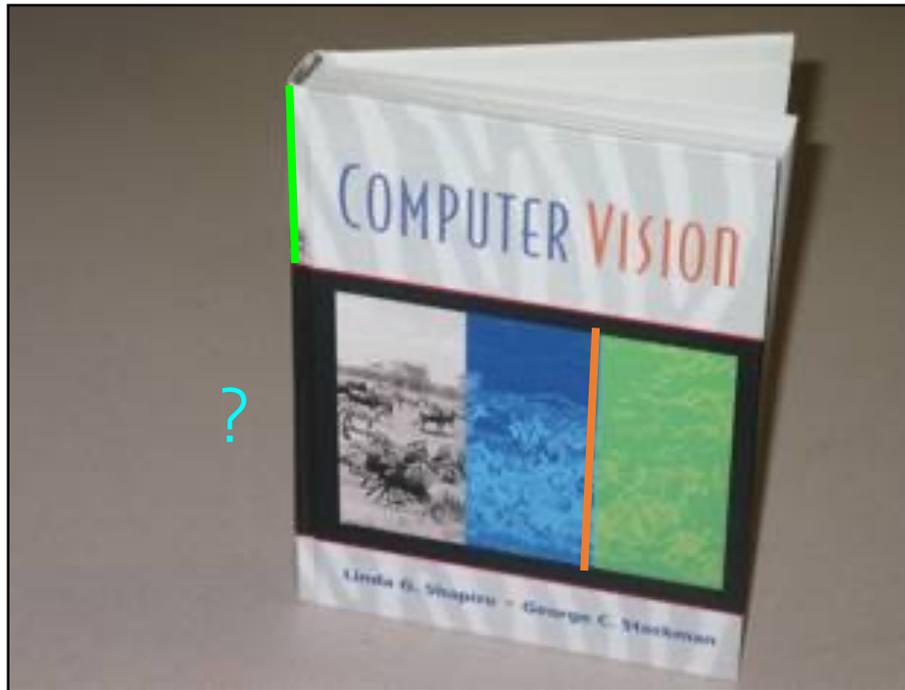
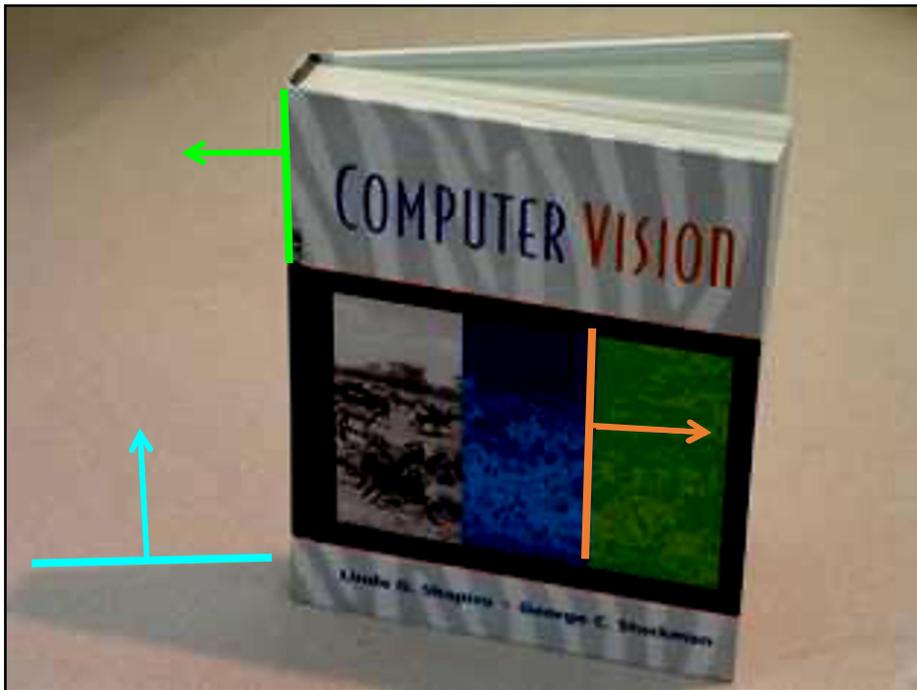
Checkerboard
outside glass
window

Reflections on
glass window

Flash/no-flash with gradient-domain processing

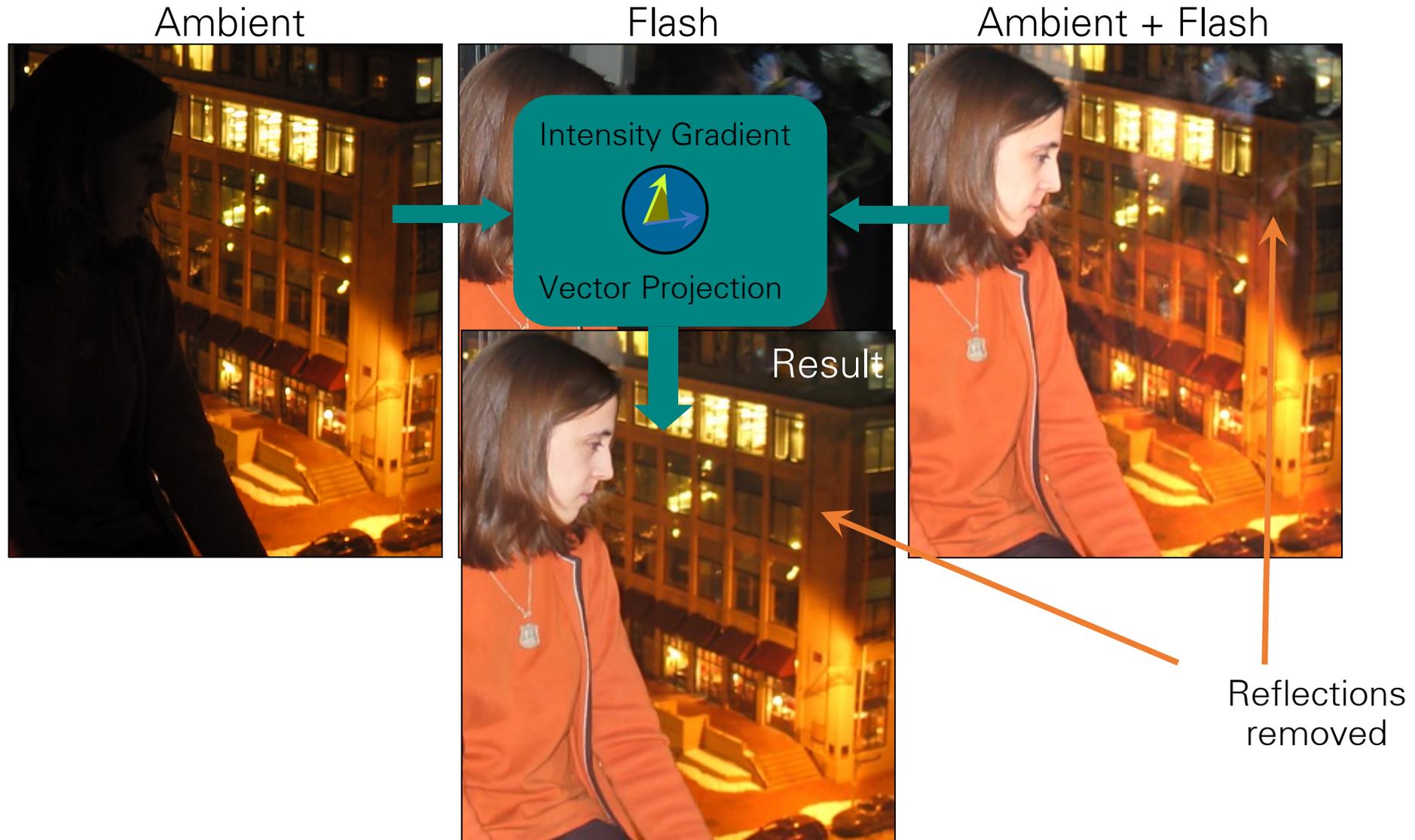


Invariance of Gradient Vectors Orientation (Gradient Orientation Coherency)

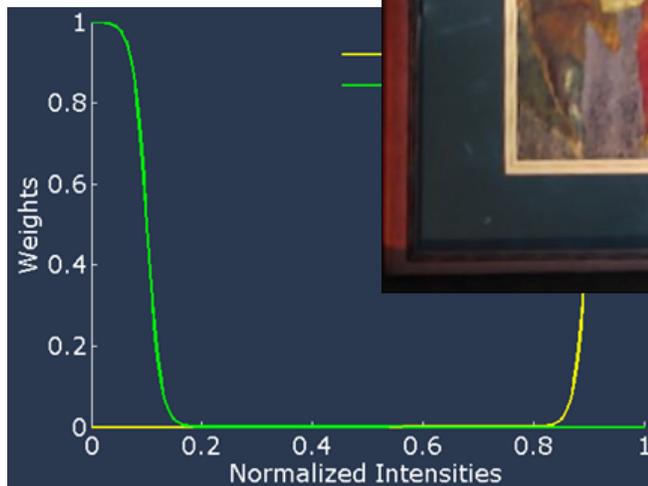
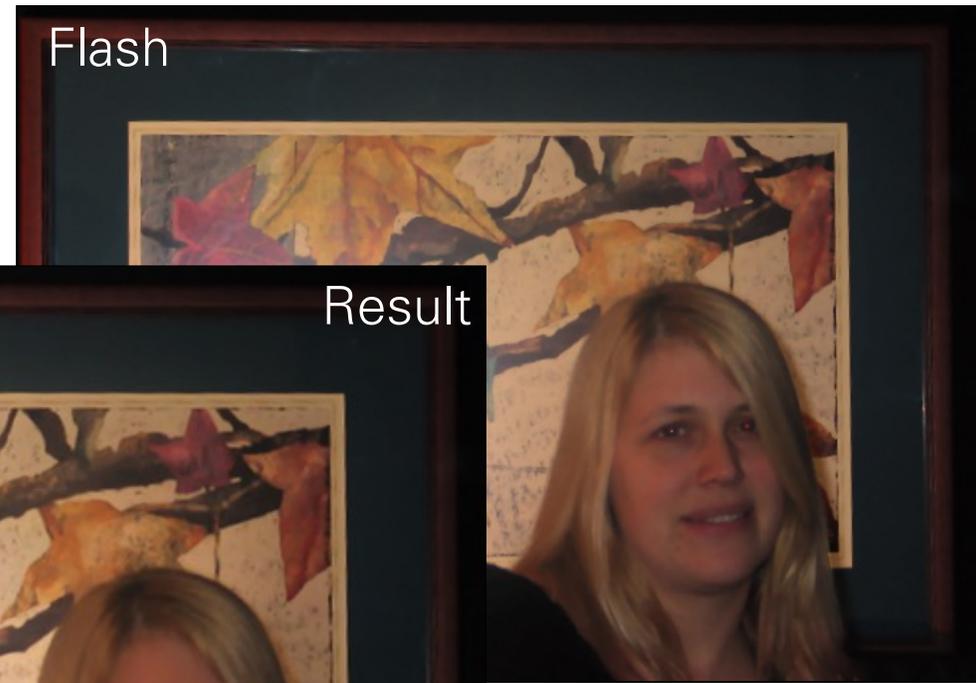
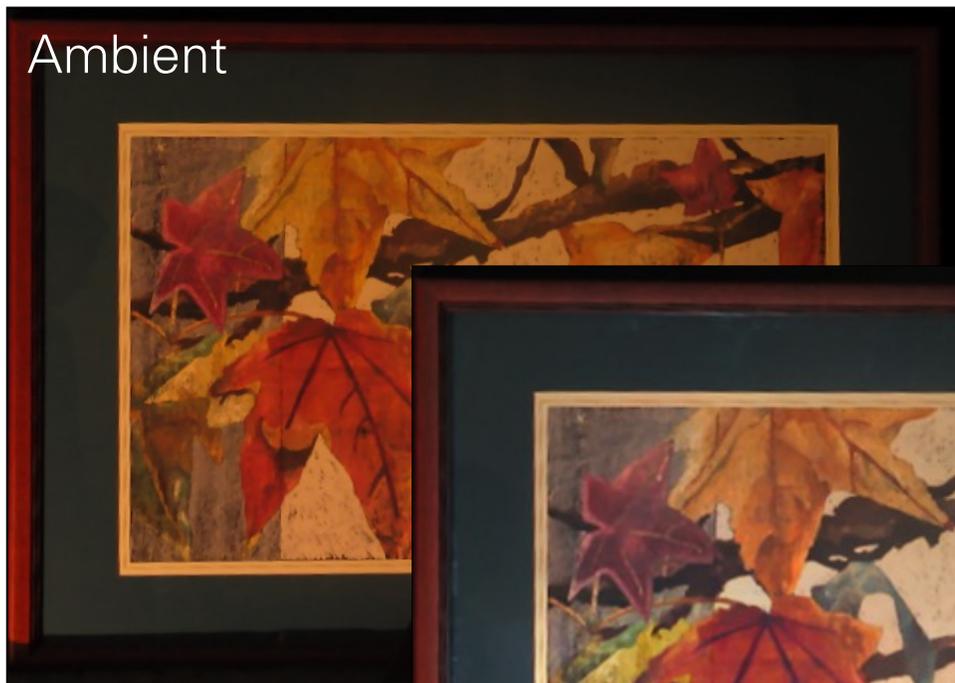


- ✓ Reflectance Edge
- $\uparrow\downarrow$ Geometric Edge
- × Illumination Edge

Removing Reflections due to Flash



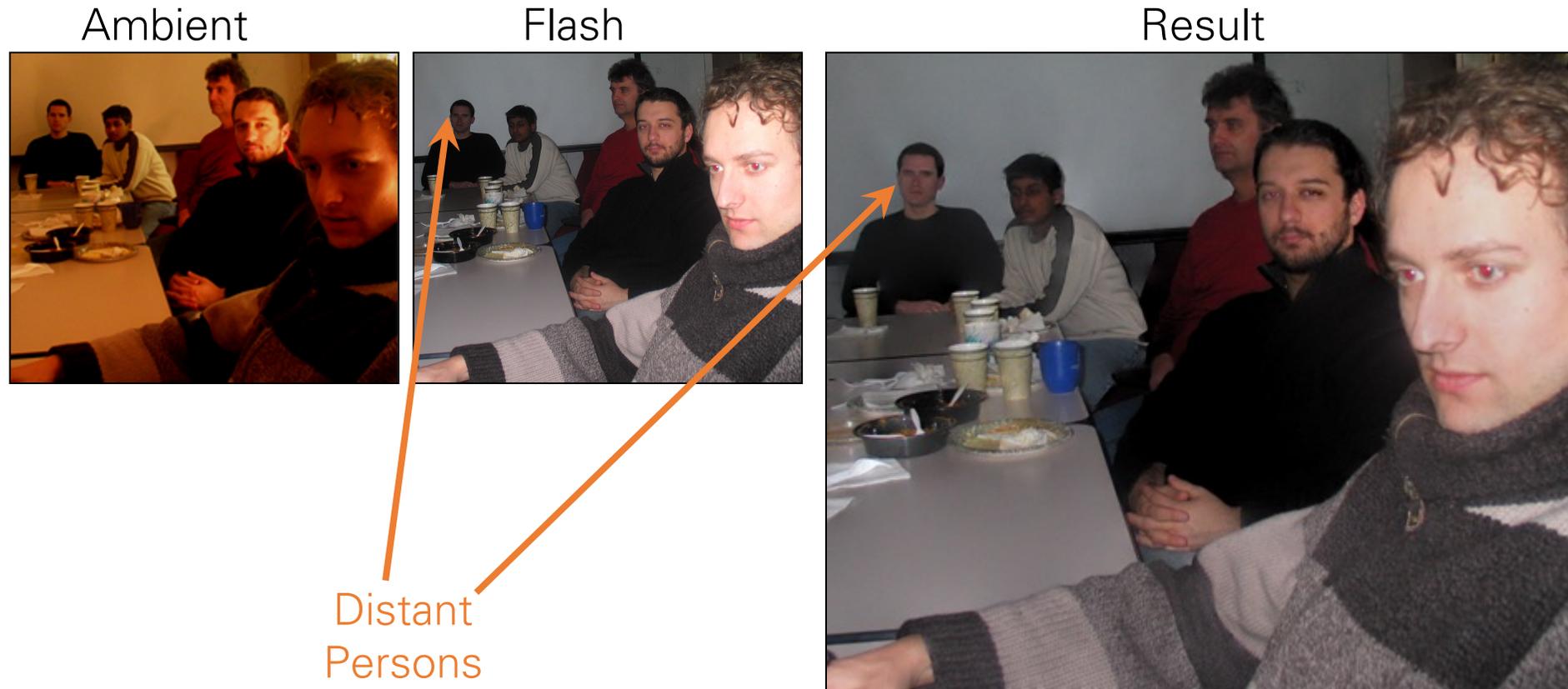
Removing Flash Hotspot



Weights W_s

and ambient image
and Gradient Coherency

Depth Compensation



Scale flash gradients using the ratio of flash and ambient images

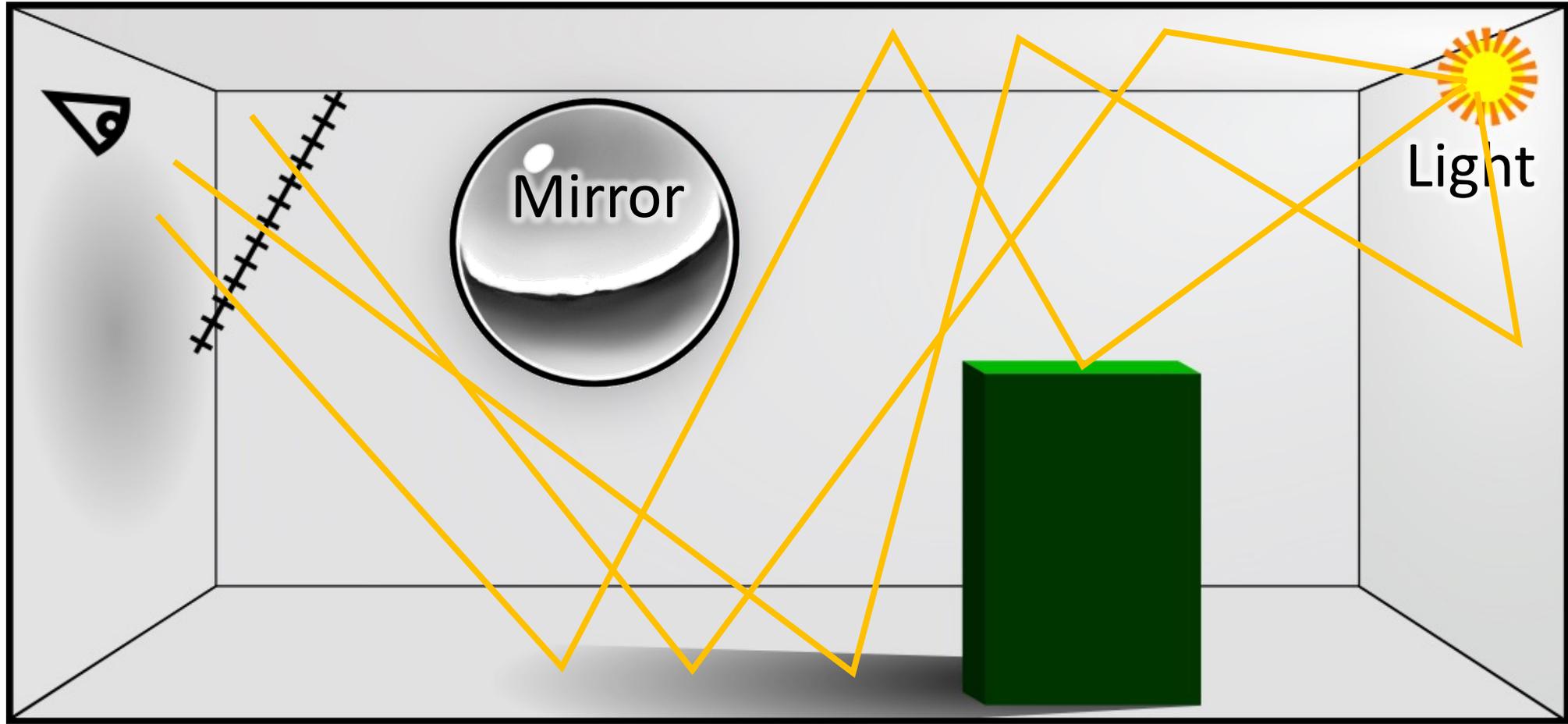
$$\frac{Flash}{Ambient} = \frac{\rho \cos \theta}{(Ambient^*) \times distance^2} \propto \frac{1}{distance^2}$$

Limitations

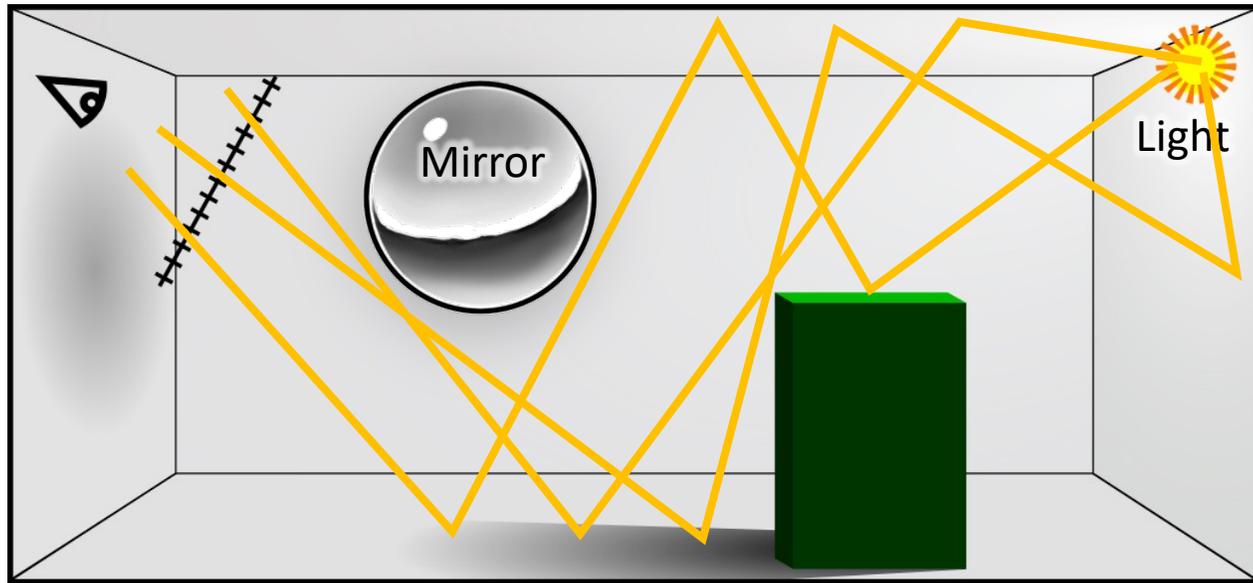
- Difficult Scenarios
 - Dynamic scenes
 - Co-located artifacts
 - Strong ambient illumination edges
- Issues
 - Lack of reliable gradients
 - Homogeneous or dark regions
 - Color coherency

Gradient-domain rendering

Rendering Equation

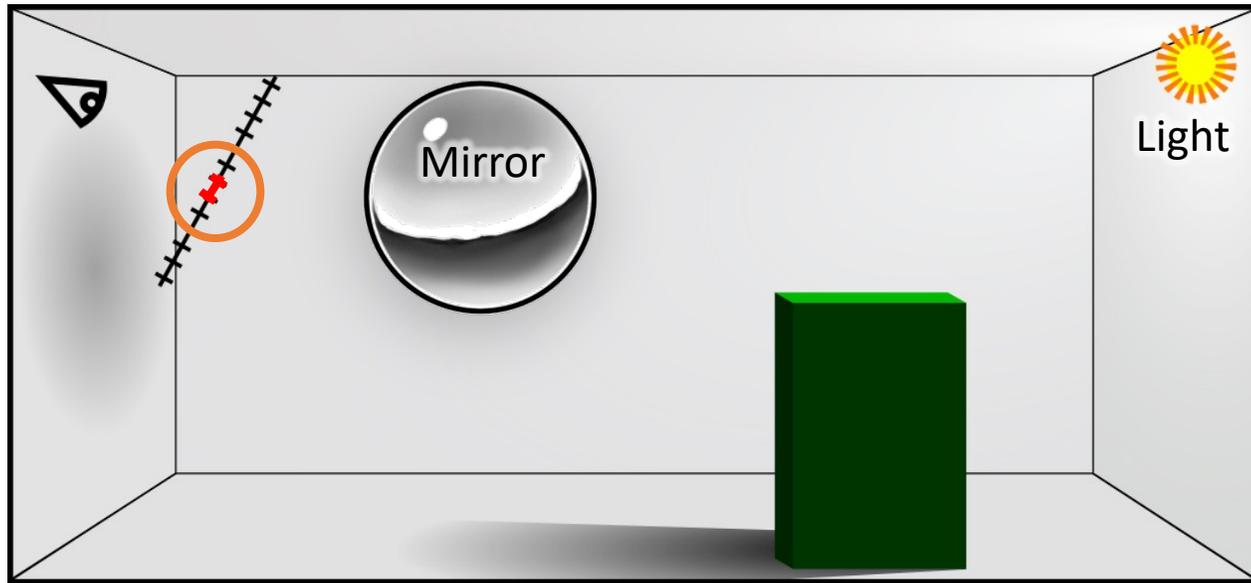


Rendering Equation



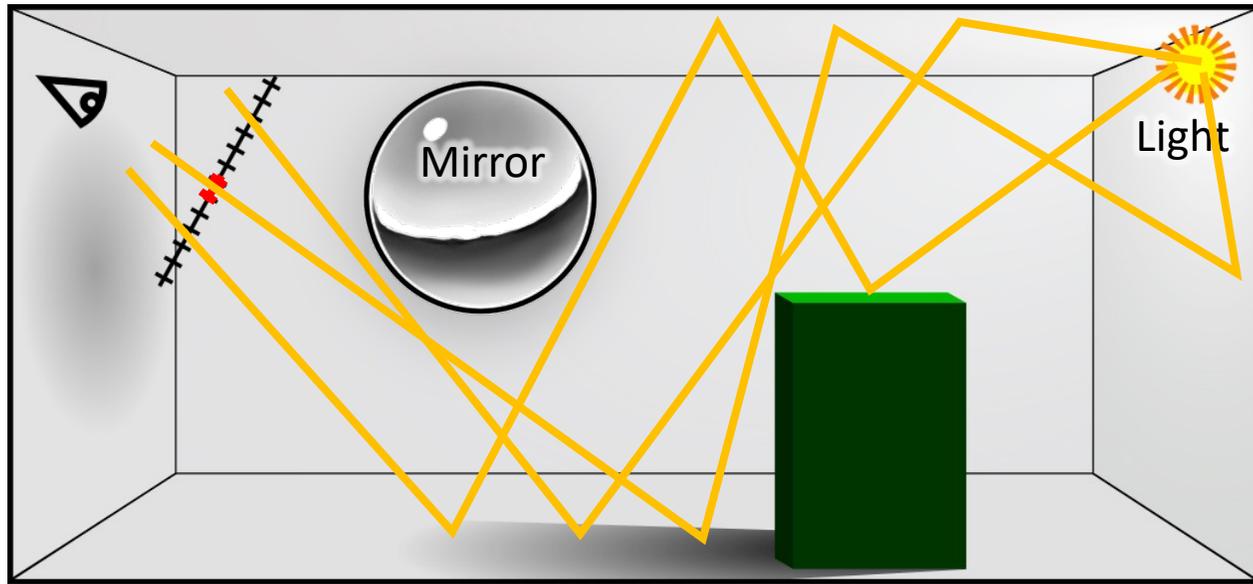
$$I_j = \int_{\Omega} f_j(\bar{x}) d\mu(\bar{x})$$

Rendering Equation



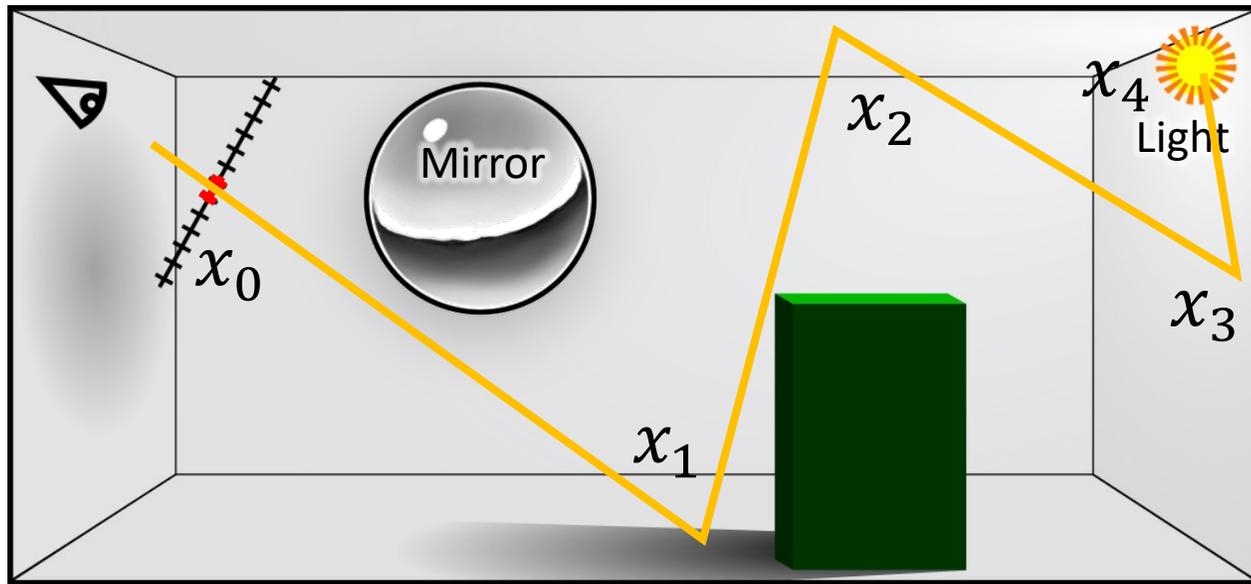
$$I_j = \int_{\Omega} f_j(\bar{x}) d\mu(\bar{x})$$

Rendering Equation



$$I_j = \int_{\Omega} f_j(\bar{x}) d\mu(\bar{x})$$

Rendering Equation



$$\bar{x} = x_0 x_1 x_2 x_3 x_4$$

$$I_j = \int_{\Omega} f_j(\bar{x}) d\mu(\bar{x})$$

$f_j(\bar{x}) =$ (Materials) x (Geometries)
x Emitted Lum. x Pixel filtering

Rendering Equation

$$I_j = \int_{\Omega} f_j(\bar{x}) d\mu(\bar{x})$$

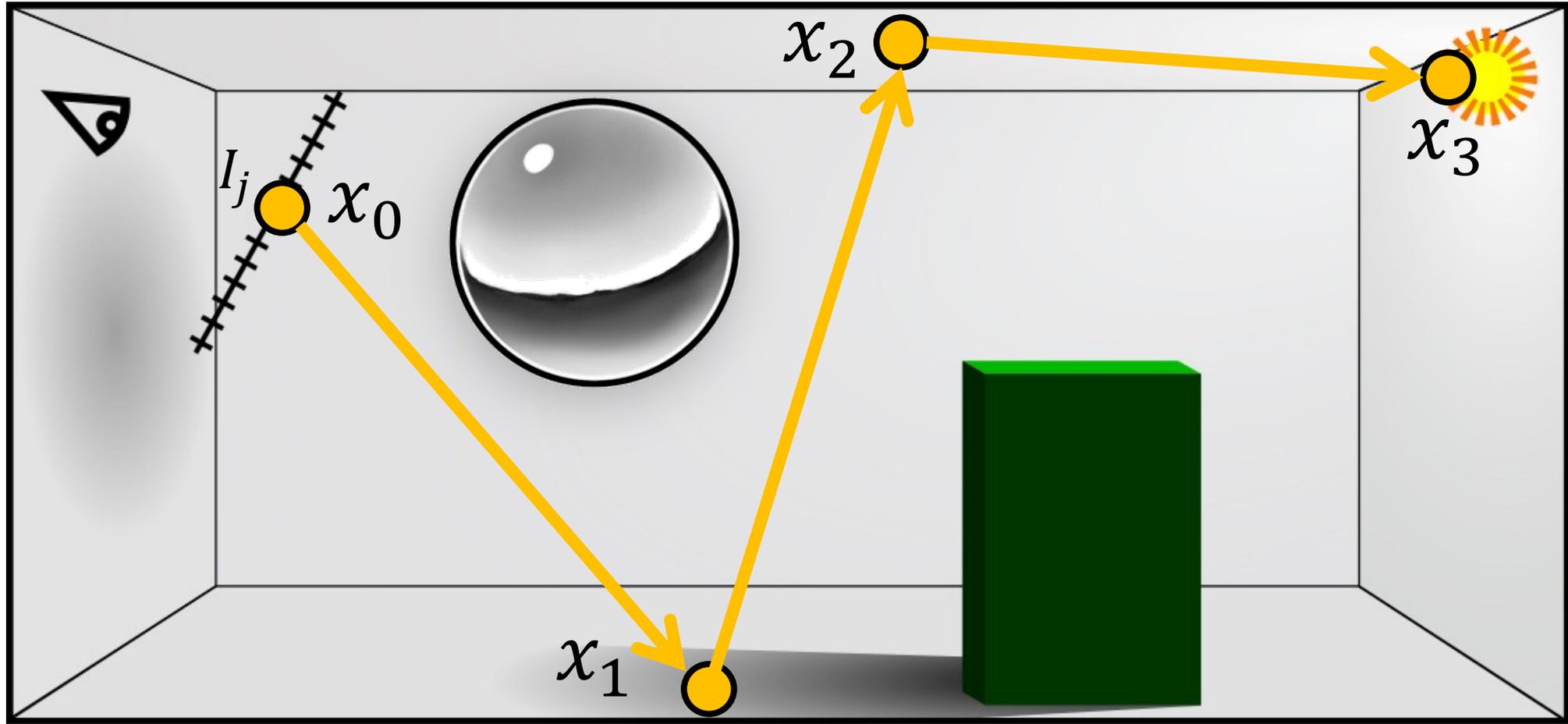


Monte Carlo estimator

$$I_j \approx \frac{1}{N} \sum_{k=1}^N \frac{f_j(\bar{x}_k)}{p(\bar{x}_k)}$$

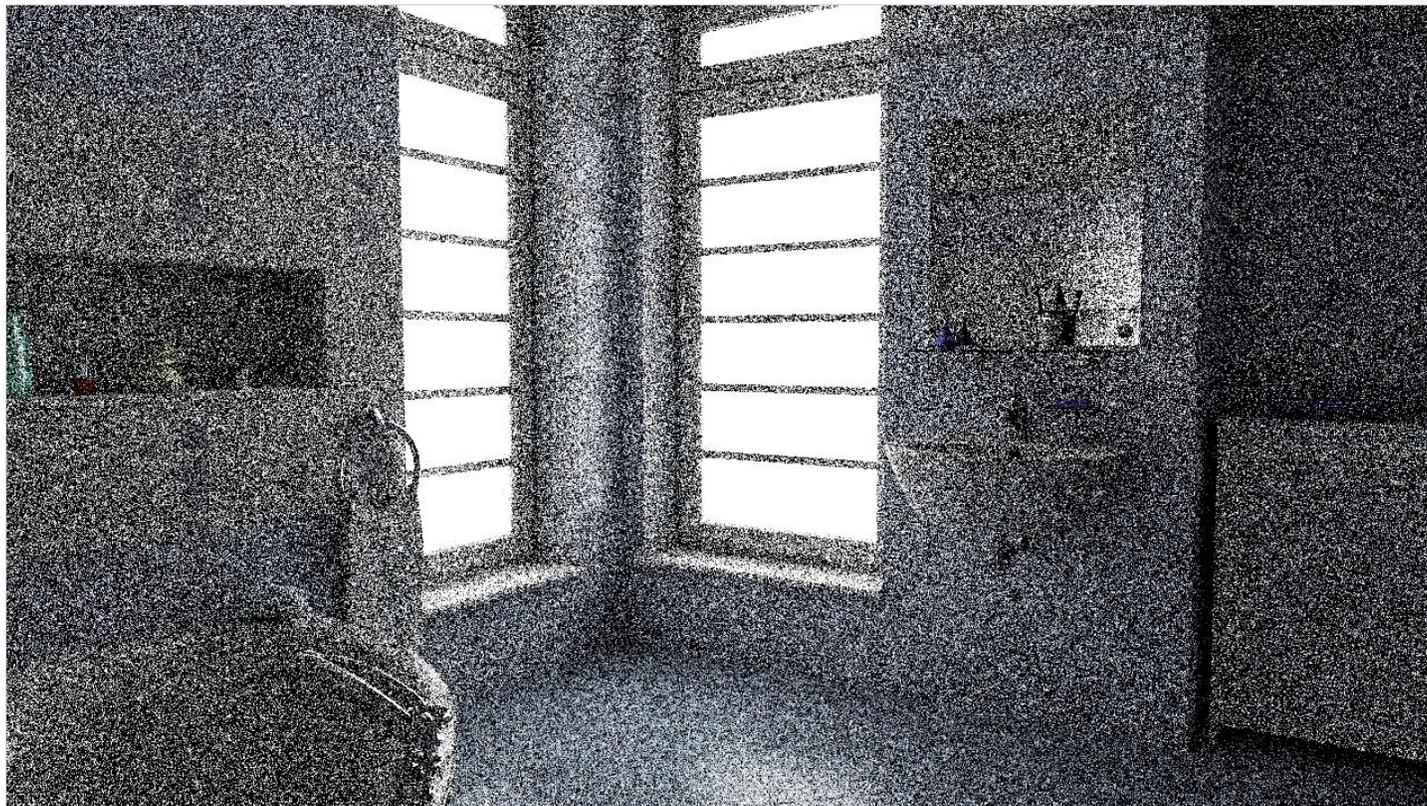
$p(\bar{x}_k)$ is the probability density to sample \bar{x}_k

Path Tracing



Motivation

$$\text{error} / 2 = \text{samples} * 4$$



15min

30min

45min

1h

Motivation

Observation

- Noise mostly proportional to signal magnitude

Idea

- Noise reduction by sampling **sparse** signal representation
 - Sparse: signal magnitude low, except in small regions
 - Wavelets, edge filters, **gradients**, etc.
 - Theoretical justification: Kettunen et al. SIGGRAPH 2015

The Basic Algorithm

1. Perform standard Monte Carlo rendering to obtain primal image
2. Sample gradients: horizontal and vertical
3. Reconstruct image from primal and gradients

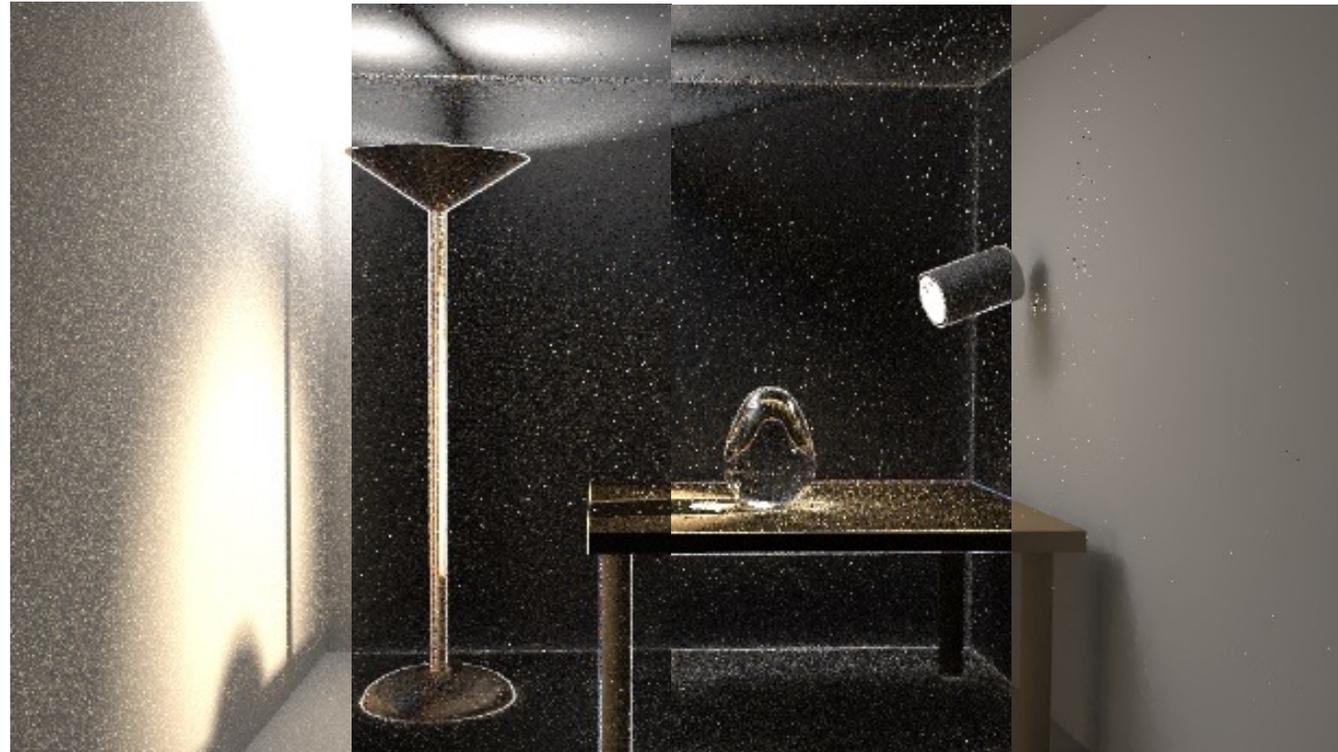
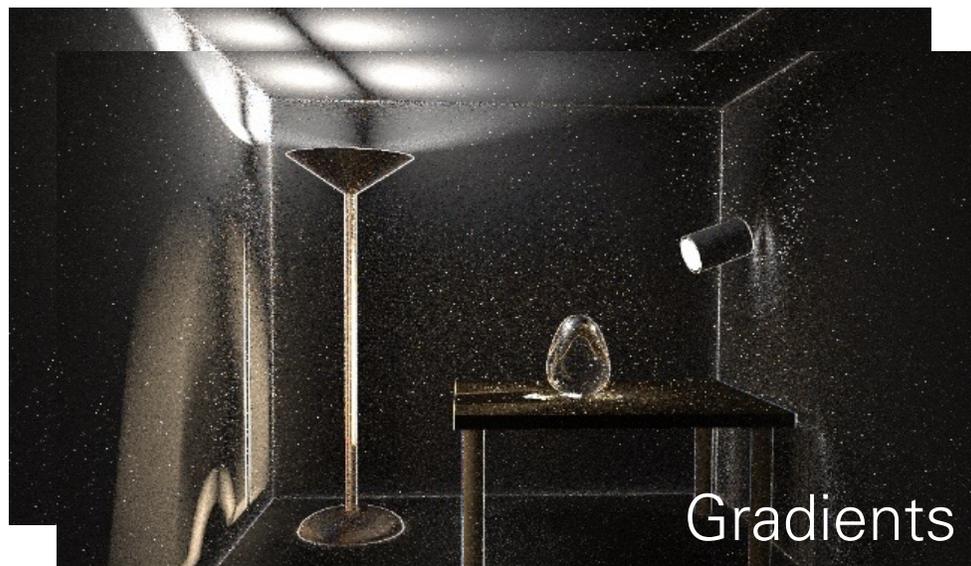


Image Reconstruction

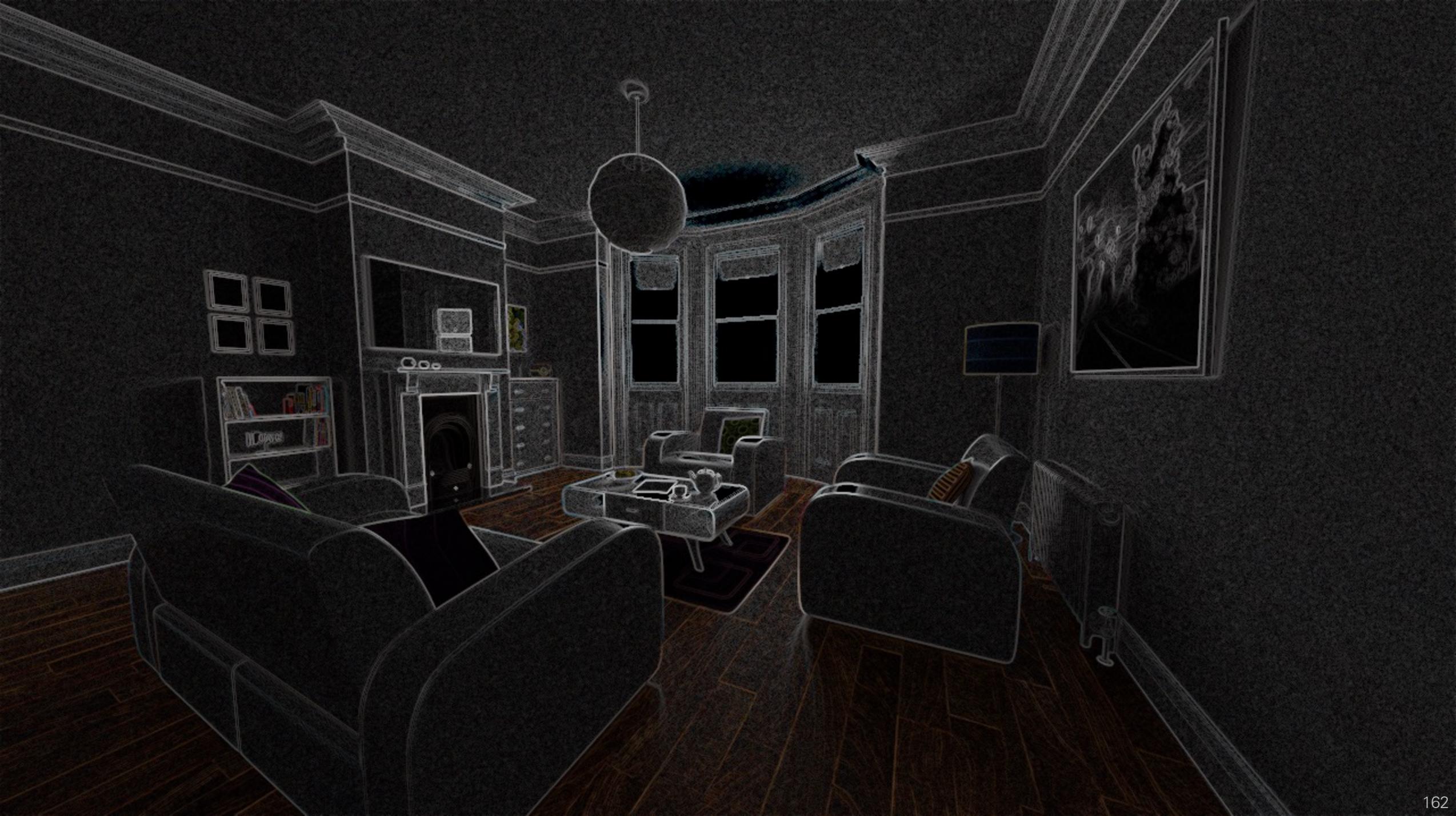


Reconstructed image



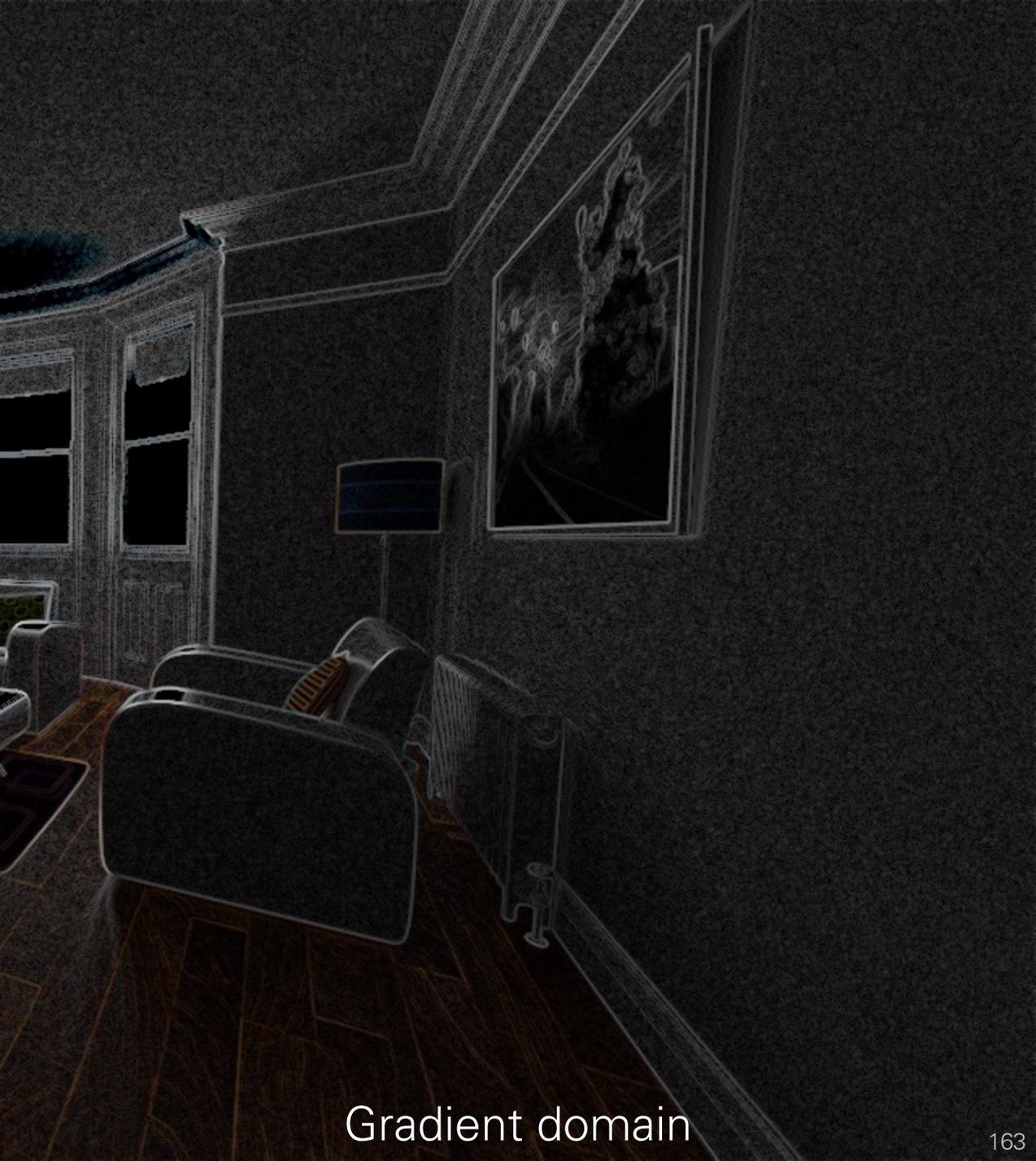
Fusing **gradients** and **primal** information inside one image







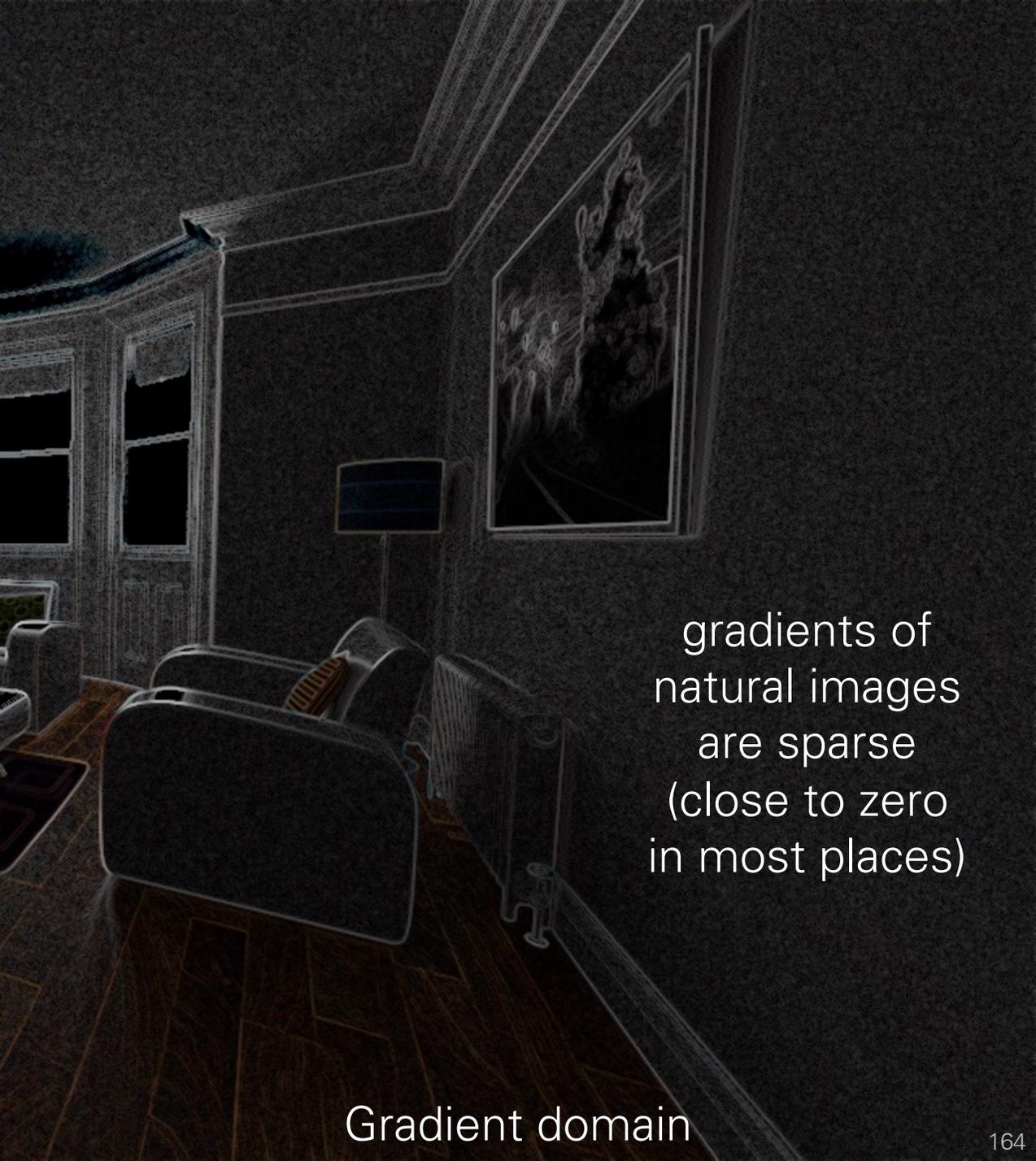
Primal domain



Gradient domain



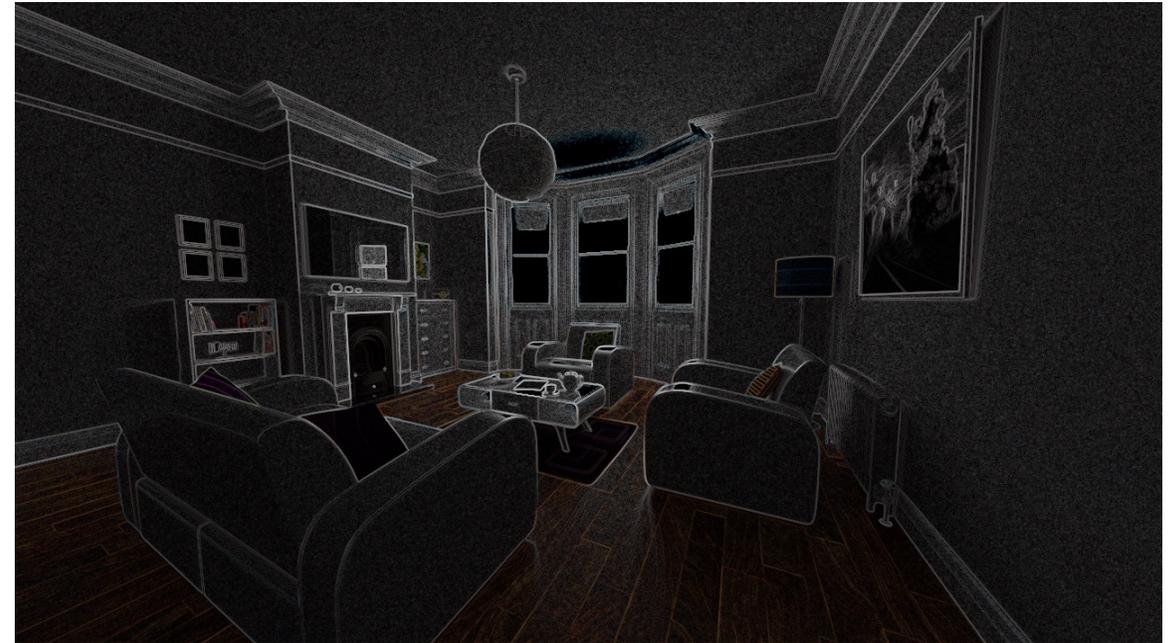
Primal domain



Gradient domain

gradients of
natural images
are sparse
(close to zero
in most places)

Can I go from one image to the other?



Can I go from one image to the other?

differentiation (e.g., convolution with forward-difference kernel)



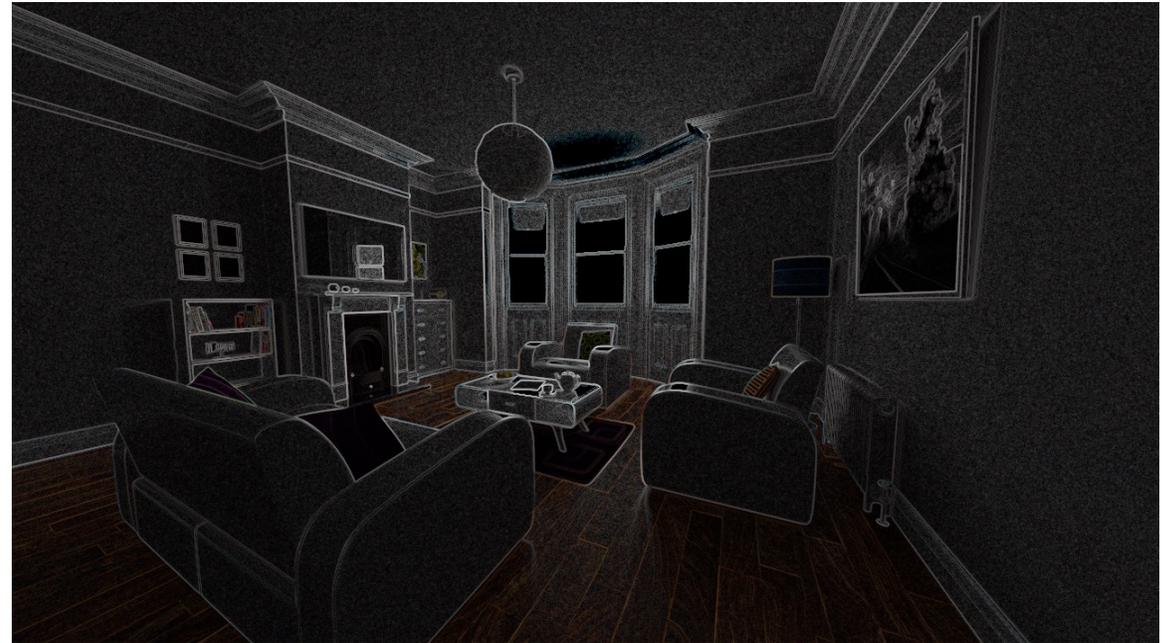
integration (e.g., Poisson solver)

Rendering

Primal-domain rendering: simulate intensities directly



Gradient-domain rendering: simulate gradients, then solve Poisson problem



Why would gradient-domain rendering make sense?

Rendering

Primal-domain rendering: simulate intensities directly



Gradient-domain rendering: simulate gradients, then solve Poisson problem



Why would gradient-domain rendering make sense?

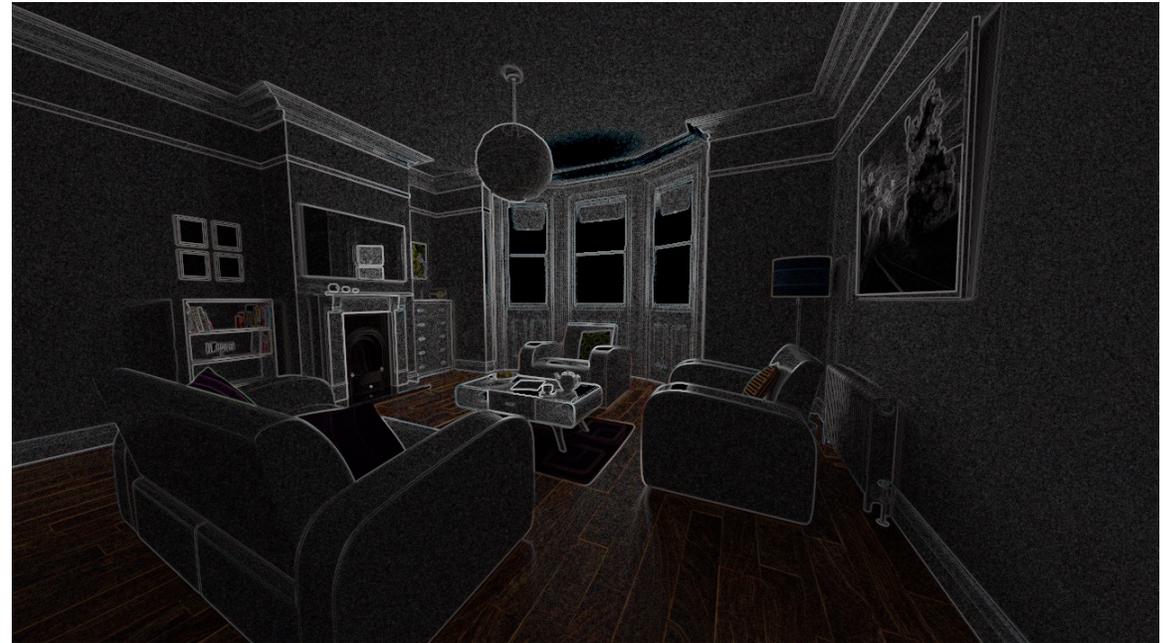
- Since gradients are sparse, I can focus most (but not all of) my resources (i.e., ray samples) on rendering the few pixels that are non-zero in gradient space, with much lower variance.
- Poisson reconstruction performs a form of “filtering” to further reduce variance.

Rendering

Primal-domain rendering: simulate intensities directly



Gradient-domain rendering: simulate gradients, then solve Poisson problem



Why would gradient-domain rendering make sense? Why not all?

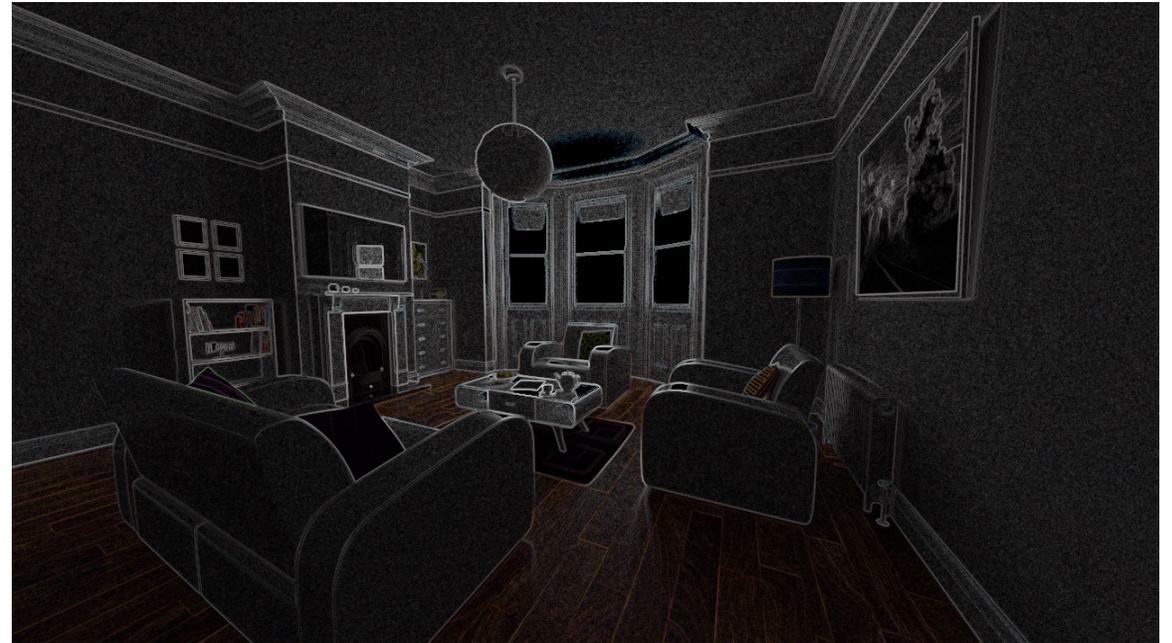
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Rendering

Primal-domain rendering: simulate intensities directly



Gradient-domain rendering: simulate gradients, then solve Poisson problem



You still need to render a few sparse pixels (roughly one per “flat” region in the image) in primal domain, to use as boundary conditions in the Poisson solver.

- In practice, do image-space stratified sampling to select these pixels.

Gradient-Domain Rendering

Gradient-Domain Metropolis Light Transport

Jaakko Lehtinen^{1,2} Tero Karras¹ Samuli Laine¹ Miika Aittala^{2,1} Frédo Durand³ Timo Aila¹

¹NVIDIA Research ²Aalto University ³MIT CSAIL

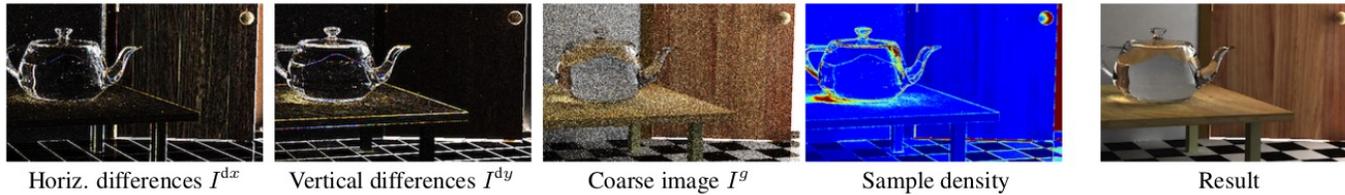


Figure 1: We compute image gradients I^{dx} , I^{dy} and a coarse image I^g using a novel Metropolis algorithm that distributes samples according to path space gradients, resulting in a distribution that mostly follows image edges. The final image is reconstructed using a Poisson solver.

Gradient-Domain Path Tracing

Markus Kettunen¹ Marco Manzi² Miika Aittala¹ Jaakko Lehtinen^{1,3} Frédo Durand⁴ Matthias Zwicker²

¹Aalto University ²University of Bern ³NVIDIA ⁴MIT CSAIL



Figure 1: Comparing gradient-domain path tracing (G-PT, L_1 reconstruction) to path tracing at equal rendering time (2 hours). In this time, G-PT draws about 2,000 samples per pixel and the path tracer about 5,000. G-PT consistently outperforms path tracing, with the rare exception of some highly specular objects. Our frequency analysis explains why G-PT outperforms conventional path tracing.

A lot of papers since SIGGRAPH 2013 (first introduction of gradient-domain rendering) that are looking to extend basically all primal-domain rendering algorithms to the gradient domain.

Does it help?



Gradient-domain path tracing (2 minutes)



Primal-domain path tracing (2 minutes)

Remember this idea (we'll come back to it)



Primal domain



Gradient domain

gradients of natural images are sparse (close to zero in most places)

Modern Gradient-Domain Rendering

Deep Convolutional Reconstruction For Gradient-Domain Rendering

MARKUS KETTUNEN, Aalto University

ERIK HÄRKÖNEN, Aalto University

JAAKKO LEHTINEN, Aalto University and Nvidia



<https://github.com/mkettune/ngpt>

Modern Gradient-Domain Rendering

GradNet: Unsupervised Deep Screened Poisson Reconstruction for Gradient-Domain Rendering

JIE GUO*, State Key Lab for Novel Software Technology, Nanjing University

MENGTIAN LI*, State Key Lab for Novel Software Technology, Nanjing University

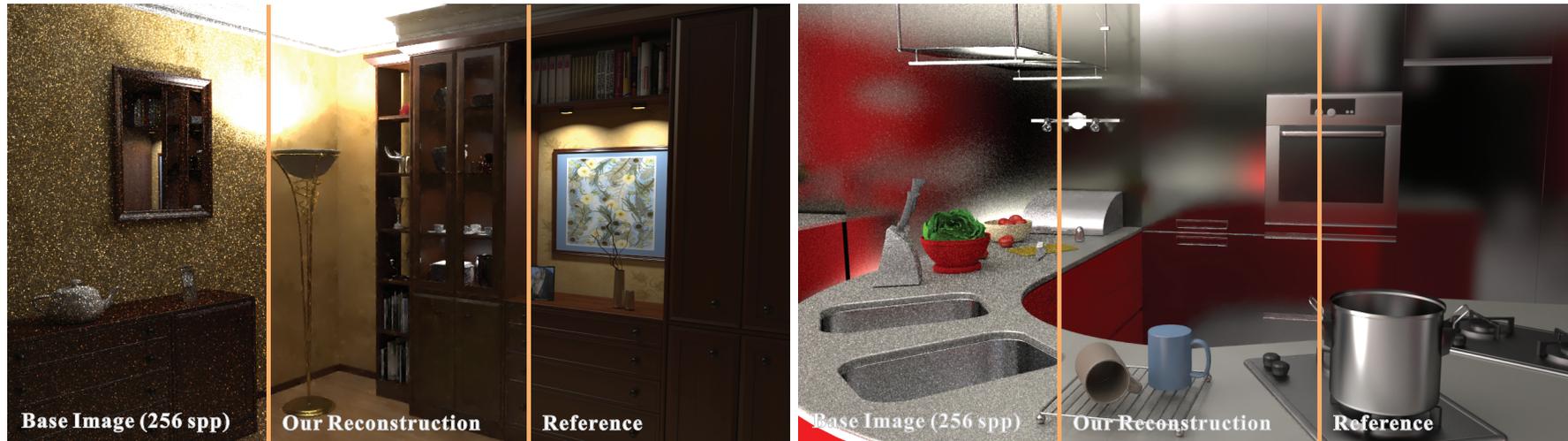
QUEWEI LI, State Key Lab for Novel Software Technology, Nanjing University

YUTING QIANG, State Key Lab for Novel Software Technology, Nanjing University

BINGYANG HU, State Key Lab for Novel Software Technology, Nanjing University

YANWEN GUO[†], State Key Lab for Novel Software Technology, Nanjing University

LING-QI YAN[†], University of California, Santa Barbara



Gradient cameras

Gradient camera

Why I want a Gradient Camera

Jack Tumblin
Northwestern University
jet@cs.northwestern.edu

Amit Agrawal
University of Maryland
aagrwal@umd.edu

Ramesh Raskar
MERL
raskar@merl.com

Why would you want a gradient camera?

Can you directly display the measurements of such a camera?

How would you build a gradient camera?

What implication would this have on a camera?



Primal domain



Gradient domain

gradients of natural images are sparse (close to zero in most places)

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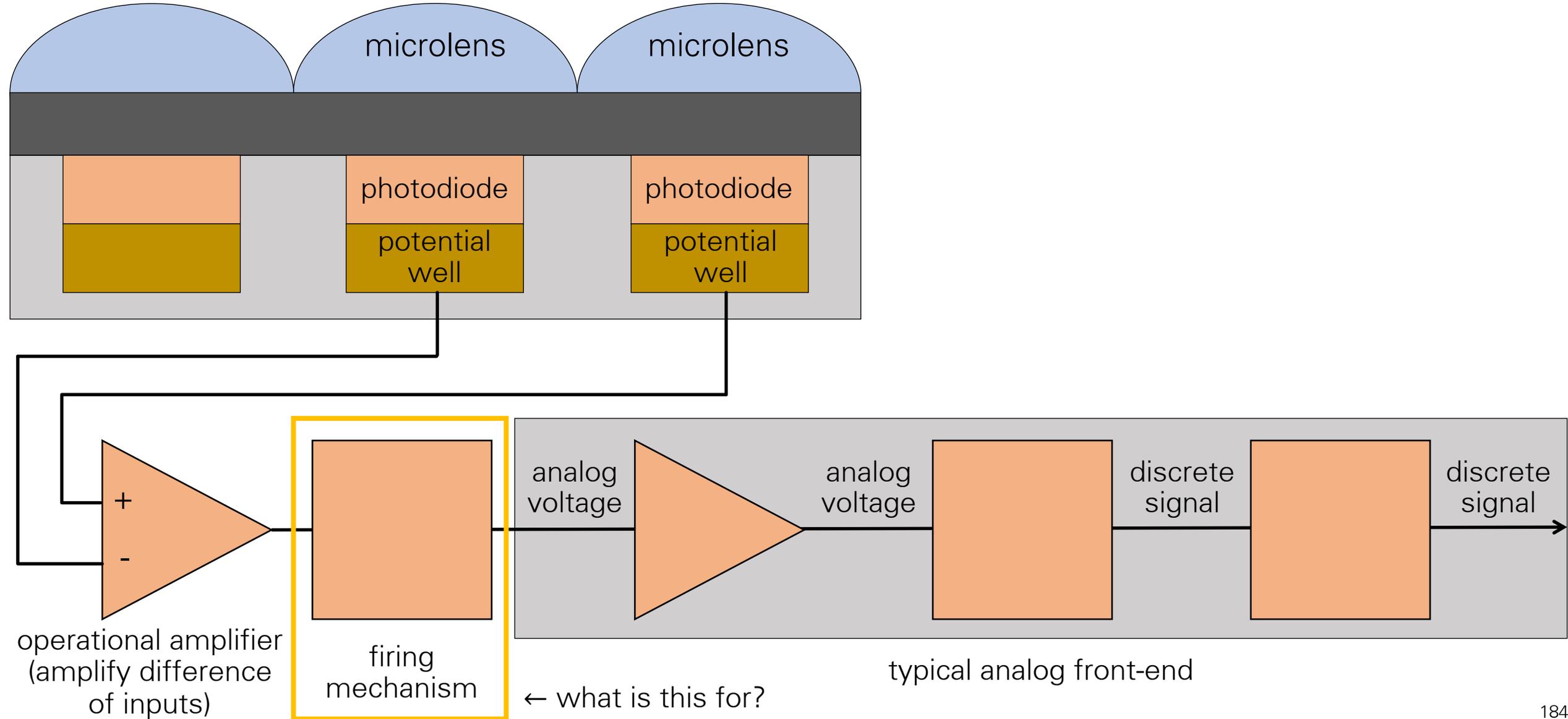
- You need to use a Poisson solver to reconstruct the image from the measured gradients.

How would you build a gradient camera?

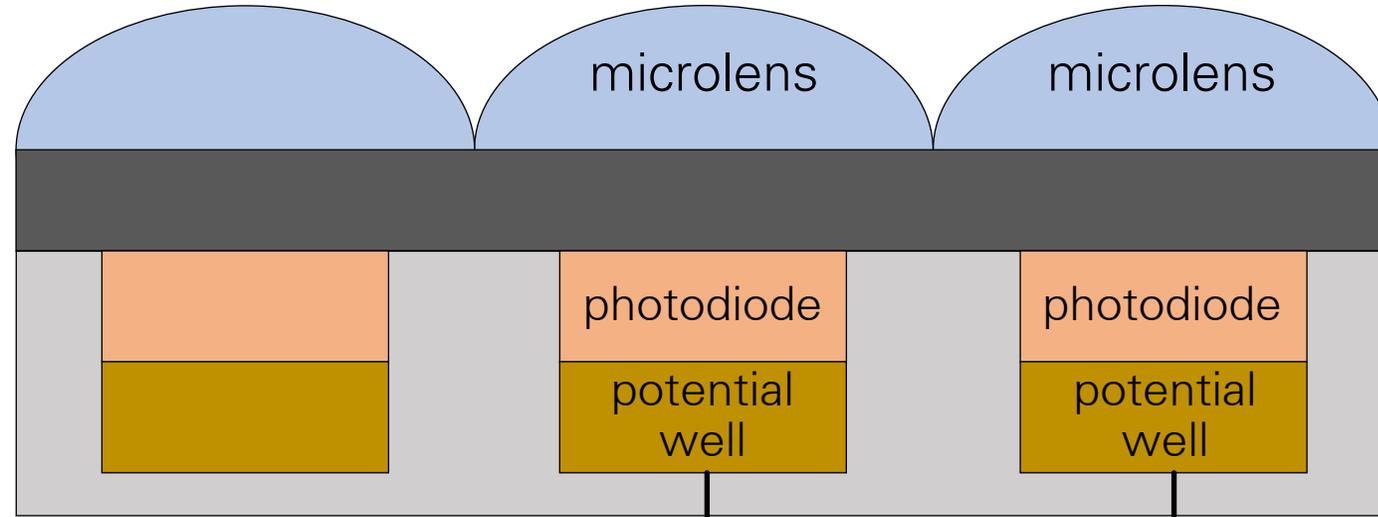
Change the sensor

Can you think how?

Change the sensor



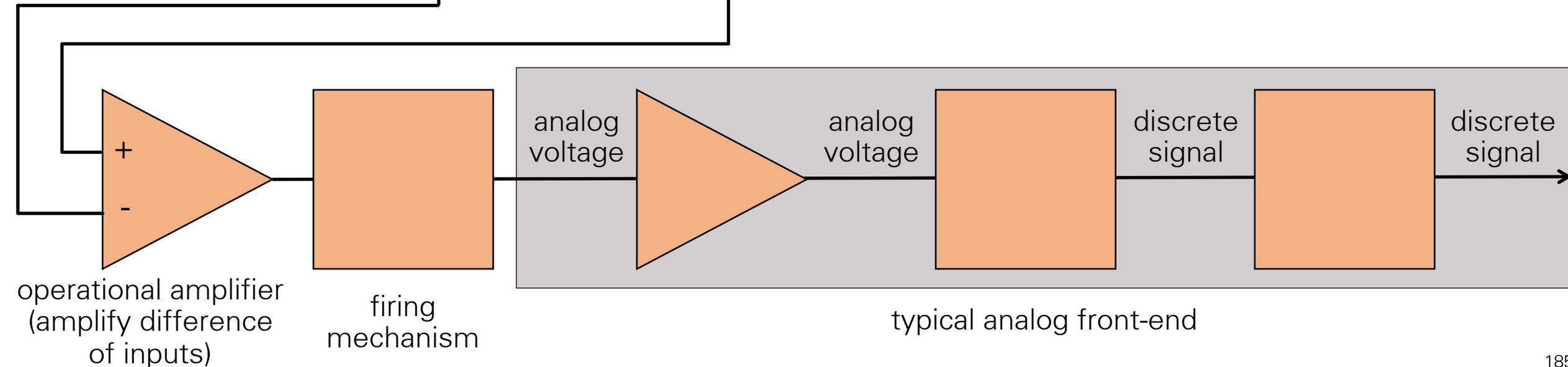
Change the sensor



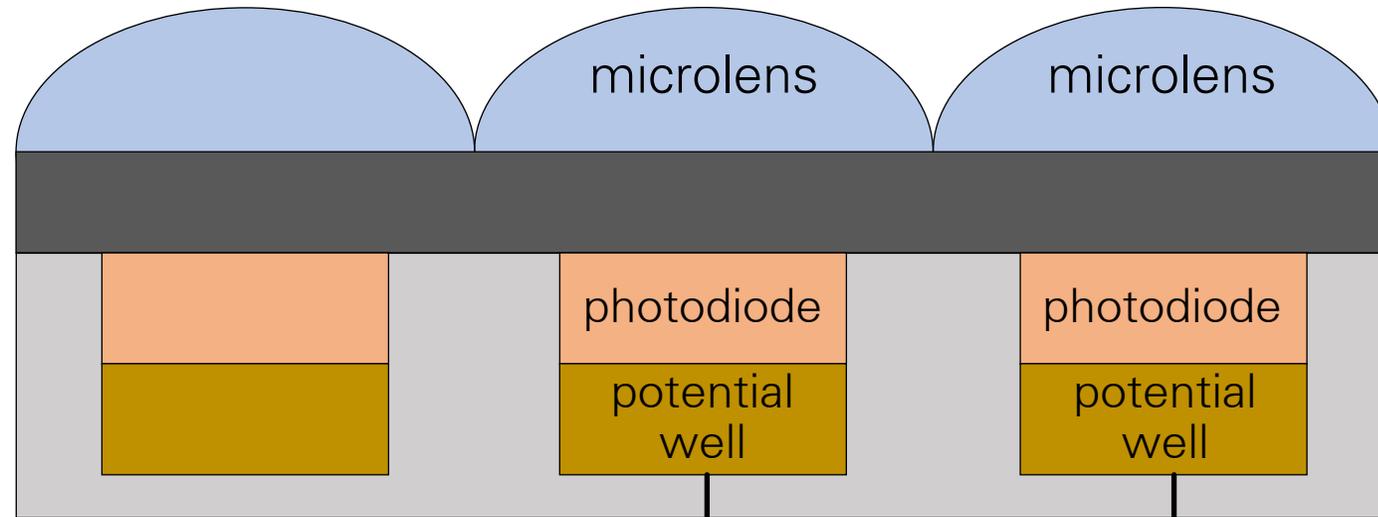
Any disadvantages of this sensor?

Why is this better than computing gradients in post-processing?

What about Poisson noise?



Change the sensor



Any disadvantages of this sensor?

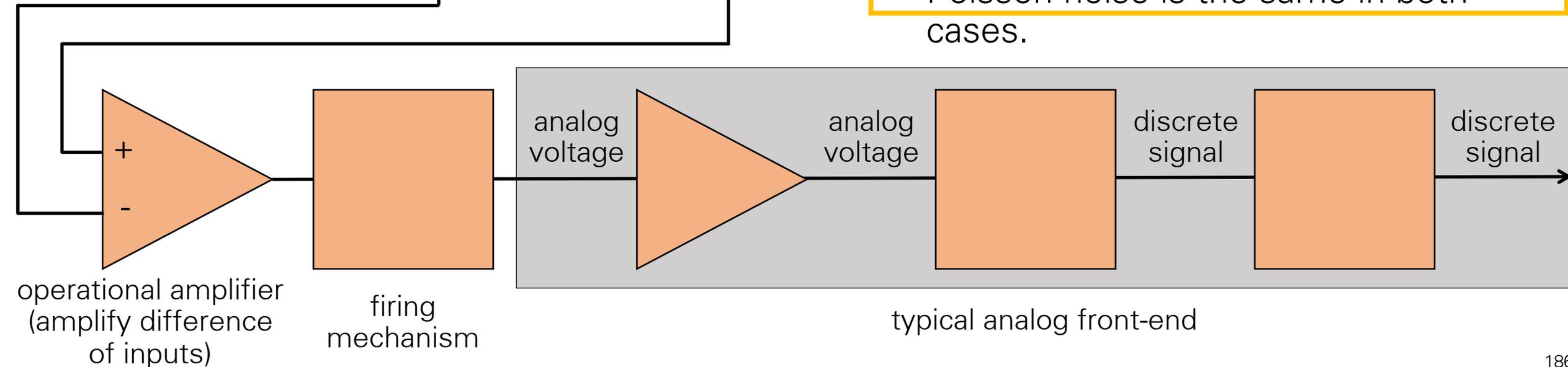
- Spatial resolution is reduced by 2x.
- Photosensitive area is reduced.

Why is this better than computing gradients in post-processing?

- Additive noise is reduced.
- Acquisition is faster thanks to the firing mechanism and sparsity of edges.

What about Poisson noise?

- Poisson noise is the same in both cases.

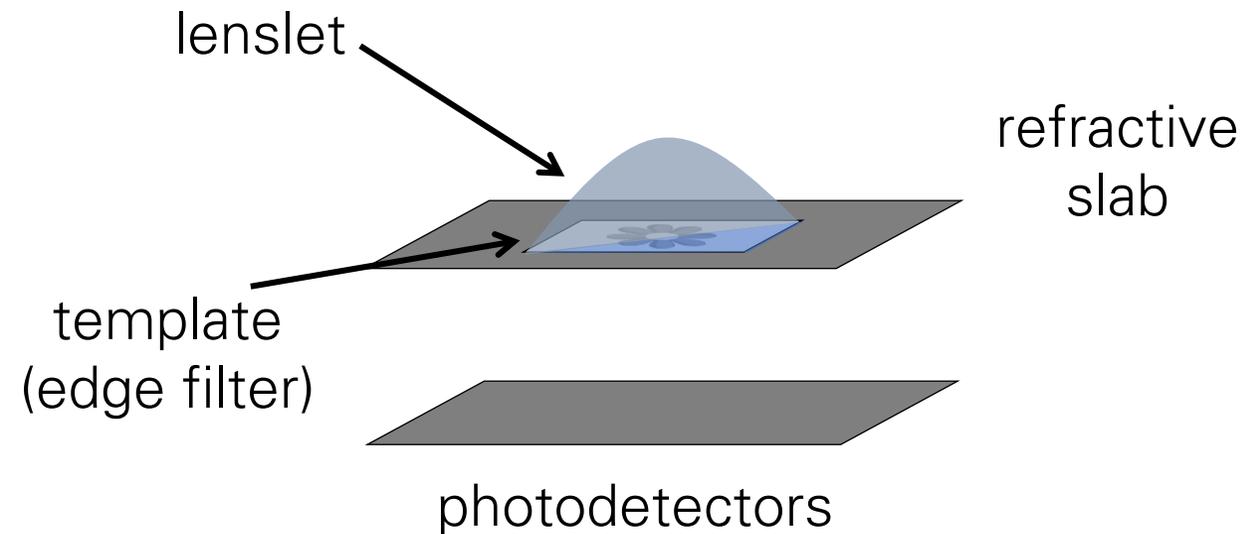


Change the optics

Can you think how?

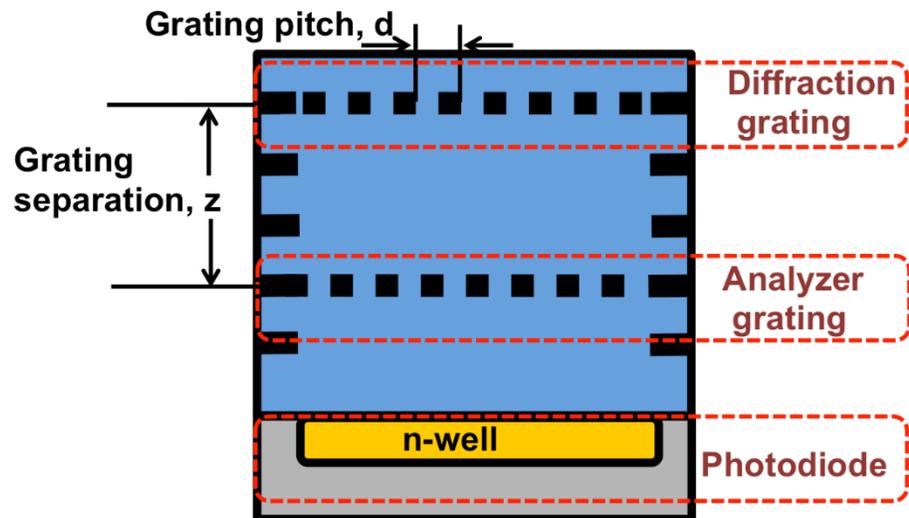
Change the optics

Optical filtering

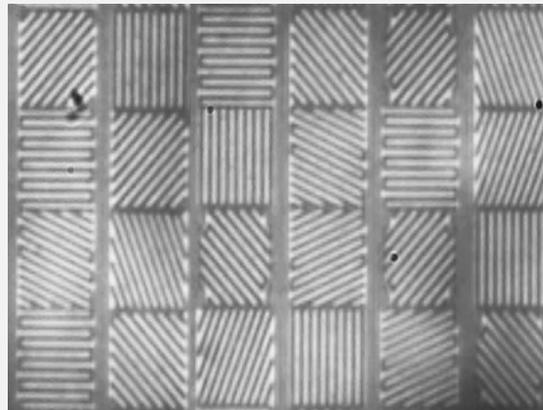


resulting image

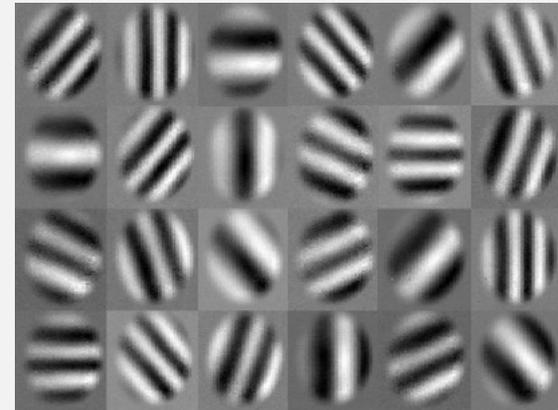
Angle-sensitive pixels



Physical Layout

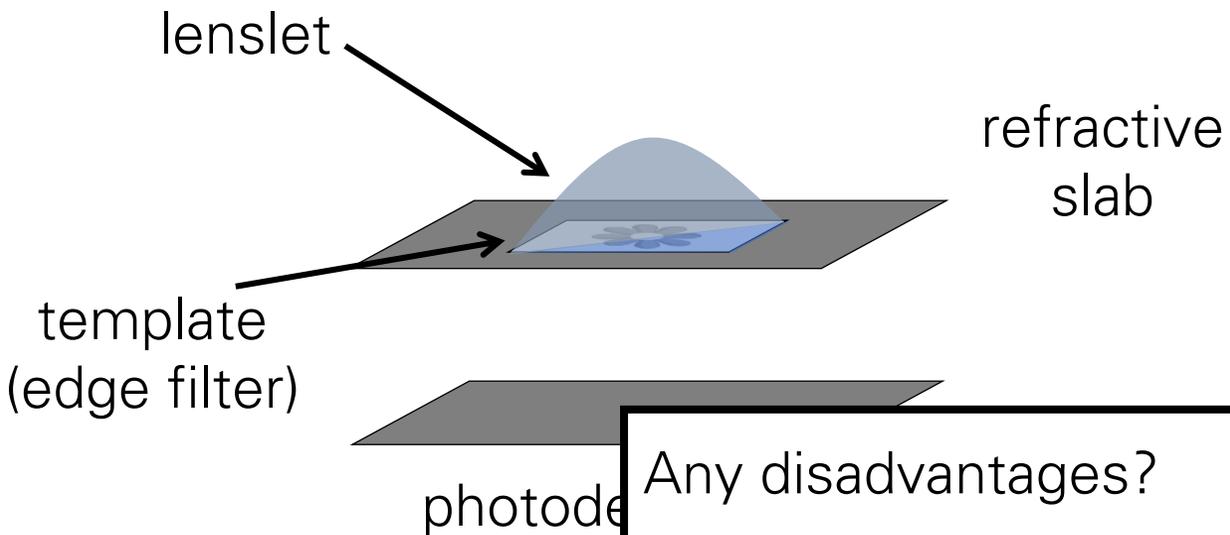


Impulse Response (2D)

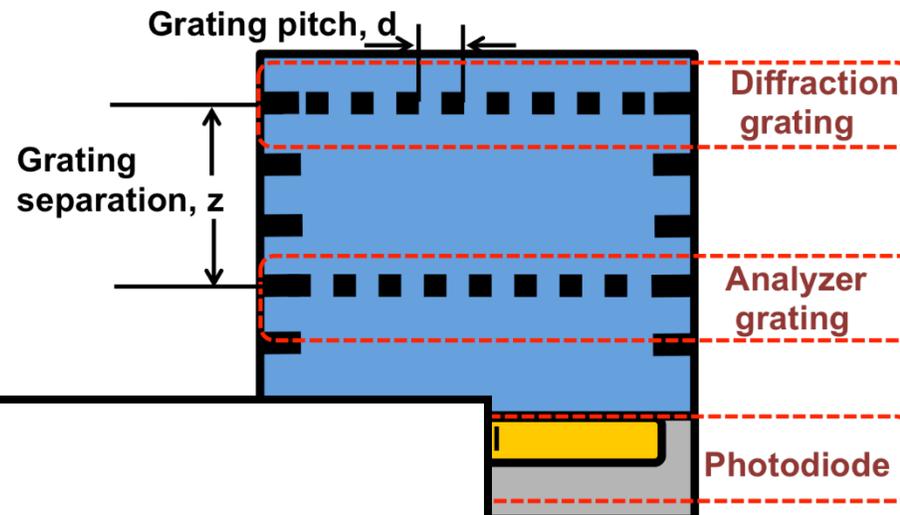


Change the optics

Optical filtering



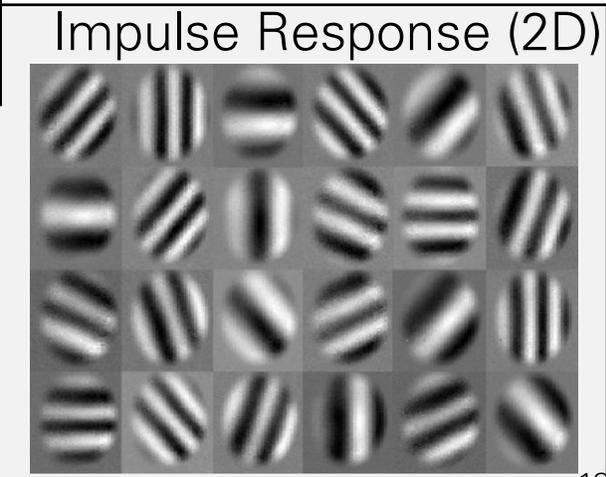
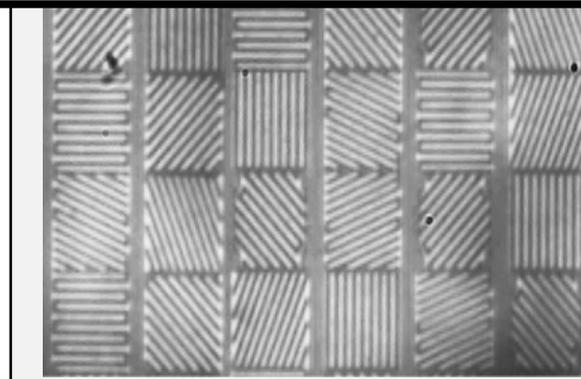
Angle-sensitive pixels



Any disadvantages?



resulting image



Change the optics

Optical filtering

lenslet

refractive slab

template
(edge filter)

photodiode



resulting image

Angle-sensitive pixels

Grating pitch, d

Grating separation, z

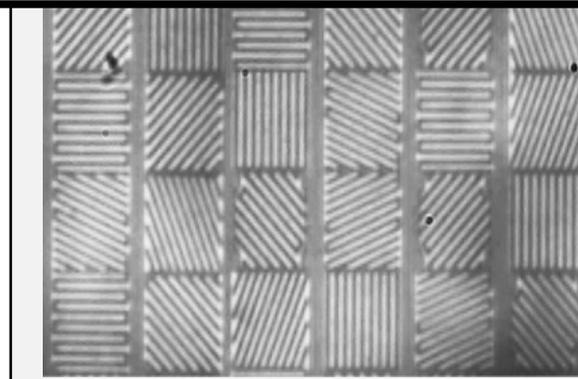
Diffraction grating

Analyzer grating

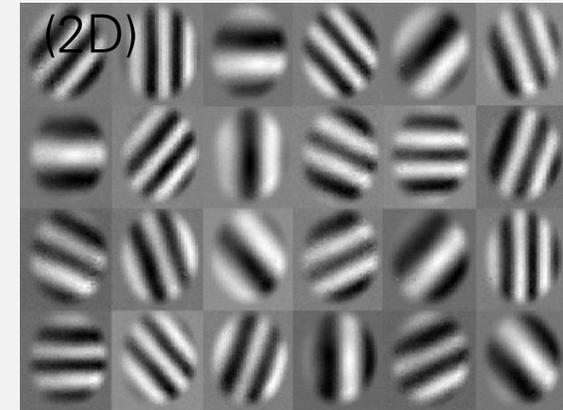
Photodiode

Any disadvantages?

- Reduced light efficiency (we block light).
- We can't do subtraction very easily in optics.



Impulse Response
(2D)



Gradient camera

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- Much higher dynamic range, if also combined with logarithmic gradients.

Can you directly display the measurements of such a camera?

- You need to use a Poisson solver to reconstruct the image from the measured gradients.

How would you build a gradient camera?

- Change the sensor.
- Change the optics.

We can also compute temporal gradients



event-based cameras (a.k.a. dynamic vision sensors, or DVS)

Concept figure for event-based camera:

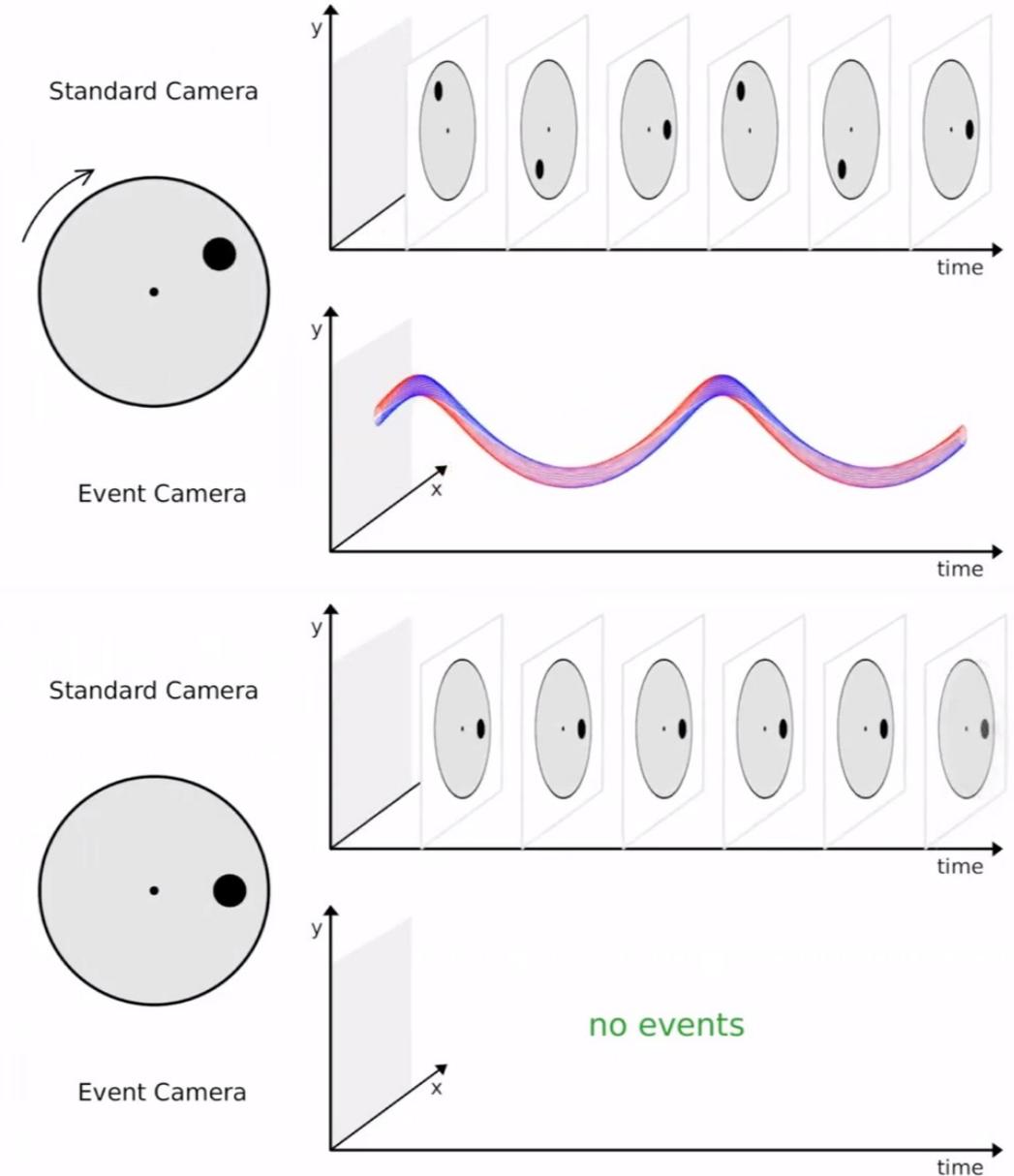
<https://www.youtube.com/watch?v=kPCZESVfHoQ>

High-speed output on a quadcopter:

<https://www.youtube.com/watch?v=LauQ6LWTkxN>

Simulator:

<http://rpg.ifi.uzh.ch/esim>



Open Challenges in Computer Vision

- The past 60 years of research have been devoted to frame-based cameras.

...but they are not good enough!

Latency & Motion blur



Dynamic Range



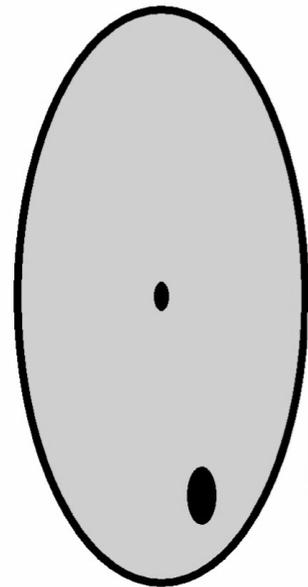
- Event cameras do not suffer from these problems!

What is an event camera?

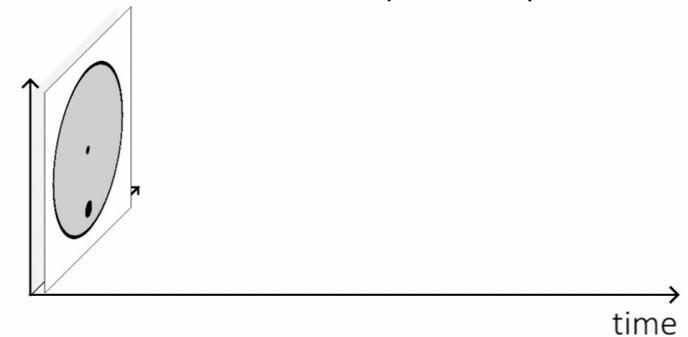
- Novel sensor that measures only **motion in the scene**
- **First commercialized in 2008** by T. Delbruck (UZHÐ) under the name of Dynamic Vision Sensor (DVS)
- **Low-latency** ($\sim 1 \mu\text{s}$)
- **No motion blur**
- **High dynamic range** (140 dB instead of 60 dB)
- **Ultra-low power** (mean: 1mW vs 1W)

Traditional vision algorithms cannot be used because:

- **Asynchronous pixels**
- **No intensity information** (only binary intensity changes)



**standard
camera
output:**



**event
camera
output:**

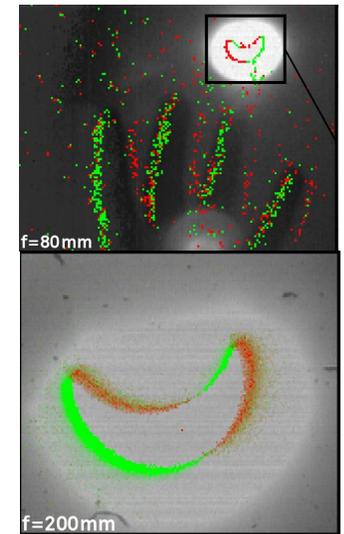
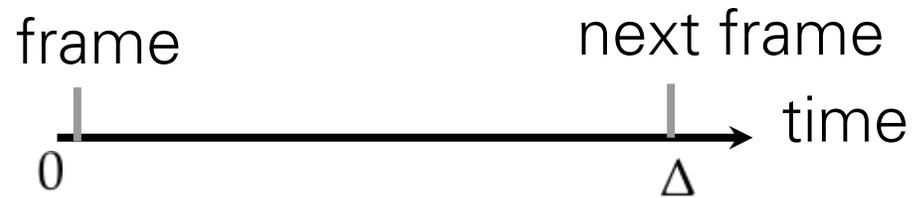


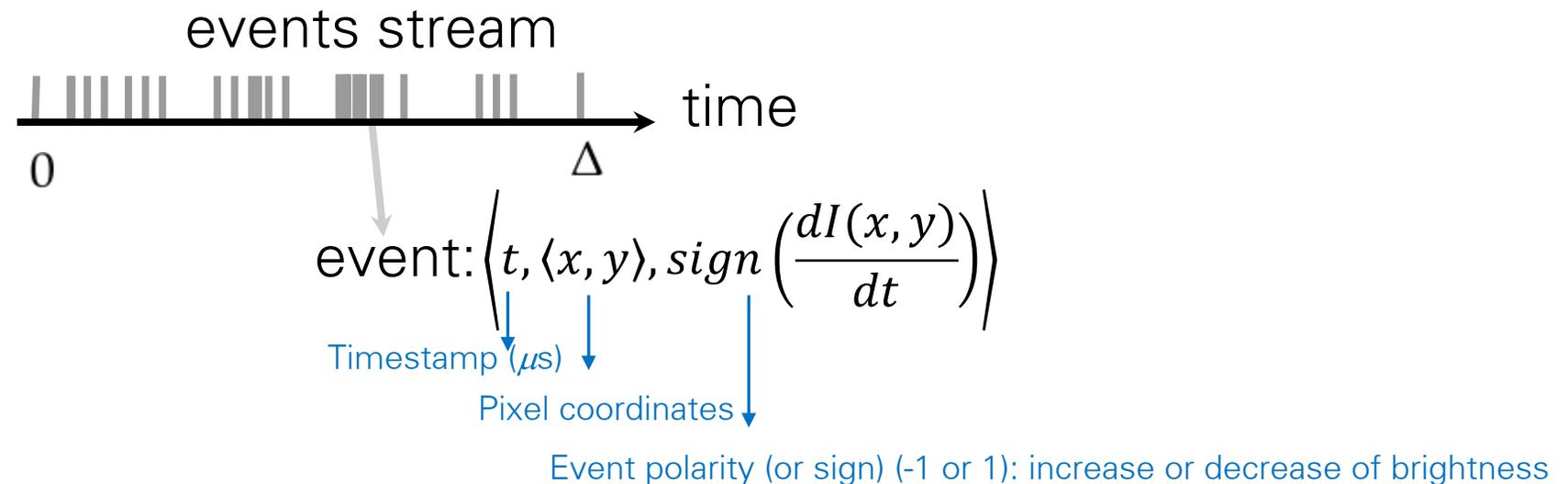
Image of the solar eclipse captured by a DVS

Camera vs Event Camera

- A traditional camera outputs frames at fixed time intervals:



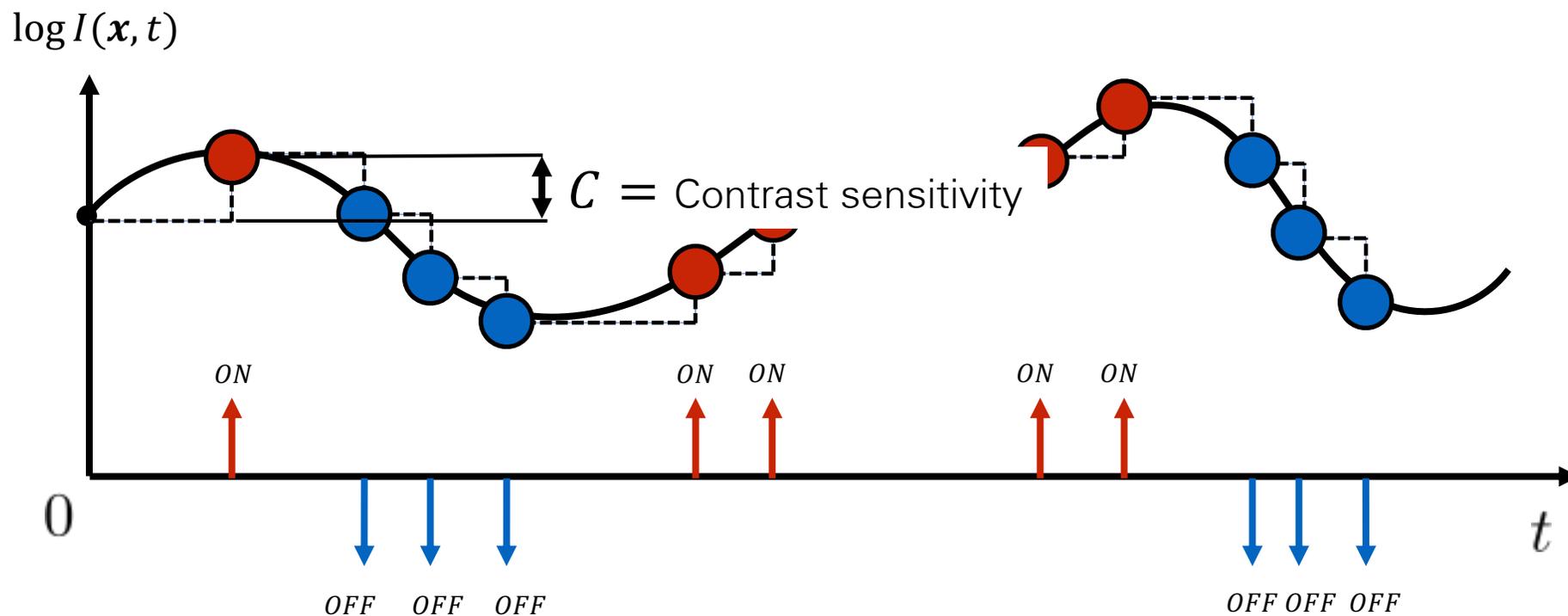
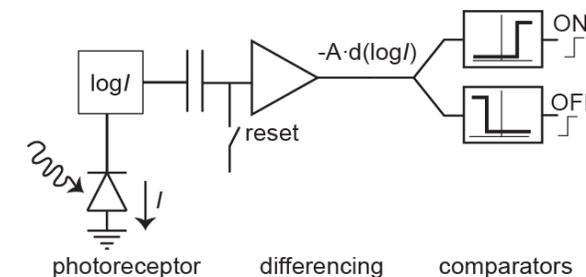
- By contrast, a **DVS** outputs **asynchronous events** at **microsecond resolution**. An event is generated each time a single pixel detects an intensity changes value



Generative Event Model

Consider the intensity at a single pixel...

$$\pm C = \log I(\mathbf{x}, t) - \log I(\mathbf{x}, t - \Delta t)$$

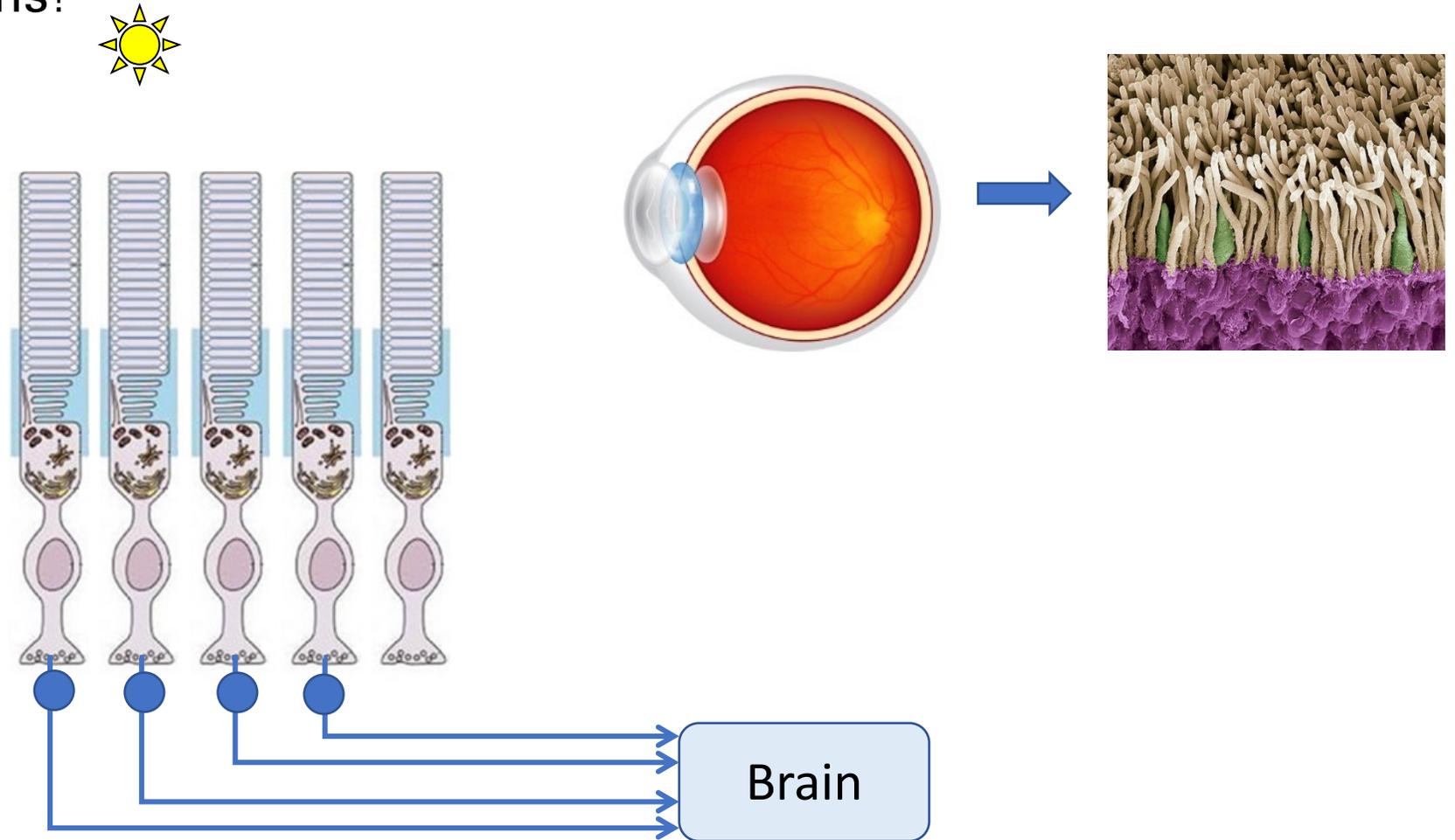


Events are triggered **asynchronously**

Event cameras are inspired by the Human Eye

Human retina:

- 130 million photoreceptors
- But only 2 million axons!

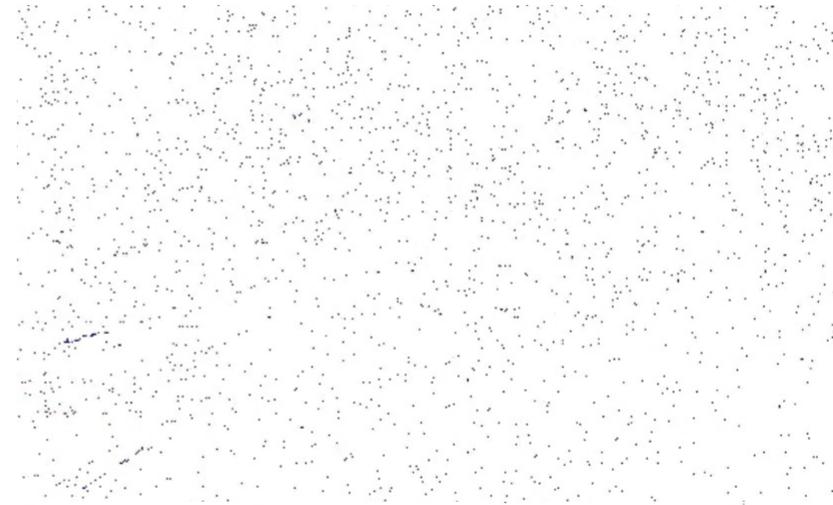


Event Camera Output with No Motion

Standard Camera



Event Camera (ON, OFF events)



$\Delta T = 40 \text{ ms}$

Without motion, only background noise is output

Event Camera Output with Relative Motion

Standard Camera



Event Camera (ON, OFF events)



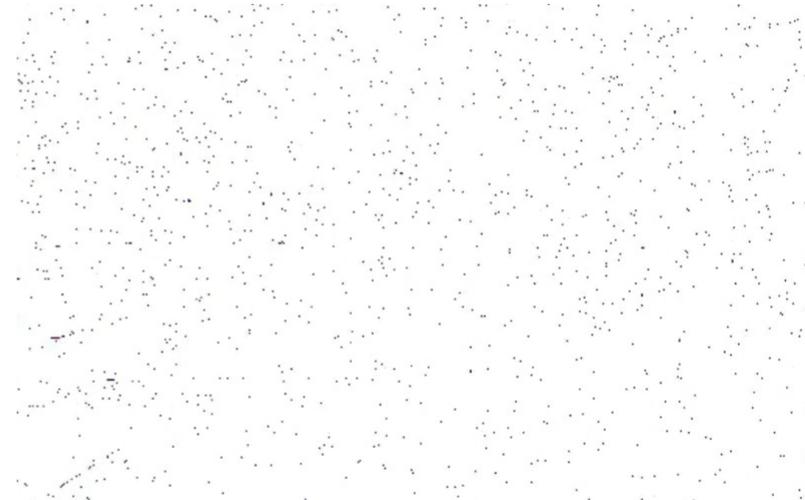
$\Delta T = 10 \text{ ms}$

Event Camera Output with Relative Motion

Standard Camera



Event Camera (ON, OFF events)



$\Delta T = 40 \text{ ms}$

Low-light Sensitivity (night drive)



GoPro Hero 6



Event Camera by Prophesee
White = Positive events
Black = Negative events

Image Reconstruction from Events

- Probabilistic simultaneous, gradient & rotation estimation from $C = -\nabla L \cdot \mathbf{u}$
- Obtain intensity from gradients via Poisson reconstruction
- The reconstructed image has super-resolution and high dynamic range (HDR)
- In real time on a GPU

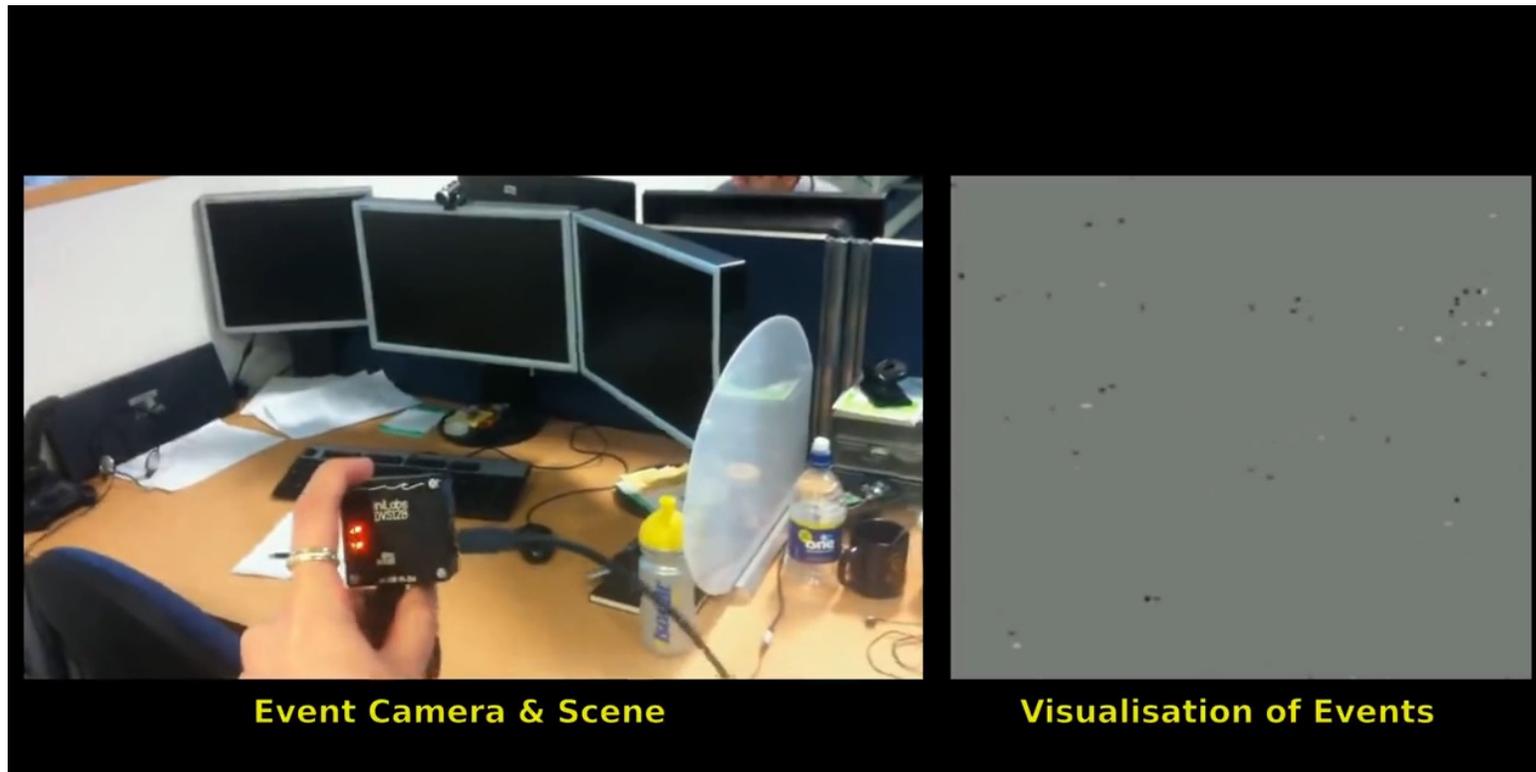
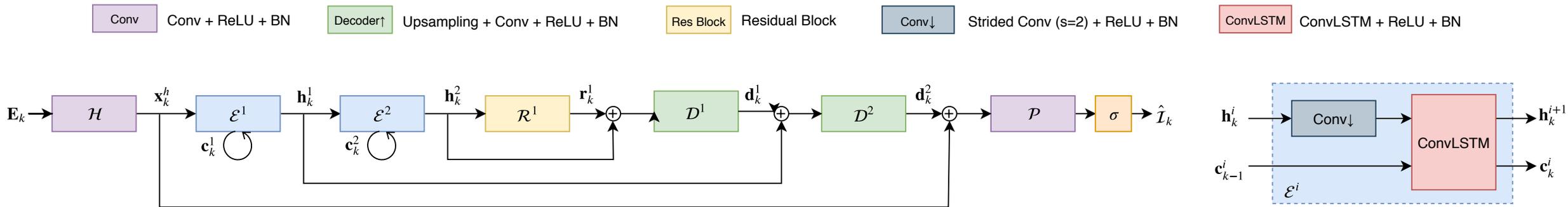
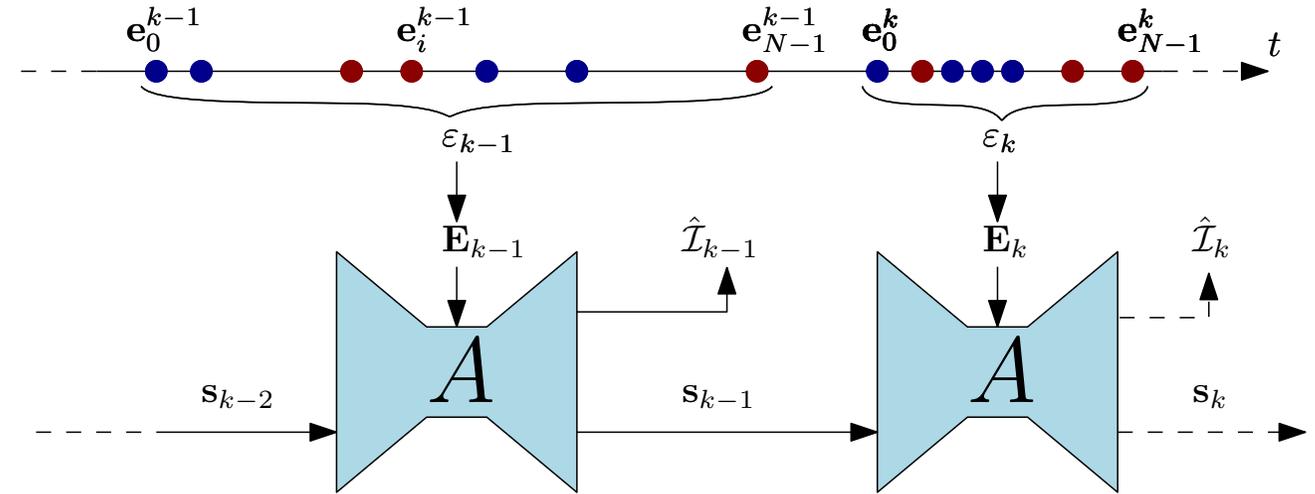


Image Reconstruction from Events – E2VID

- A fully convolutional, UNet-like architecture composed of recurrent encoder layers, followed by residual blocks and decoder layers, with skip connections between symmetric layers.



Rebecq et al., "Events-to-Video: Bringing Modern Computer Vision to Event Cameras", CVPR19.

Rebecq et al., "High Speed and High Dynamic Range Video with an Event Camera", PAMI, 2019.

Image Reconstruction from Events – E2VID

Events



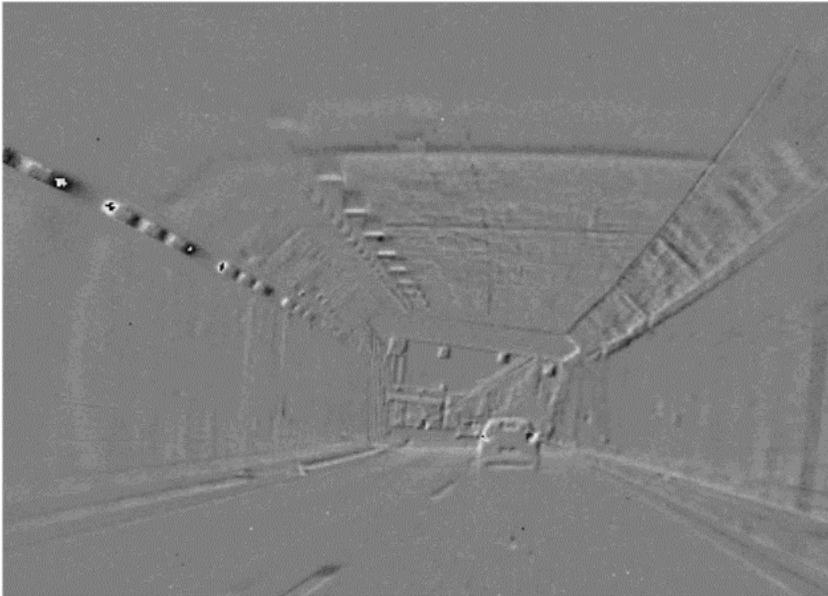
Reconstructed image from events
(Samsung DVS)



Rebecq et al., "Events-to-Video: Bringing Modern Computer Vision to Event Cameras", CVPR19.

Rebecq et al., "High Speed and High Dynamic Range Video with an Event Camera", PAMI, 2019.

HDR Video: Driving out of a tunnel



Events



Our reconstruction



Phone camera

Rebecq et al., "Events-to-Video: Bringing Modern Computer Vision to Event Cameras", CVPR19.

Rebecq et al., "High Speed and High Dynamic Range Video with an Event Camera", PAMI, 2019.

HDR Video: Night Drive



Our reconstruction from events



GoPro Hero 6

Rebecq et al., "Events-to-Video: Bringing Modern Computer Vision to Event Cameras", CVPR19.
Rebecq et al., "High Speed and High Dynamic Range Video with an Event Camera", PAMI, 2019.

Image Reconstruction from Events - HyperE2VID

- A dynamic network architecture for the task of video reconstruction from events, where existing static architectures are extended with hypernetworks, dynamic convolutional layers, and a context fusion block.

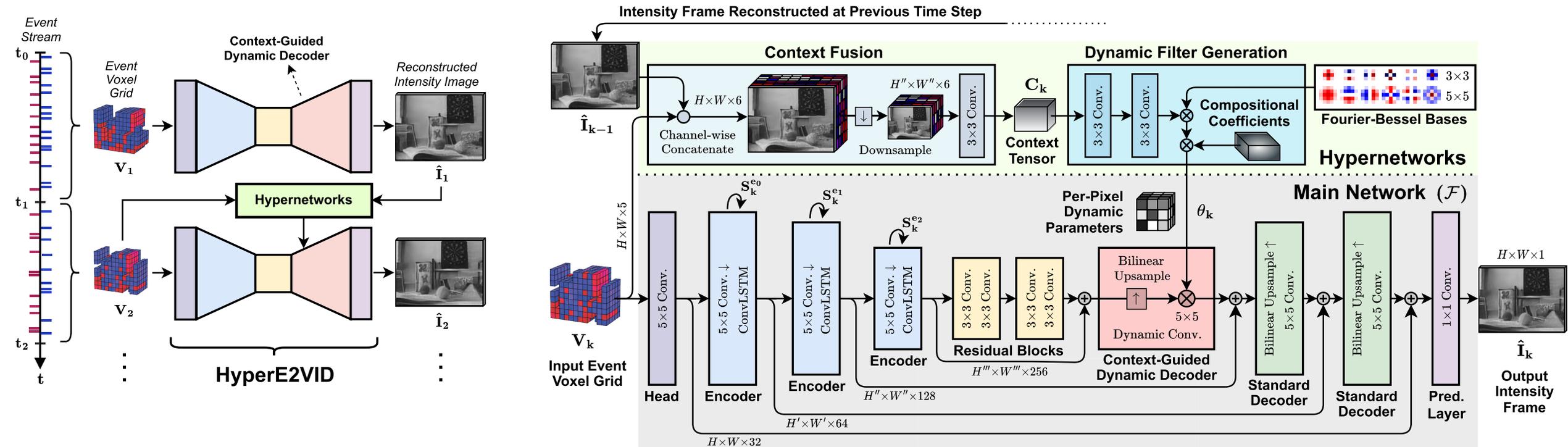


Image Reconstruction from Events - HyperE2VID

boxes_6dof sequence from ECD dataset



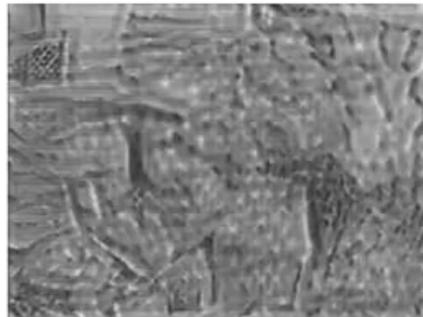
E2VID



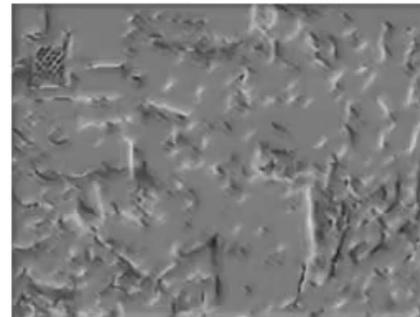
FireNet



SSL_E2VID



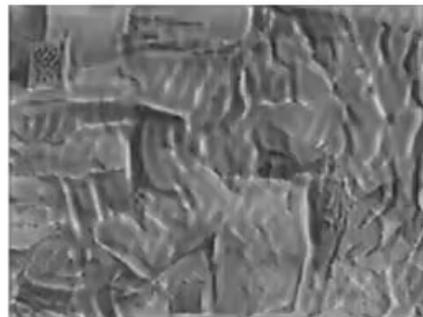
E2VID+



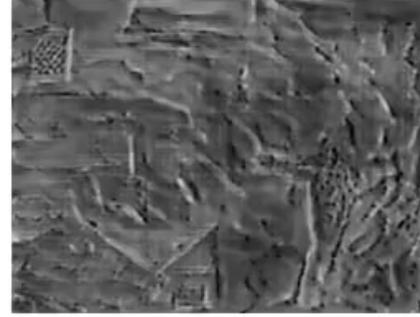
FireNet+



SPADE_E2VID



ET-Net



HyperE2VID

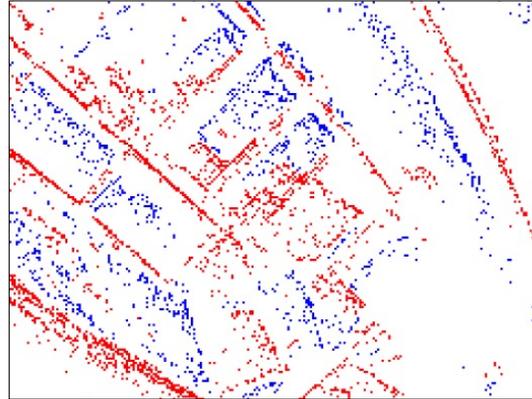


Ground Truth

What if we combined the complementary advantages of event and standard cameras?

Why combining them?

< 10 years research



Event Camera

> 60 years of research!

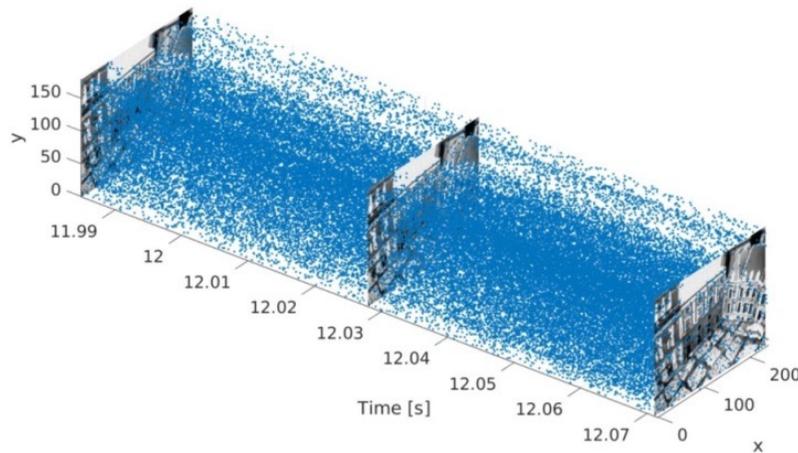


Standard Camera

Update rate	High (asynchronous): 1 MHz	Low (synchronous)
Dynamic Range	High (140 dB)	Low (60 dB)
Motion Blur	No	Yes
Static motion	No (event camera is a high pass filter)	Yes
Absolute intensity	No (reconstructable up to a constant)	Yes

DAVIS sensor: Events + Images + IMU

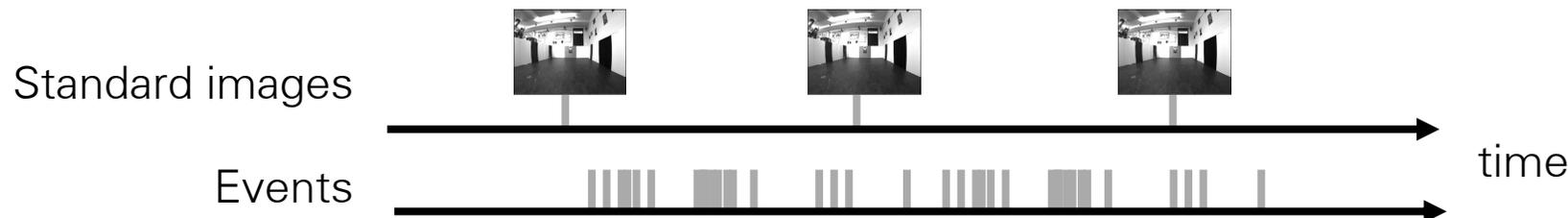
- Combines an event and a standard camera in the same pixel array (→ the same pixel can both trigger events and integrate light intensity).
- It also has an IMU



Spatio-temporal visualization of the output of a DAVIS sensor



Temporal aggregation of events overlaid on a DAVIS frame



Deblurring a blurry video

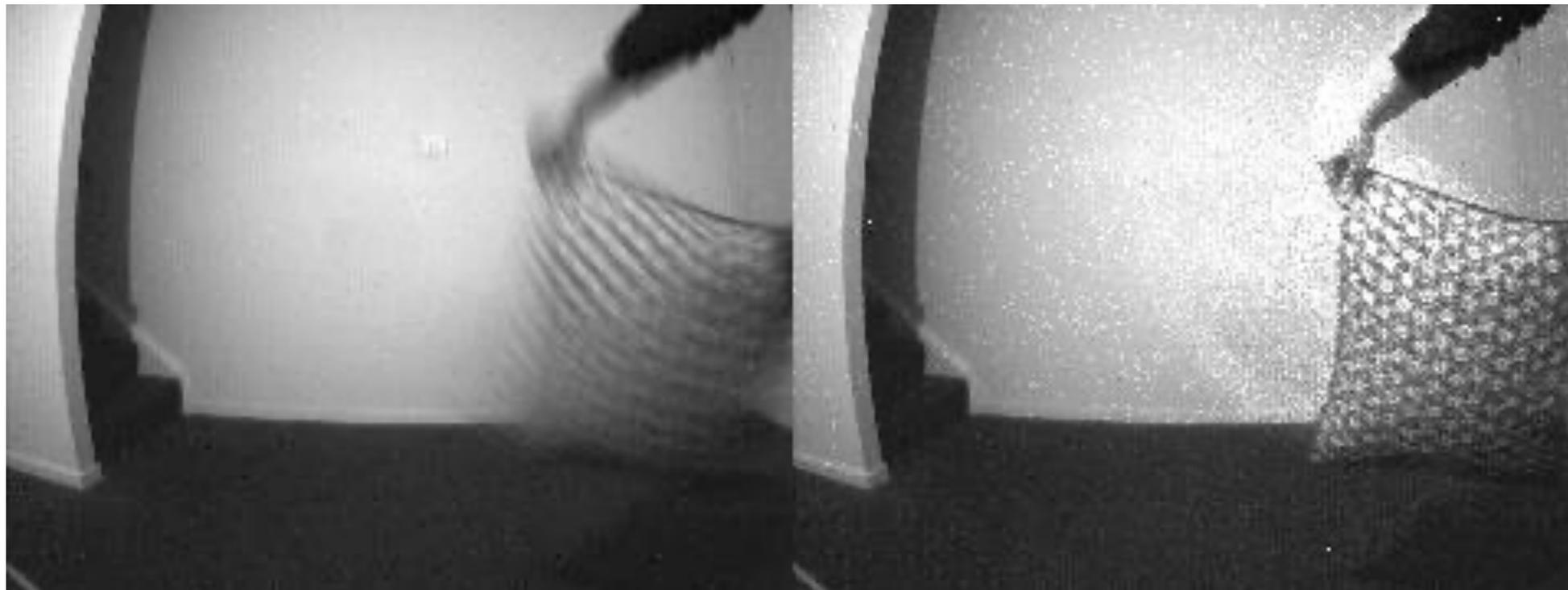
- A **blurry image** can be regarded as the **integral of a sequence of latent images** during the exposure time, while the **events** indicate the **changes between the latent images**.
- **Finding:** sharp image obtained by subtracting the double integral of event from input image

$$\log \left[\text{Input blur image} \right] - \iint \left[\text{Input events} \right] = \log \left[\text{Output sharp image} \right]$$

Input blur image Input events Output sharp image

Deblurring a blurry video

- A **blurry image** can be regarded as the **integral of a sequence of latent images** during the exposure time, while the **events** indicate the **changes between the latent images**.
- **Finding:** sharp image obtained by subtracting the double integral of event from input image

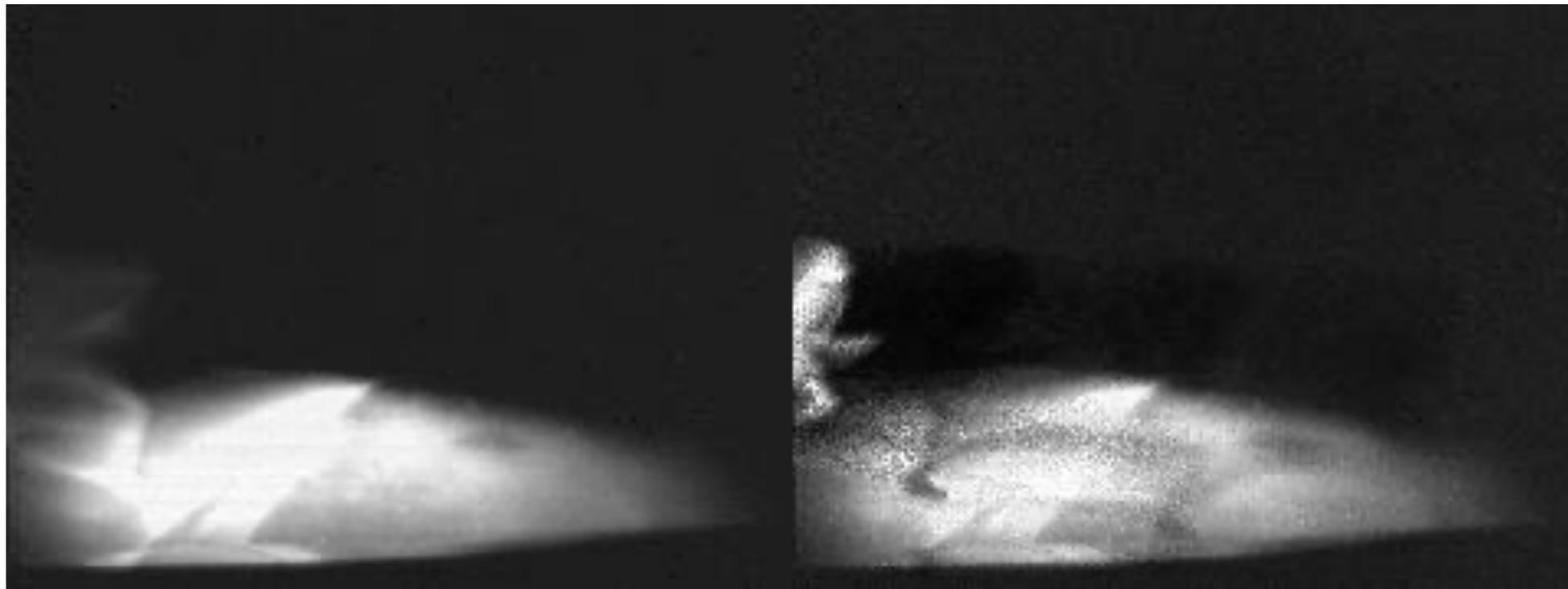


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Video Frame Interpolation

- Video frame interpolation methods aims at **generating intermediate frames by inferring object motions** in the image from **consecutive keyframes**.
- Motion is generally modelled with first-order approximations like **optical flow**.
 - This choice restricts the types of motions, leading to errors in highly dynamic scenarios.
- **Event cameras** provides **auxiliary visual information** in the blind-time **between frames**.

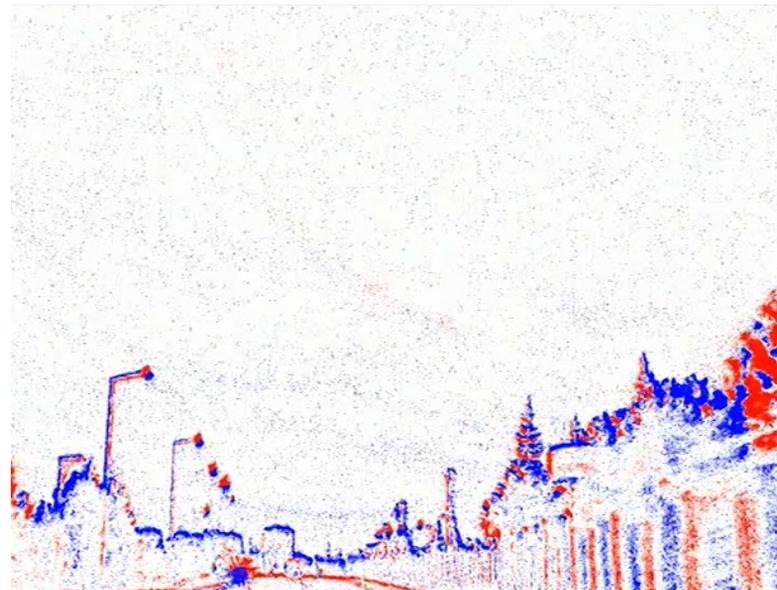


Low-Light Video Enhancement

- Low-light video enhancement is a **challenging task** due to issues like severe noise, reduced contrast, and poor temporal consistency.
- Unlike traditional sensors that capture redundant or noisy information, **event cameras focus on meaningful changes**, enabling robust performance in **low-light scenarios**, and thus can be used as complementary source of information about the scenes.



Input dark video



Input events



Output enhanced video

Next Lecture:
Focal Stacks and Lightfields