# BBM 202 - ALGORITHMS

HACETTEPE UNIVERSITY

DEPT. OF COMPUTER ENGINEERING

ERKUT ERDEM

# BALANCED TREES

Apr. 4, 2013

**Acknowledgement:** The course slides are adapted from the slides prepared by R. Sedgewick and K. Wayne of Princeton University.

# Balanced Search Trees

- 2-3 search trees
- Red-black BSTs
- B-trees
- Geometric applications of BSTs

# Text

| implementation | worst-case cost (after N inserts) | | | average case (after N random inserts) | | | ordered iteration? | key interface |
|---|---|---|---|---|---|---|---|---|
| | search | insert | delete | search hit | insert | delete | | |
| sequential search (unordered list) | N | N | N | N/2 | N | N/2 | no | `equals()` |
| binary search (ordered array) | lg N | N | N | lg N | N/2 | N/2 | yes | `compareTo()` |
| BST | N | N | N | 1.39 lg N | 1.39 lg N | ? | yes | `compareTo()` |
| goal | log N | log N | log N | log N | log N | log N | yes | `compareTo()` |

- **Challenge.** **Guarantee performance.**

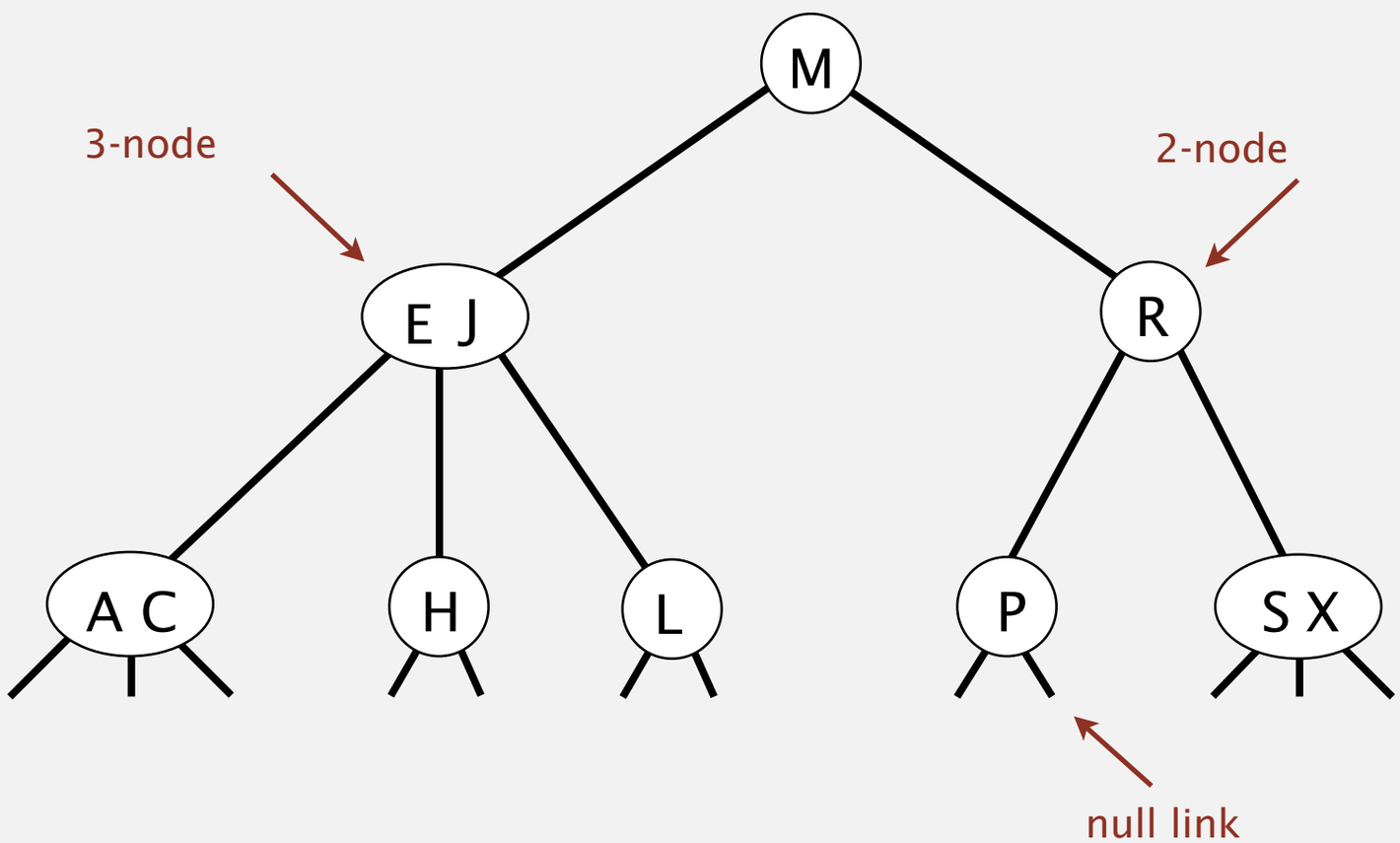
# Balanced Search Trees

- 2-3 search trees
- Red-black BSTs
- B-trees
- Geometric applications of BSTs

# 2-3 tree

Allow 1 or 2 keys per node.

- 2-node: one key, two children.
- 3-node: two keys, three children.



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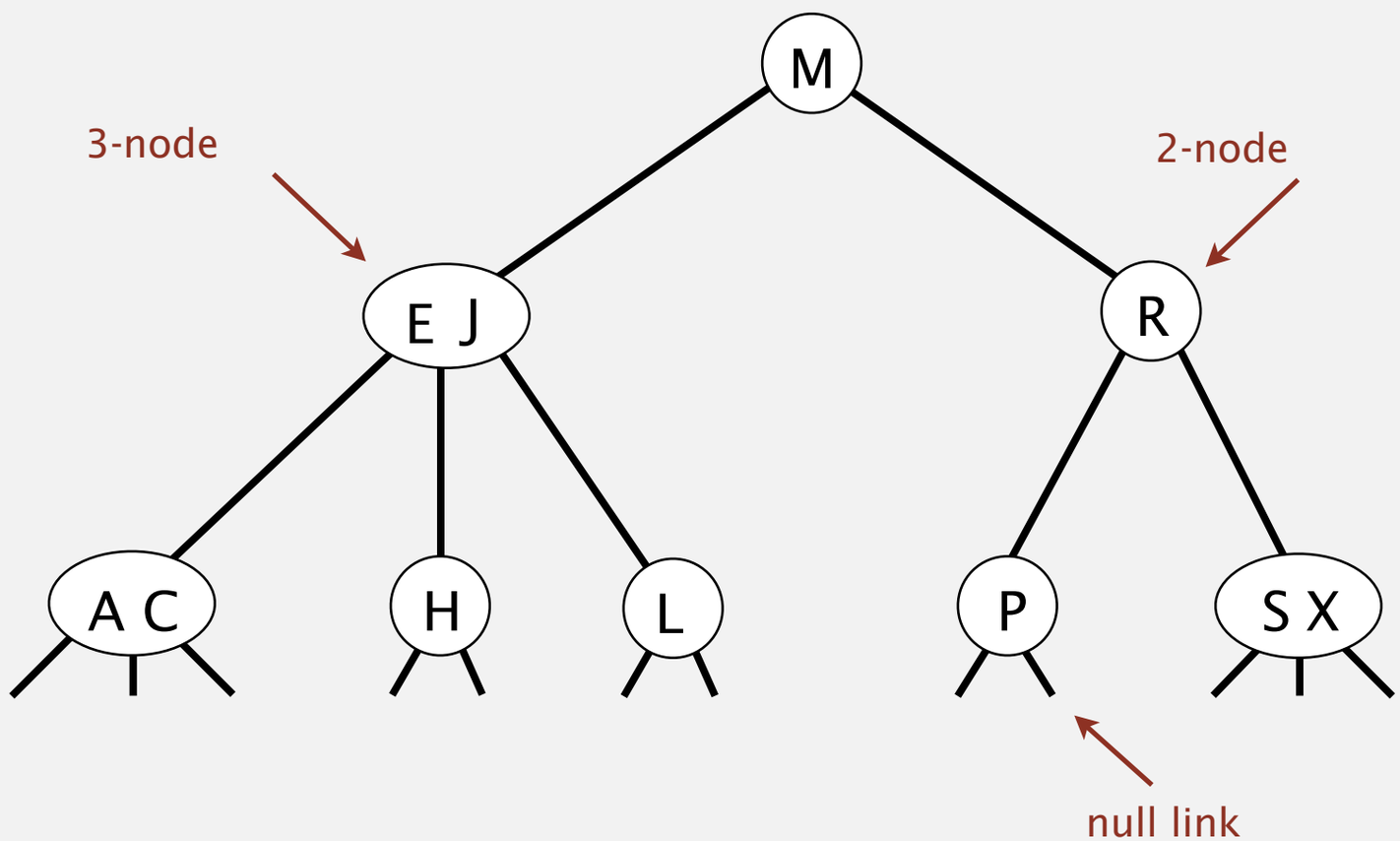
Perfect balance. Every path from root to null link has same length.



# 2-3 tree

Allow 1 or 2 keys per node.

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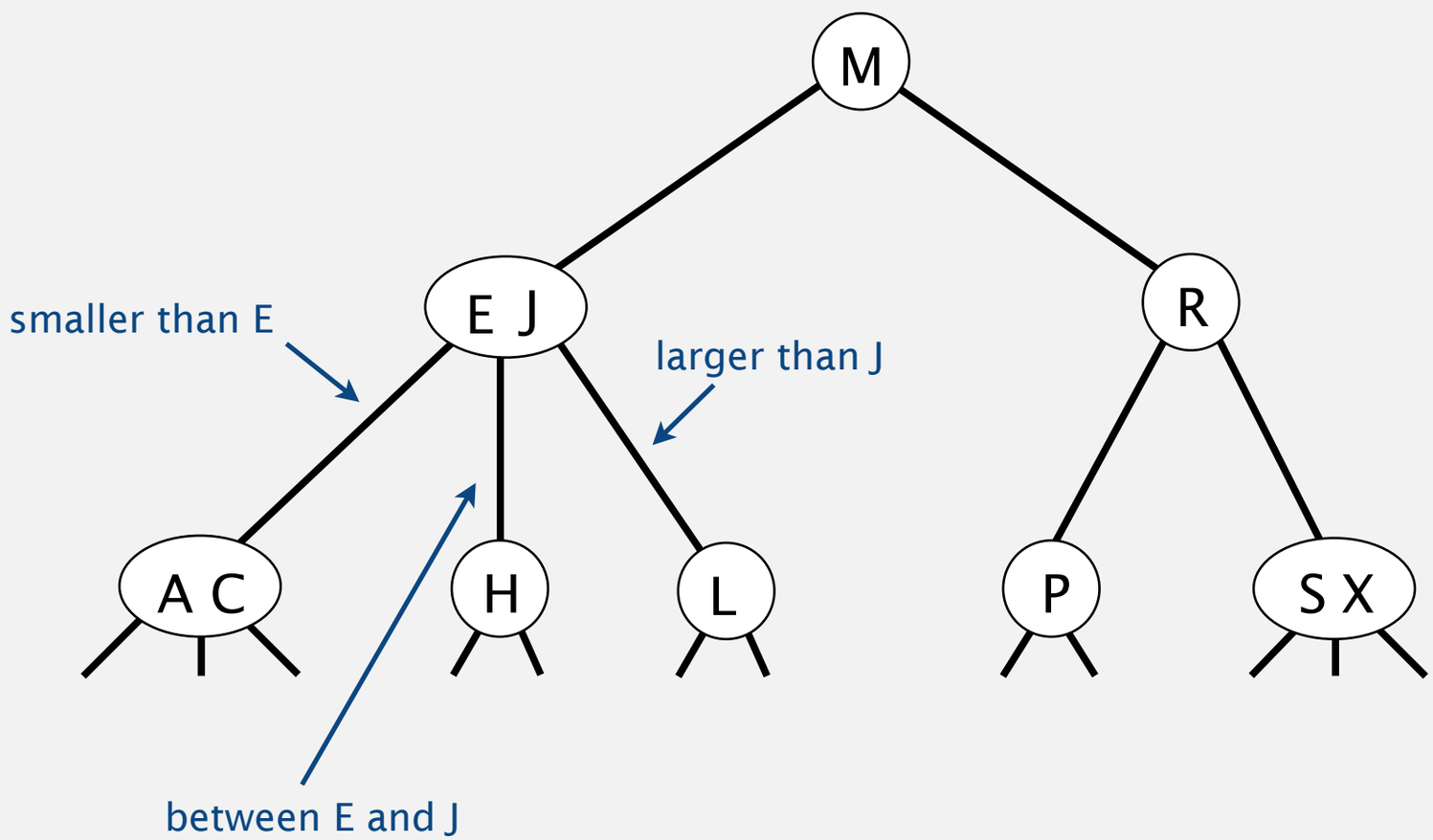
Perfect balance. Every path from root to null link has same length.

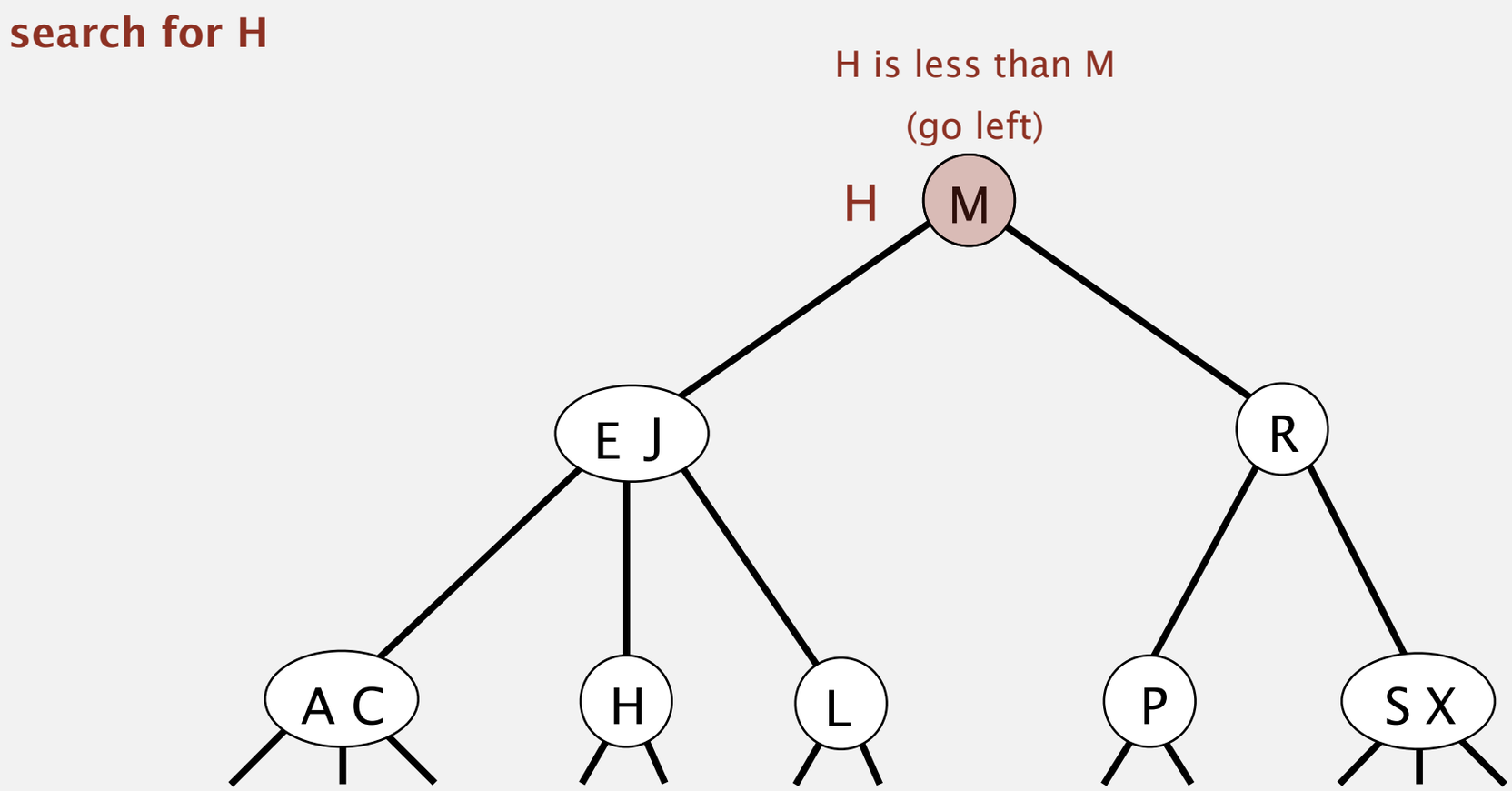
Symmetric order. Inorder traversal yields keys in ascending order.



# 2-3 tree demo

## Search.

- Compare search key against keys in node.
- Find interval containing search key.
- Follow associated link (recursively).



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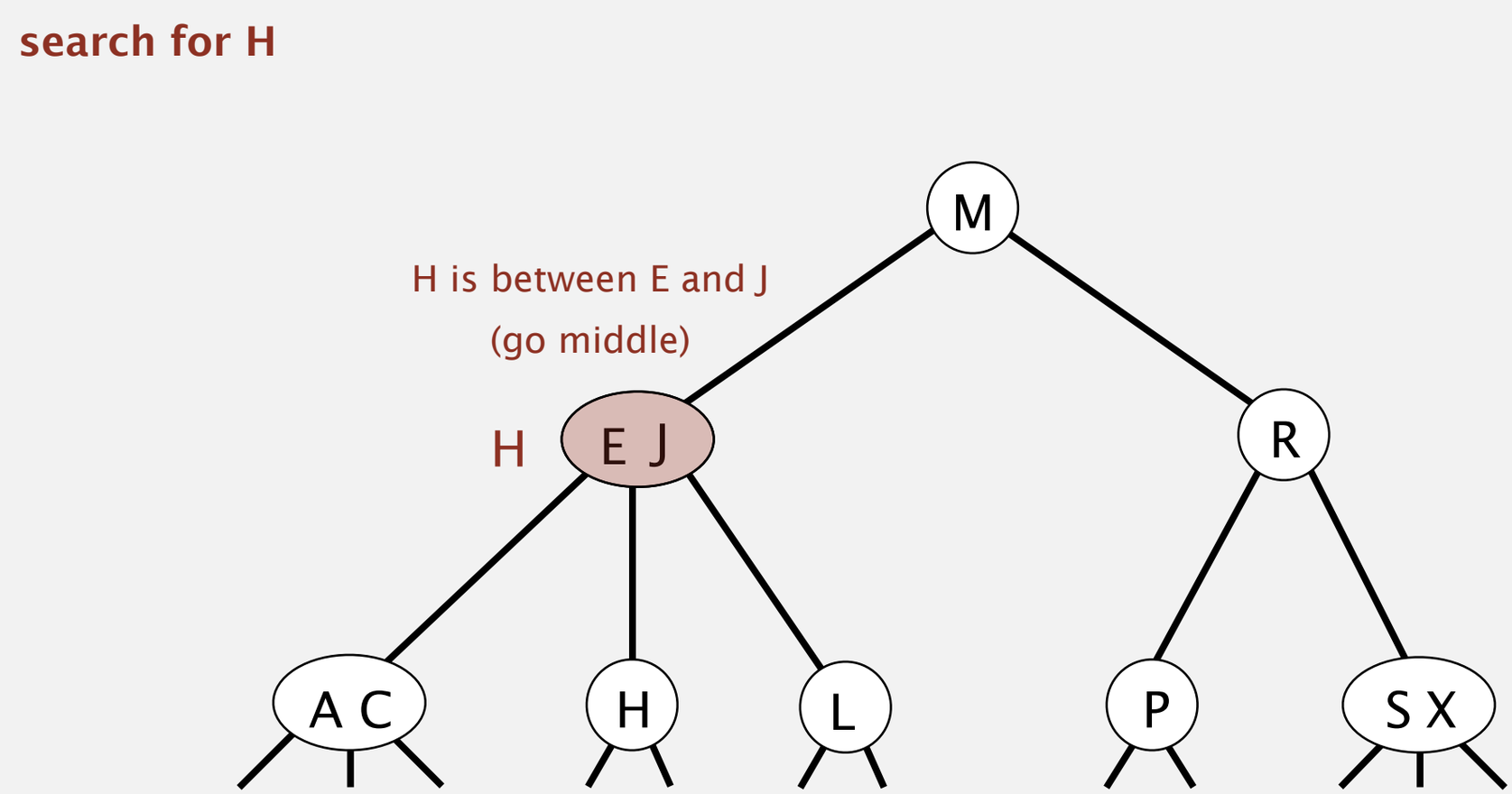

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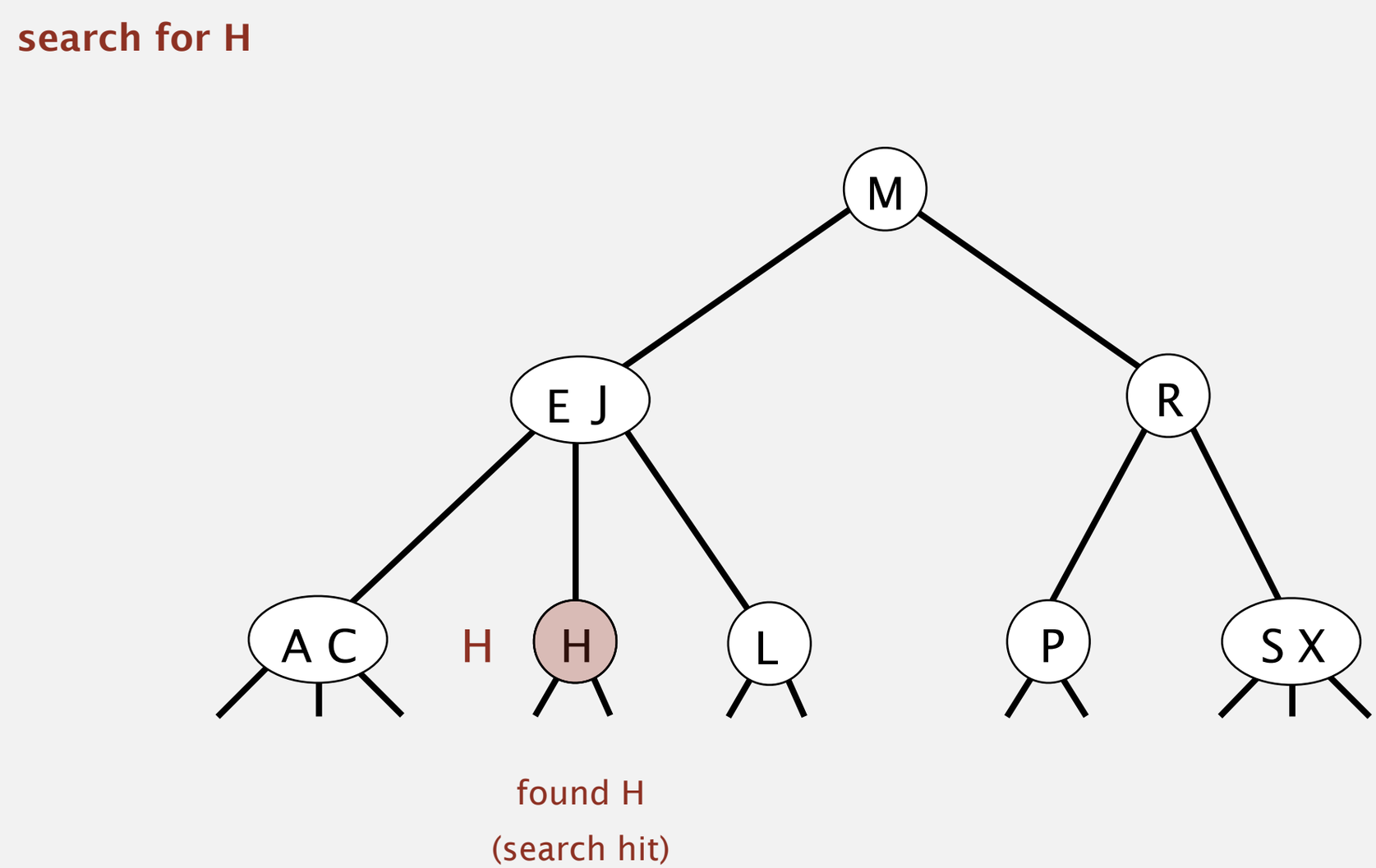

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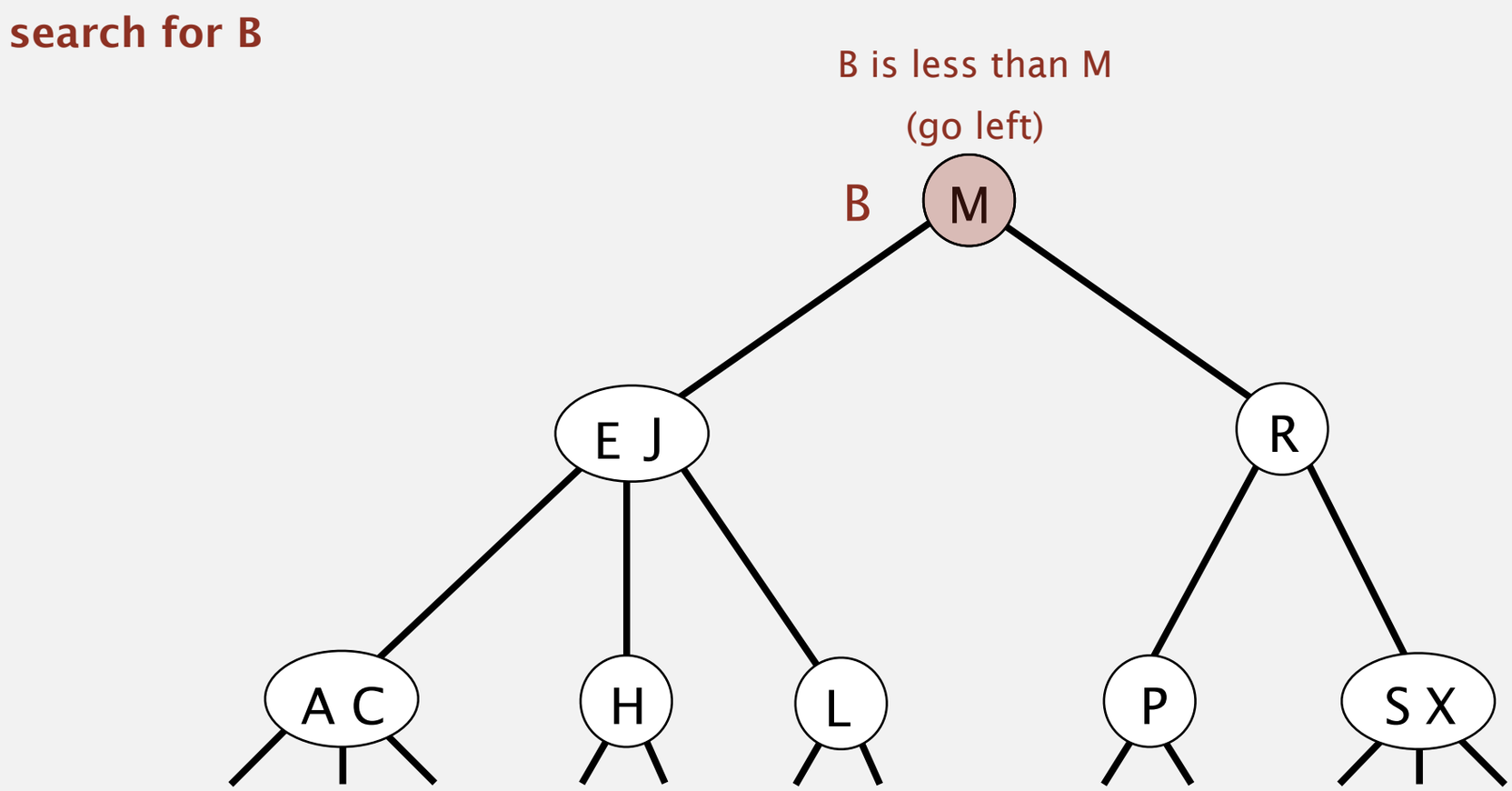

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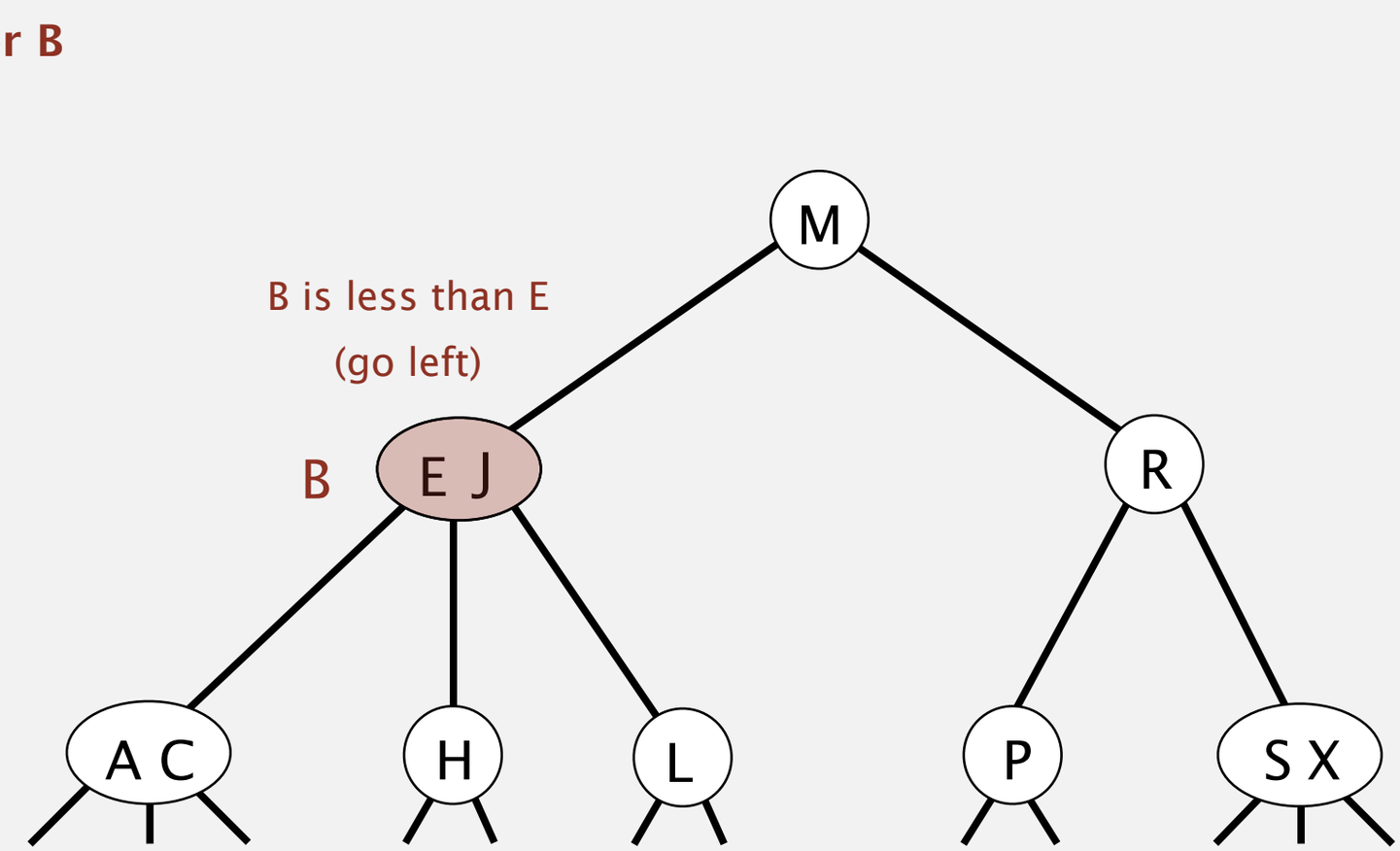

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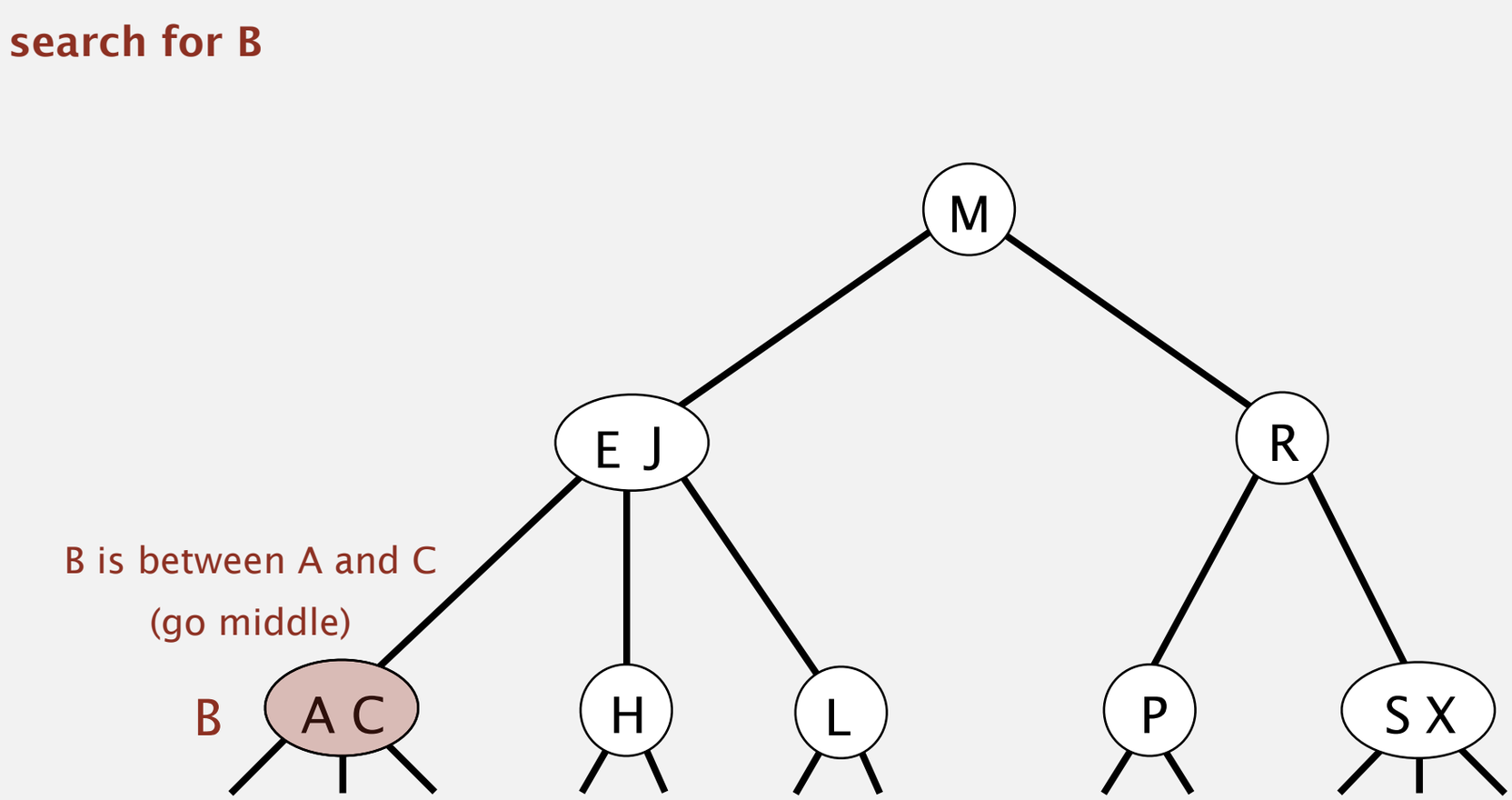
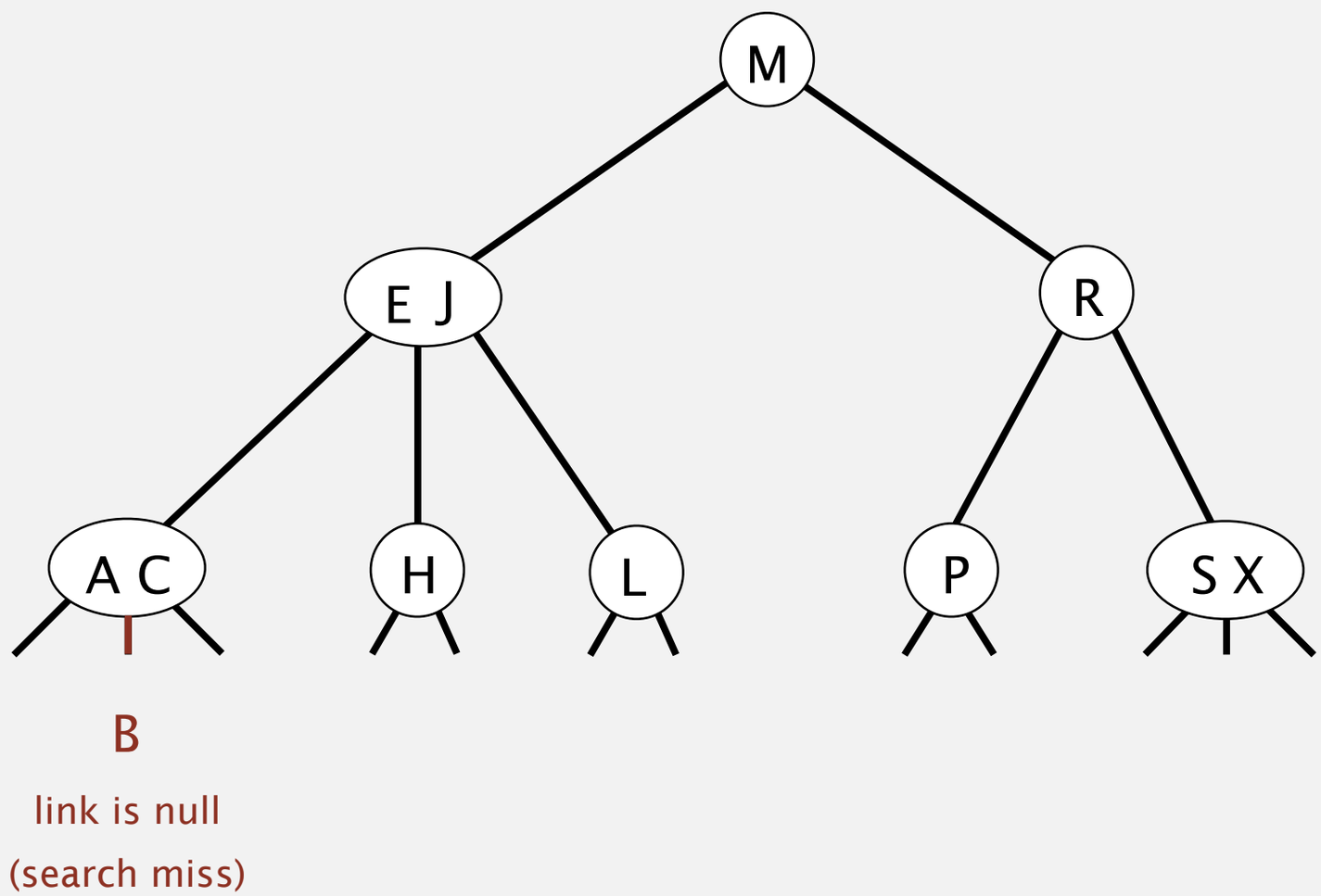


# 2-3 tree demo

Search.

- Compare search key against keys in node.
- Find interval containing search key.
- Follow associated link (recursively).

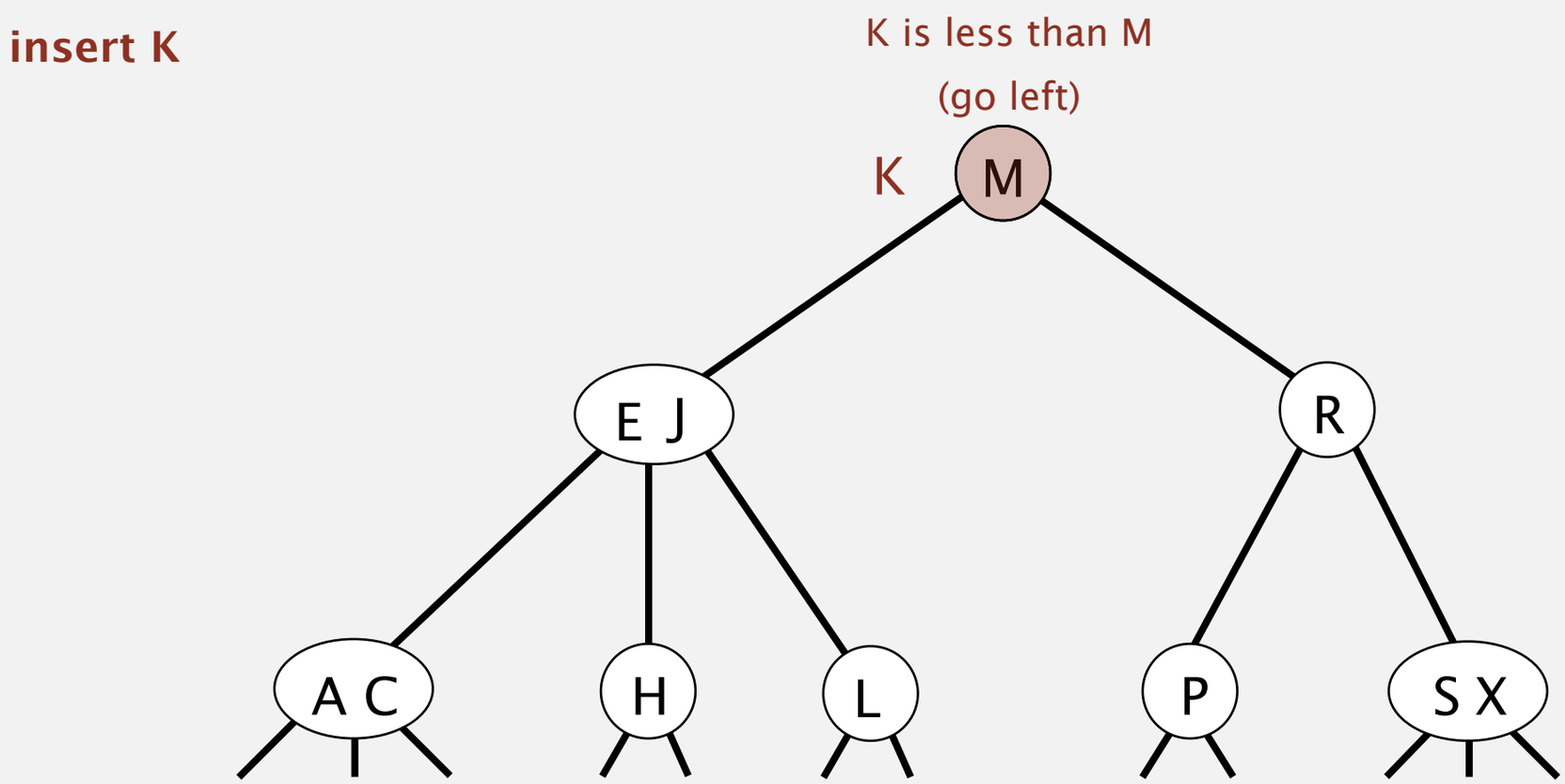
search for B



# 2-3 tree demo

## Insert into a 2-node at bottom.

- Search for key, as usual.
- Replace 2-node with 3-node.



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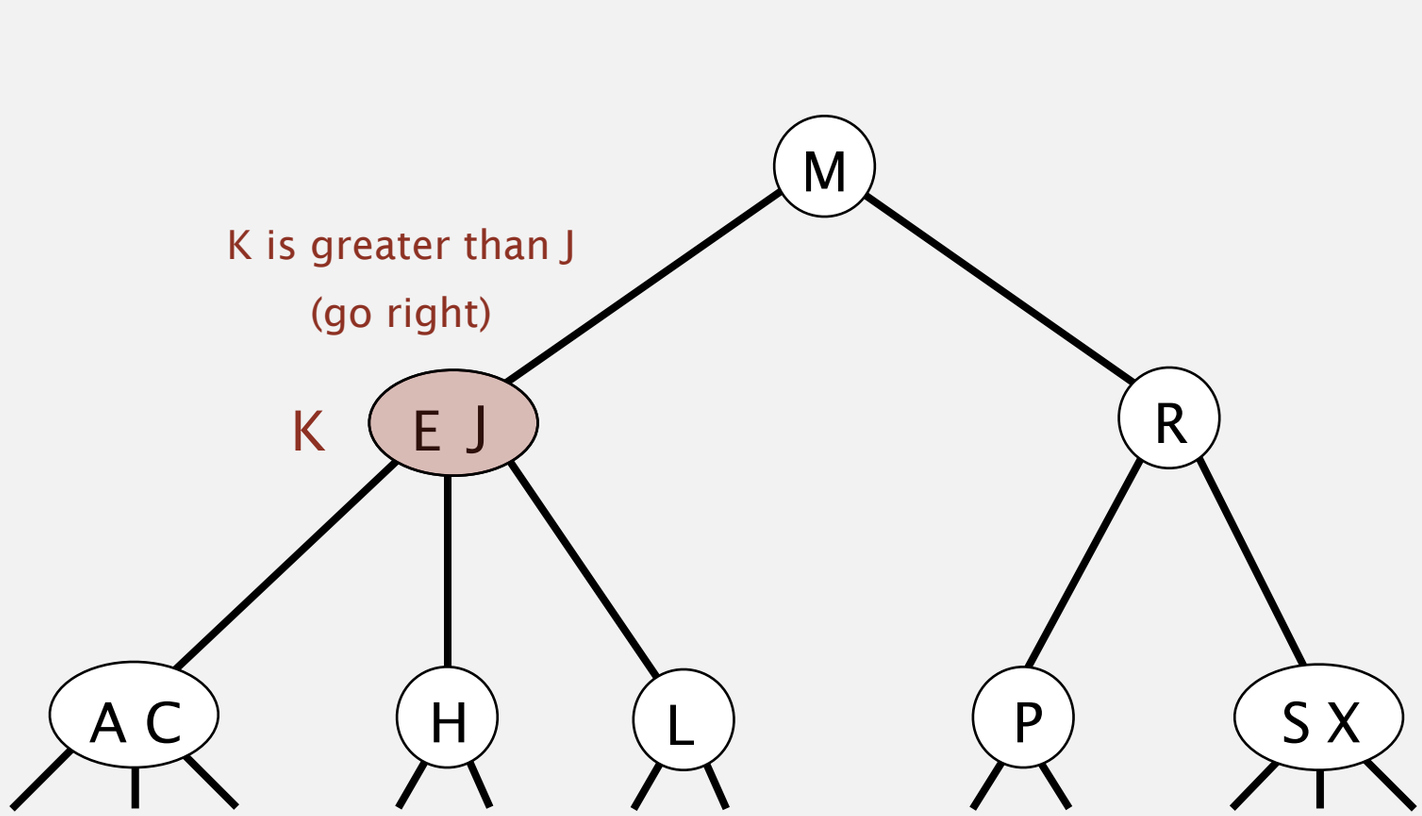

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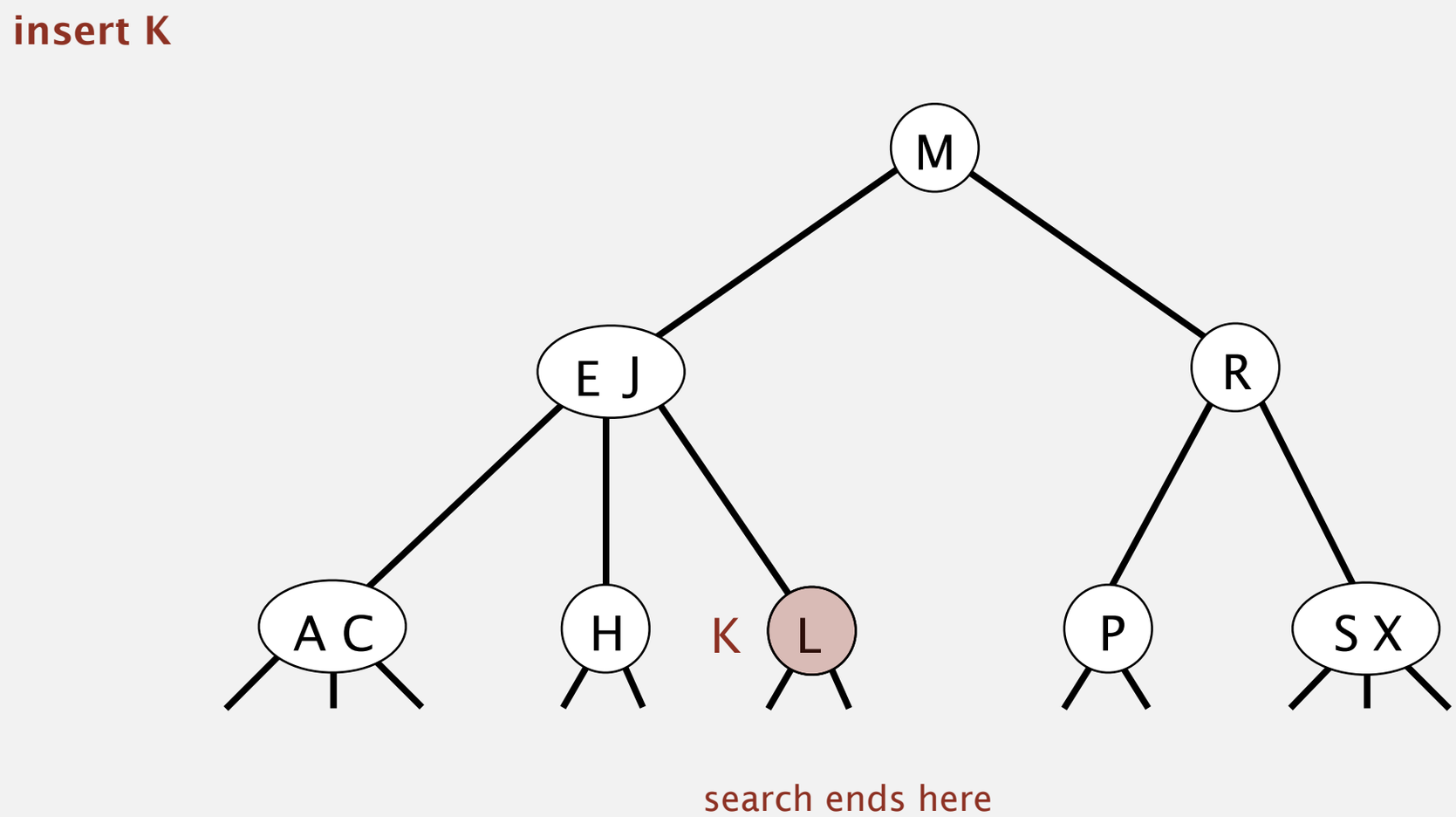

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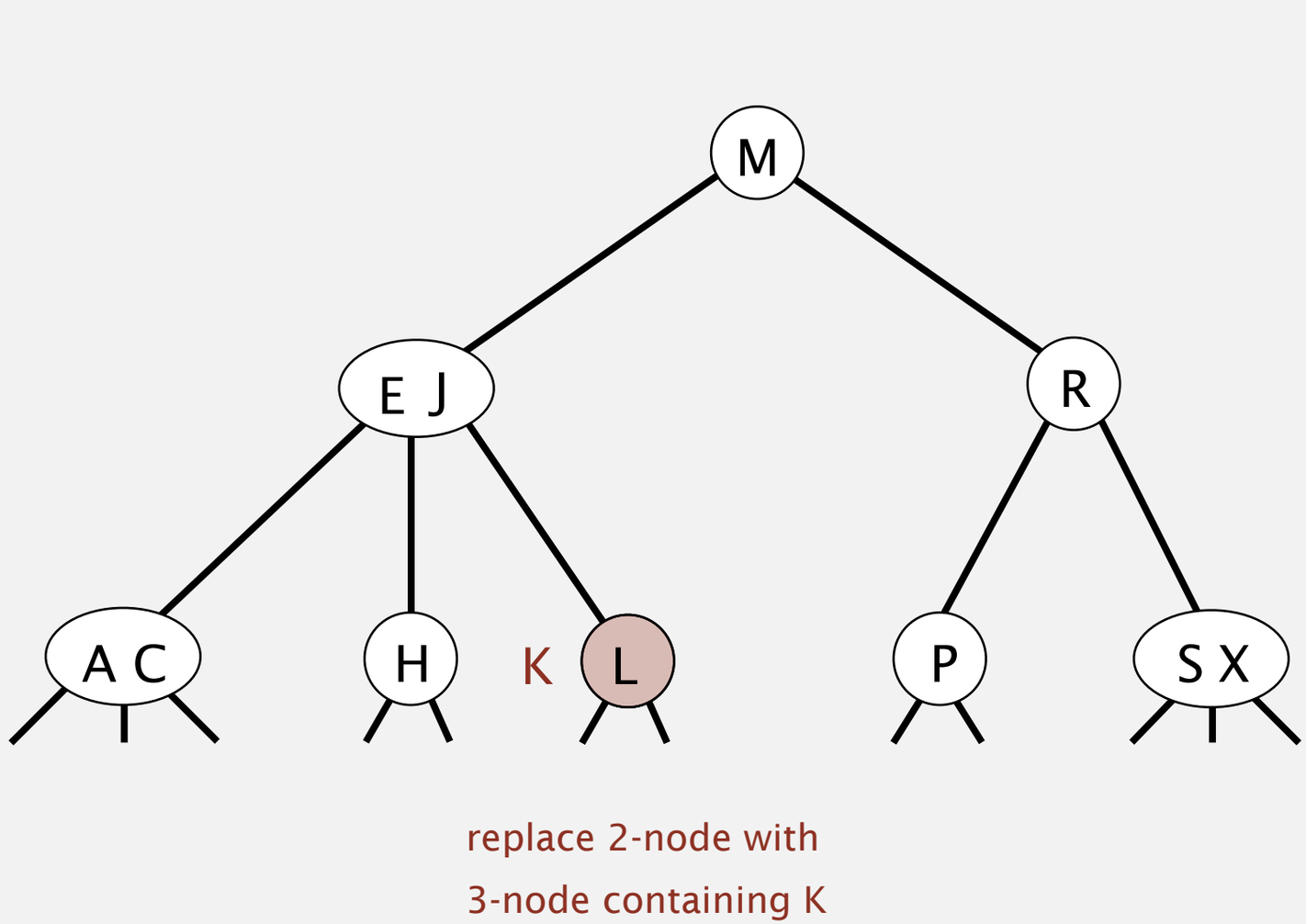

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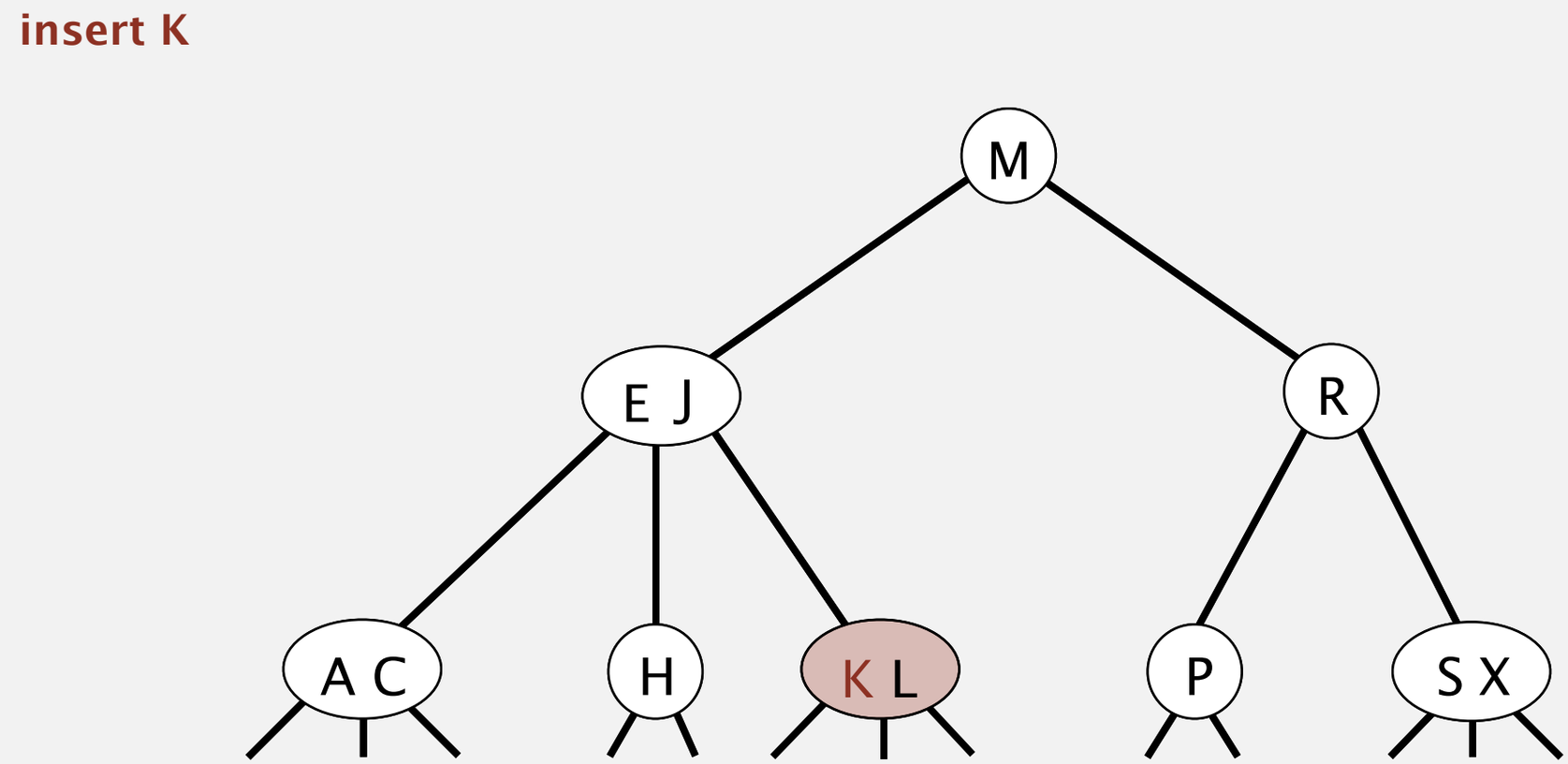
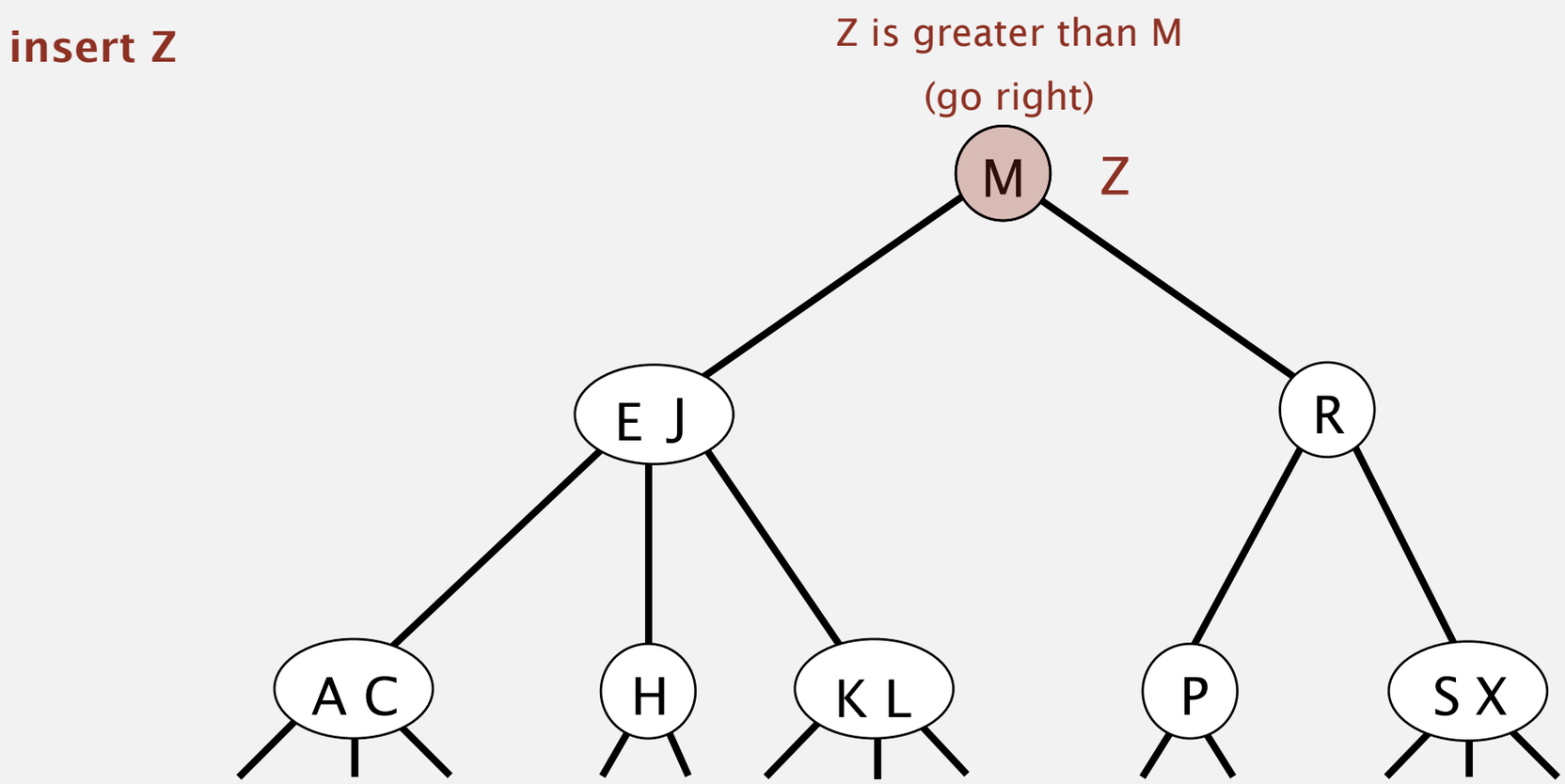


# 2-3 tree demo

## Insert into a 3-node at bottom.

- Add new key to 3-node to create temporary 4-node.
- Move middle key in 4-node into parent.



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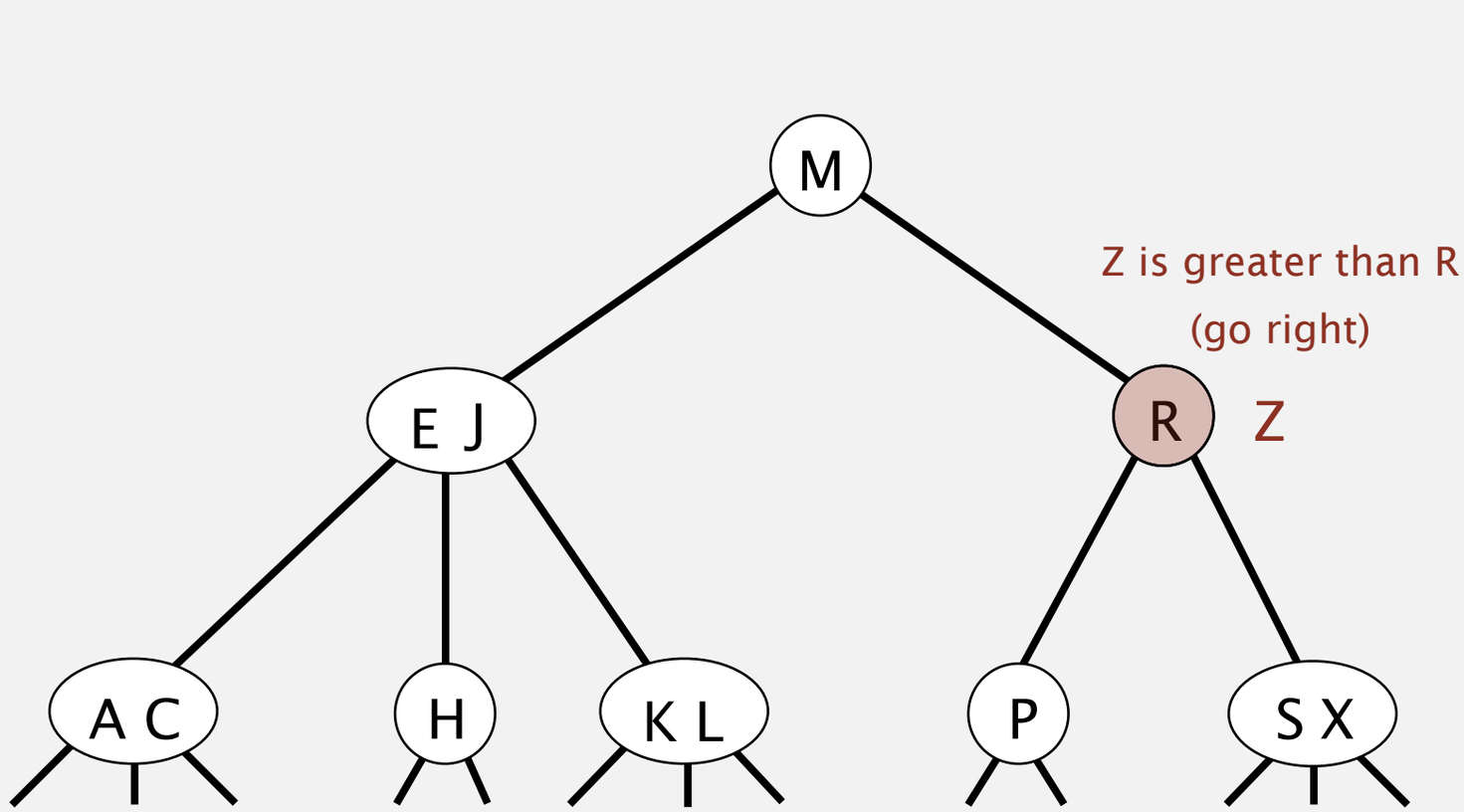
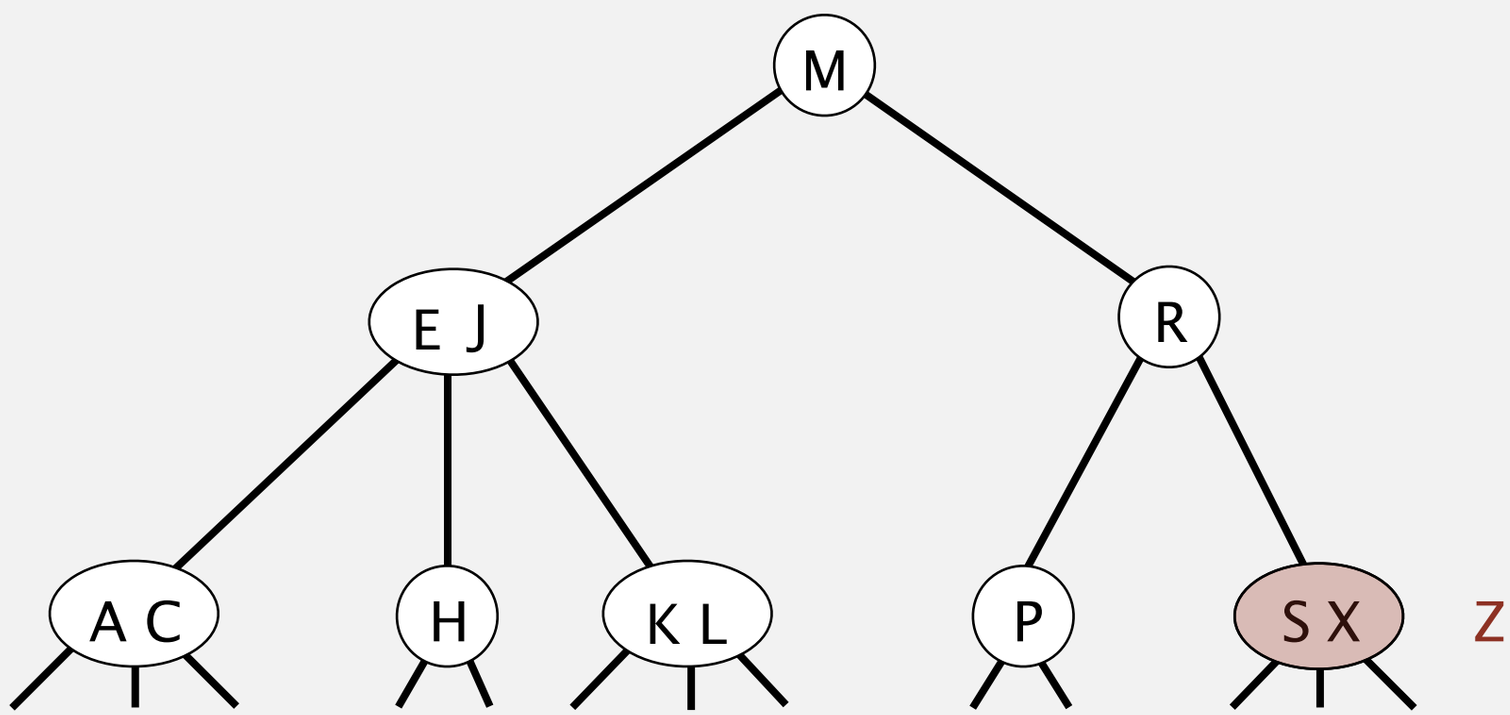


# 2-3 tree demo

## Insert into a 3-node at bottom.

- Add new key to 3-node to create temporary 4-node.
- Move middle key in 4-node into parent.

insert Z




search ends here

# 2-3 tree demo

## Insert into a 3-node at bottom.

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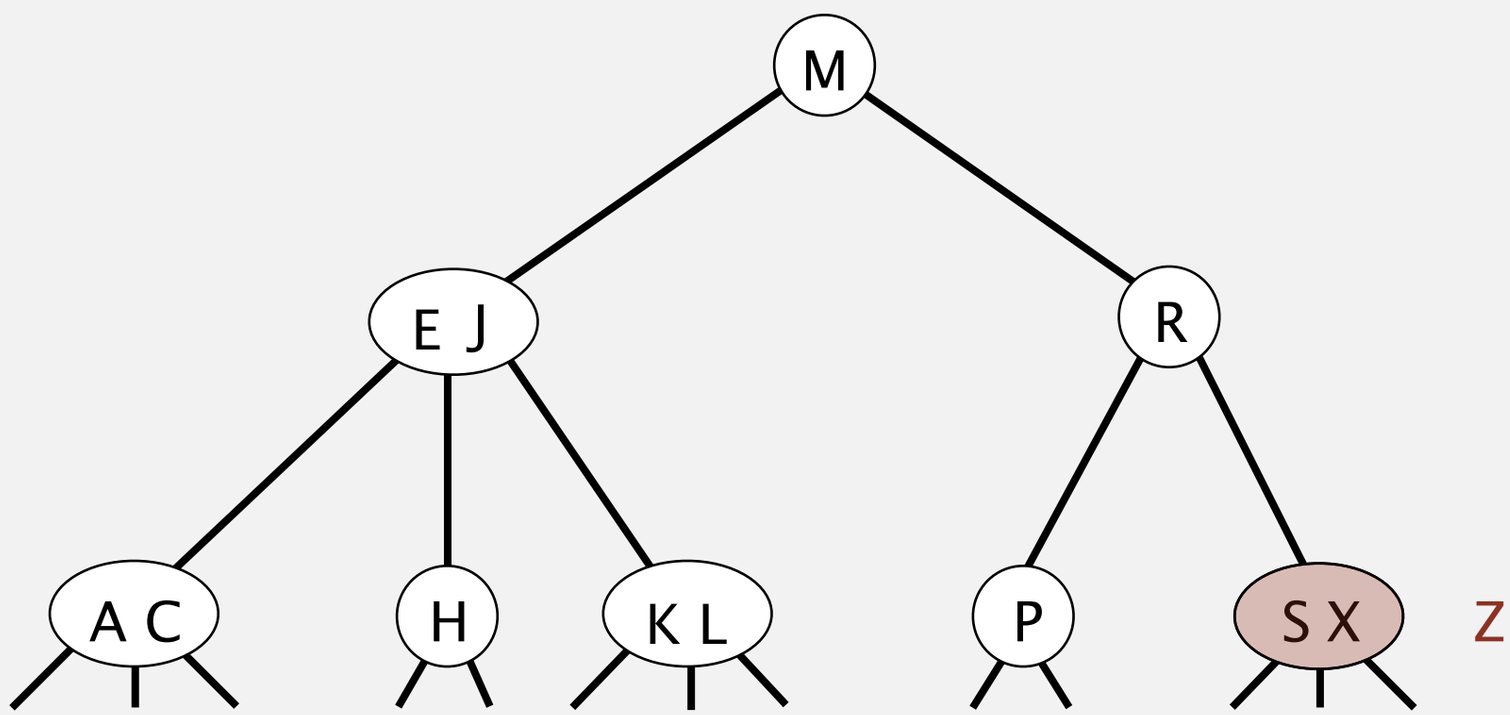
insert Z




replace 3-node with
temporary 4-node containing Z

# 2-3 tree demo

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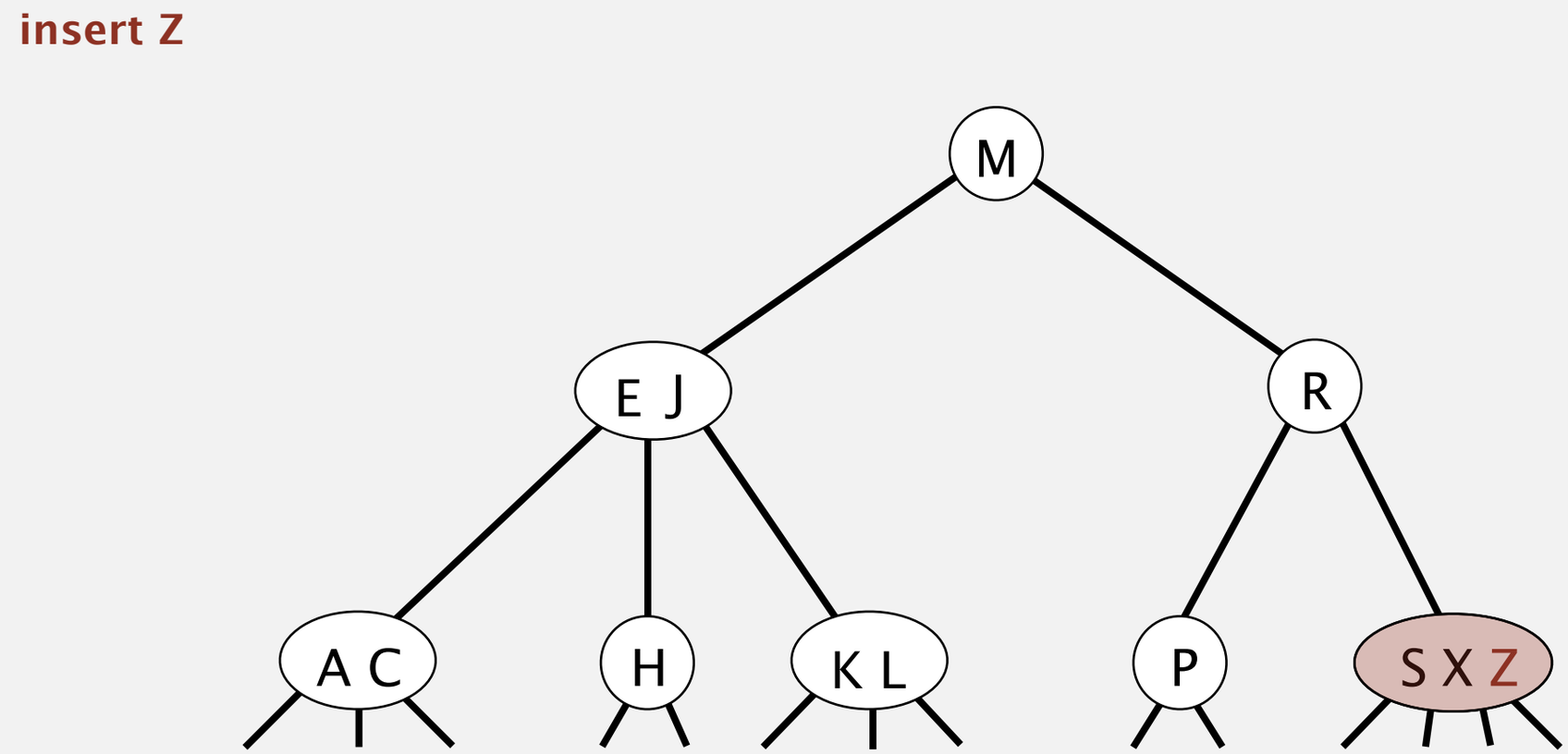

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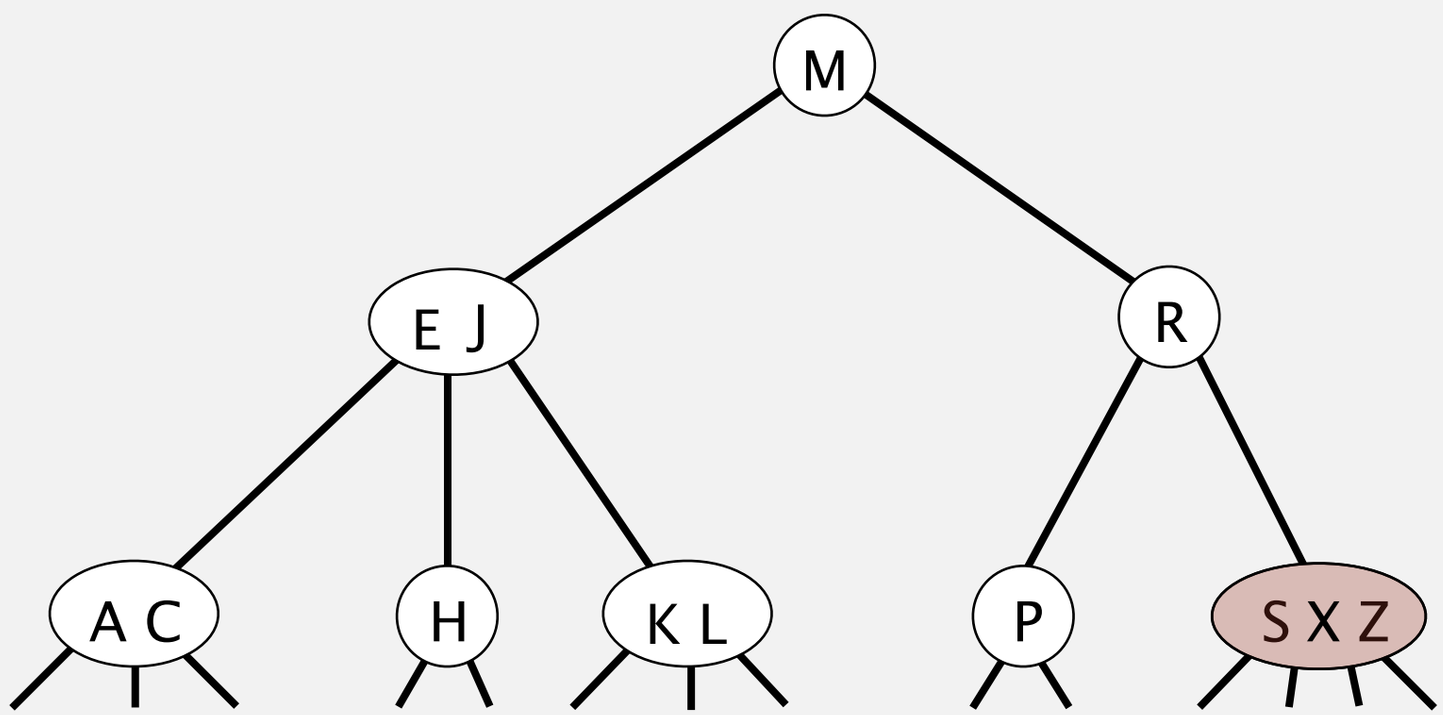


split 4-node into two 2-nodes
(pass middle key to parent)

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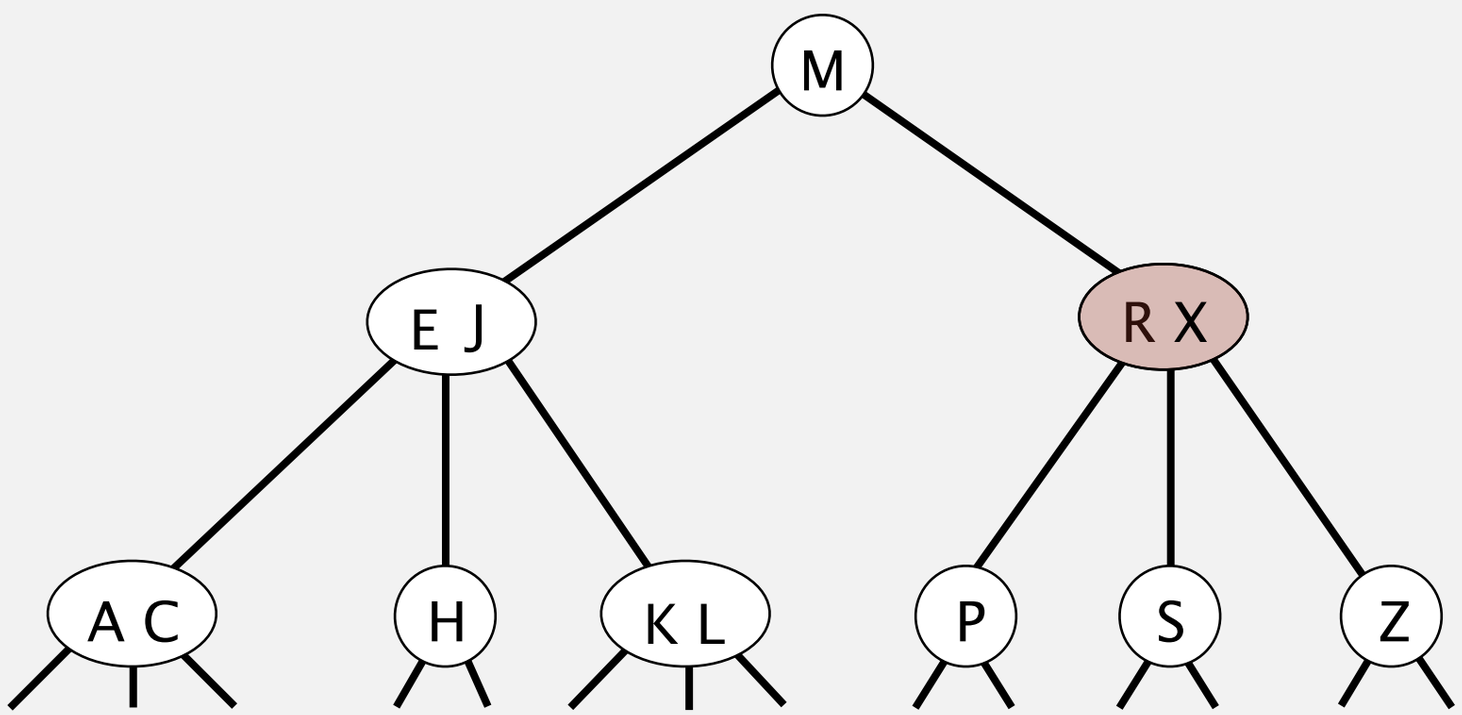
insert Z



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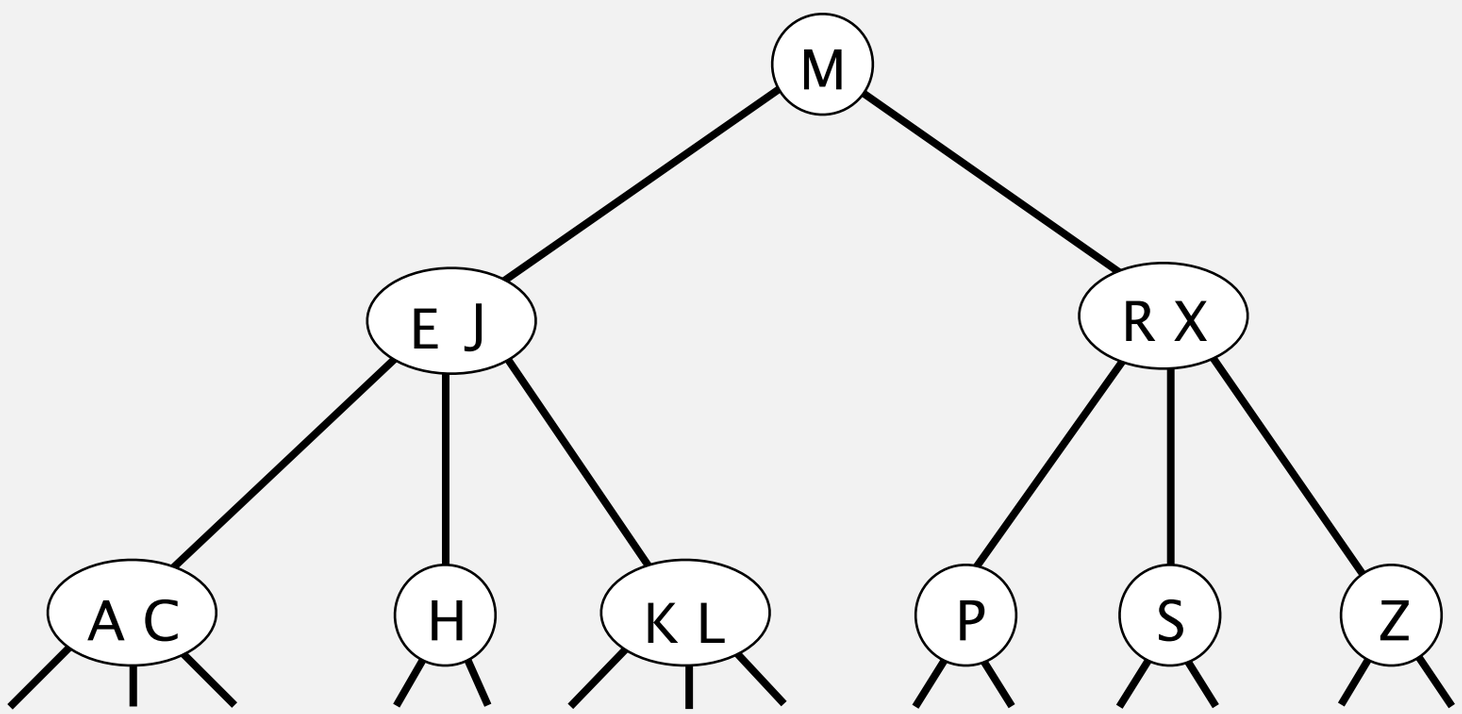
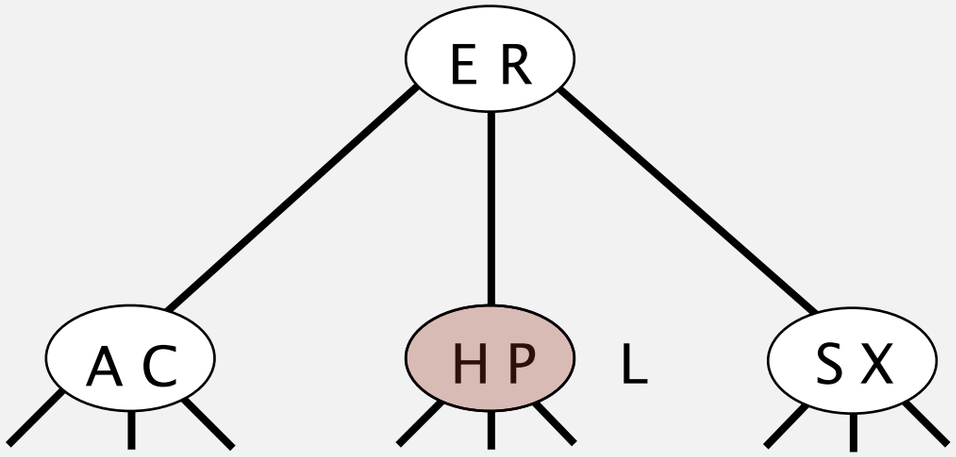


# 2-3 tree demo

## Insert into a 3-node at bottom.

- Add new key to 3-node to create temporary 4-node.
- Move middle key in 4-node into parent.
- Repeat up the tree, as necessary.
- If you reach the root and it's a 4-node, split it into three 2-nodes.

**insert L**




convert 3-node into 4-node

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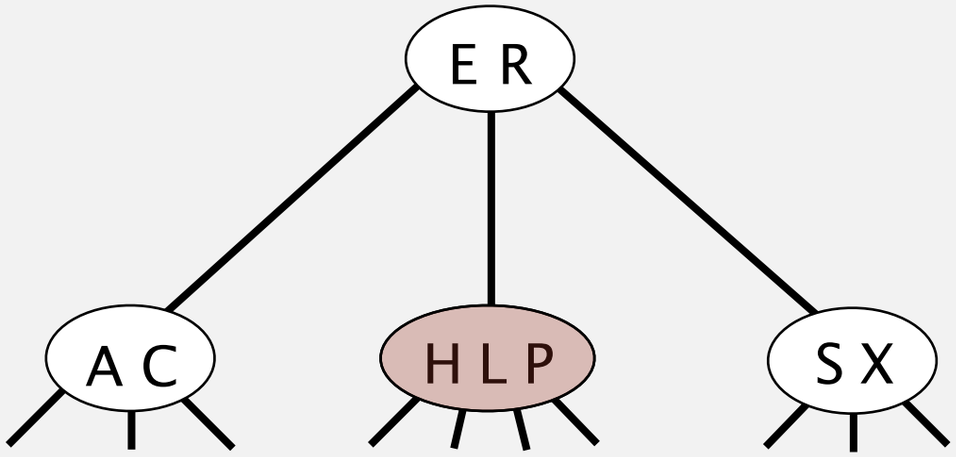

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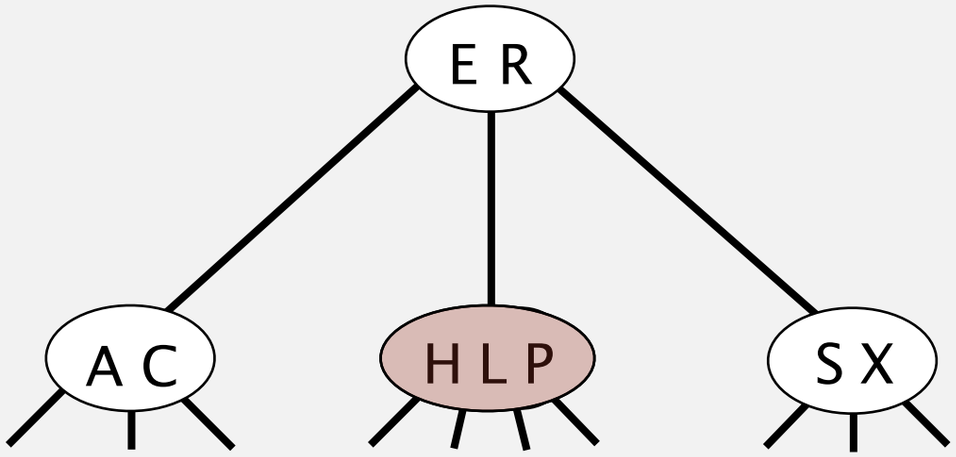


split 4-node

(move L to parent)

# 2-3 tree demo

## Insert into a 3-node at bottom.

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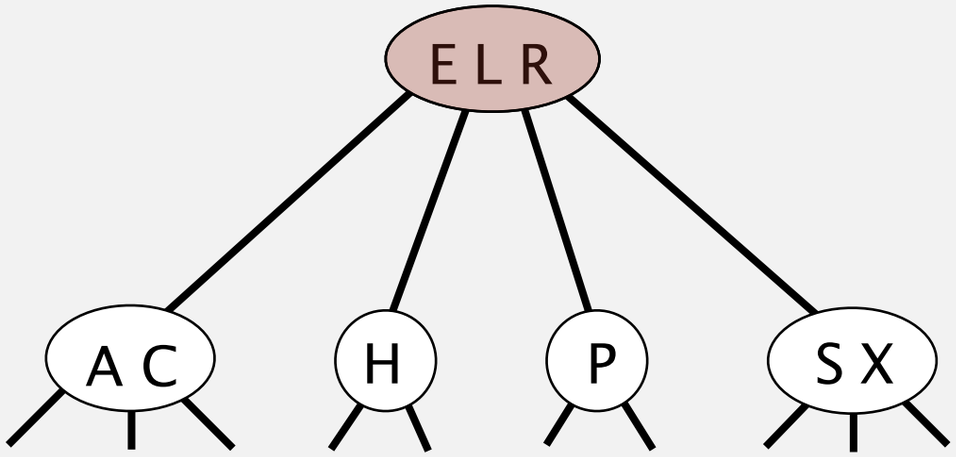

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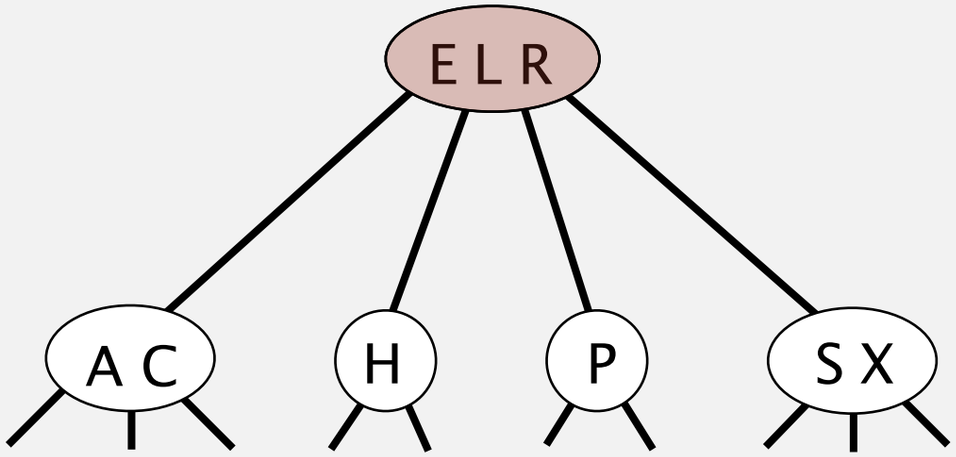
**insert L**

split 4-node

(move L to parent)



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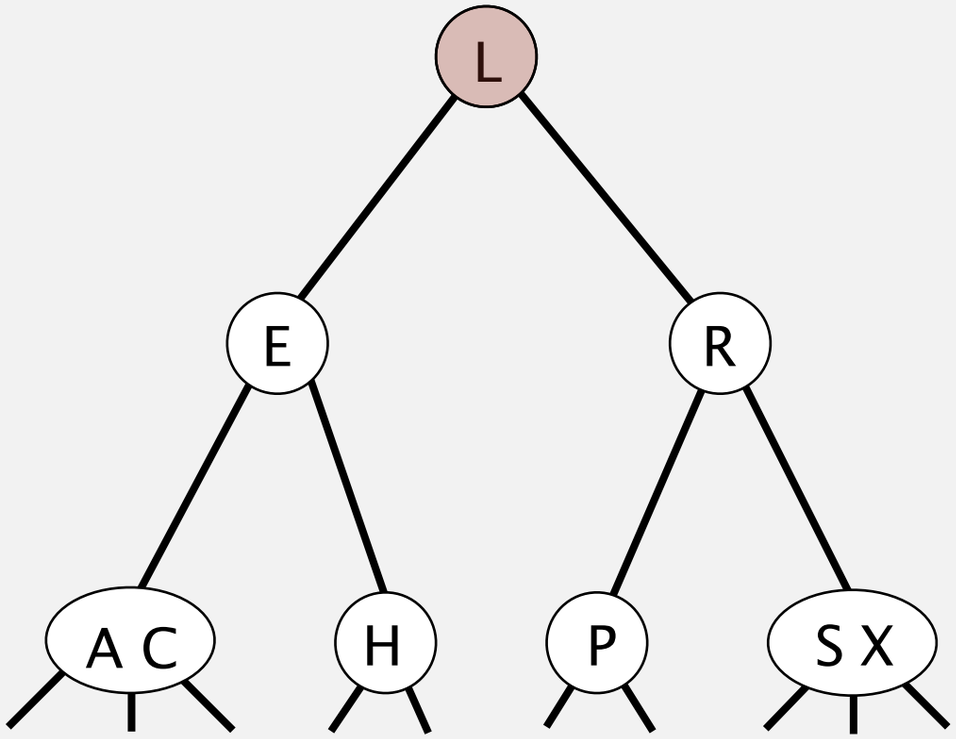
height of tree increases by 1

**insert L**



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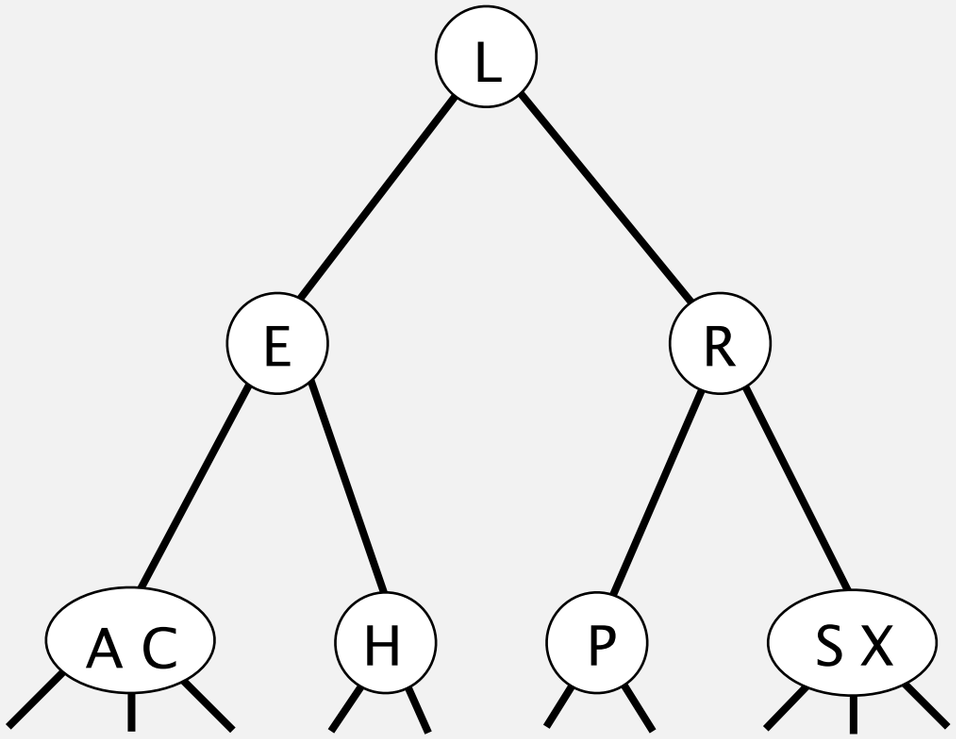
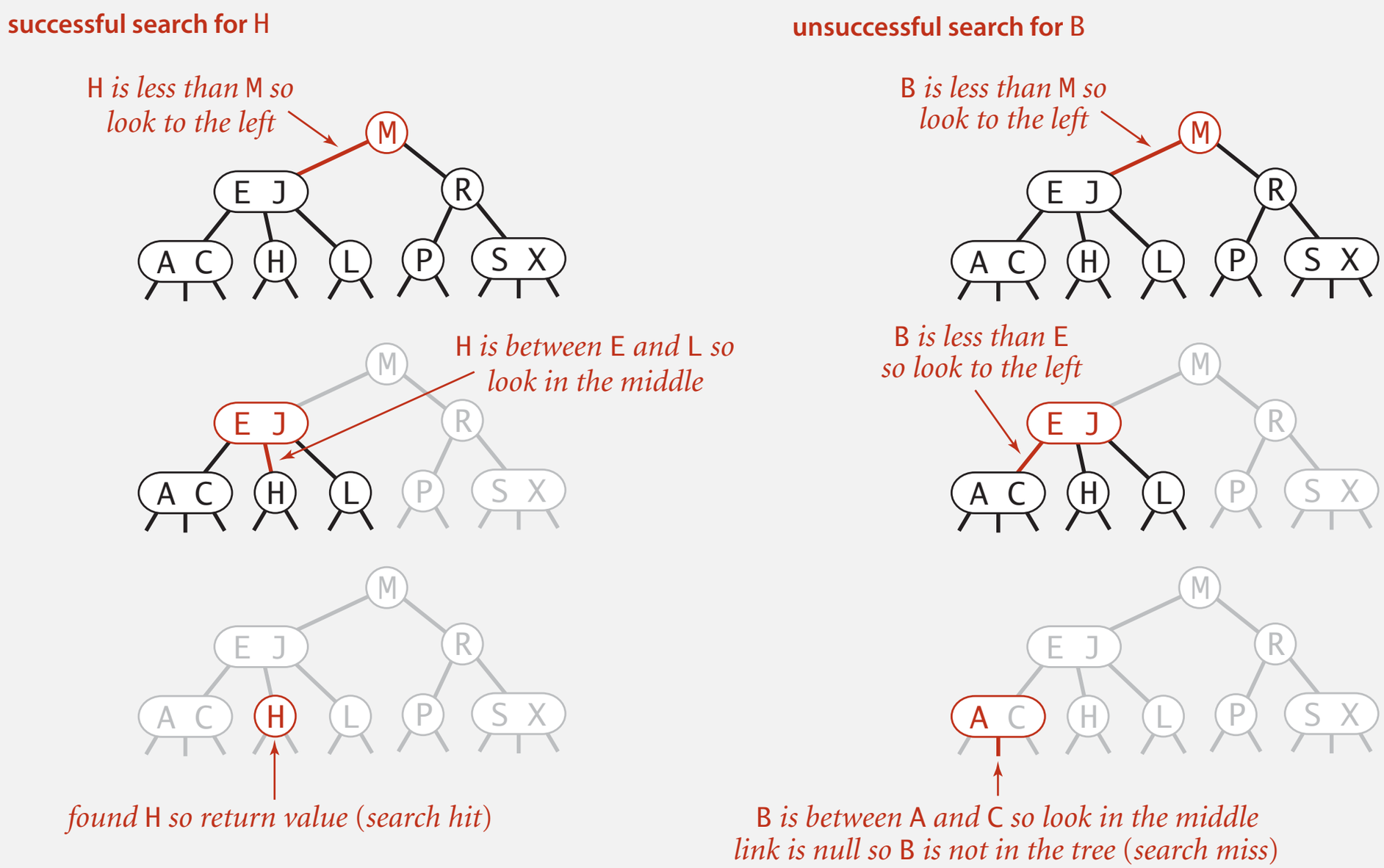


# Search in a 2-3 tree

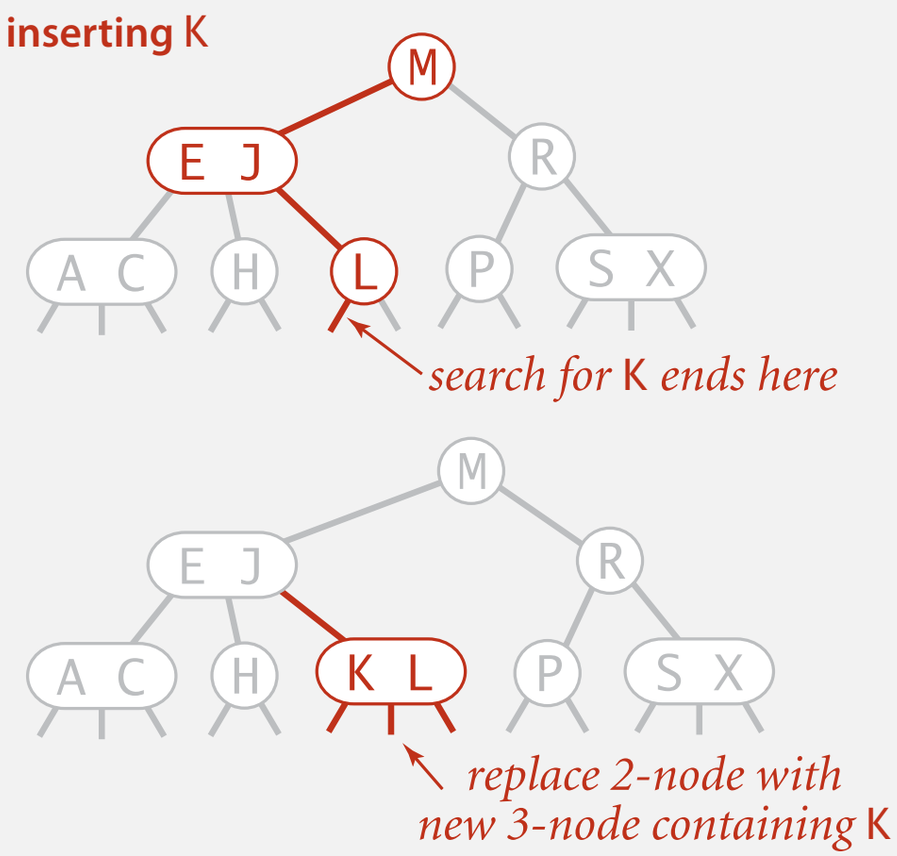
- Compare search key against keys in node.
- Find interval containing search key.
- Follow associated link (recursively).



# Insertion in a 2-3 tree

Case 1. Insert into a 2-node at bottom.

- Search for key, as usual.
- Replace 2-node with 3-node.



# Insertion in a 2-3 tree

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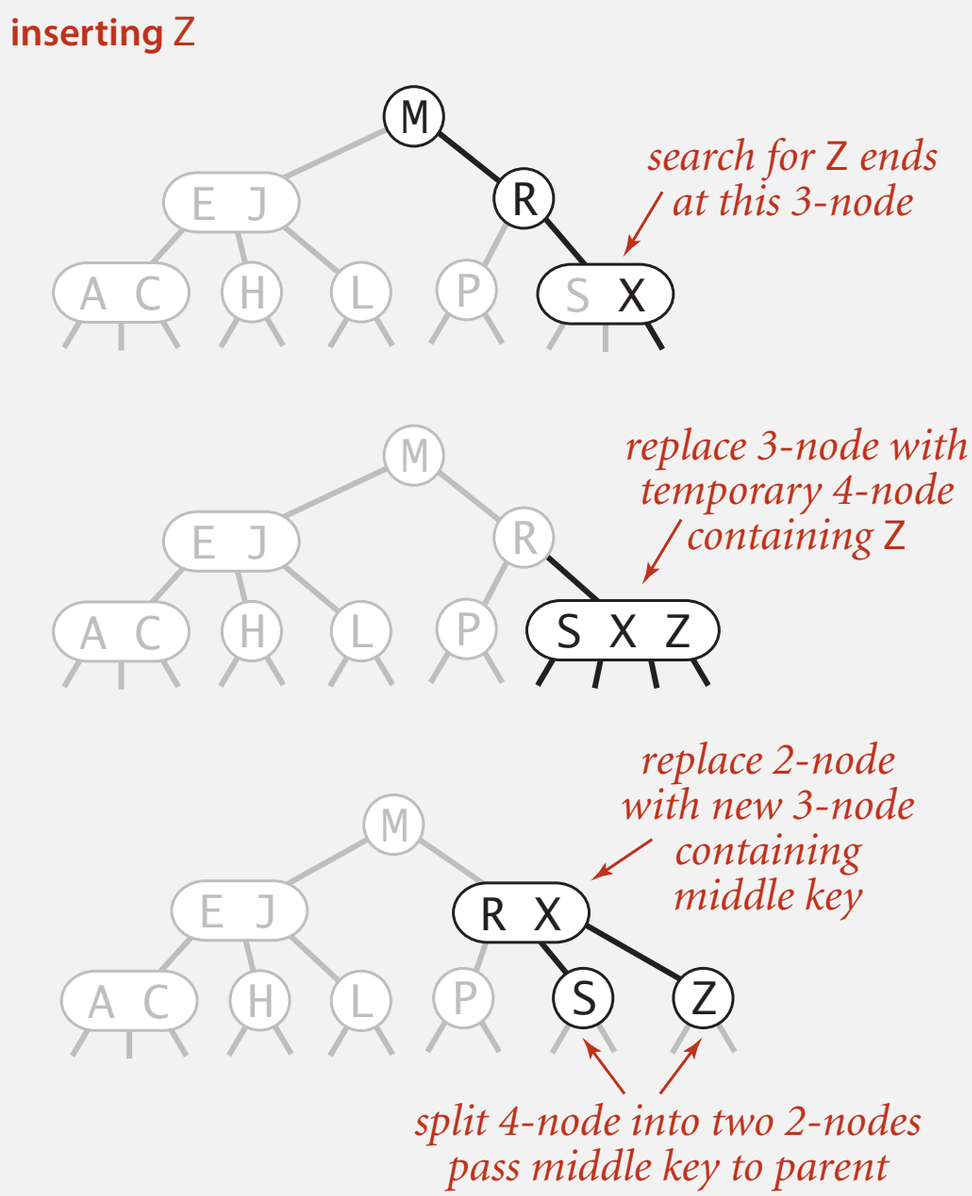
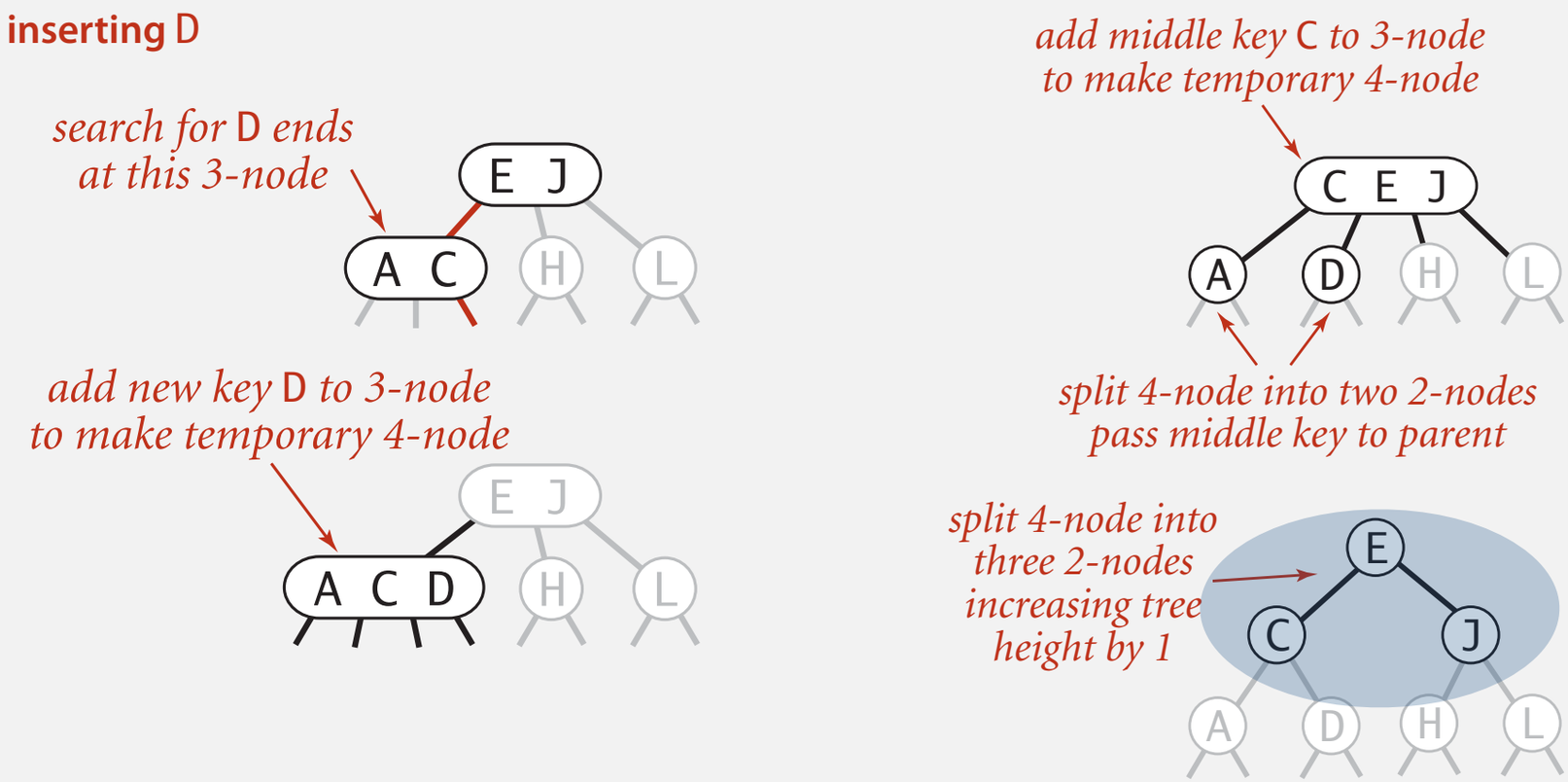


# Insertion in a 2-3 tree

Case 2. Insert into a 3-node at bottom.

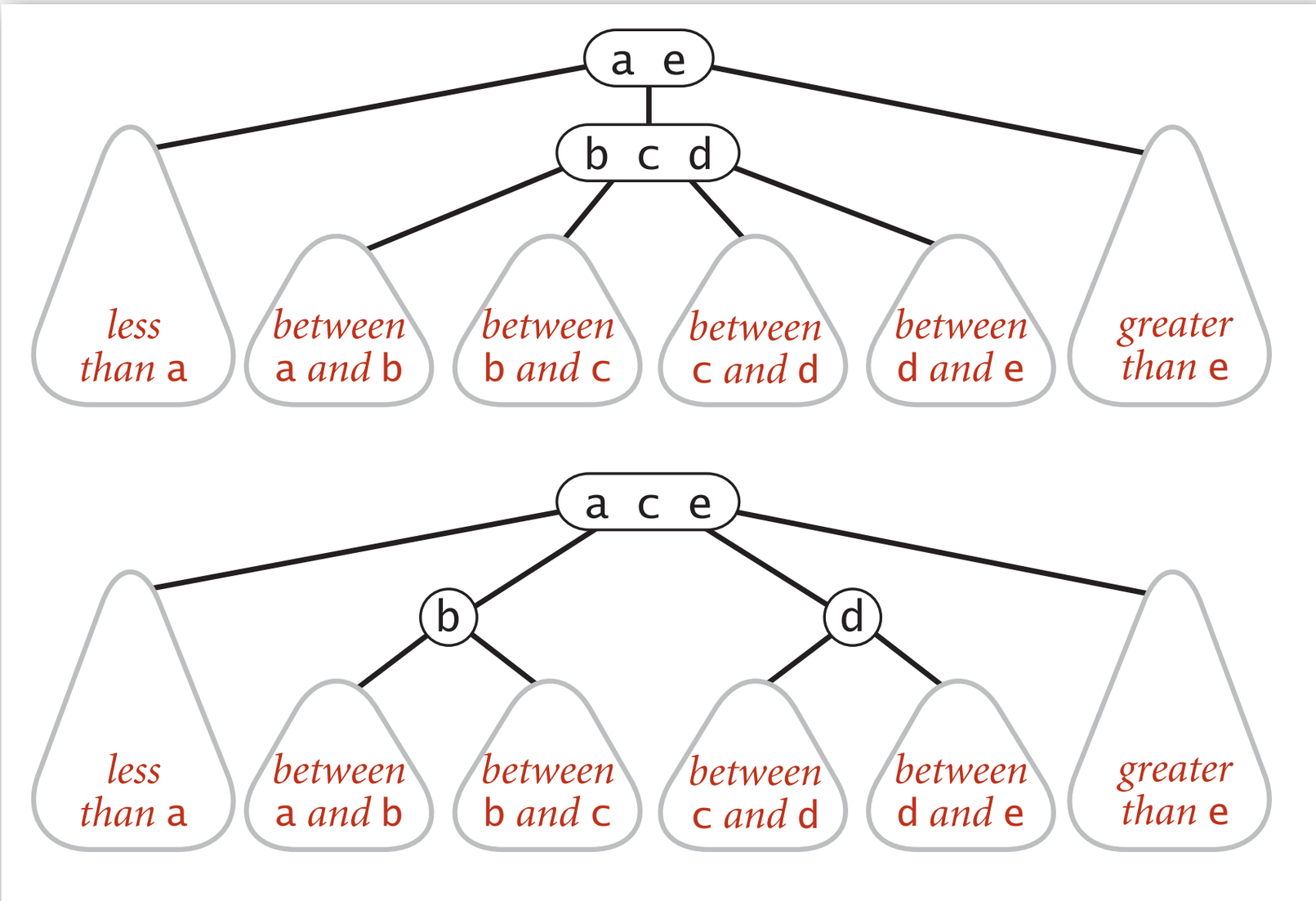
- Add new key to 3-node to create temporary 4-node.
- Move middle key in 4-node into parent.
- Repeat up the tree, as necessary.
- If you reach the root and it's a 4-node, split it into three 2-nodes.

increases height by 1



# Local transformations in a 2-3 tree

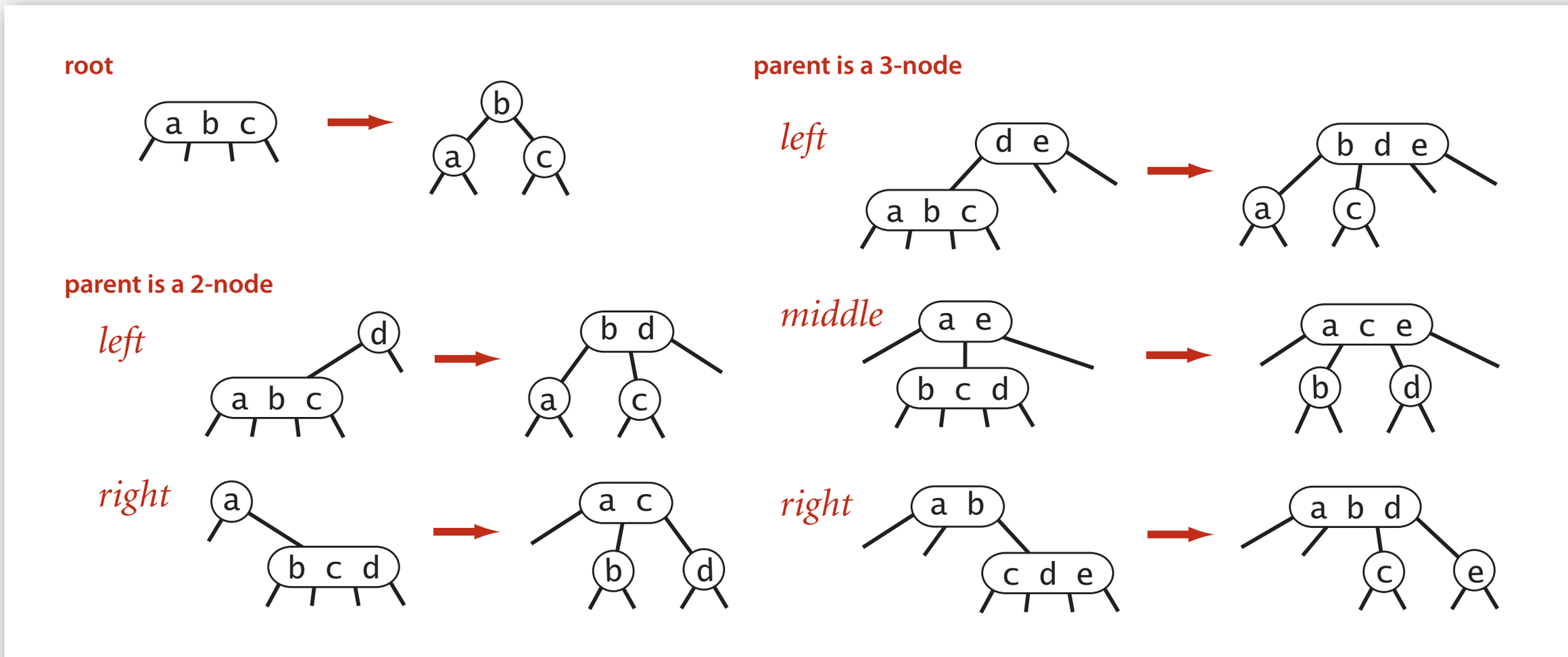
Splitting a 4-node is a local transformation: constant number of operations.



# Global properties in a 2-3 tree

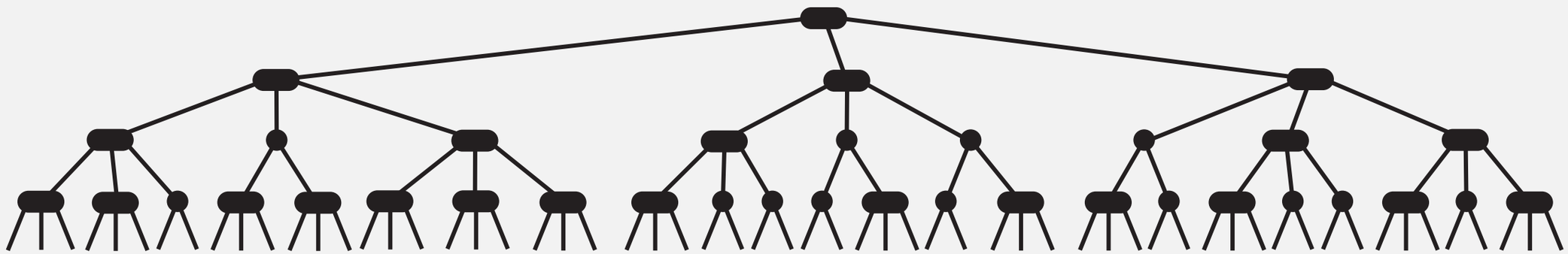
Invariants. Maintains symmetric order and perfect balance.

Pf. Each transformation maintains symmetric order and perfect balance.



# 2-3 tree: performance

Perfect balance. Every path from root to null link has same length.

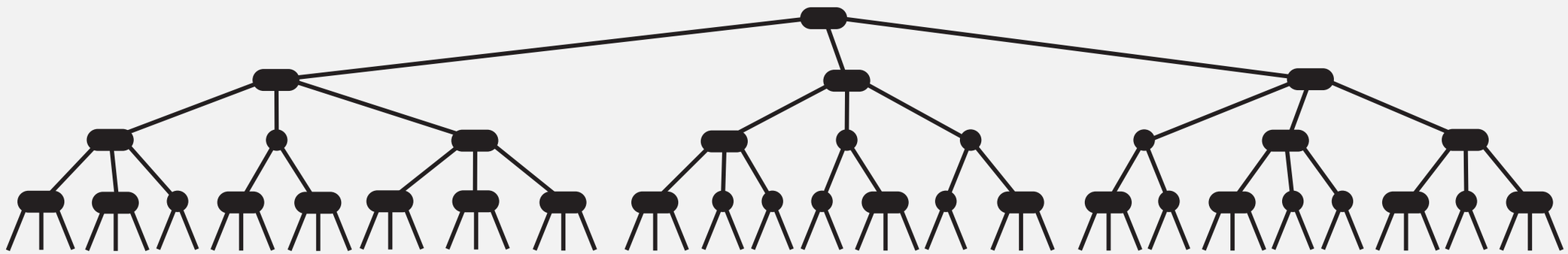


Tree height.

- Worst case:
- Best case:

# 2-3 tree: performance

Perfect balance. Every path from root to null link has same length.



Tree height.

- Worst case: $\lg N$. [all 2-nodes]
- Best case: $\log_3 N \approx .631 \lg N$. [all 3-nodes]
- Between 12 and 20 for a million nodes.
- Between 18 and 30 for a billion nodes.

Guaranteed logarithmic performance for search and insert.

# ST implementations: summary

| implementation | worst-case cost (after N inserts) | | | average case (after N random inserts) | | | ordered iteration? | key interface |
|---|---|---|---|---|---|---|---|---|
| | search | insert | delete | search hit | insert | delete | | |
| sequential search (unordered list) | N | N | N | N/2 | N | N/2 | no | `equals()` |
| binary search (ordered array) | lg N | N | N | lg N | N/2 | N/2 | yes | `compareTo()` |
| BST | N | N | N | 1.39 lg N | 1.39 lg N | ? | yes | `compareTo()` |
| 2-3 tree | c lg N | c lg N | c lg N | c lg N | c lg N | c lg N | yes | `compareTo()` |

constants depend upon implementation

# 2-3 tree: implementation?

Direct implementation is complicated, because:

- Maintaining multiple node types is cumbersome.
- Need multiple compares to move down tree.
- Need to move back up the tree to split 4-nodes.
- Large number of cases for splitting.

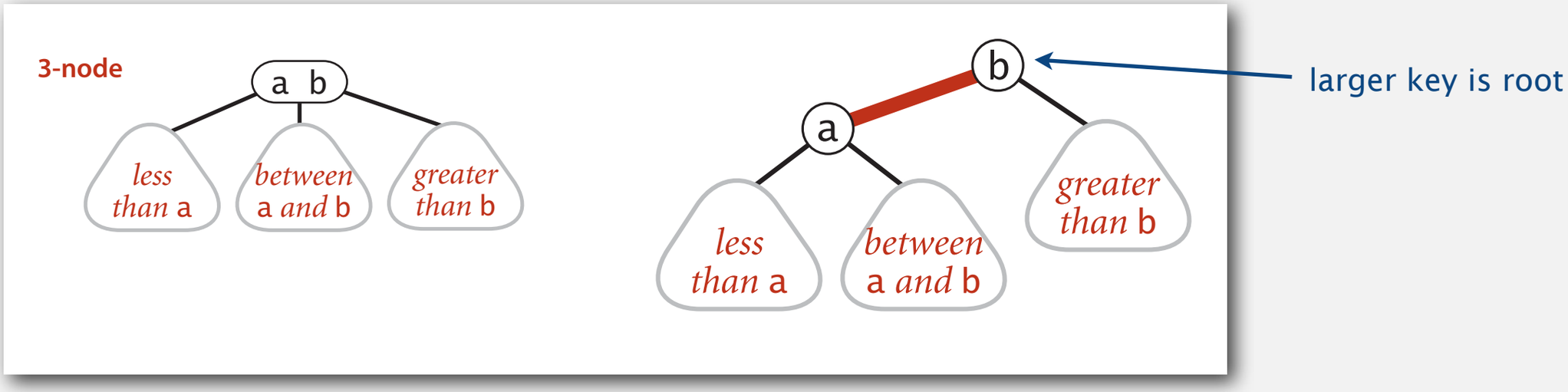
Bottom line. Could do it, but there's a better way.

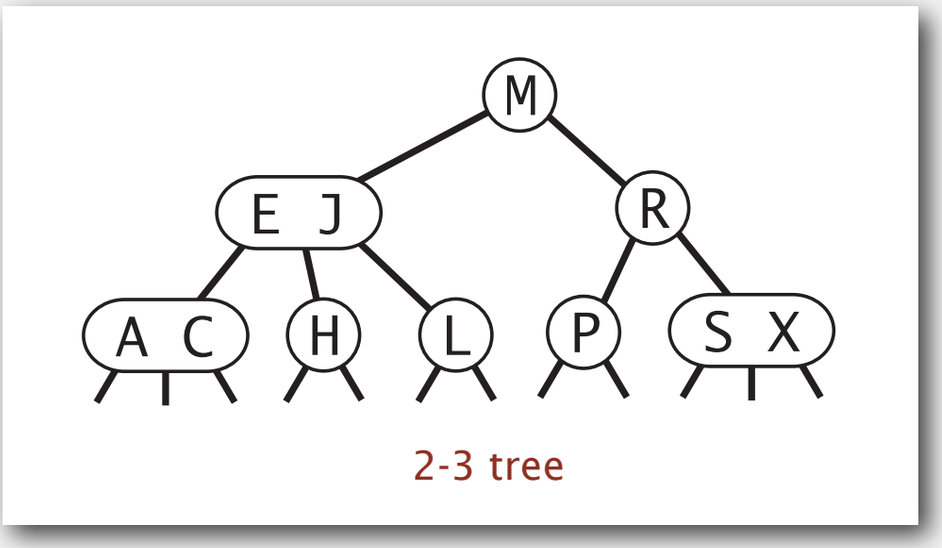
# Balanced Search Trees

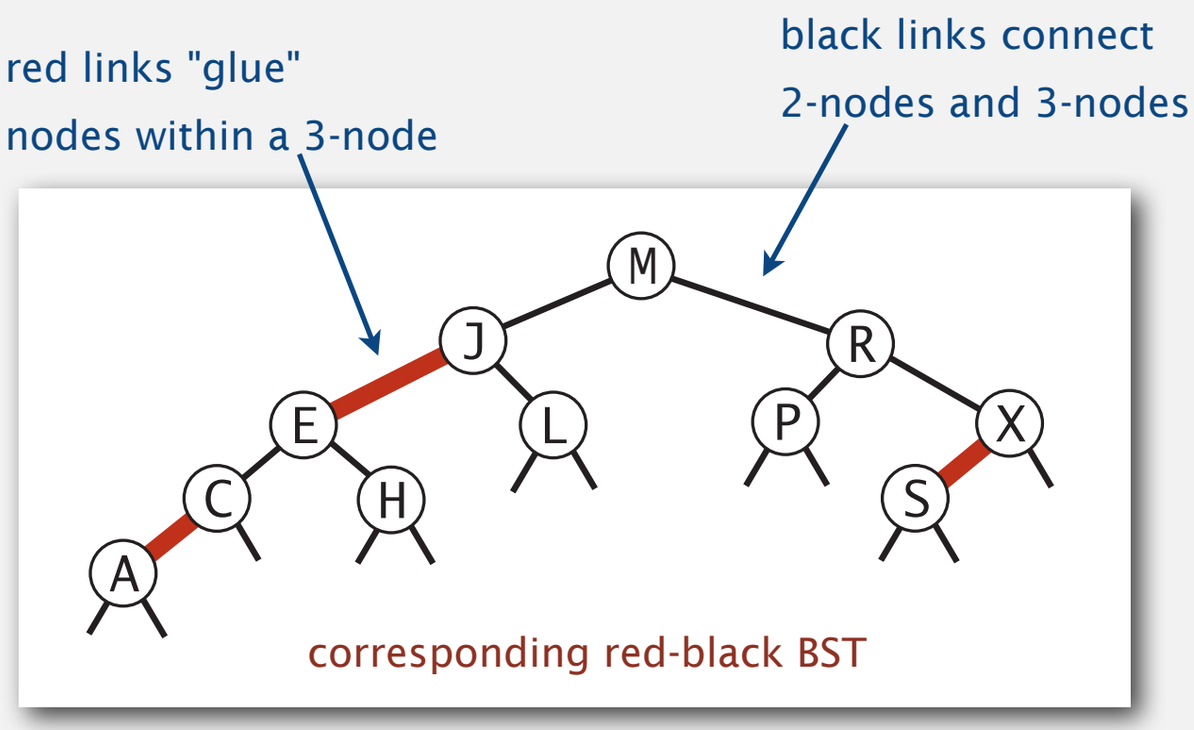
- 2-3 search trees
- **Red-black BSTs**
- B-trees
- Geometric applications of BSTs

# Left-leaning red-black BSTs (Guibas-Sedgewick 1979 and Sedgewick 2007)

1. Represent 2–3 tree as a BST.
2. Use "internal" left-leaning links as "glue" for 3–nodes.



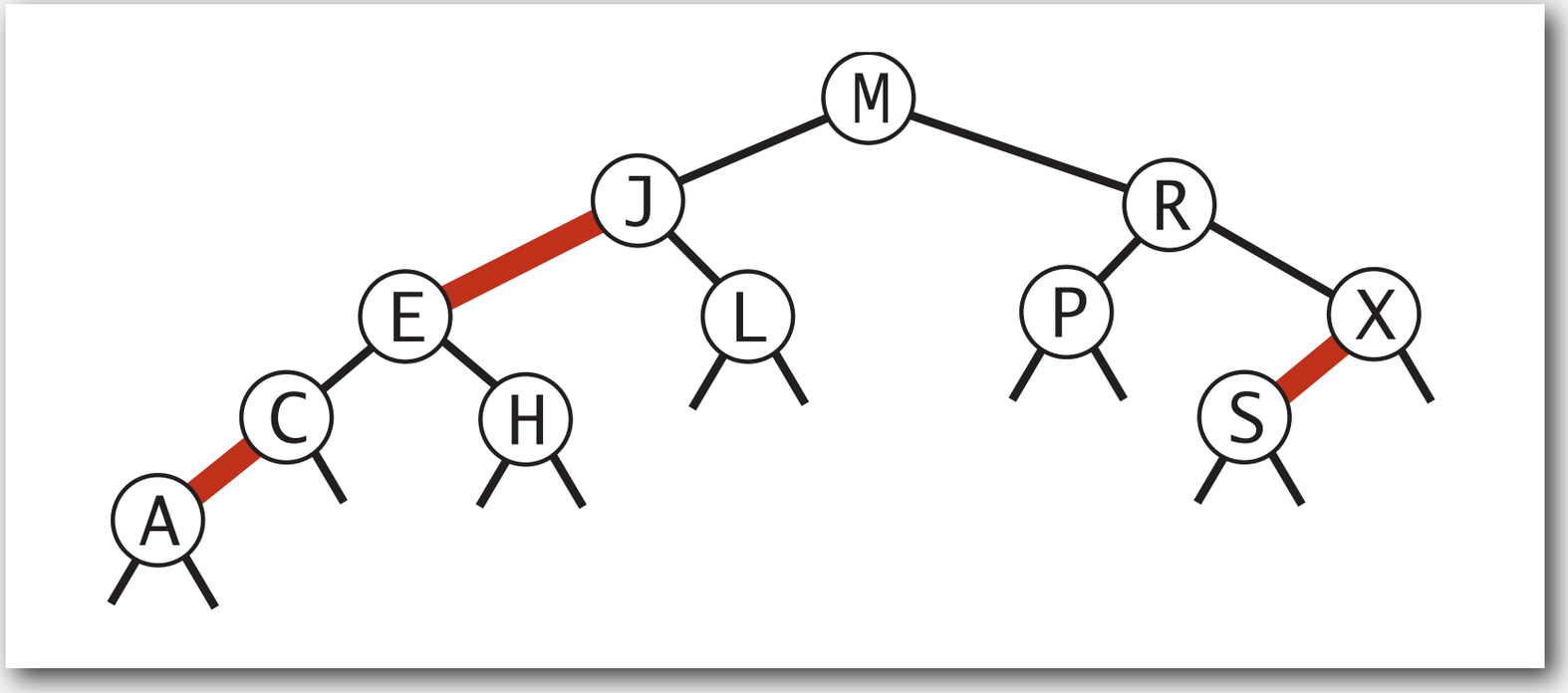



# An equivalent definition

A BST such that:

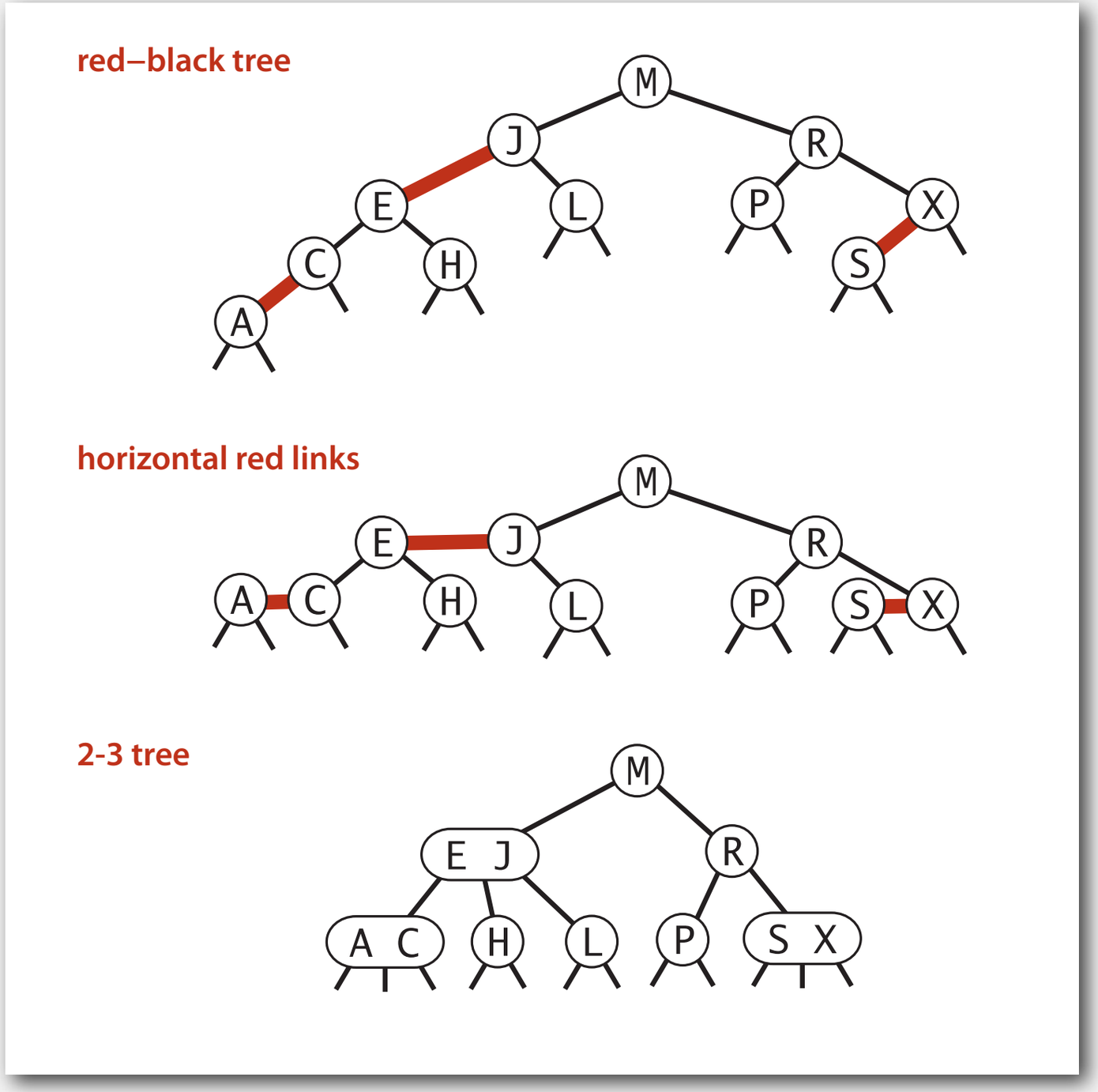
- No node has two red links connected to it.
- Every path from root to null link has the same number of black links.
- Red links lean left.

"perfect black balance"



# Left-leaning red-black BSTs: 1-1 correspondence with 2-3 trees

Key property. 1–1 correspondence between 2–3 and LLRB.

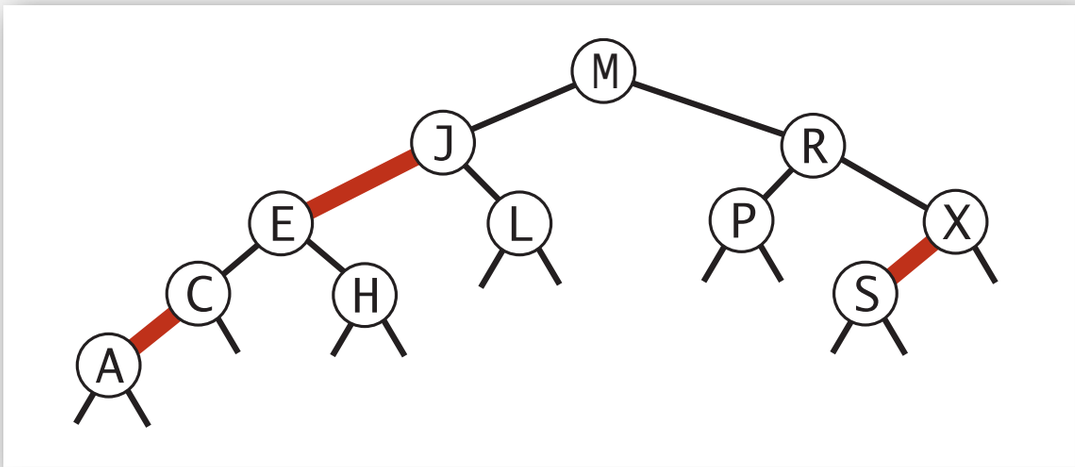


# Search implementation for red-black BSTs

Observation. Search is the same as for elementary BST (ignore color).

but runs faster because of better balance

```
public Val get(Key key)
{
   Node x = root;
   while (x != null)
   {
      int cmp = key.compareTo(x.key);
      if      (cmp  < 0) x = x.left;
      else if (cmp  > 0) x = x.right;
      else if (cmp == 0) return x.val;
   }
   return null;
}
```



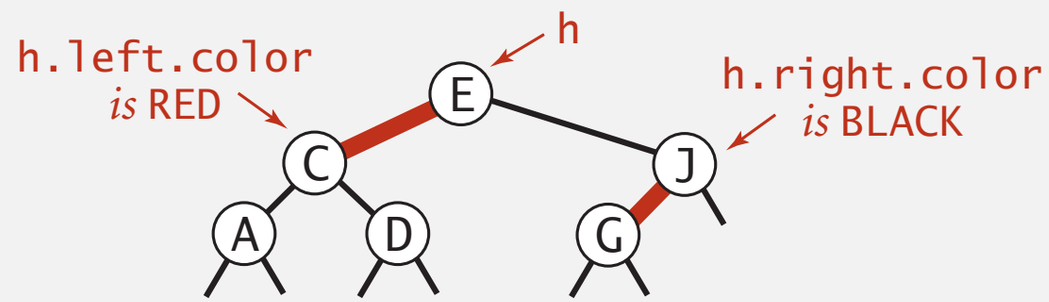

Remark. Most other ops (e.g., ceiling, selection, iteration) are also identical.

# Red-black BST representation

Each node is pointed to by precisely one link (from its parent) $\Rightarrow$ can encode color of links in nodes.

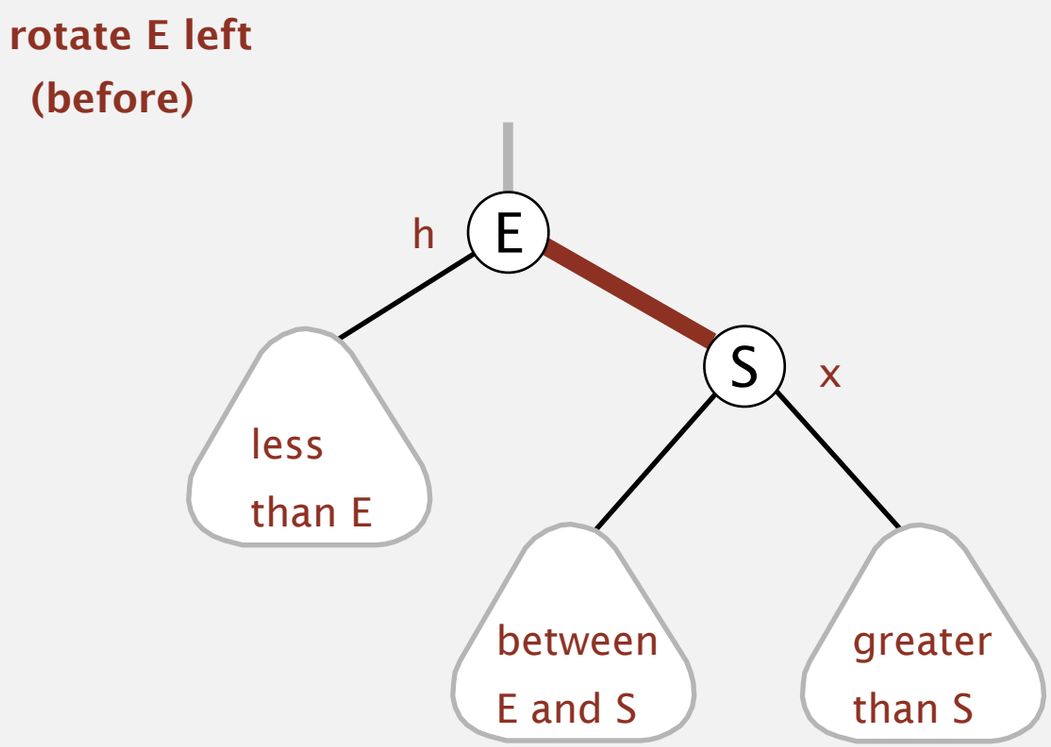
```
private static final boolean RED   = true;
private static final boolean BLACK = false;

private class Node
{
   Key key;
   Value val;
   Node left, right;
   boolean color;   // color of parent link
}

private boolean isRed(Node x)
{
   if (x == null) return false;
   return x.color == RED;
}
```

null links are black



# Elementary red-black BST operations

Left rotation. Orient a (temporarily) right-leaning red link to lean left.

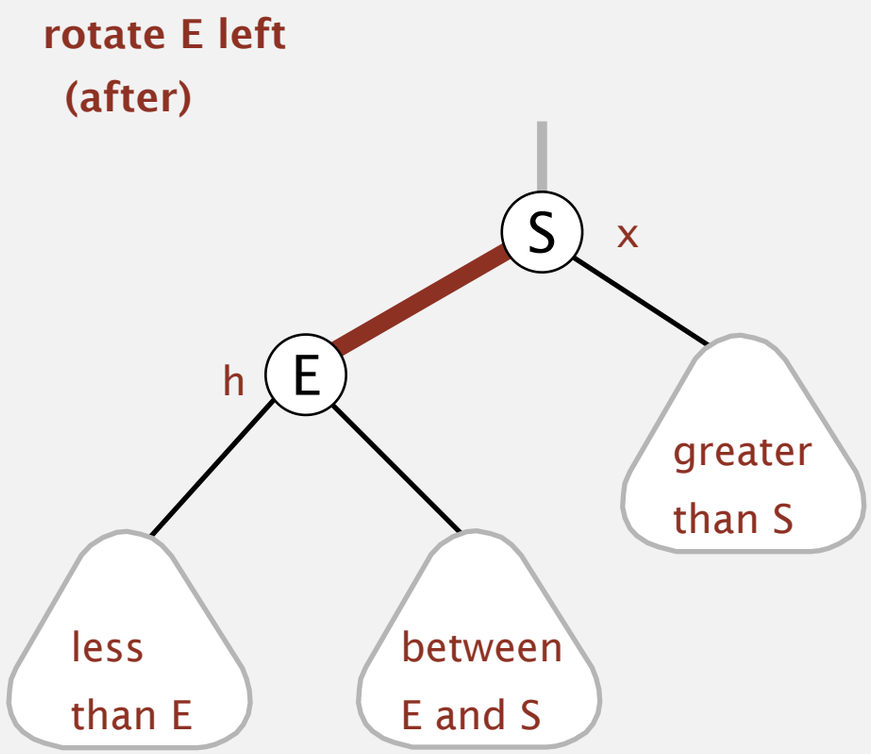



```
private Node rotateLeft(Node h)
{
   assert isRed(h.right);
   Node x = h.right;
   h.right = x.left;
   x.left = h;
   x.color = h.color;
   h.color = RED;
   return x;
}
```

Invariants. Maintains symmetric order and perfect black balance.

# Elementary red-black BST operations

Left rotation. Orient a (temporarily) right-leaning red link to lean left.

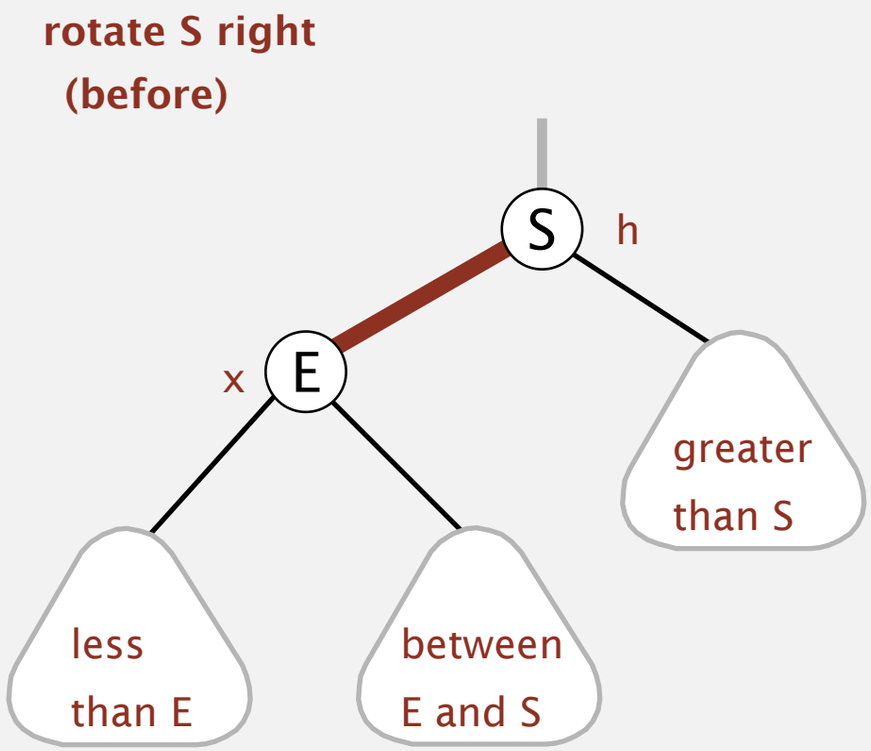



```
private Node rotateLeft(Node h)
{
   assert isRed(h.right);
   Node x = h.right;
   h.right = x.left;
   x.left = h;
   x.color = h.color;
   h.color = RED;
   return x;
}
```

Invariants. Maintains symmetric order and perfect black balance.

# Elementary red-black BST operations

Right rotation. Orient a left-leaning red link to (temporarily) lean right.

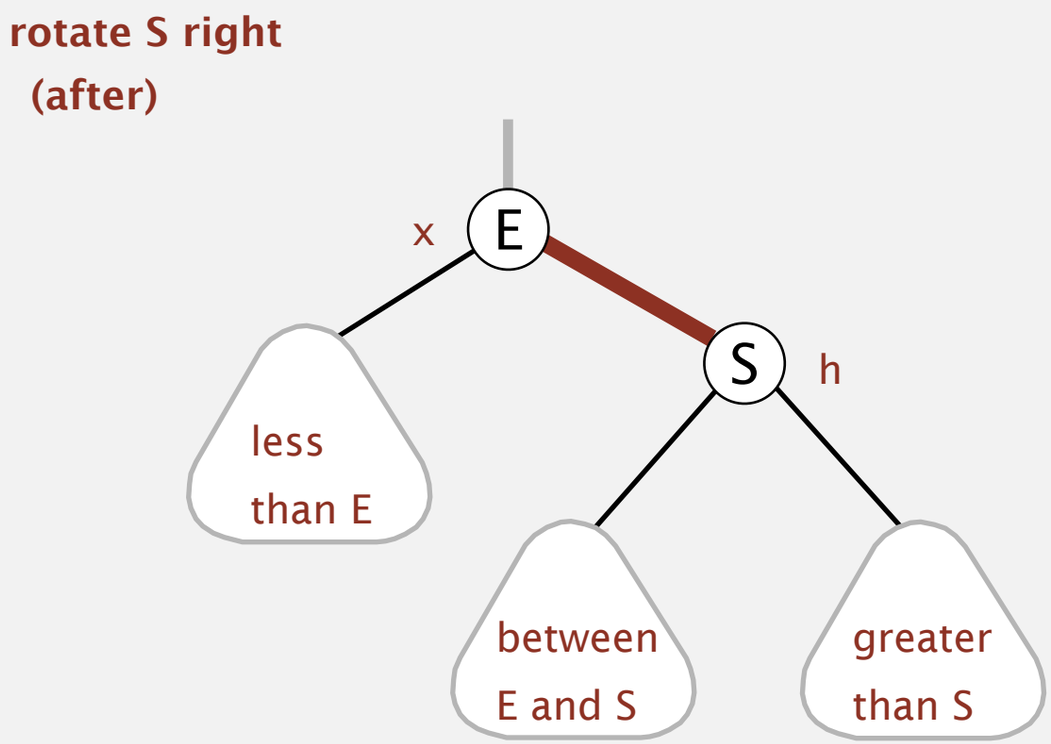



```
private Node rotateRight(Node h)
{
   assert isRed(h.left);
   Node x = h.left;
   h.left = x.right;
   x.right = h;
   x.color = h.color;
   h.color = RED;
   return x;
}
```

Invariants. Maintains symmetric order and perfect black balance.

# Elementary red-black BST operations

Right rotation. Orient a left-leaning red link to (temporarily) lean right.

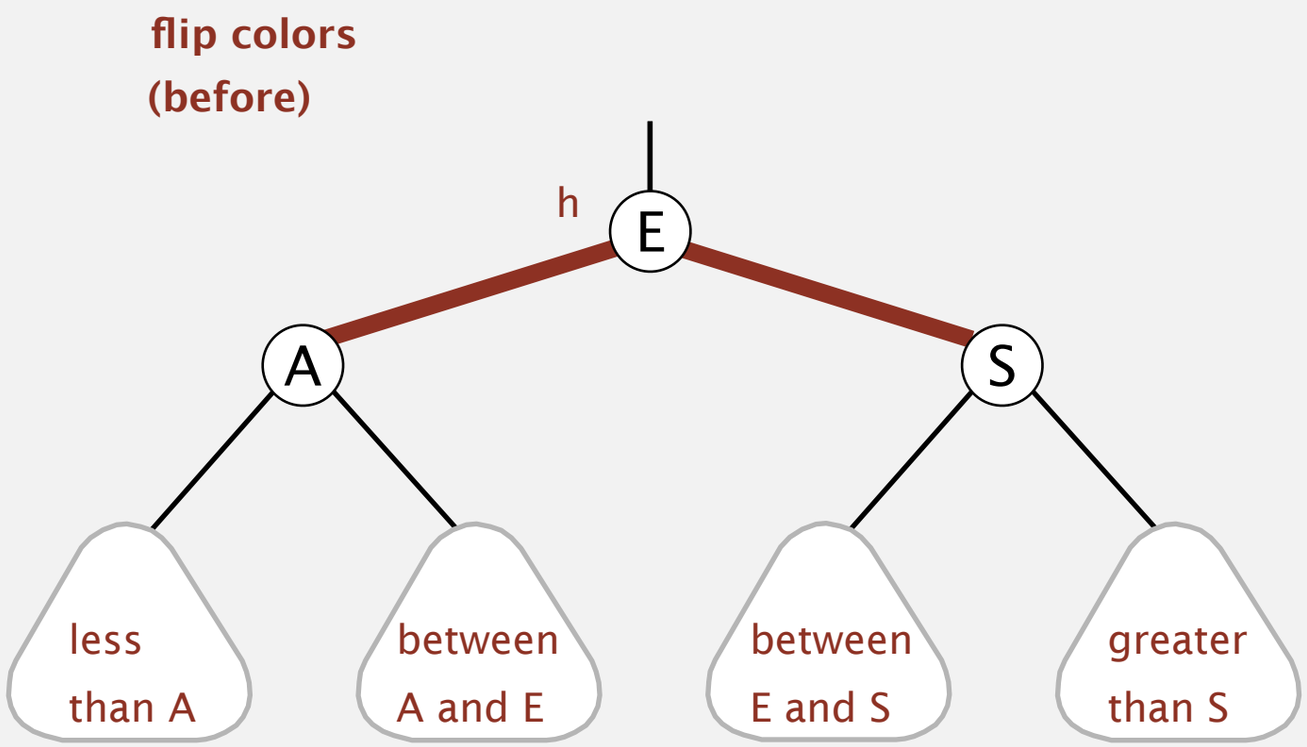



```
private Node rotateRight(Node h)
{
   assert isRed(h.left);
   Node x = h.left;
   h.left = x.right;
   x.right = h;
   x.color = h.color;
   h.color = RED;
   return x;
}
```

Invariants. Maintains symmetric order and perfect black balance.

# Elementary red-black BST operations

Color flip. Recolor to split a (temporary) 4-node.

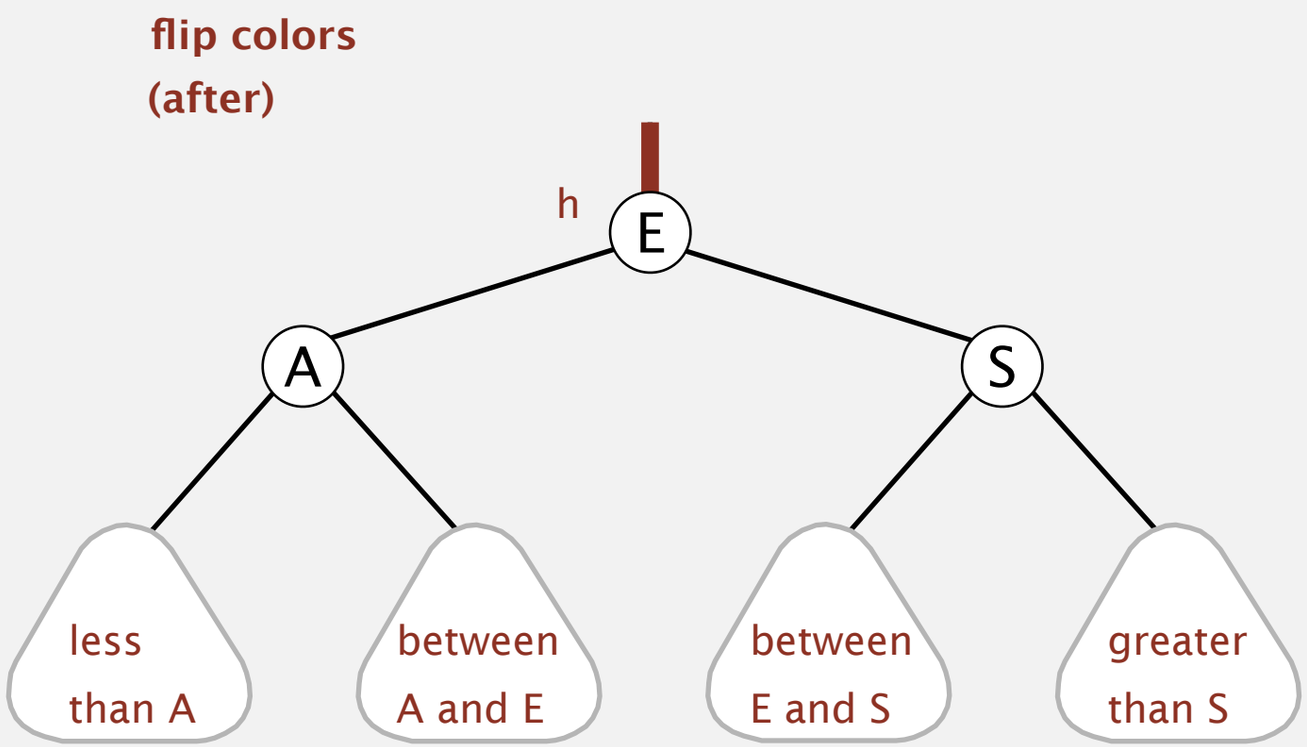



```
private void flipColors(Node h)
{
   assert !isRed(h);
   assert isRed(h.left);
   asset isRed(h.right);
   h.color = RED;
   h.left.color = BLACK;
   h.right.color = BLACK;
}
```

Invariants. Maintains symmetric order and perfect black balance.

# Elementary red-black BST operations

Color flip. Recolor to split a (temporary) 4-node.

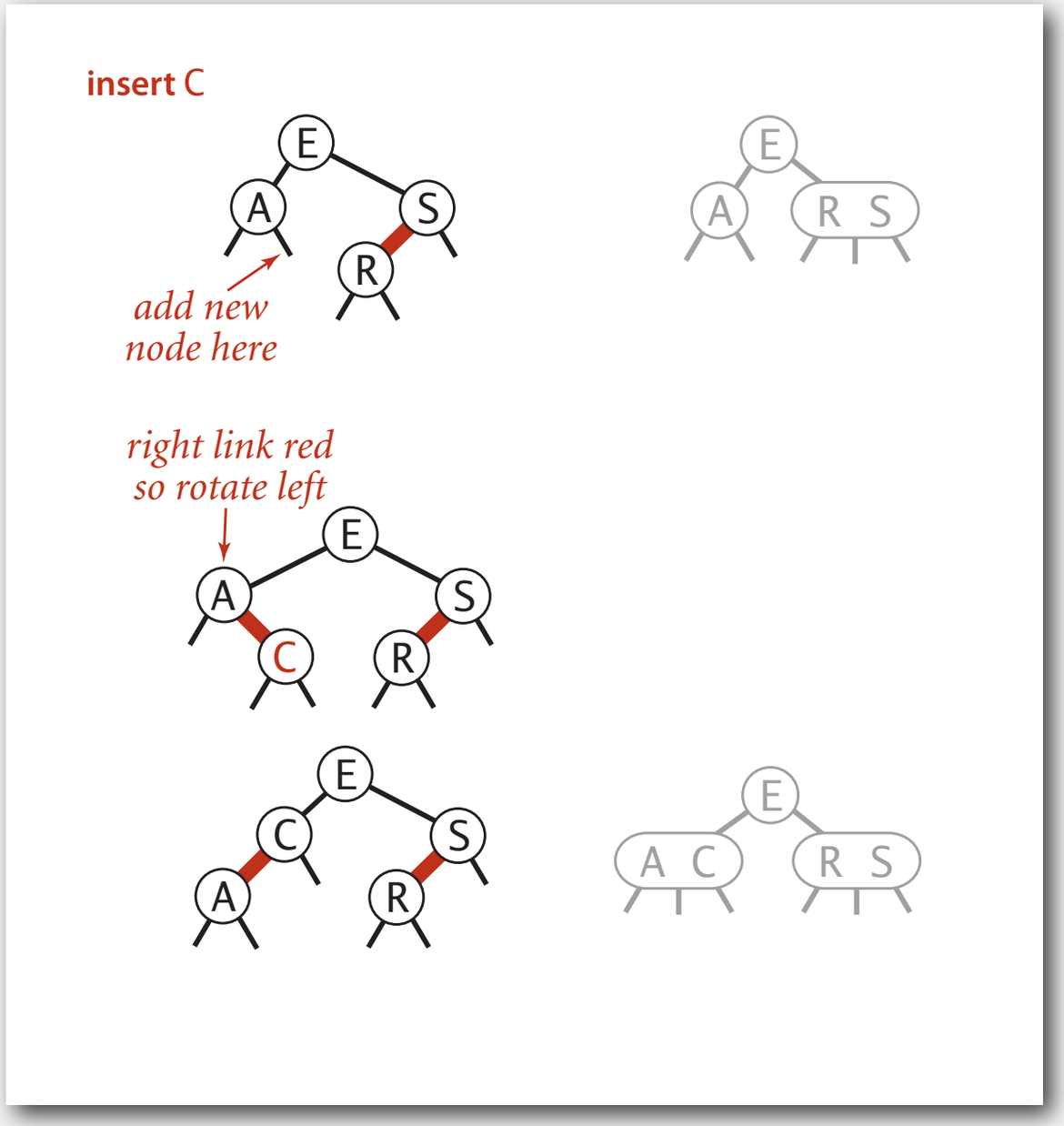



```
private void flipColors(Node h)
{
   assert !isRed(h);
   assert isRed(h.left);
   asset isRed(h.right);
   h.color = RED;
   h.left.color = BLACK;
   h.right.color = BLACK;
}
```

Invariants. Maintains symmetric order and perfect black balance.

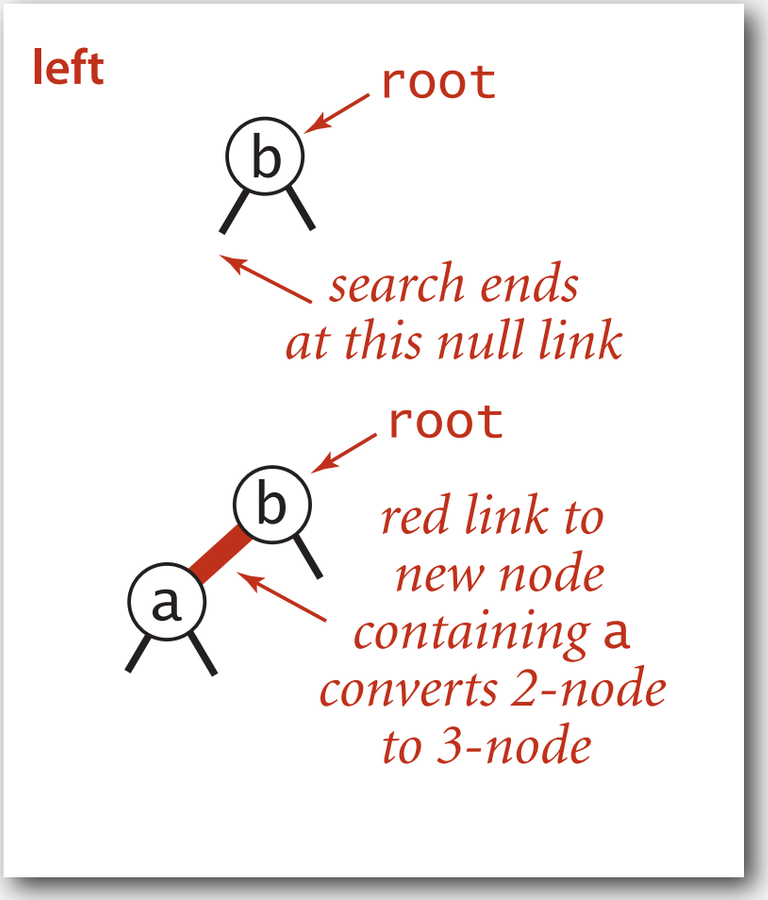
# Insertion in a LLRB tree: overview

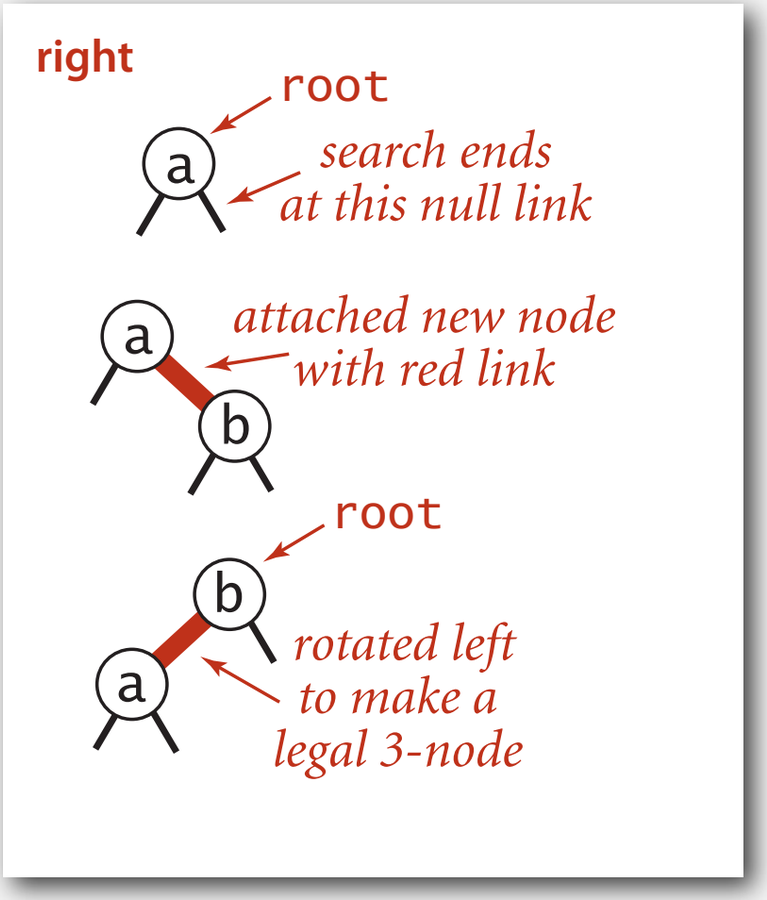
Basic strategy. Maintain 1-1 correspondence with 2-3 trees by applying elementary red-black BST operations.



# Insertion in a LLRB tree

Warmup 1. Insert into a tree with exactly 1 node.

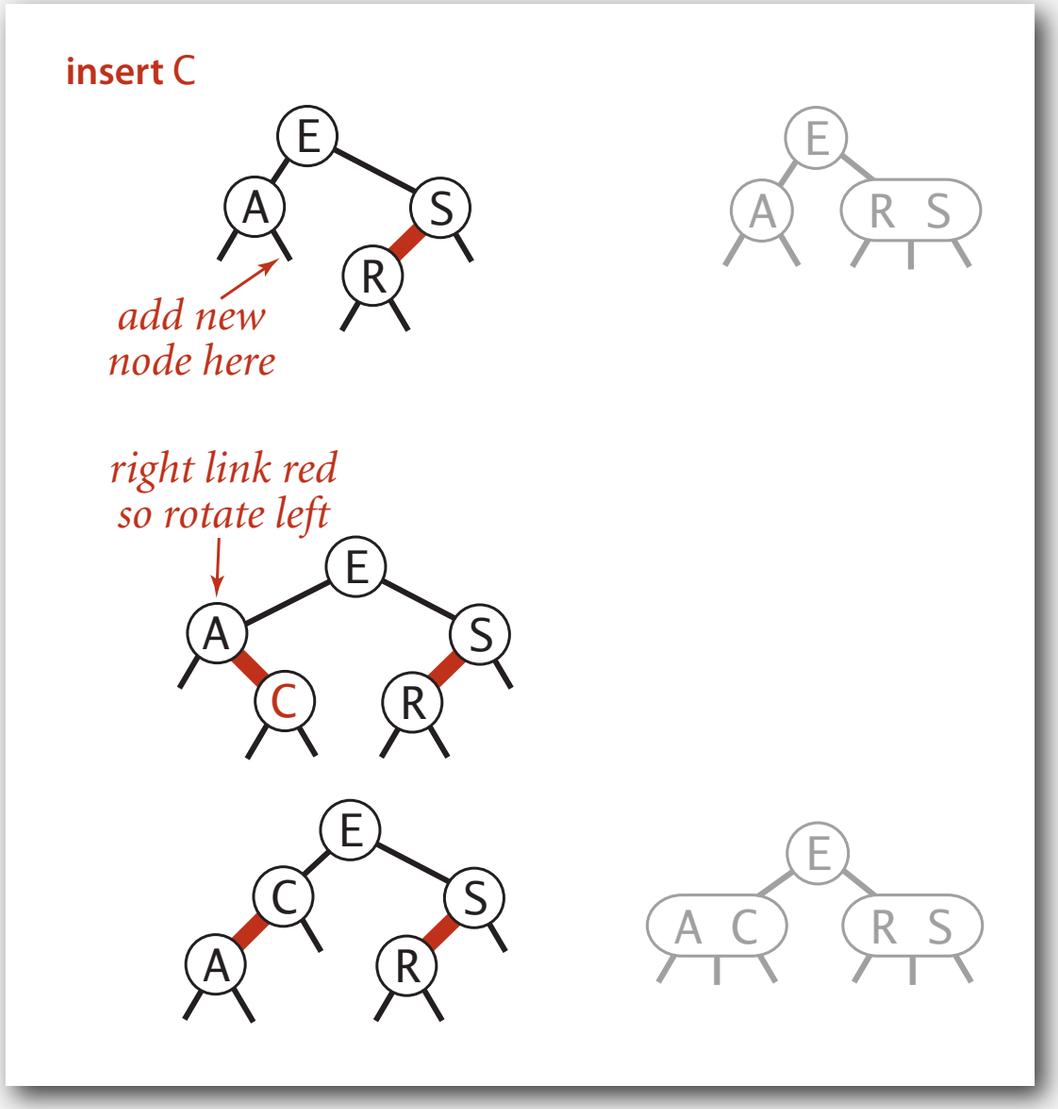


# Insertion in a LLRB tree

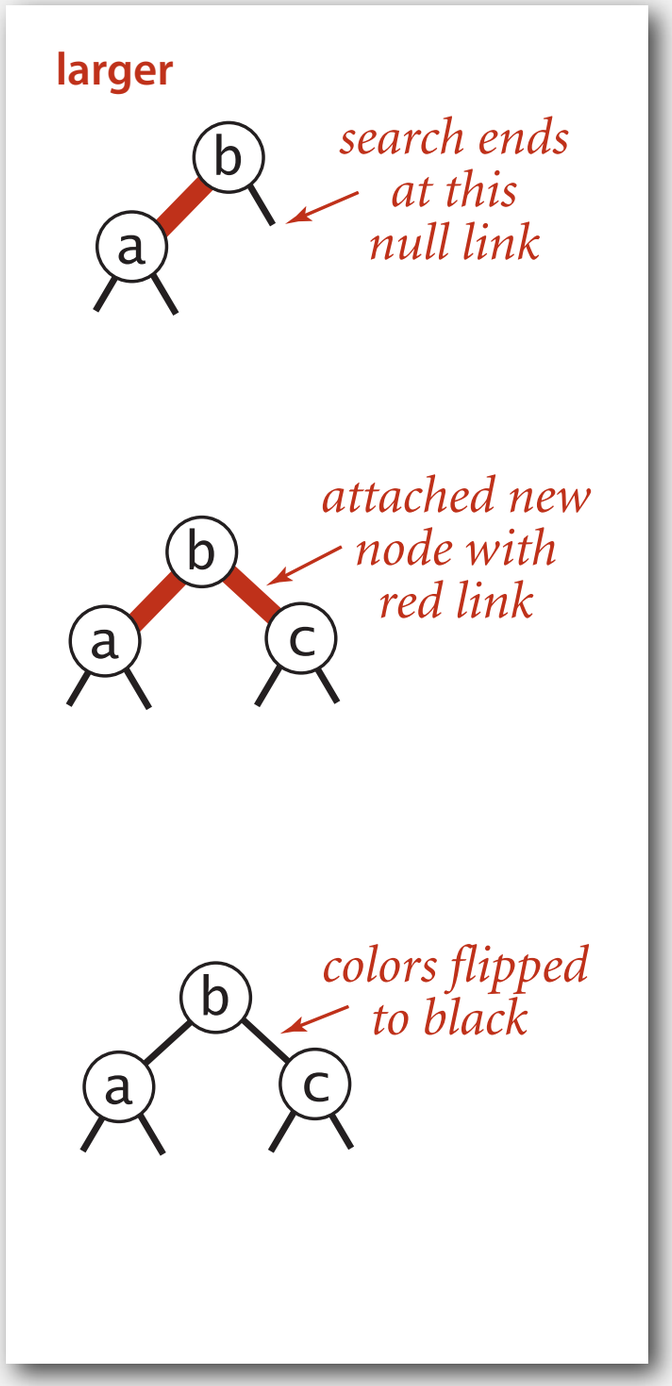
Case 1. Insert into a 2-node at the bottom.

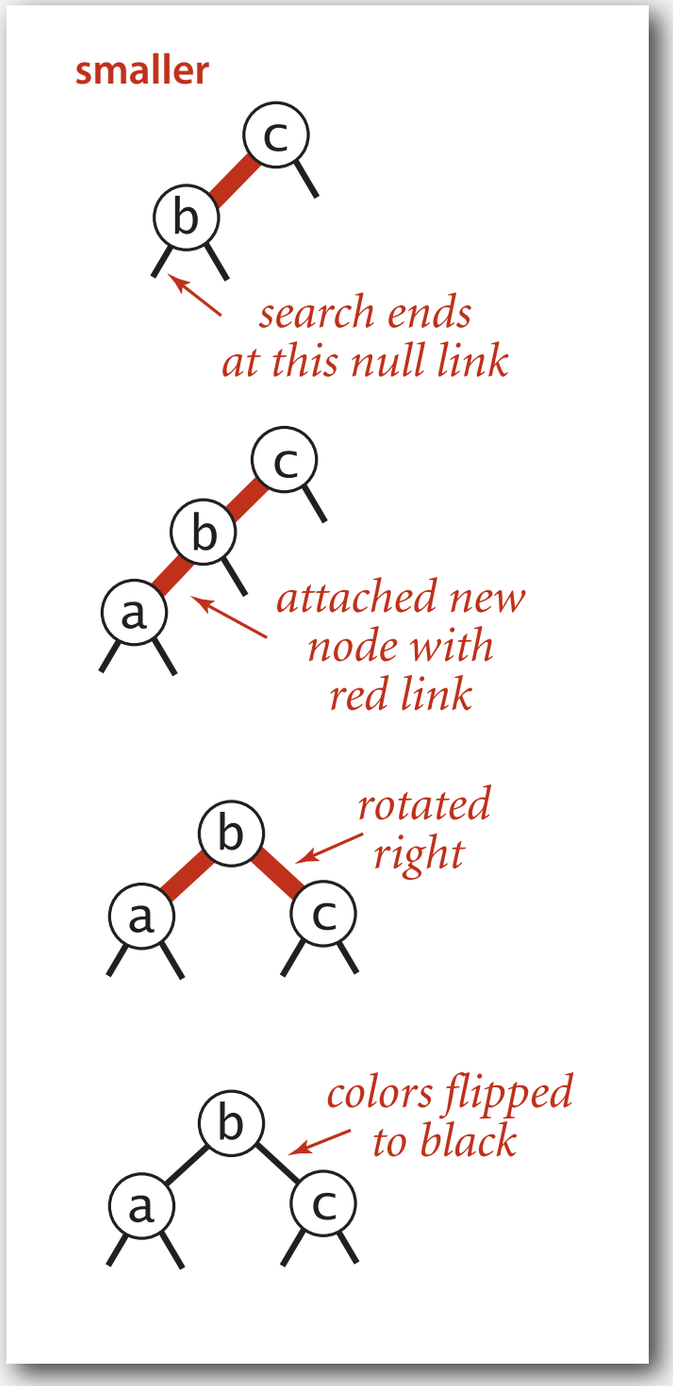
- Do standard BST insert; color new link red.
- If new red link is a right link, rotate left.

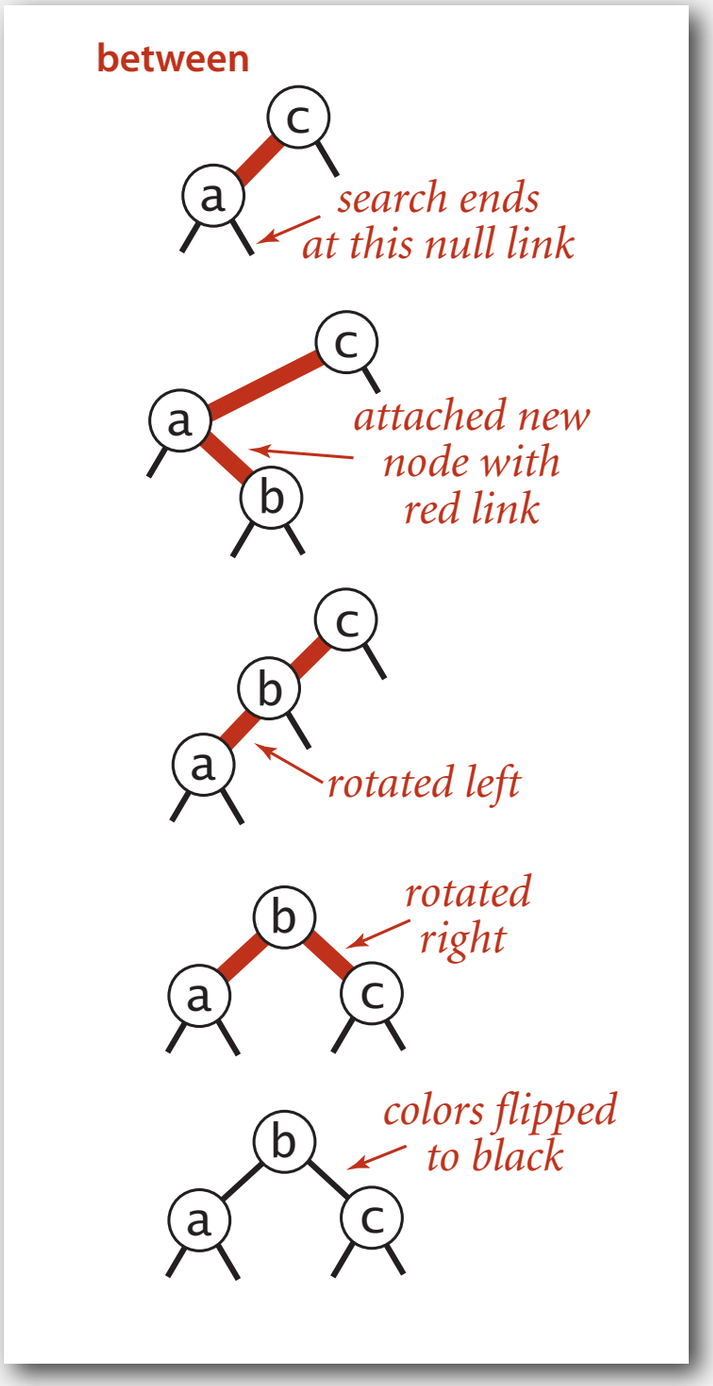


# Insertion in a LLRB tree

Warmup 2. Insert into a tree with exactly 2 nodes.



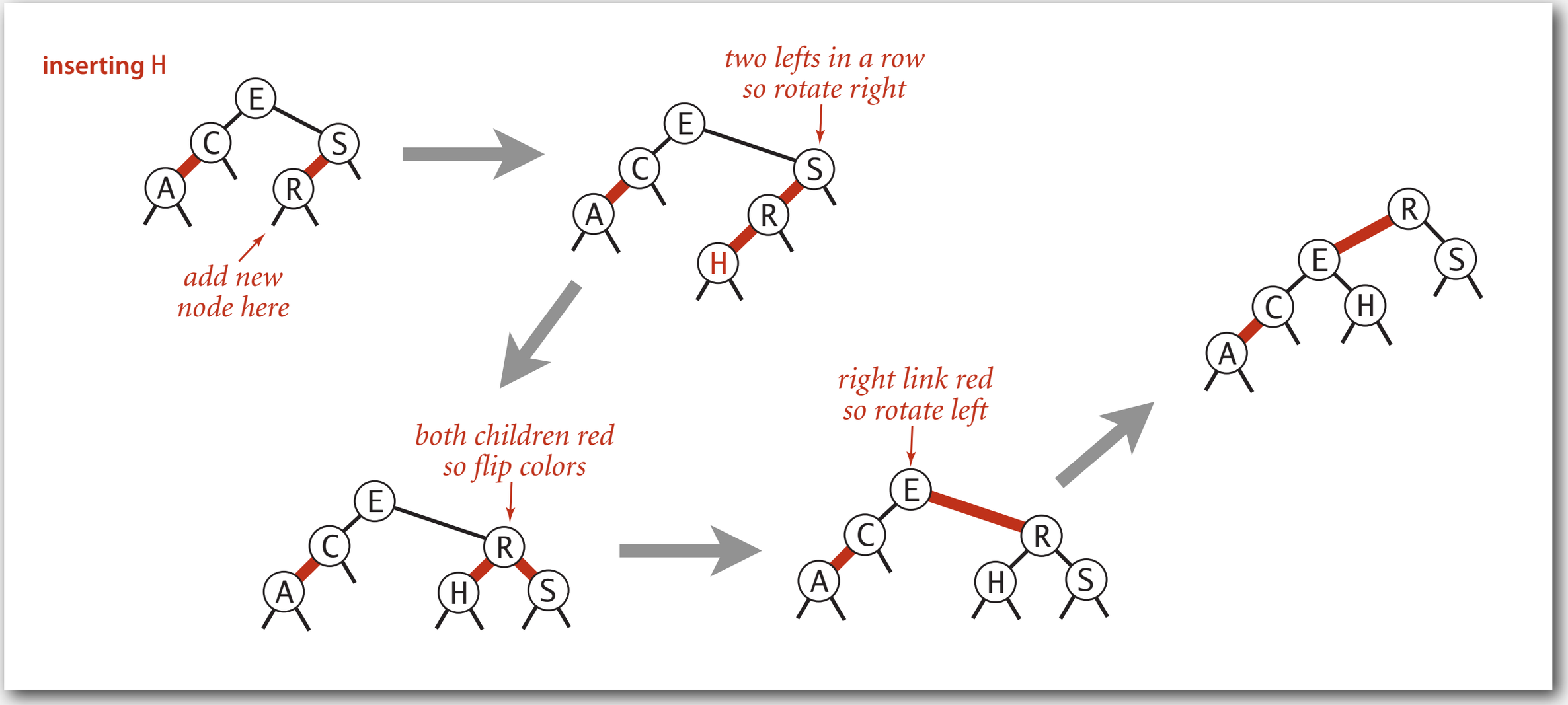



# Insertion in a LLRB tree

Case 2. Insert into a 3-node at the bottom.

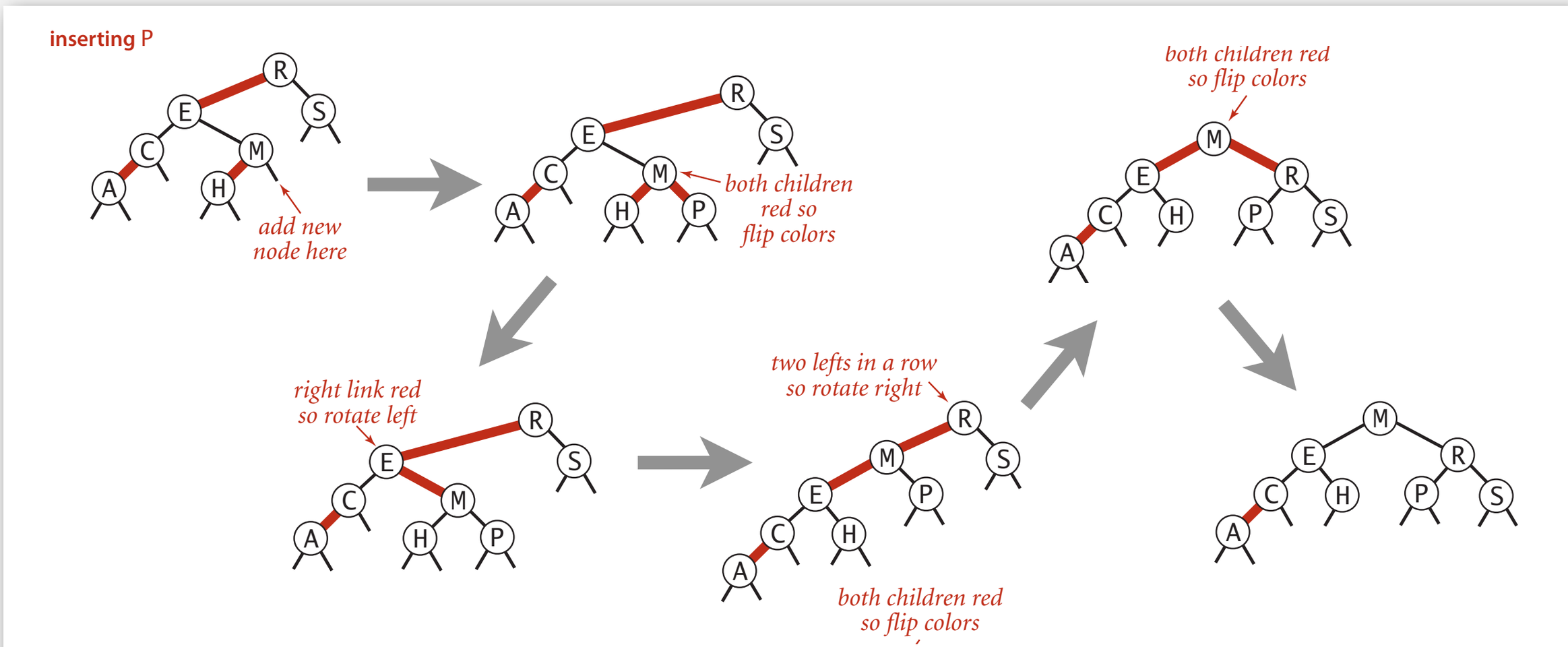
- Do standard BST insert; color new link red.
- Rotate to balance the 4-node (if needed).
- Flip colors to pass red link up one level.
- Rotate to make lean left (if needed).



# Insertion in a LLRB tree: passing red links up the tree

Case 2. Insert into a 3-node at the bottom.

- Do standard BST insert; color new link red.
- Rotate to balance the 4-node (if needed).
- Flip colors to pass red link up one level.
- Rotate to make lean left (if needed).
- Repeat case 1 or case 2 up the tree (if needed).

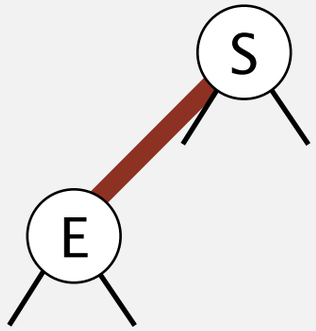


# Red-black BST insertion

insert S

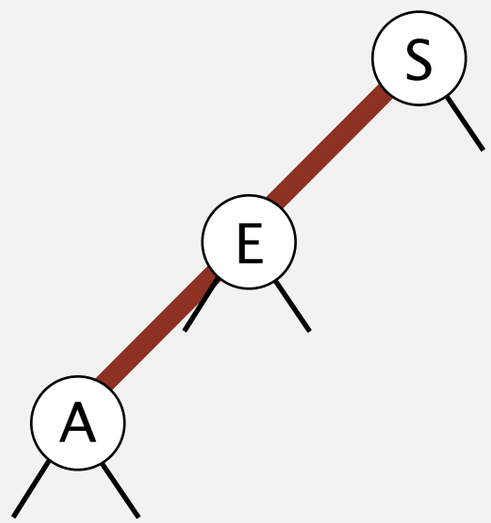
# Red-black BST insertion

insert E



# Red-black BST insertion

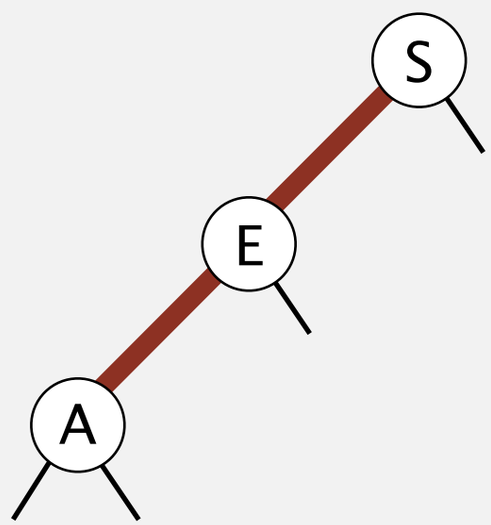
insert A



# Red-black BST insertion

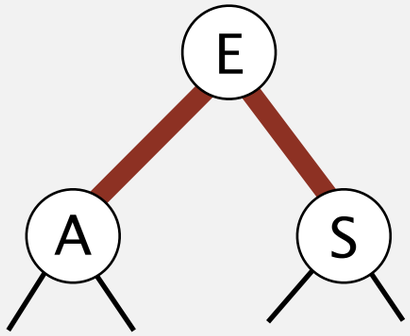
**insert A**

two left reds in a row
(rotate S right)



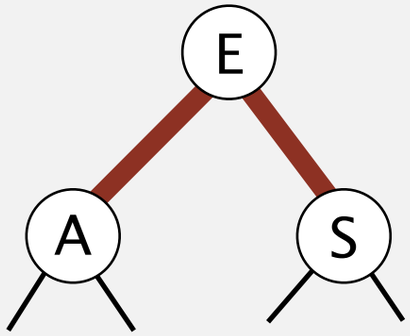
# Red-black BST insertion

both children red
(flip colors)



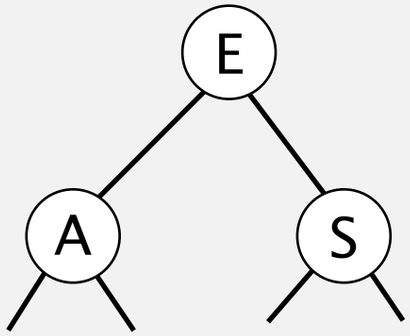
# Red-black BST insertion

both children red
(flip colors)



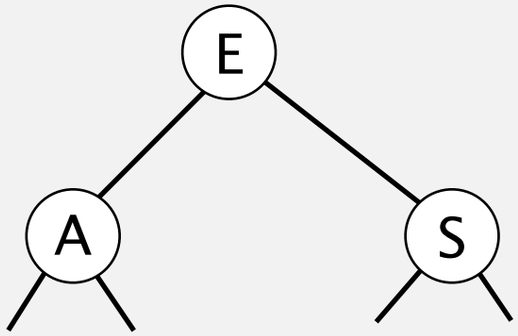
# Red-black BST insertion

red-black BST



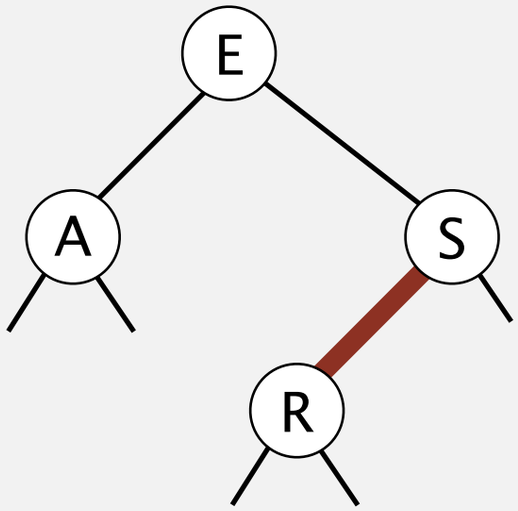
# Red-black BST insertion

red-black BST



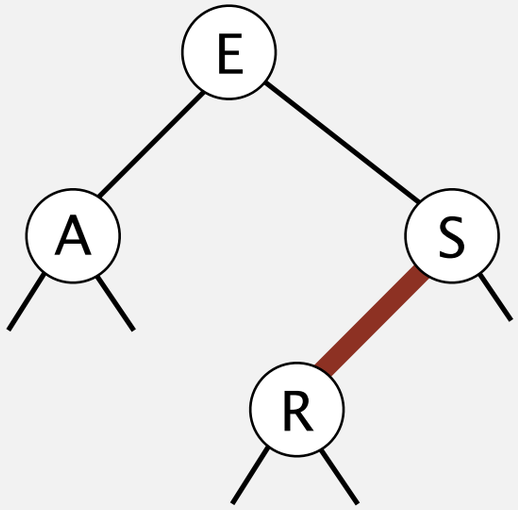
# Red-black BST insertion

insert R



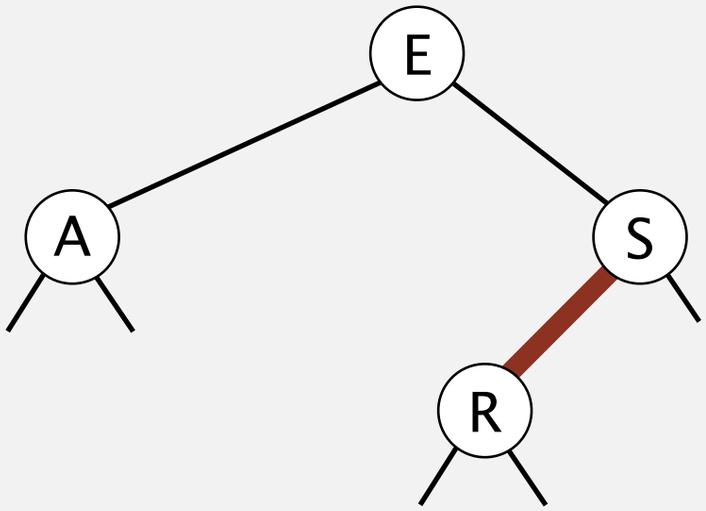
# Red-black BST insertion

red-black BST



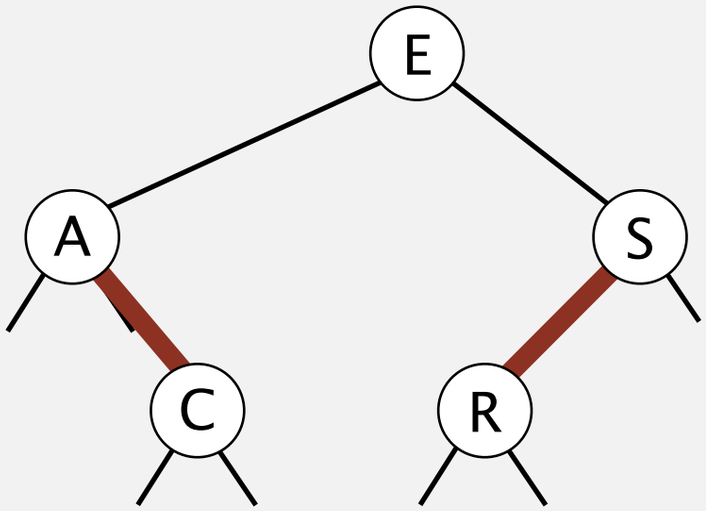
# Red-black BST insertion

red-black BST

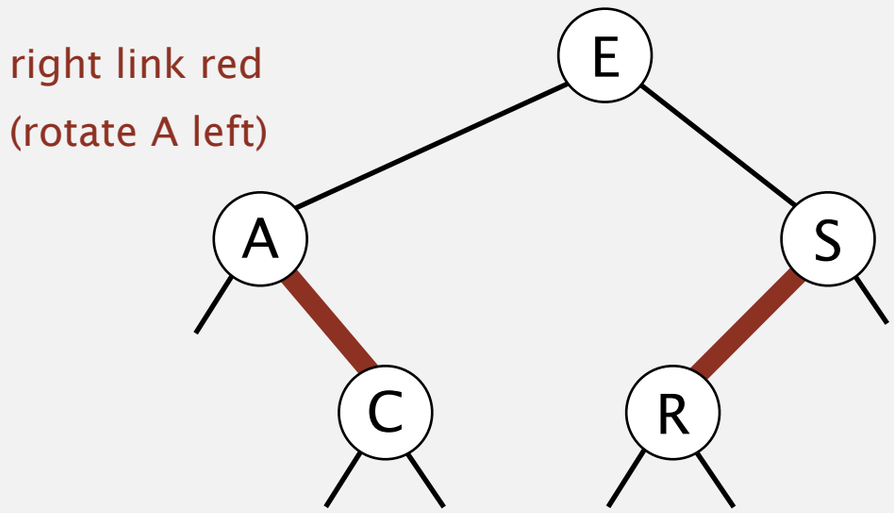


# Red-black BST insertion

insert C

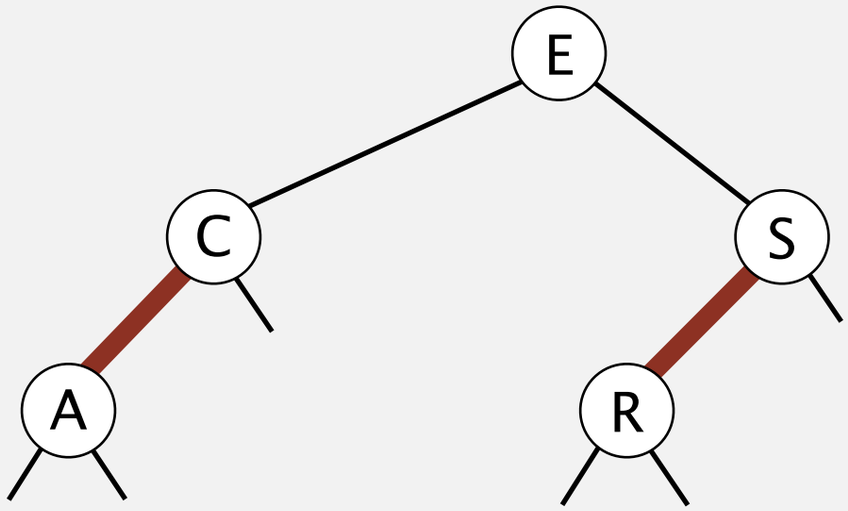


# Red-black BST insertion



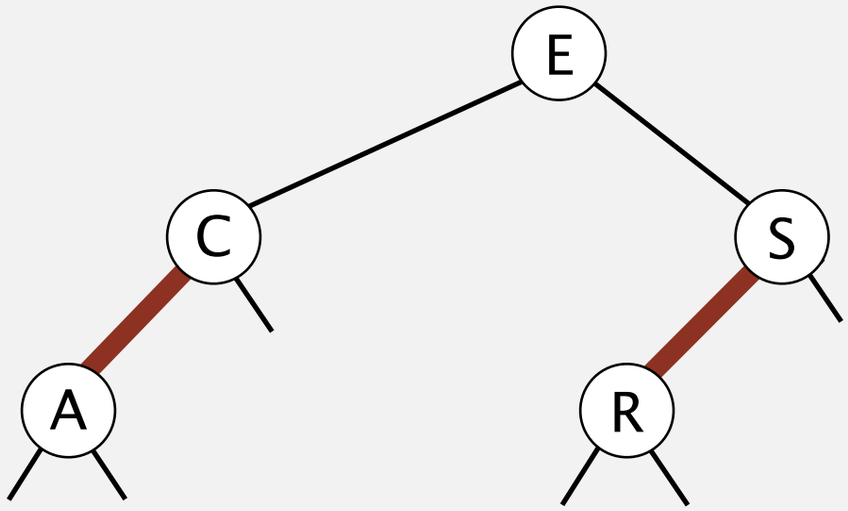
# Red-black BST insertion

red-black BST



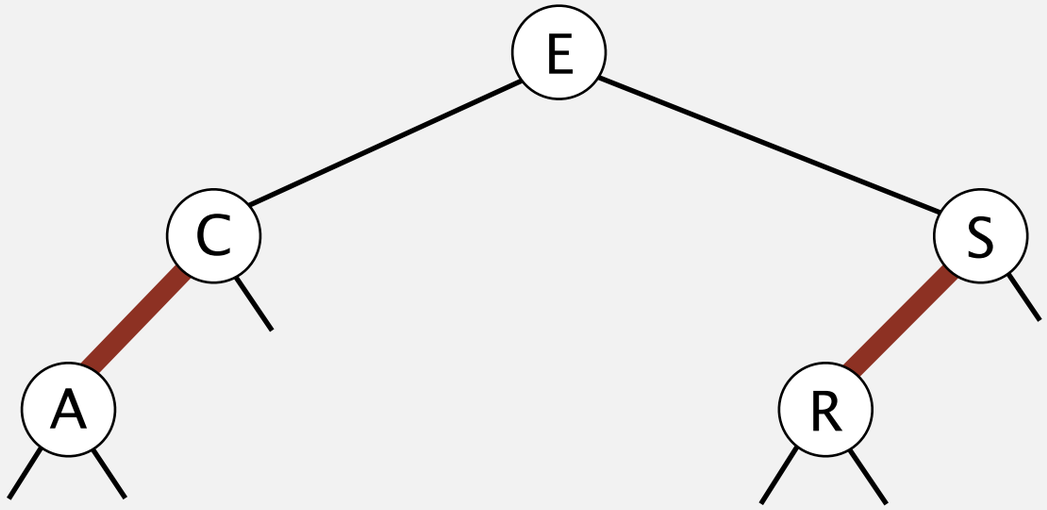
# Red-black BST insertion

red-black BST



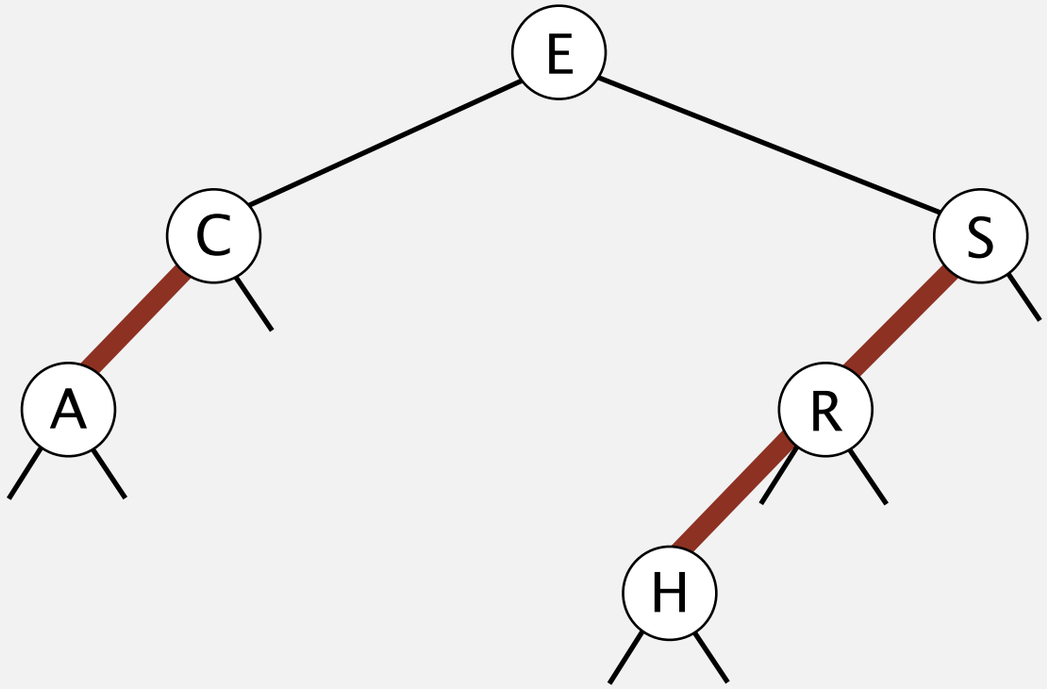
# Red-black BST insertion

red-black BST

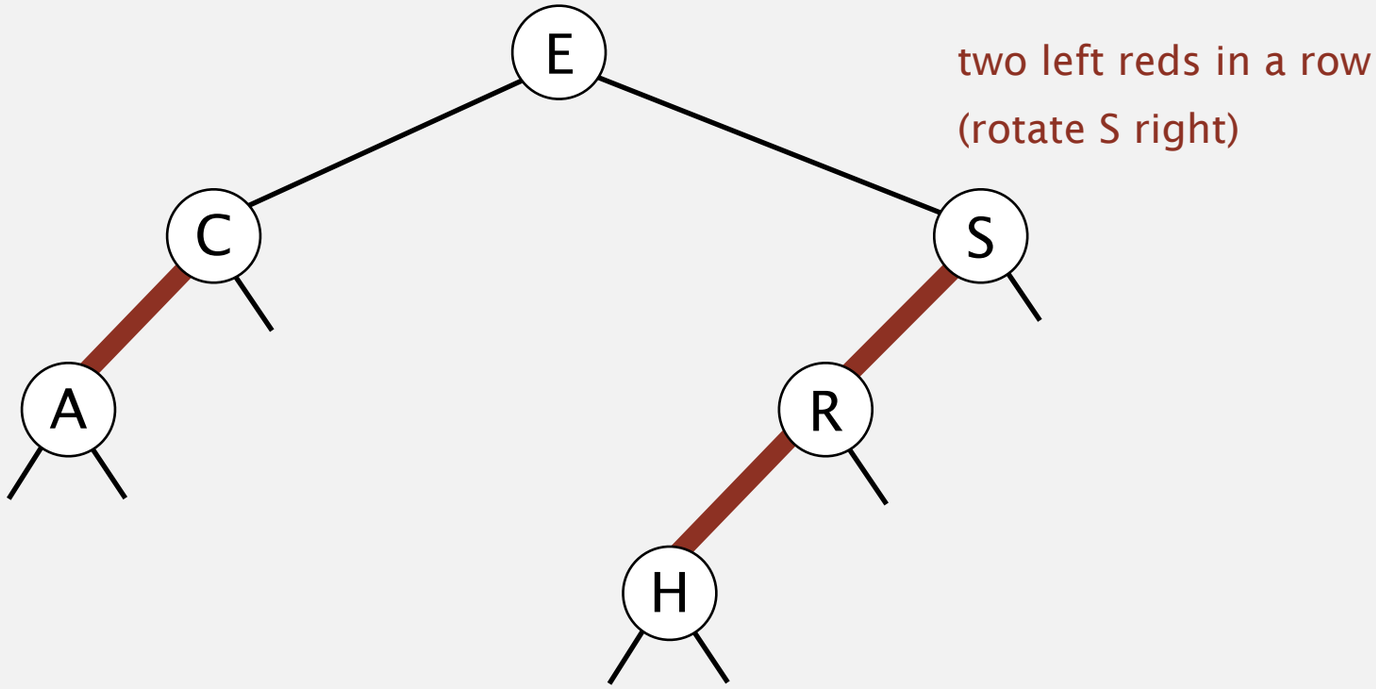


# Red-black BST insertion

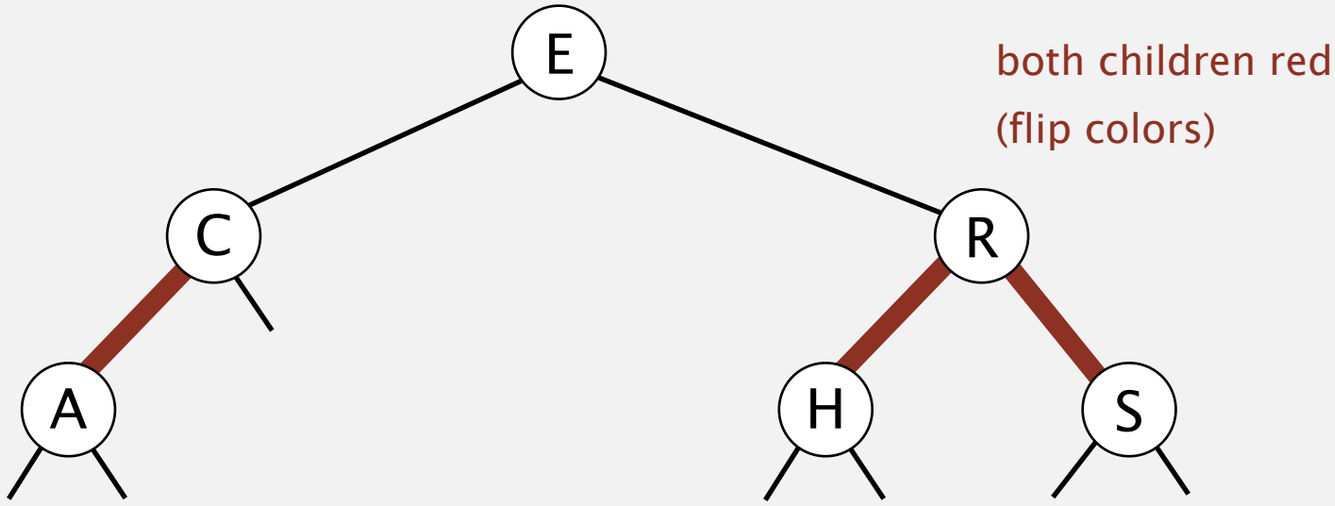
insert H



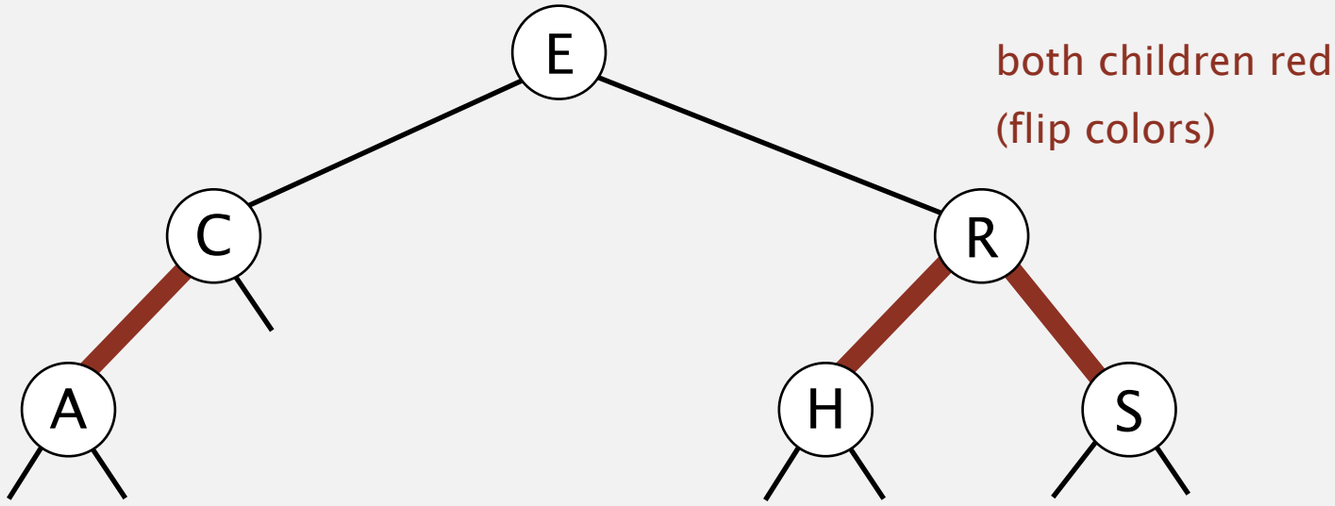
# Red-black BST insertion



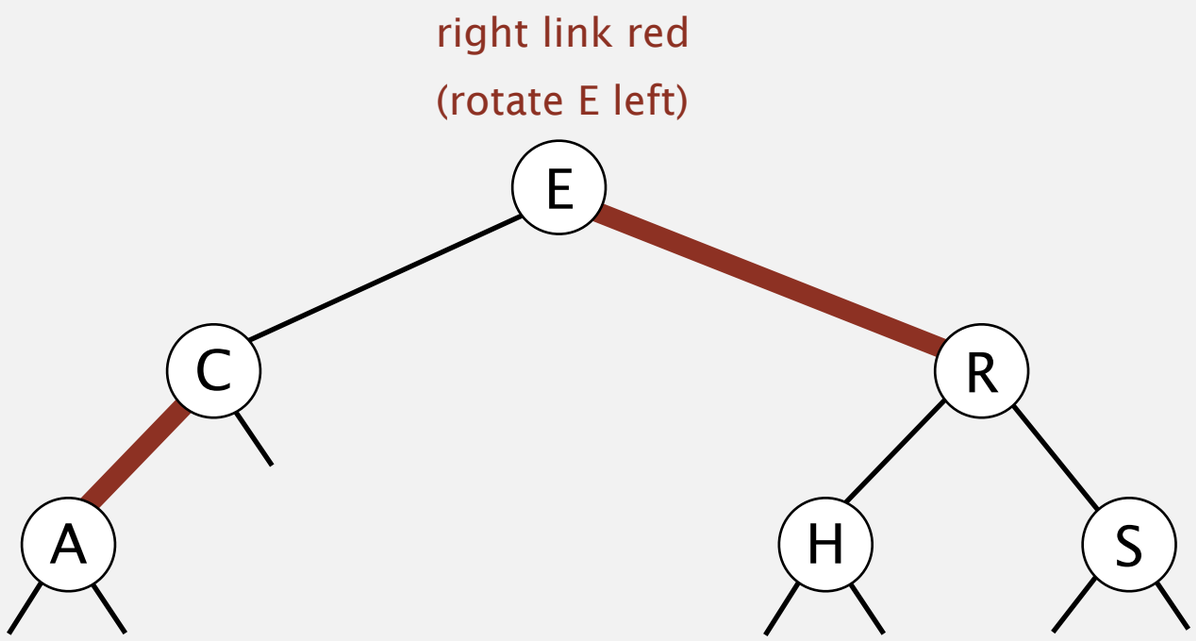
# Red-black BST insertion



# Red-black BST insertion

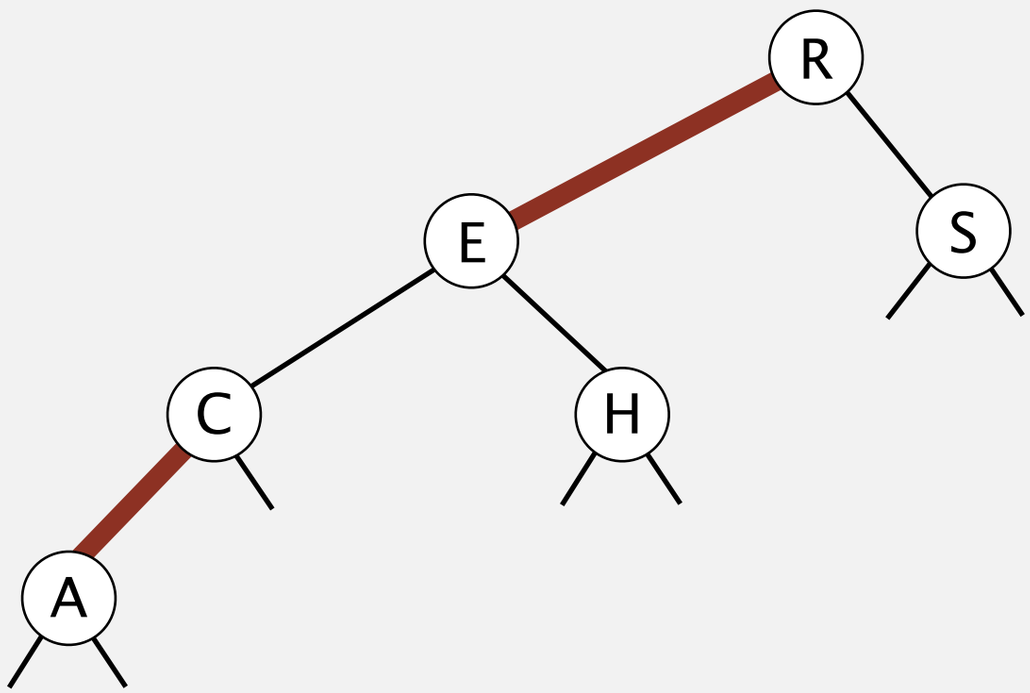


# Red-black BST insertion



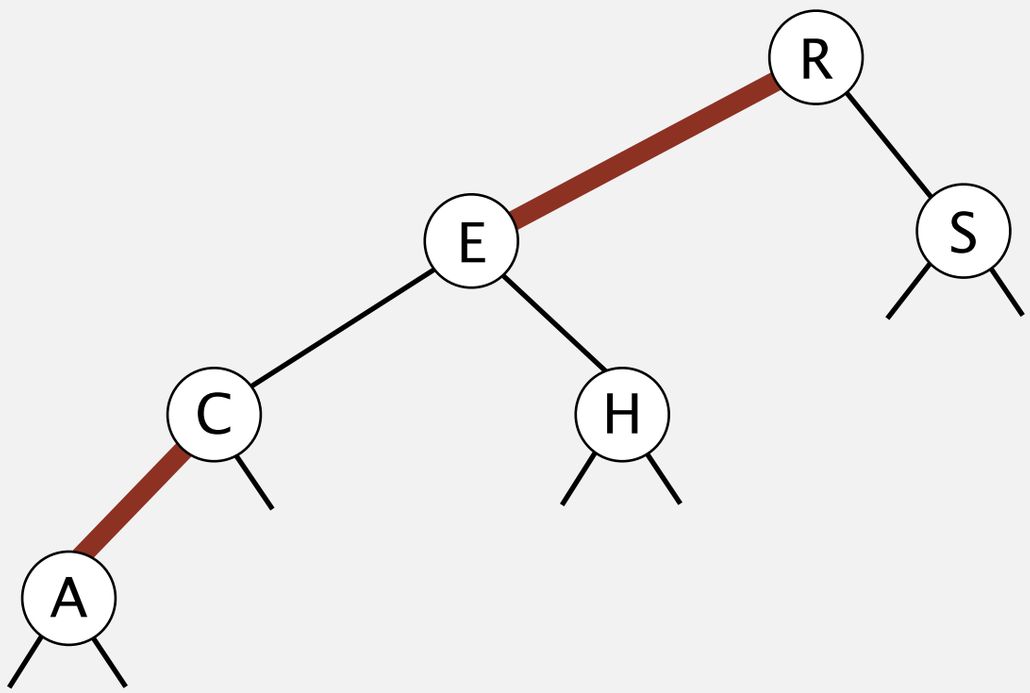
# Red-black BST insertion

red-black BST



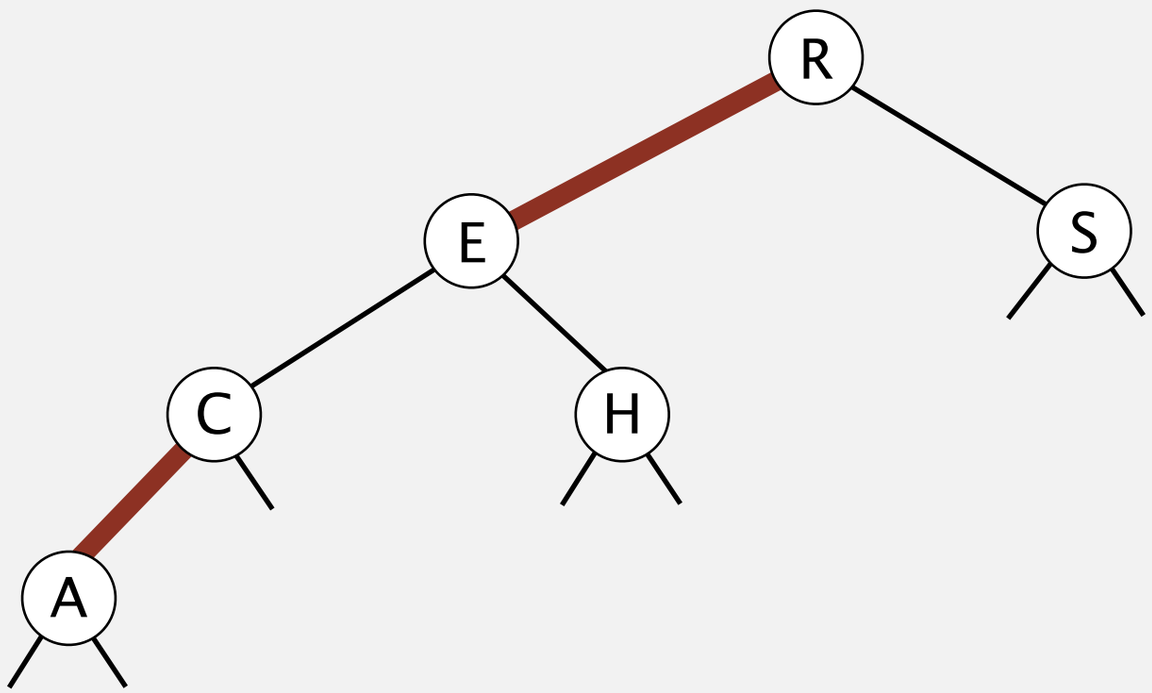
# Red-black BST insertion

red-black BST



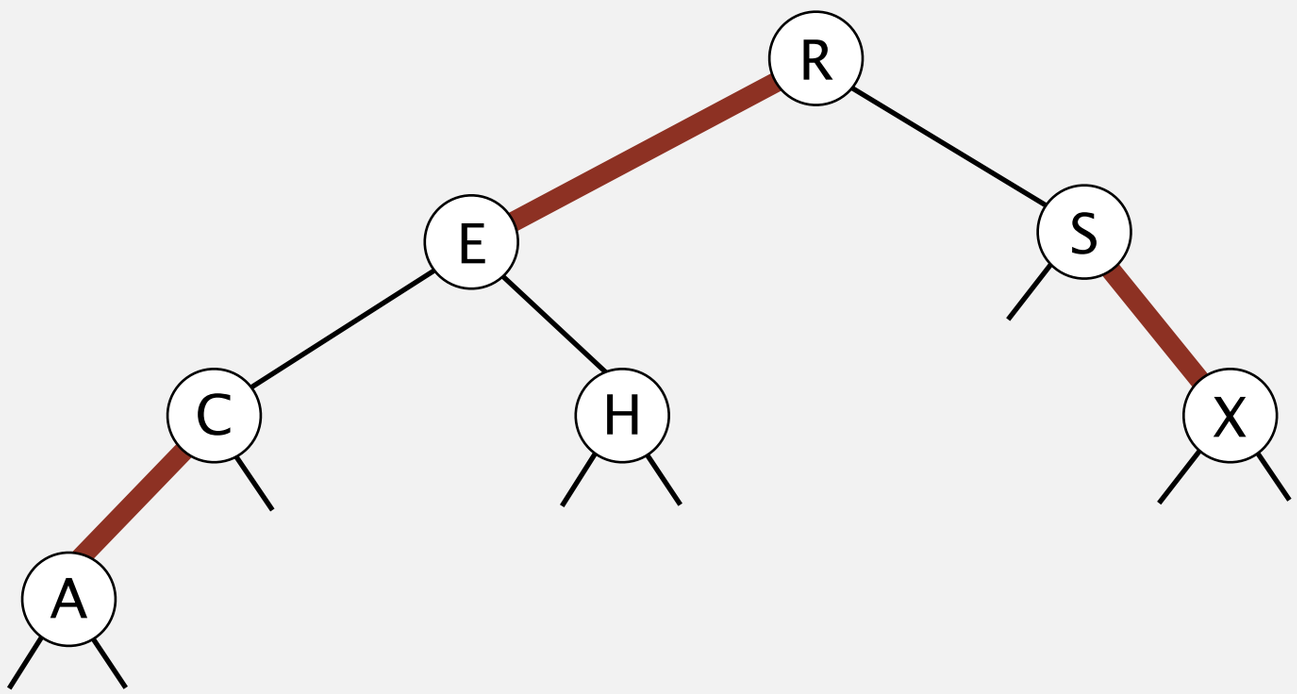
# Red-black BST insertion

red-black BST



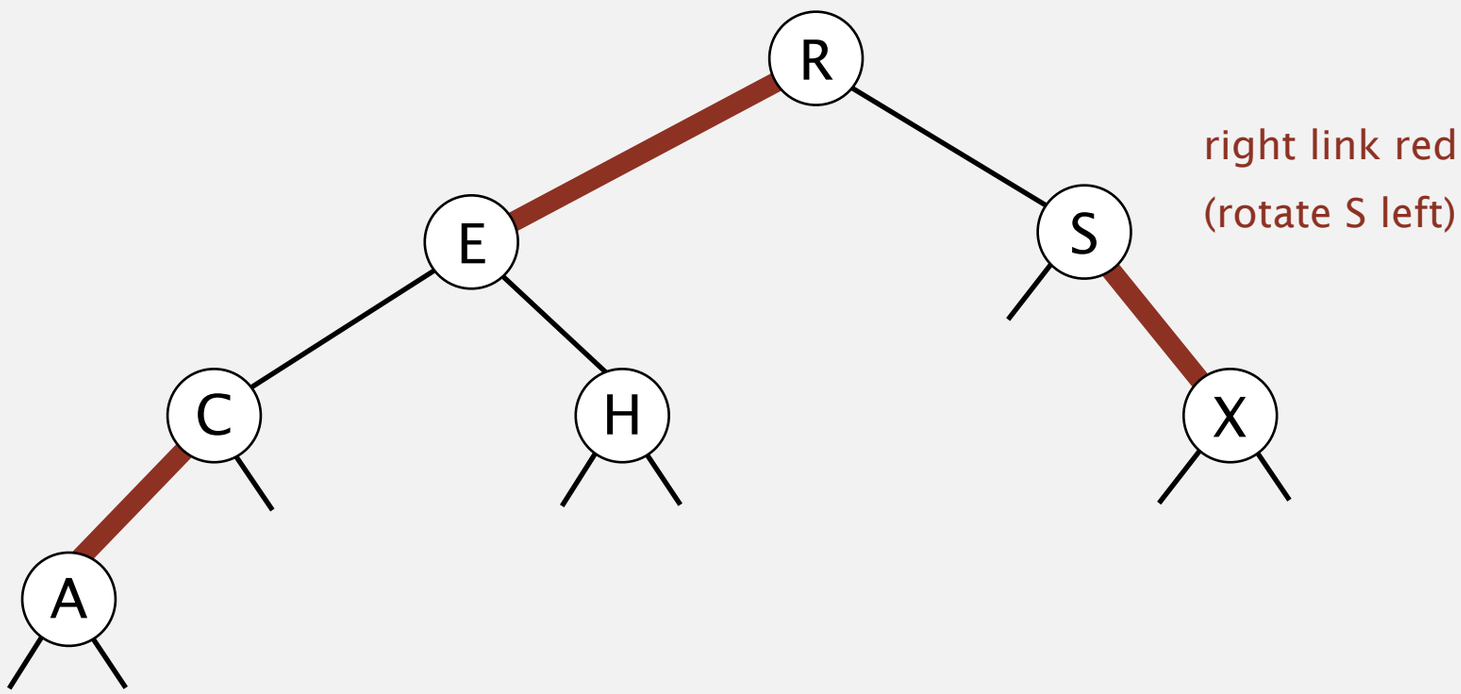
# Red-black BST insertion

insert X



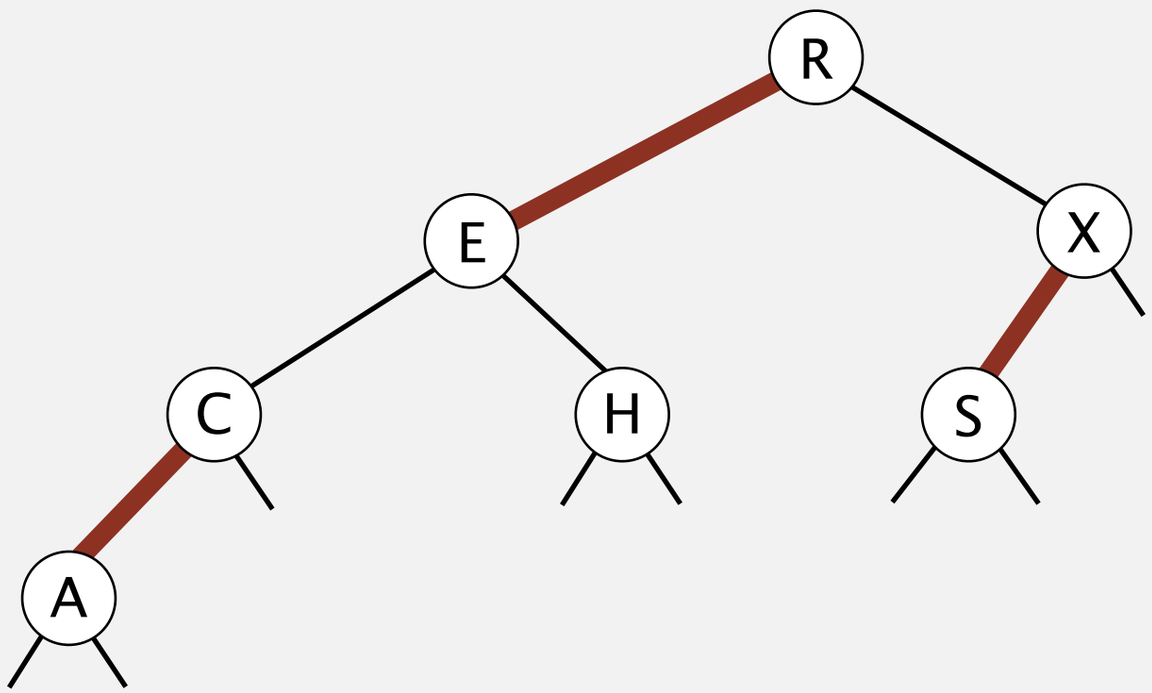
# Red-black BST insertion

insert X



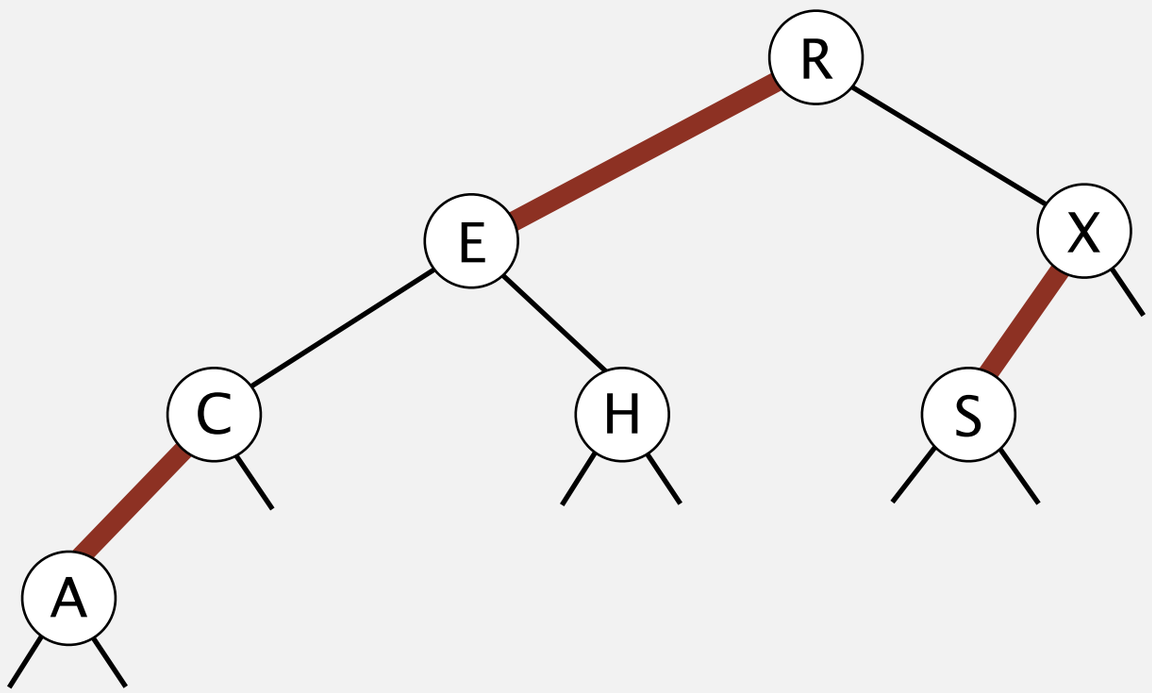
# Red-black BST insertion

red-black BST



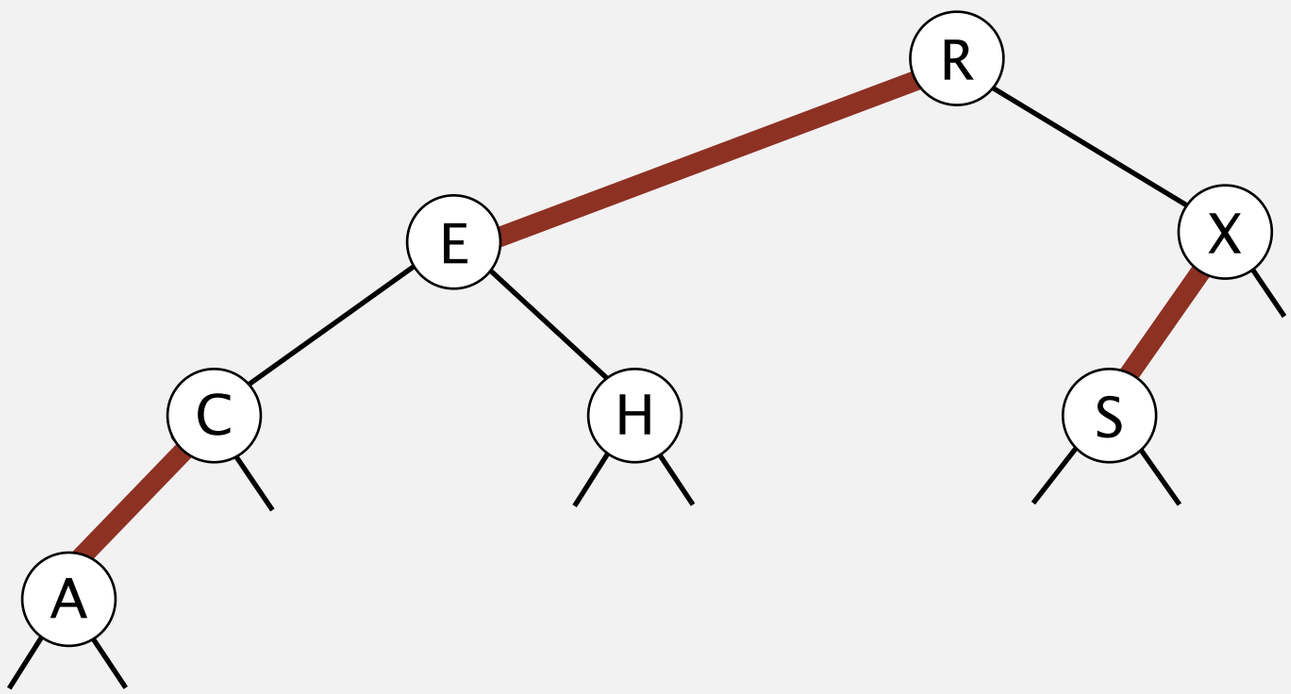
# Red-black BST insertion

red-black BST



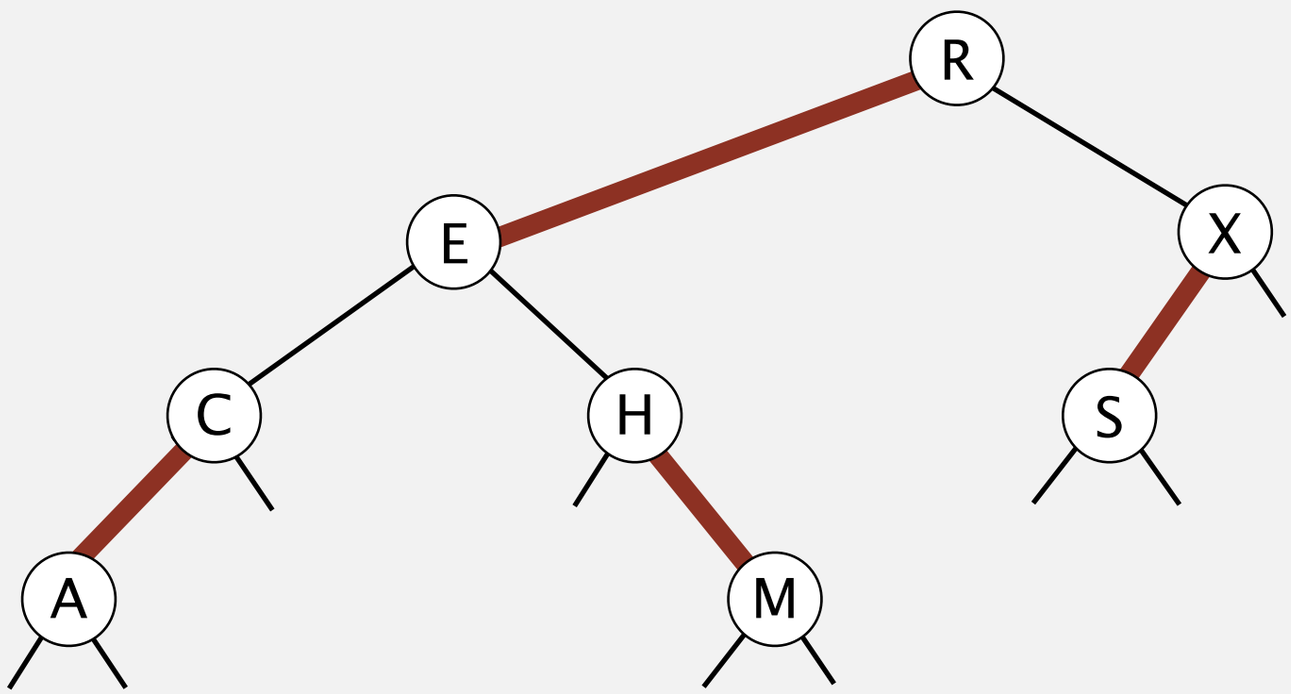
# Red-black BST insertion

red-black BST



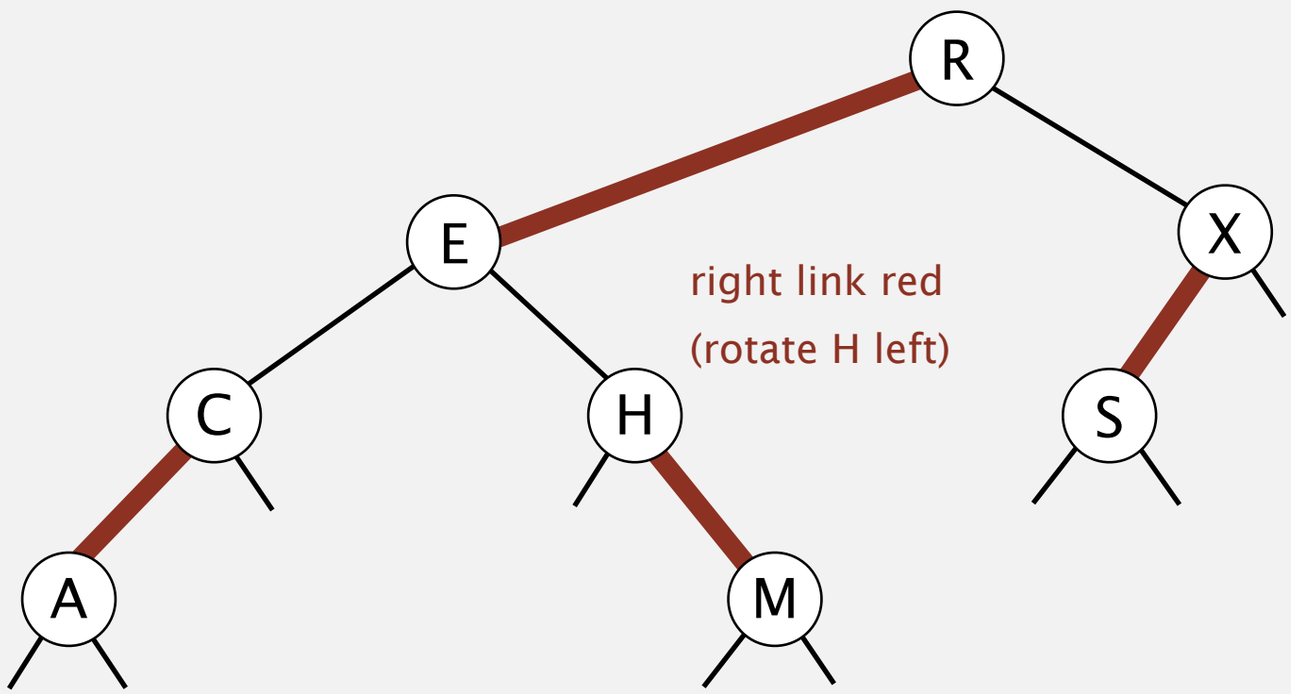
# Red-black BST insertion

insert M



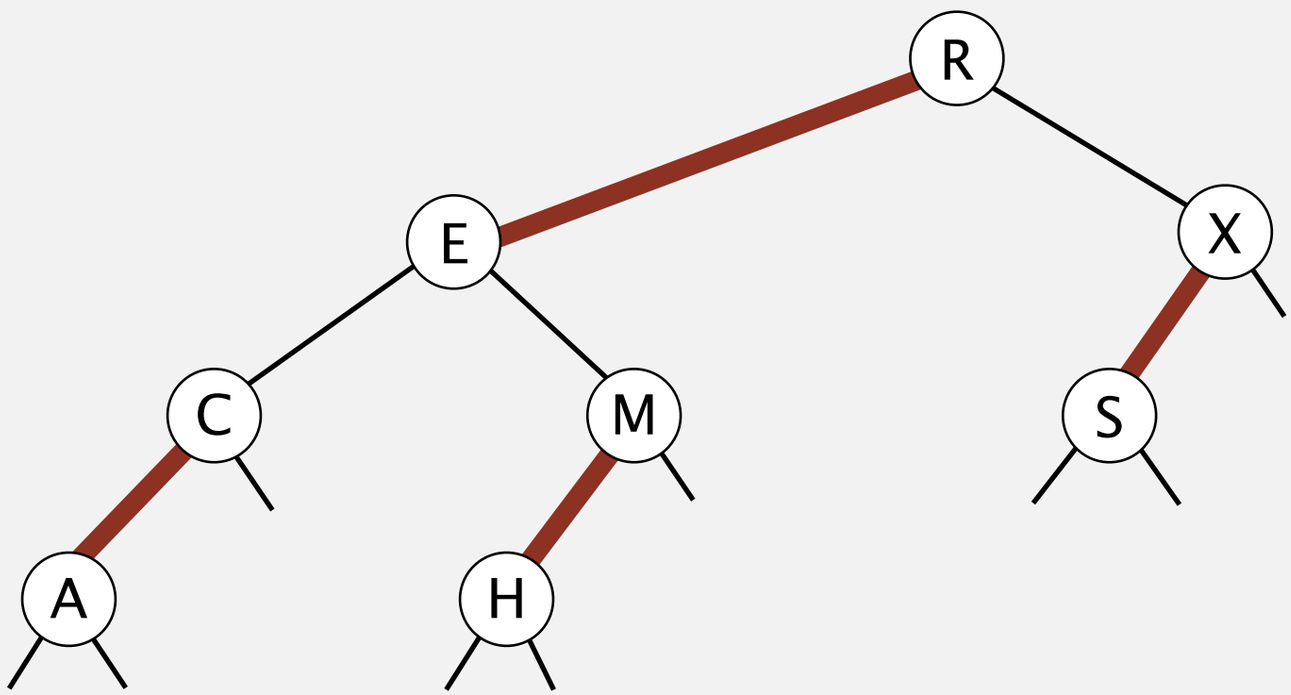
# Red-black BST insertion

insert M



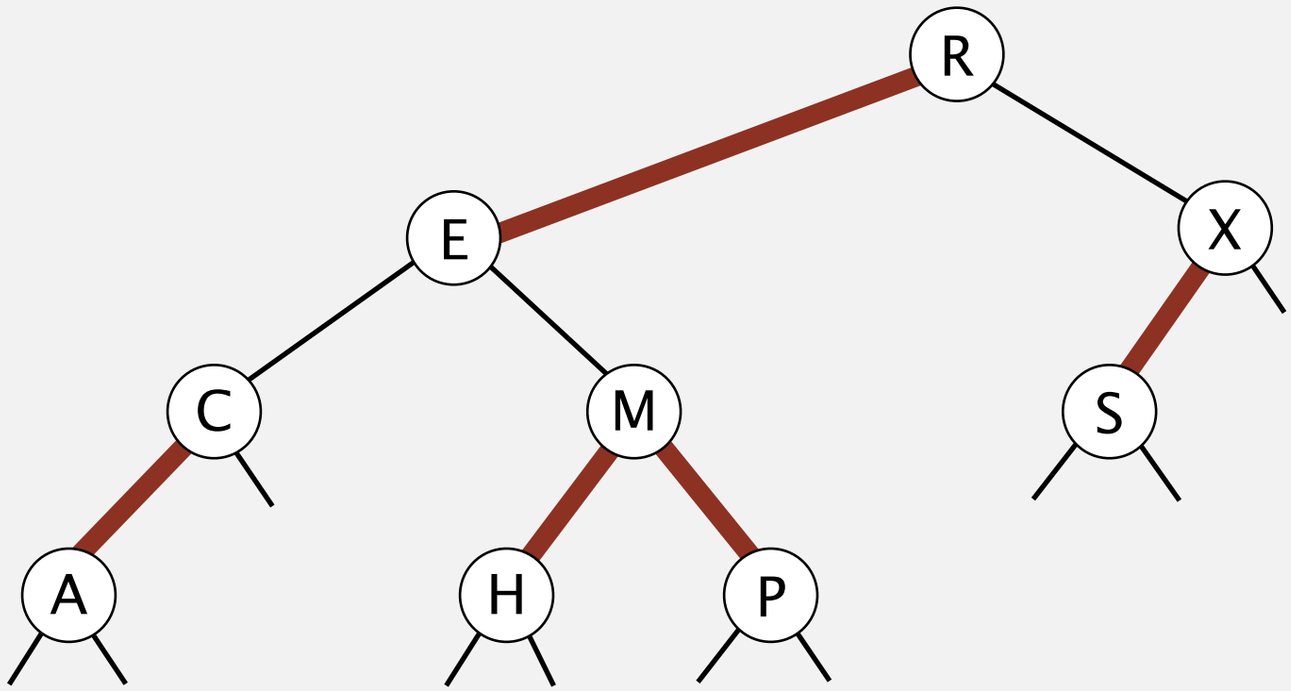
# Red-black BST insertion

red-black BST



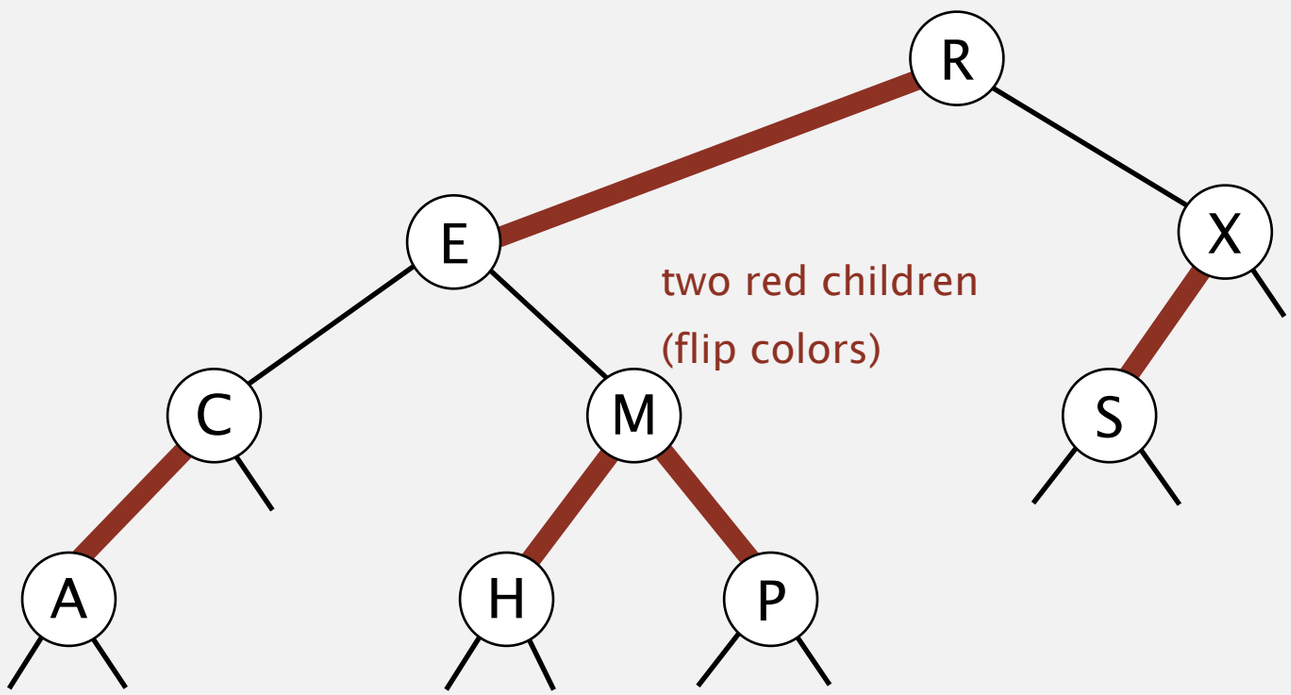
# Red-black BST insertion

insert P



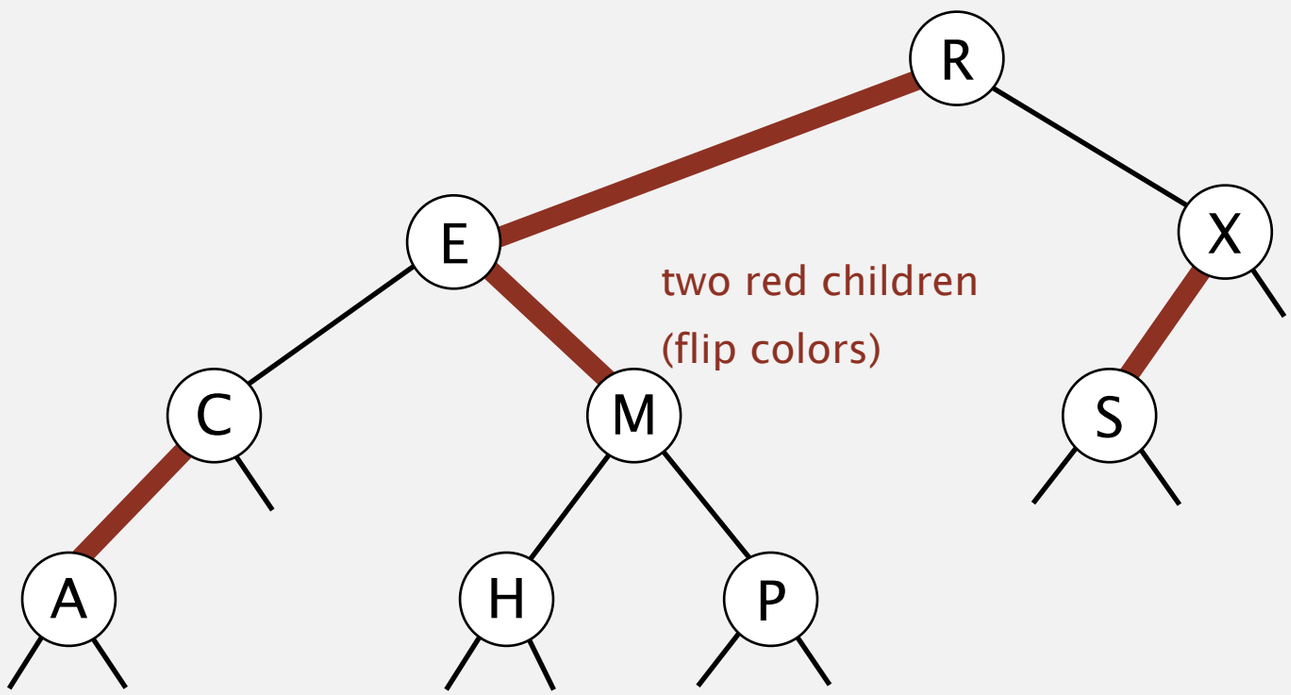
# Red-black BST insertion

insert P

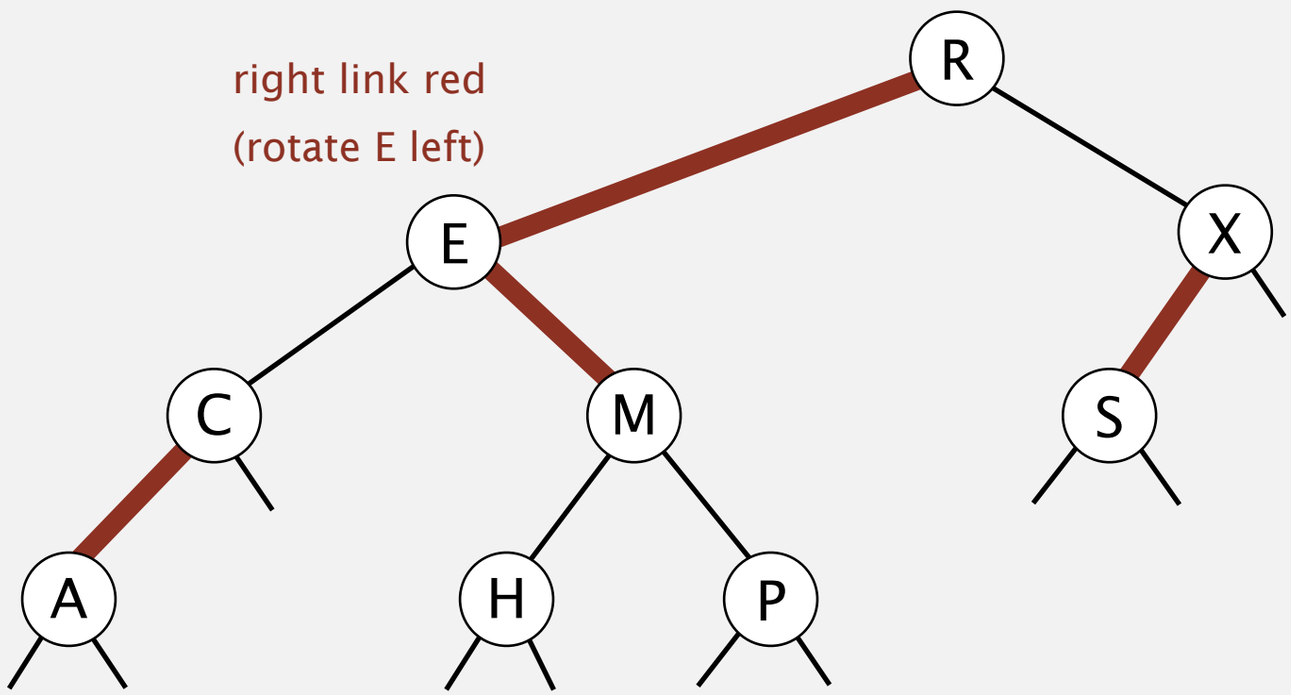


# Red-black BST insertion

insert P

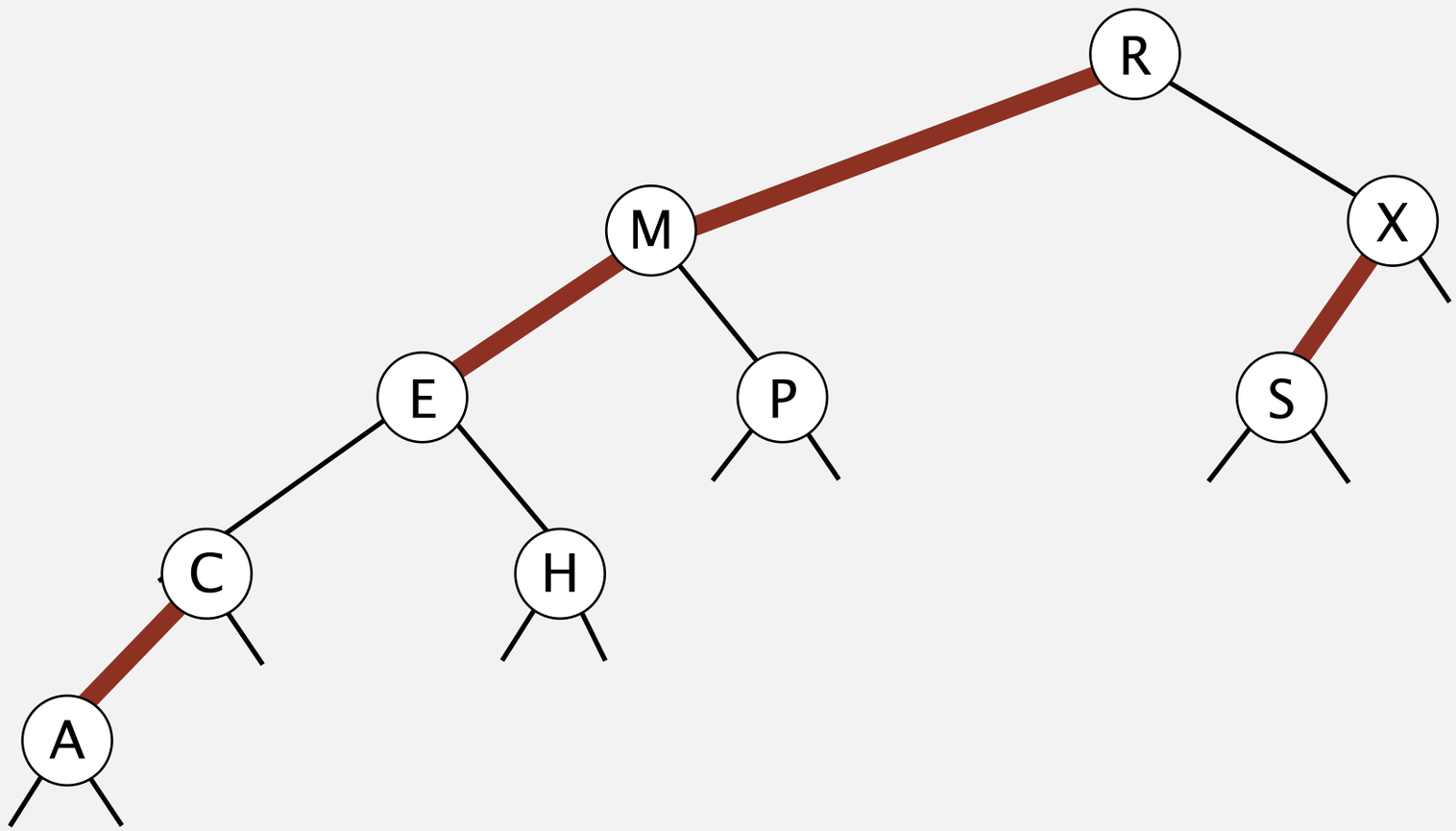


# Red-black BST insertion

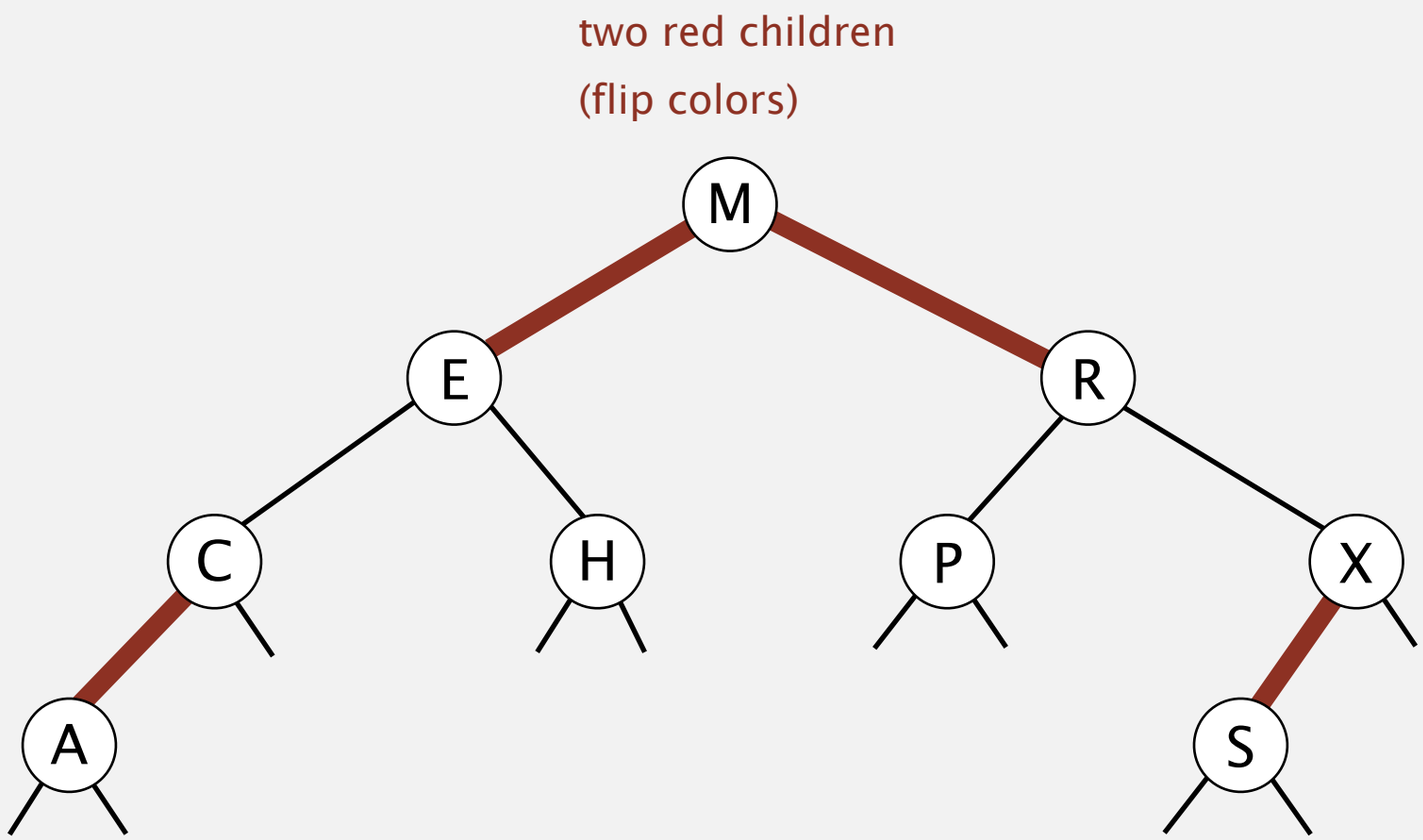


# Red-black BST insertion

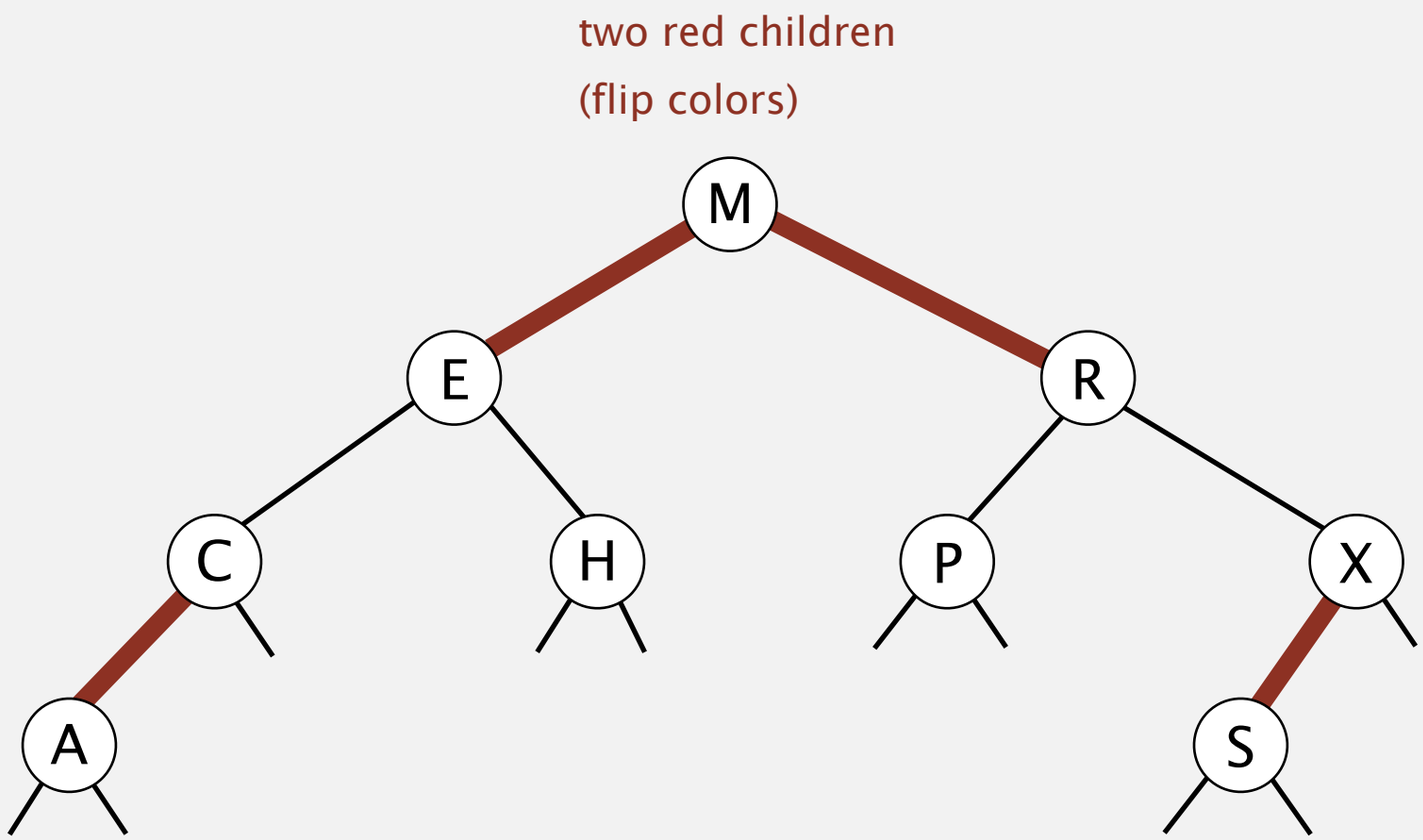
two left reds in a row
(rotate R right)



# Red-black BST insertion

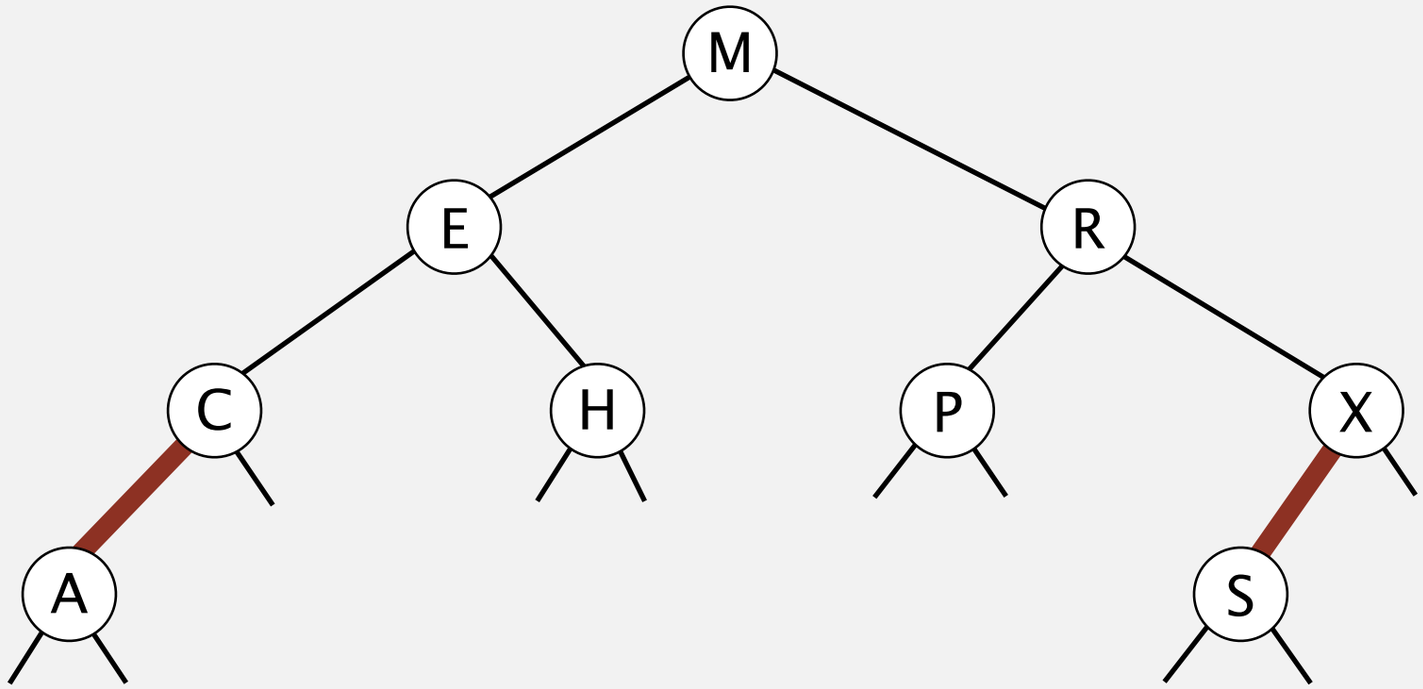


# Red-black BST insertion



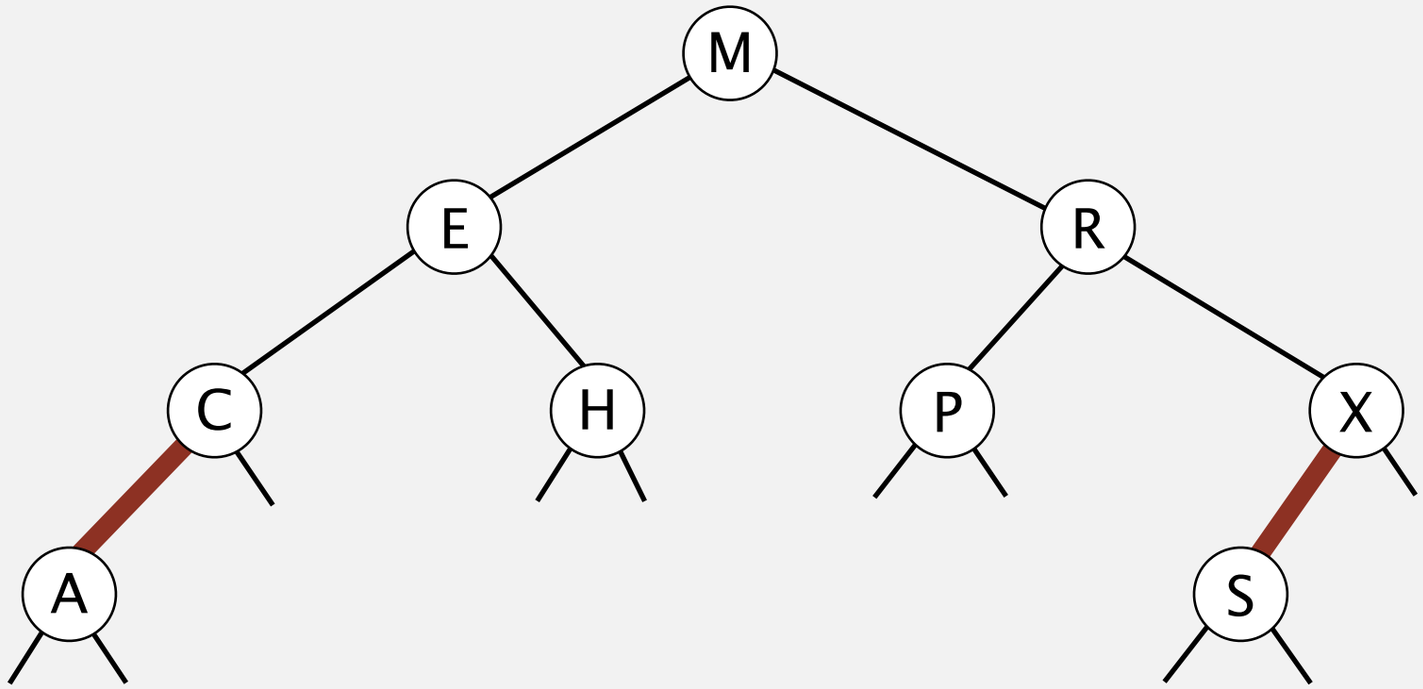
# Red-black BST insertion

red-black BST



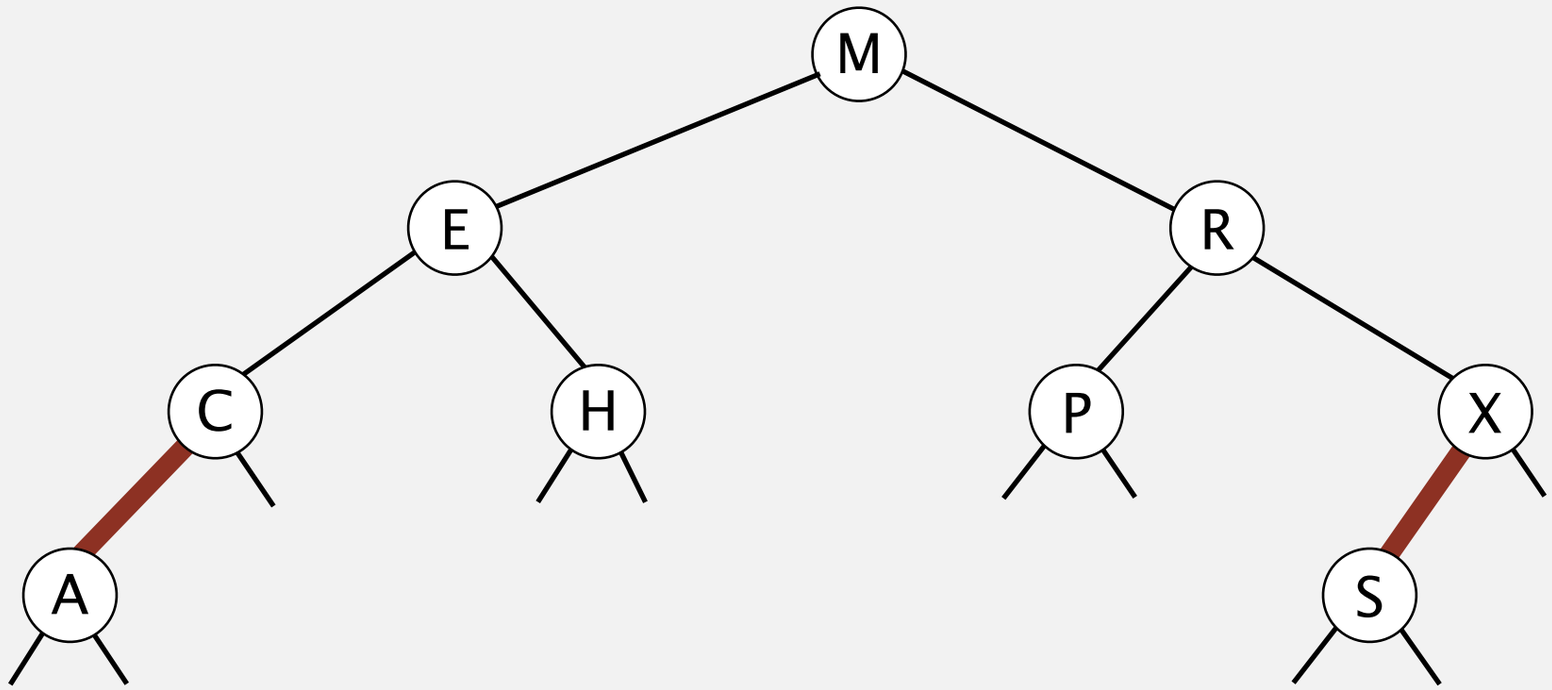
# Red-black BST insertion

red-black BST



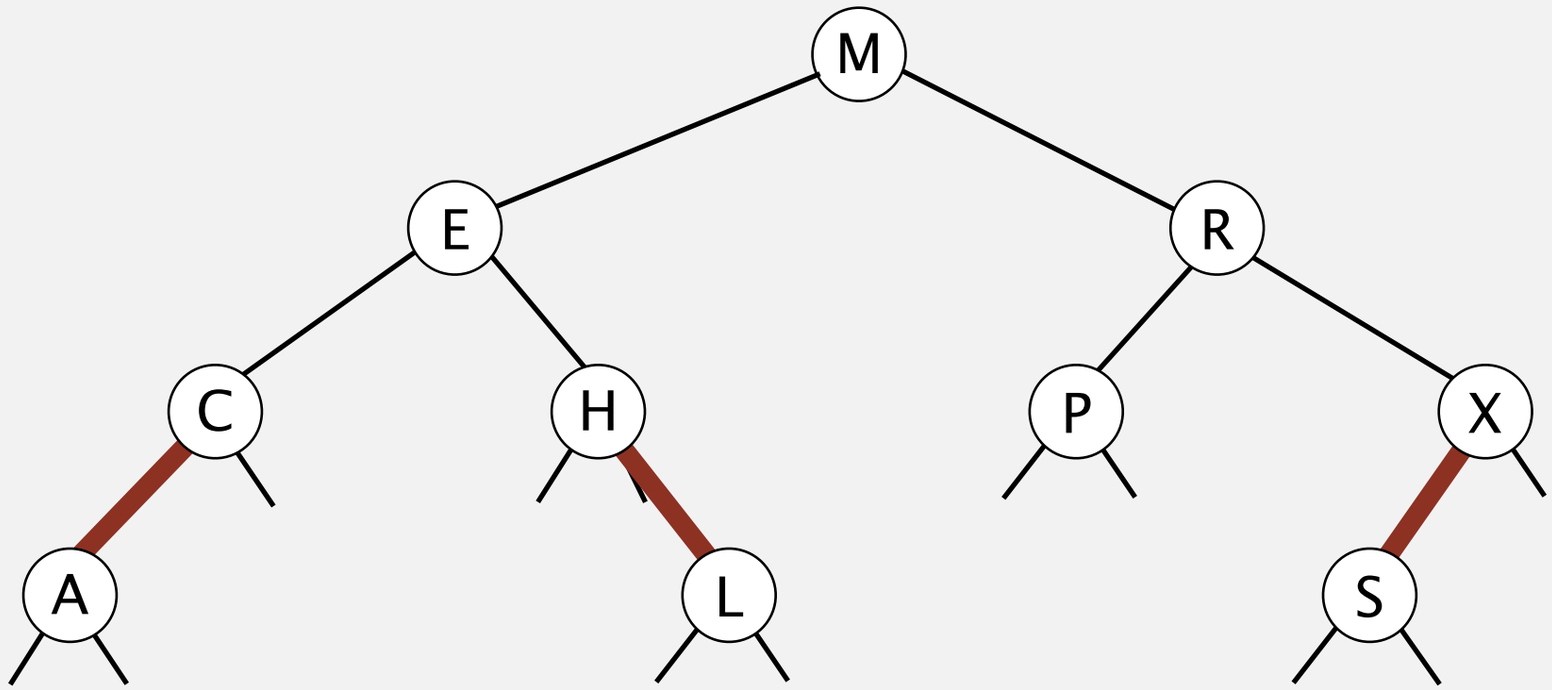
# Red-black BST insertion

red-black BST



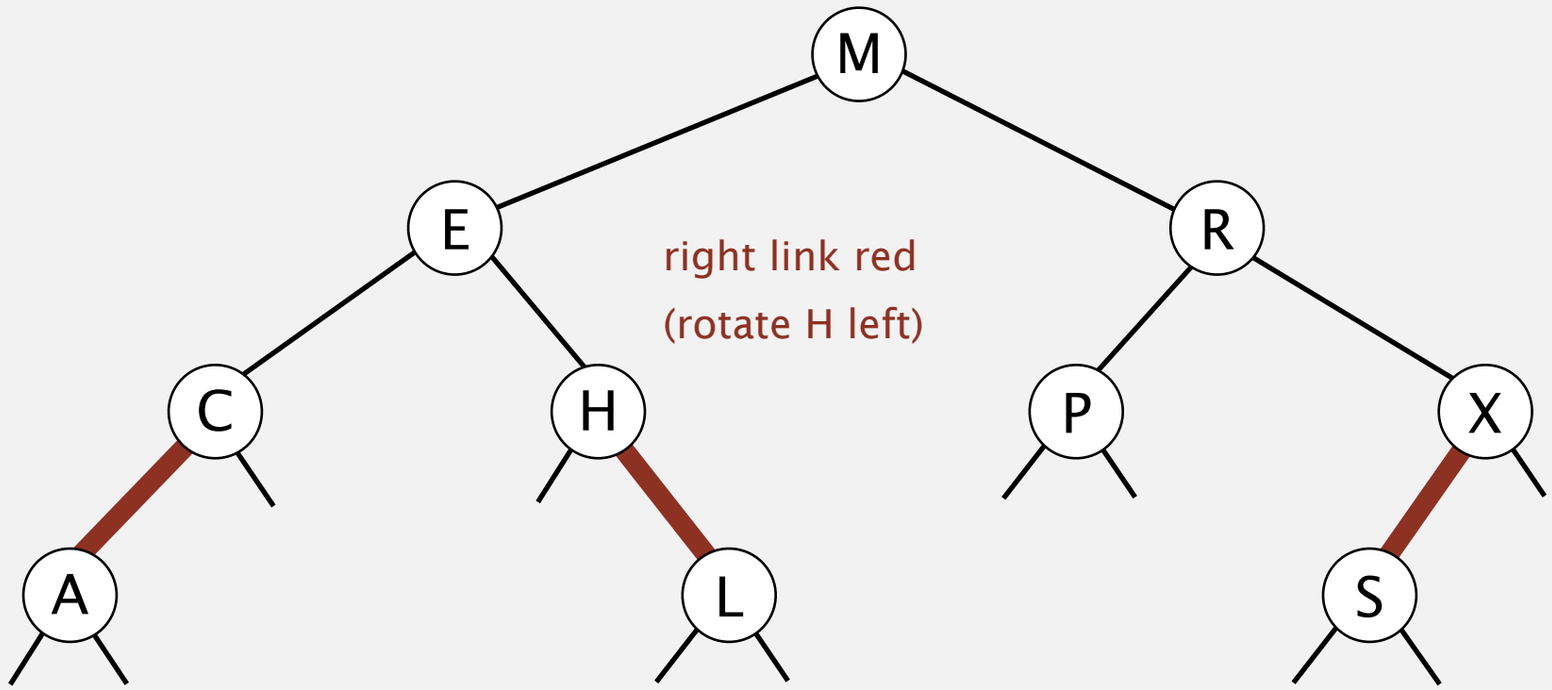
# Red-black BST insertion

insert L



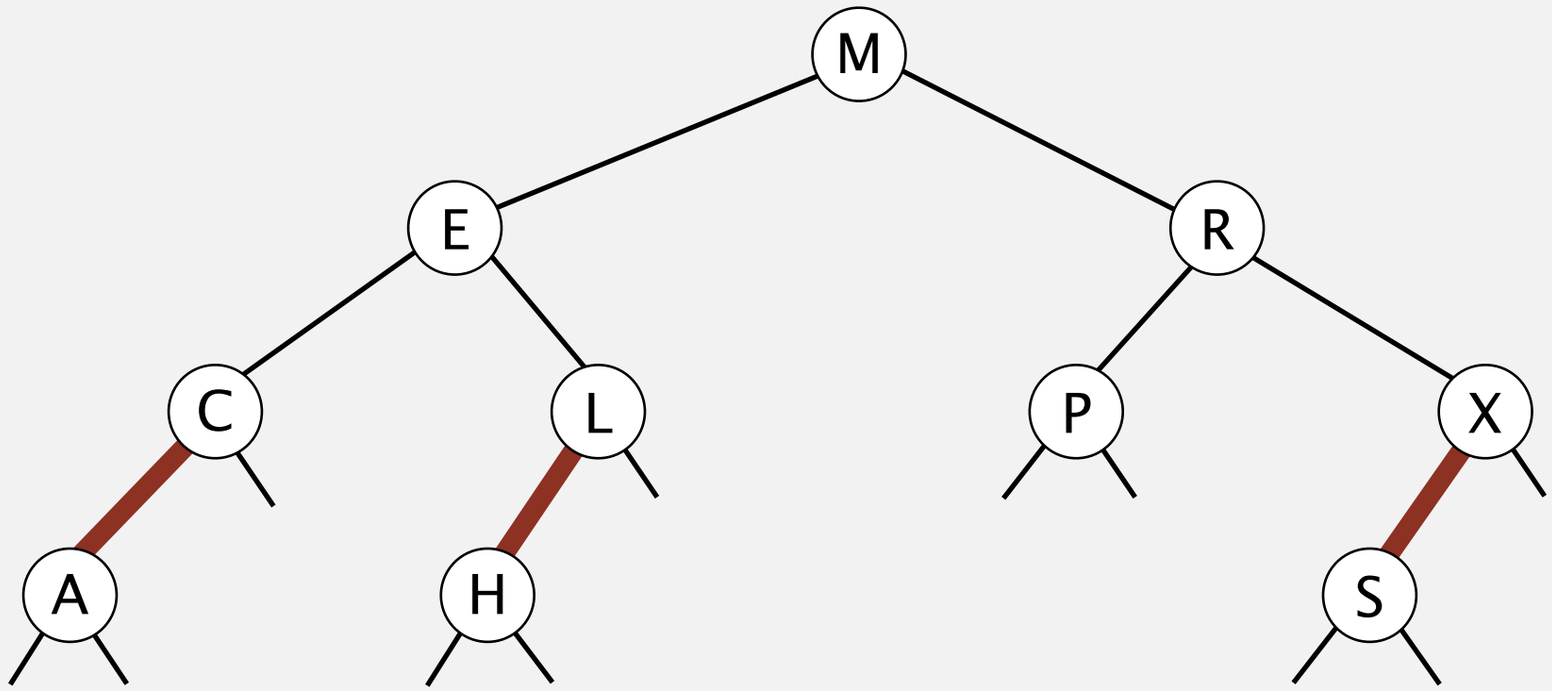
# Red-black BST insertion

insert L



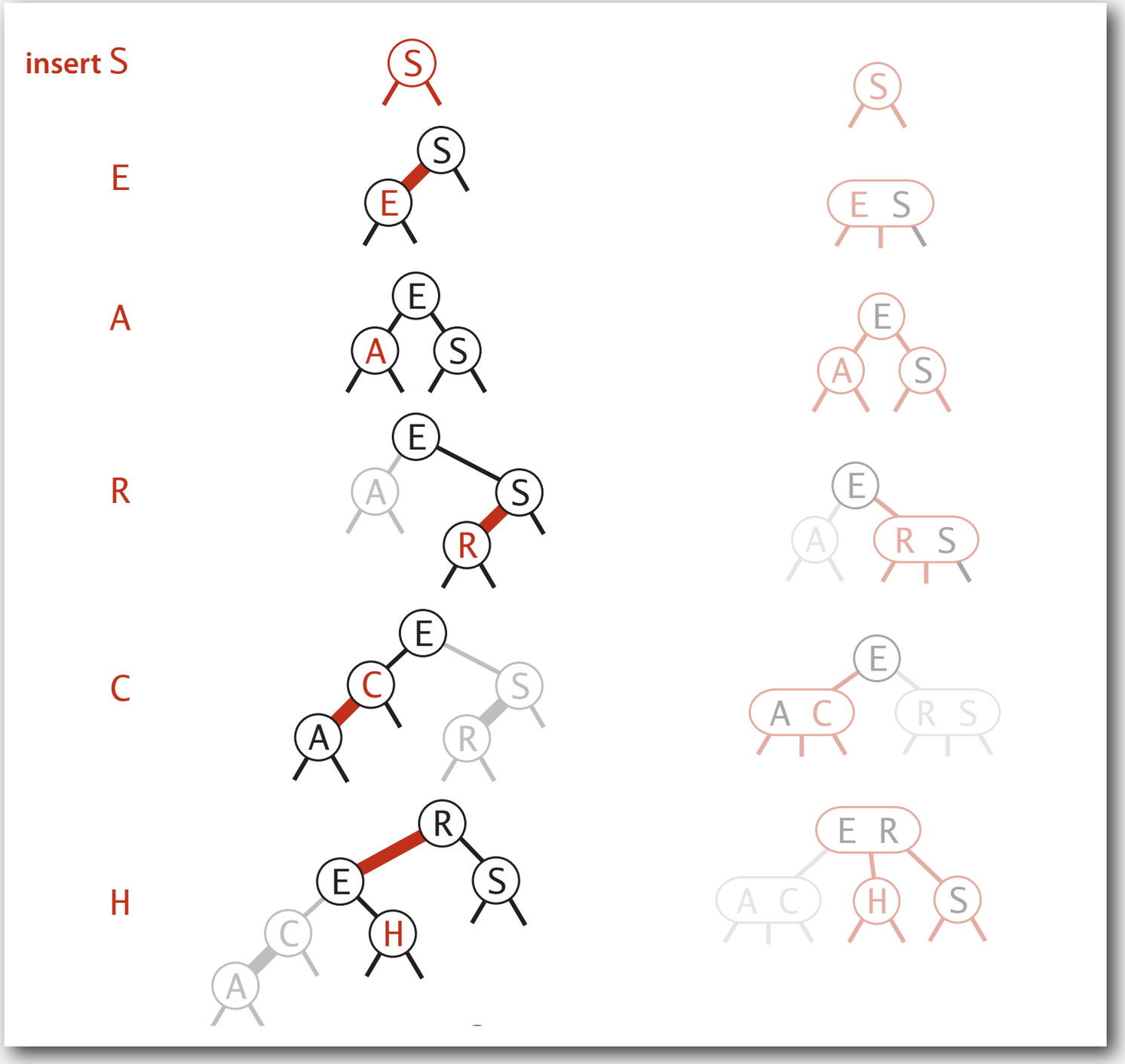
# Red-black BST insertion

red-black BST



# LLRB tree insertion trace

Standard indexing client.

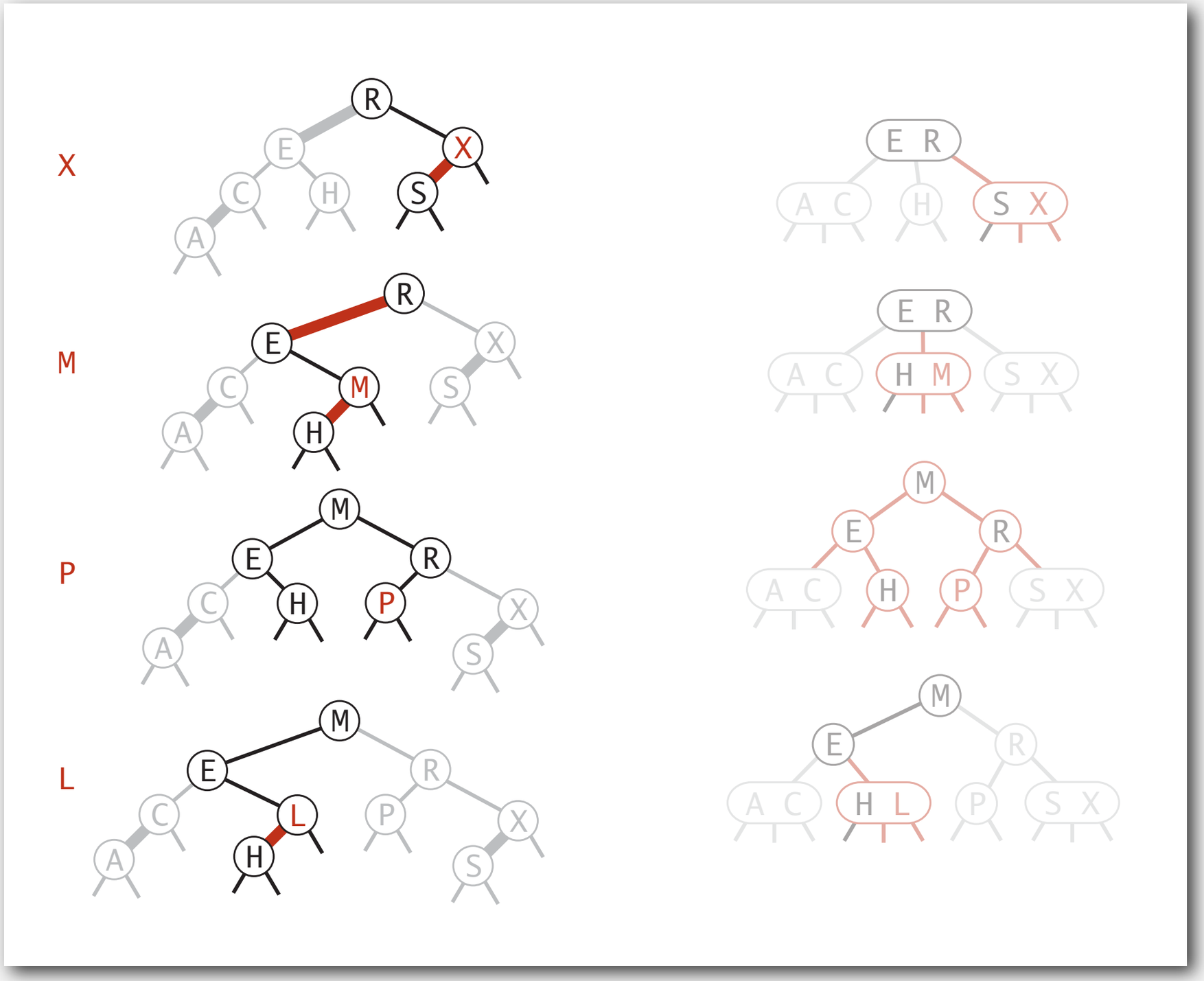



red-black BST

corresponding 2-3 tree

# LLRB tree insertion trace

Standard indexing client (continued).



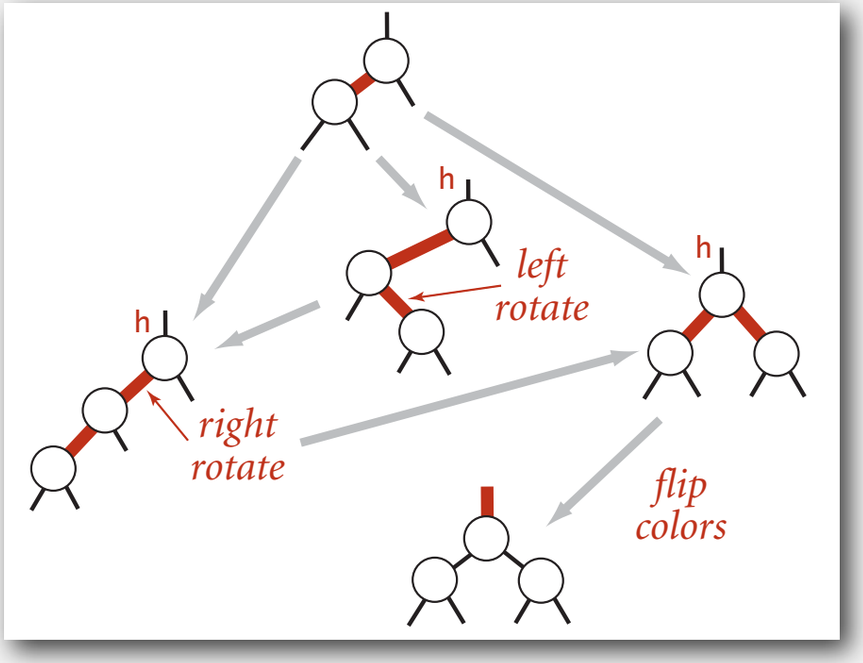

red-black BST

corresponding 2-3 tree

# Insertion in a LLRB tree: Java implementation

Same code for both cases.

- Right child red, left child black: rotate left.
- Left child, left-left grandchild red: rotate right.
- Both children red: flip colors.




```
private Node put(Node h, Key key, Value val)
{
   if (h == null) return new Node(key, val, RED);
   int cmp = key.compareTo(h.key);
   if      (cmp  < 0) h.left  = put(h.left,  key, val);
   else if (cmp  > 0) h.right = put(h.right, key, val);
   else if (cmp == 0) h.val = val;

   if (isRed(h.right) && !isRed(h.left))     h = rotateLeft(h);
   if (isRed(h.left)  && isRed(h.left.left)) h = rotateRight(h);
   if (isRed(h.left)  && isRed(h.right))     flipColors(h);

   return h;
}
```

insert at bottom (and color red)

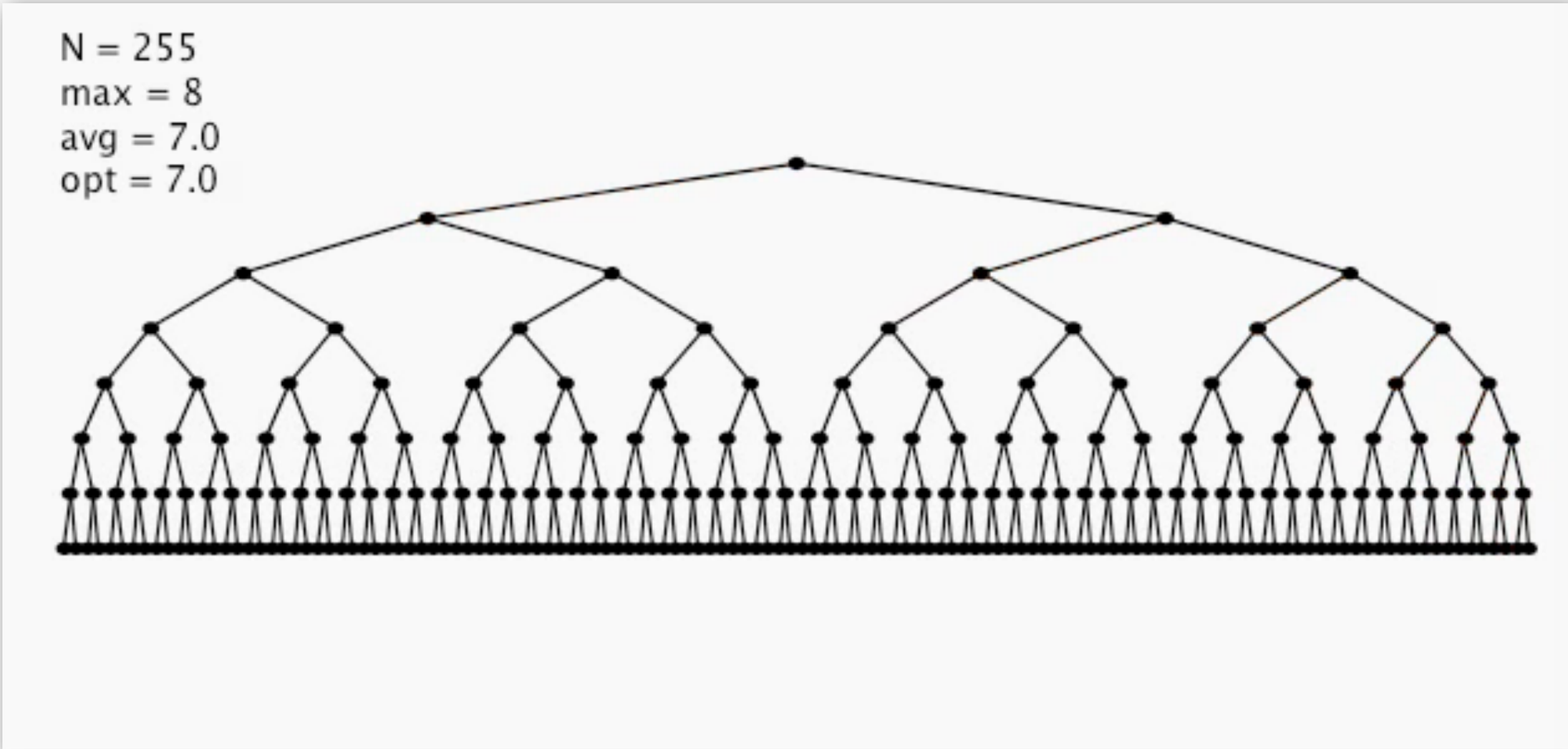
lean left

balance 4-node

split 4-node

only a few extra lines of code provides near-perfect balance

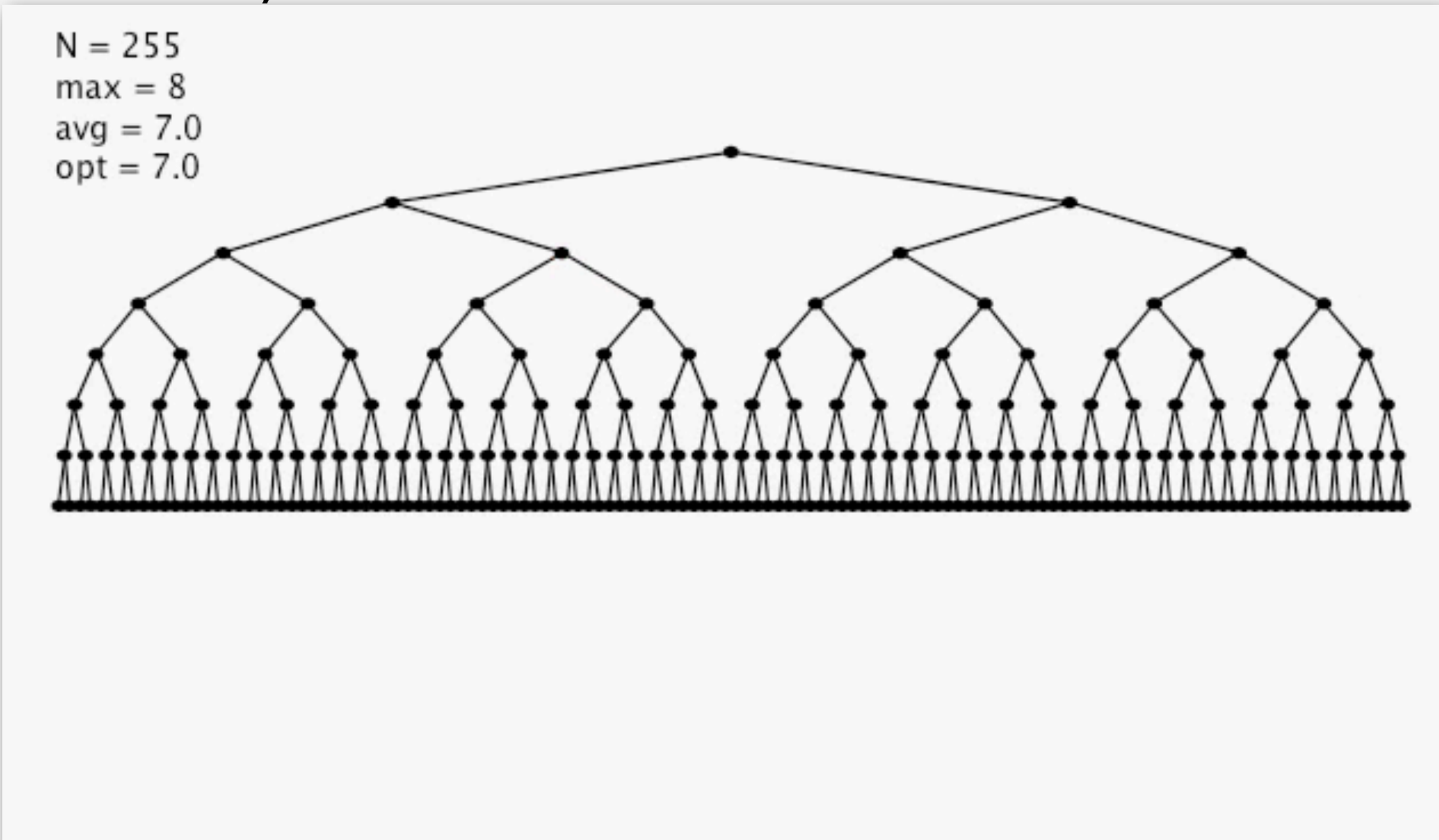
# Insertion in a LLRB tree: visualization




**255 insertions in ascending order**

# Insertion in a LLRB tree: visualization

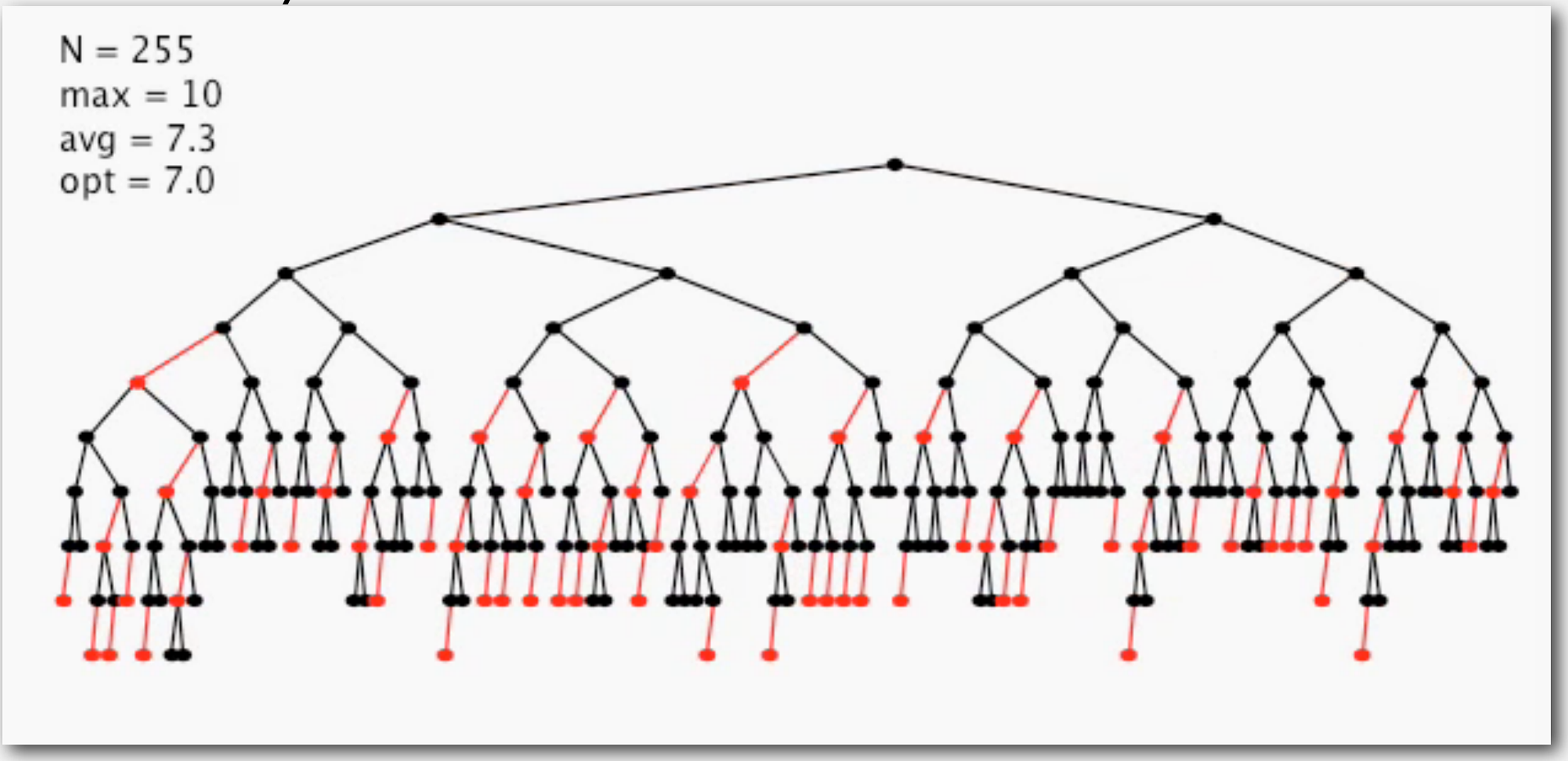
Remark. Only a few extra lines of code to standard BST insert.




**255 insertions in descending order**

# Insertion in a LLRB tree: visualization

Remark. Only a few extra lines of code to standard BST insert.



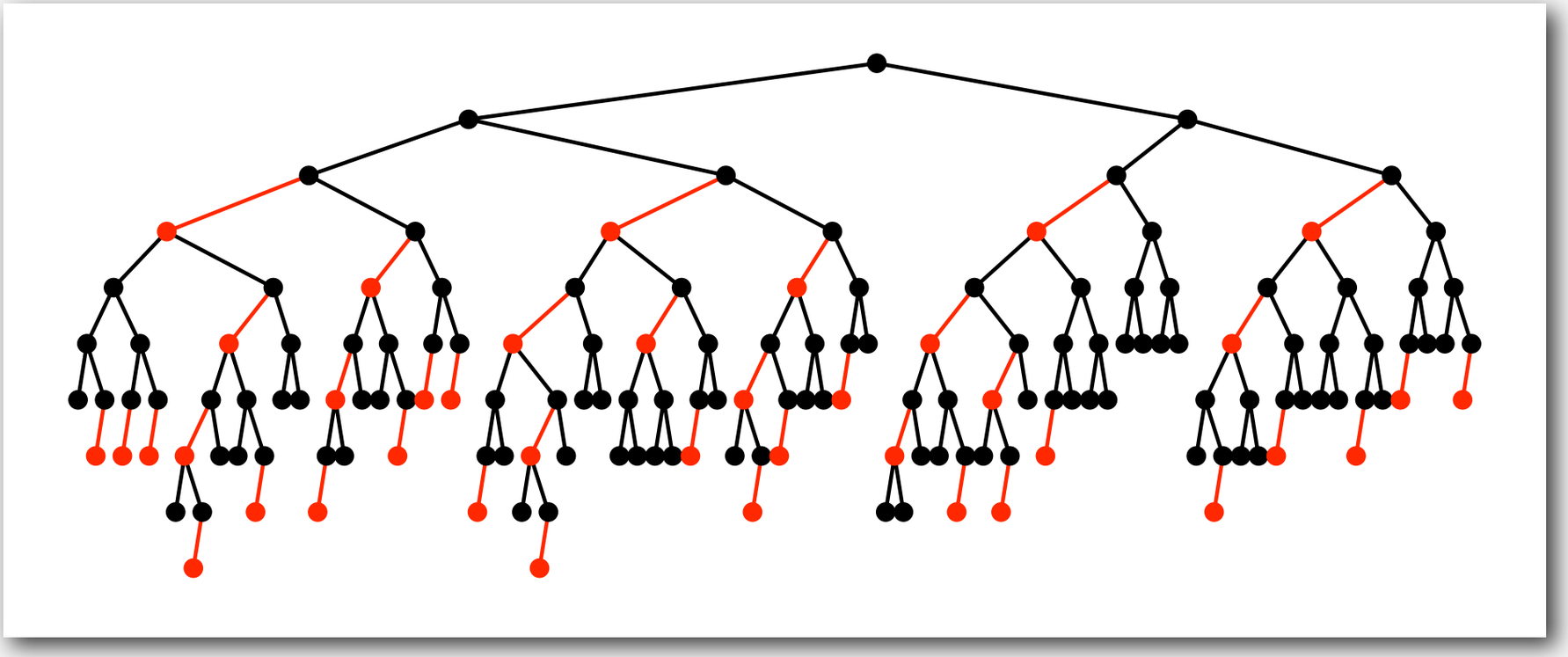

**255 random insertions**

# Balance in LLRB trees

Proposition. Height of tree is $\leq 2 \lg N$ in the worst case.

Pf.

- Every path from root to null link has same number of black links.
- Never two red links in-a-row.



Property. Height of tree is ~ $1.00 \lg N$ in typical applications.

# ST implementations: summary

| implementation | worst-case cost (after N inserts) | | | average case (after N random inserts) | | | ordered iteration? | key interface |
|---|---|---|---|---|---|---|---|---|
| | search | insert | delete | search hit | insert | delete | | |
| sequential search (unordered list) | N | N | N | N/2 | N | N/2 | no | `equals()` |
| binary search (ordered array) | lg N | N | N | lg N | N/2 | N/2 | yes | `compareTo()` |
| BST | N | N | N | 1.39 lg N | 1.39 lg N | ? | yes | `compareTo()` |
| 2-3 tree | c lg N | c lg N | c lg N | c lg N | c lg N | c lg N | yes | `compareTo()` |
| red-black BST | 2 lg N | 2 lg N | 2 lg N | 1.00 lg N * | 1.00 lg N * | 1.00 lg N * | yes | `compareTo()` |

* exact value of coefficient unknown but extremely close to 1

# War story: why red-black?

Xerox PARC innovations. [ 1970s ]

- Alto.
- GUI.
- Ethernet.
- Smalltalk.
- InterPress.
- Laser printing.
- Bitmapped display.
- WYSIWYG text editor.
- ...

Xerox Alto

A DICHROMATIC FRAMEWORK FOR BALANCED TREES

Leo J. Guibas
*Xerox Palo Alto Research Center,*
Palo Alto, California, and
*Carnegie-Mellon University*

and

Robert Sedgewick*
Program in Computer Science
*Brown University*
Providence, R. I.

ABSTRACT

In this paper we present a uniform framework for the implementation and study of balanced tree algorithms. We show how to imbed in this

the way down towards a leaf. As we will see, this has a number of significant advantages over the older methods. We shall examine a number of variations on a common theme and exhibit full implementations which are notable for their brevity. One implementation is examined carefully, and some properties about its

# Balanced Search Trees

- 2-3 search trees
- Red-black BSTs
- **B-trees**
- Geometric applications of BSTs

# File system model

Page. Contiguous block of data (e.g., a file or 4,096-byte chunk).

Probe. First access to a page (e.g., from disk to memory).

slow

fast

Property. Time required for a probe is much larger than time to access data within a page.

Cost model. Number of probes.

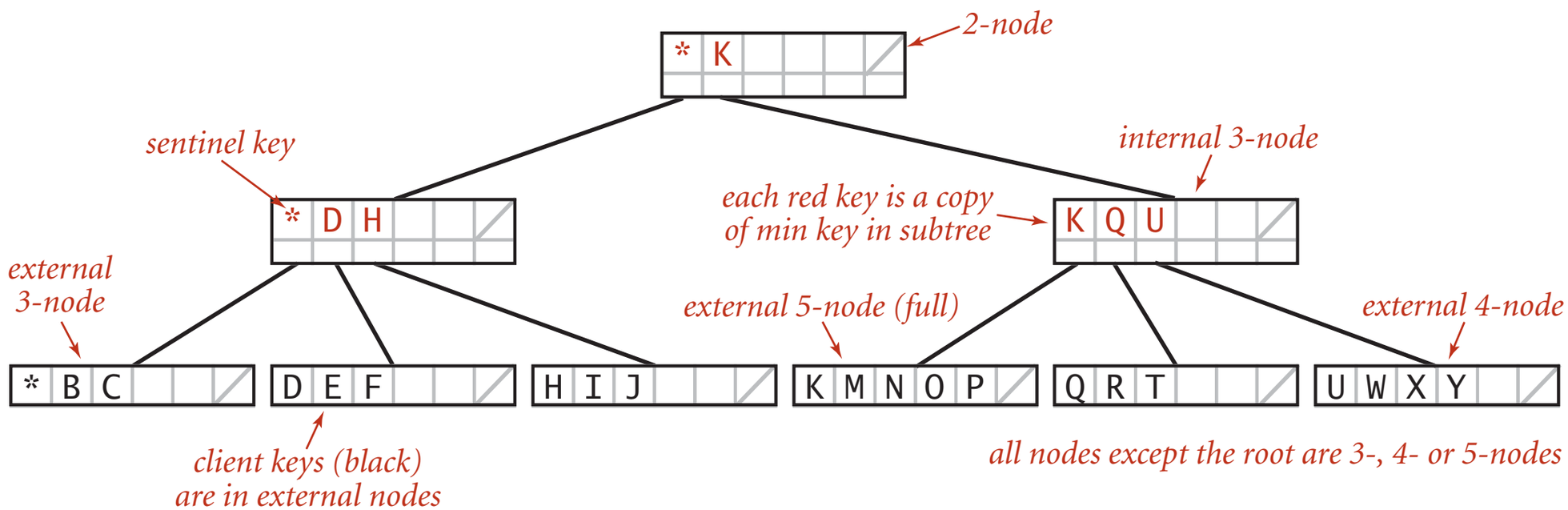
Goal. Access data using minimum number of probes.

# B-trees (Bayer-McCreight, 1972)

B-tree. Generalize 2-3 trees by allowing up to $M - 1$ key-link pairs per node.

choose M as large as possible so that M links fit in a page, e.g., M = 1024

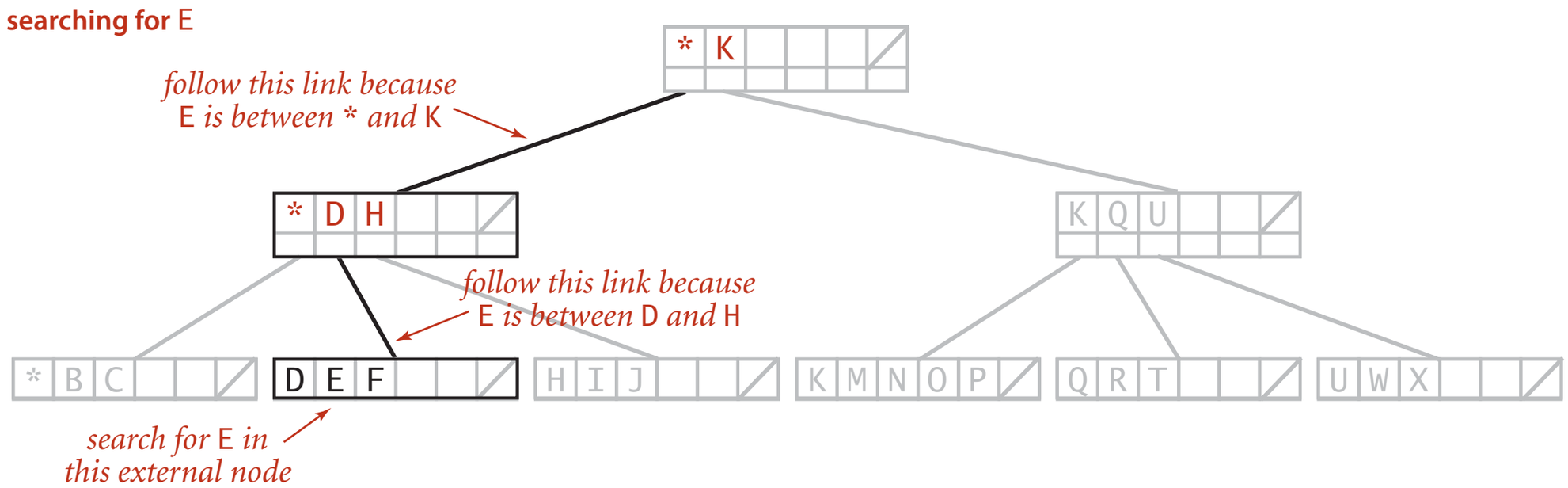
- At least 2 key-link pairs at root.
- At least $M / 2$ key-link pairs in other nodes.
- External nodes contain client keys.
- Internal nodes contain copies of keys to guide search.




Anatomy of a B-tree set (M = 6)

# Searching in a B-tree

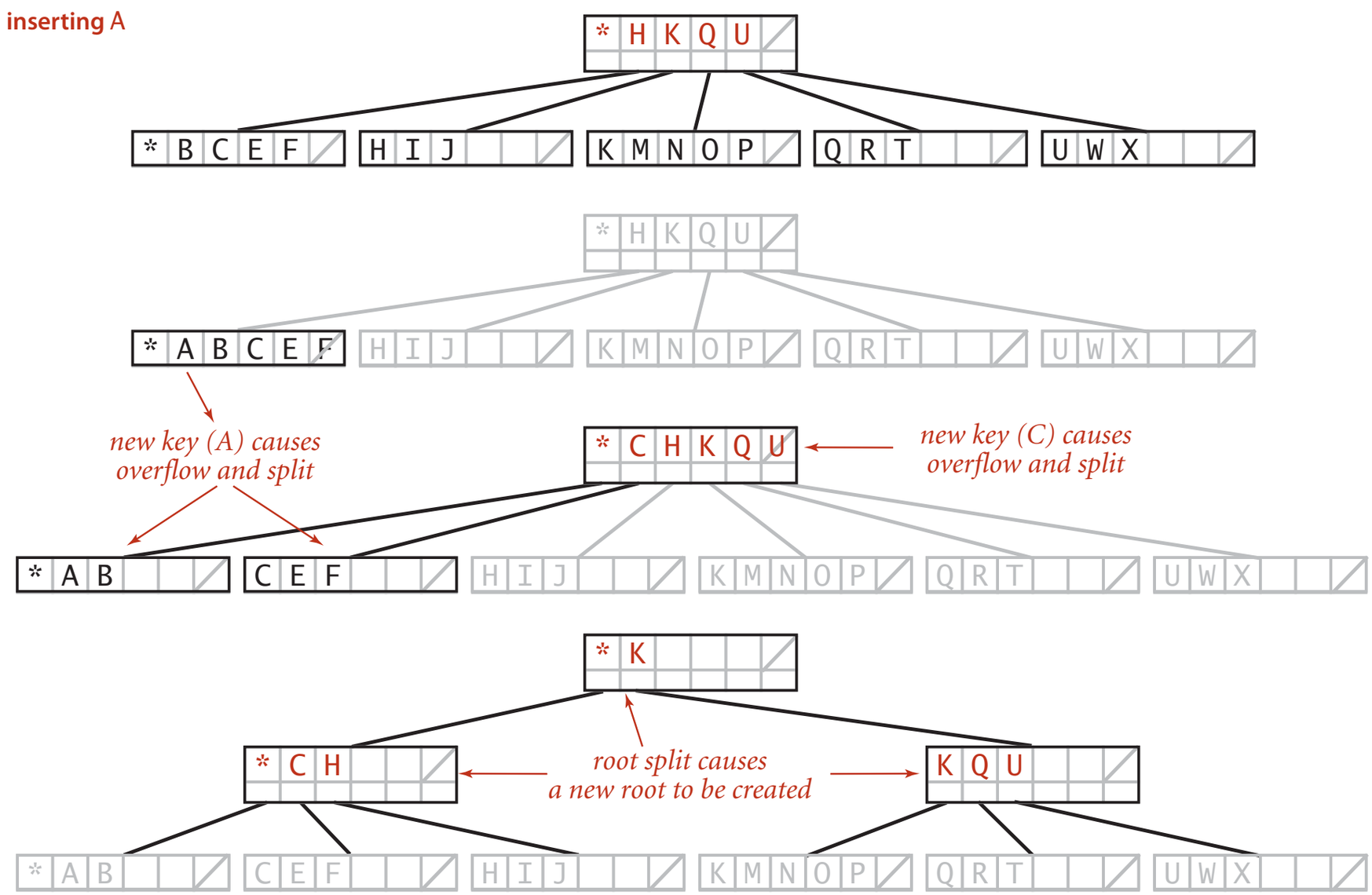
- Start at root.
- Find interval for search key and take corresponding link.
- Search terminates in external node.




Searching in a B-tree set (M = 6)

# Insertion in a B-tree

- Search for new key.
- Insert at bottom.
- Split nodes with $M$ key-link pairs on the way up the tree.




Inserting a new key into a B-tree set

# Balance in B-tree

Proposition. A search or an insertion in a B-tree of order $M$ with $N$ keys requires between $\log_{M-1} N$ and $\log_{M/2} N$ probes.

Pf. All internal nodes (besides root) have between $M / 2$ and $M - 1$ links.

In practice. Number of probes is at most 4. ← M = 1024; N = 62 billion

$\log_{M/2} N \leq 4$

Optimization. Always keep root page in memory.

# Building a large B tree

# Balanced trees in the wild

Red-black trees are widely used as system symbol tables.

- Java: `java.util.TreeMap`, `java.util.TreeSet`.
- C++ STL: map, multimap, multiset.
- Linux kernel: completely fair scheduler, `linux/rbtree.h`.

B-tree variants. B+ tree, B*tree, B# tree, …

B-trees (and variants) are widely used for file systems and databases.

- Windows: HPFS.
- Mac: HFS, HFS+.
- Linux: ReiserFS, XFS, Ext3FS, JFS.
- Databases: ORACLE, DB2, INGRES, SQL, PostgreSQL.

# Balanced Search Trees

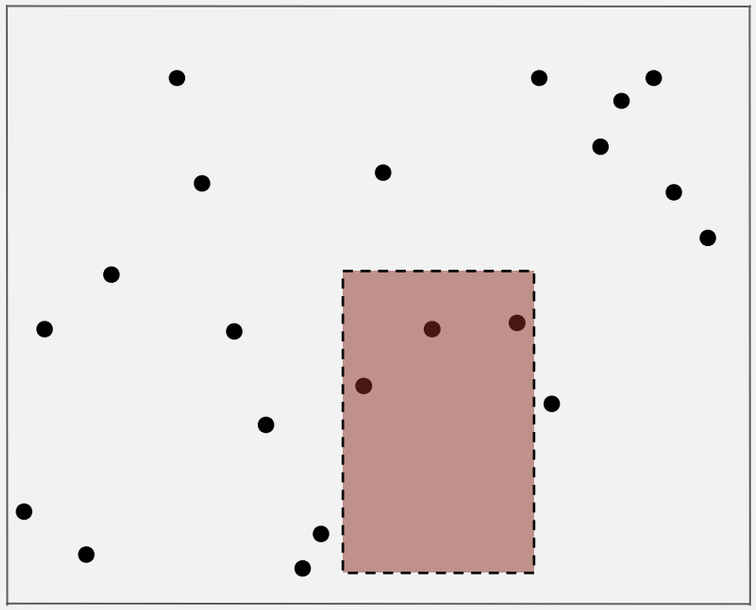
- 2-3 search trees
- Red-black BSTs
- B-trees
- **Geometric applications of BSTs**

# Geometric applications of BSTs

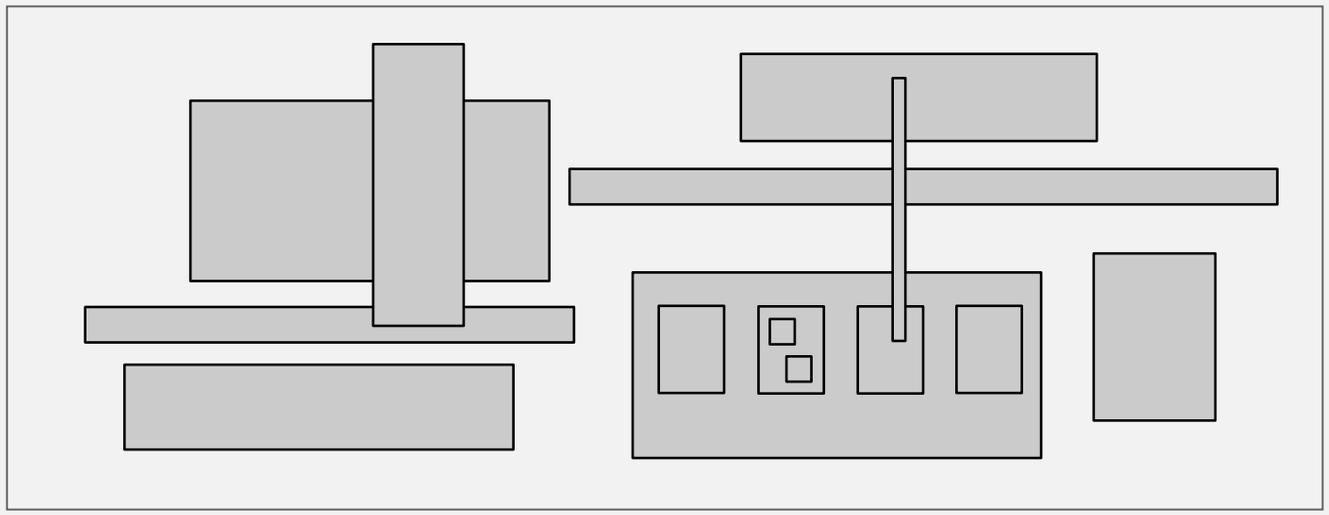
- 1D range search
- Line segment intersection
- kd trees
- Rectangle intersection

# Overview

This lecture. Intersections among geometric objects.



2d orthogonal range search



orthogonal rectangle intersection

Applications. CAD, games, movies, virtual reality, VLSI design, databases, ...

Efficient solutions. Binary search trees (and extensions).

# Geometric applications of BSTs

- 1D range search
- Line segment intersection
- kd trees

# 1d range search

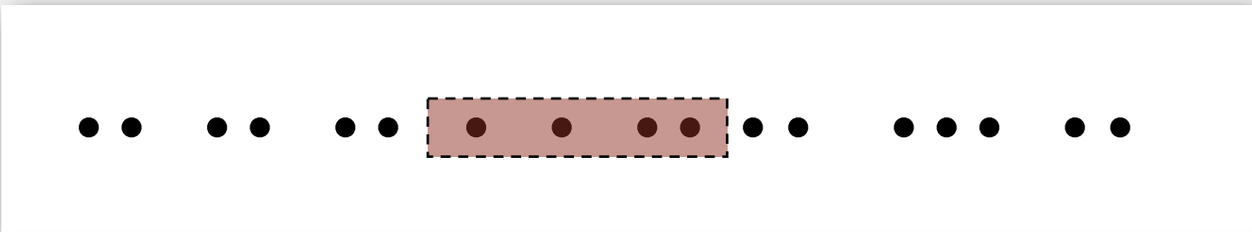
## Extension of ordered symbol table.

- Insert key-value pair.
- Search for key $k$.
- Delete key $k$.
- Range search: find all keys between $k_1$ and $k_2$.
- Range count: number of keys between $k_1$ and $k_2$.

## Application. Database queries.

## Geometric interpretation.

- Keys are point on a line.
- Find/count points in a given 1d interval.



| | |
|---|---|
| insert B | B |
| insert D | B D |
| insert A | A B D |
| insert I | A B D I |
| insert H | A B D H I |
| insert F | A B D F H I |
| insert P | A B D F H I P |
| count G to K | 2 |
| search G to K | H I |

# 1d range search: implementations

Unordered array. Fast insert, slow range search.

Ordered array. Slow insert, binary search for $k_1$ and $k_2$ to do range search.

**order of growth of running time for 1d range search**

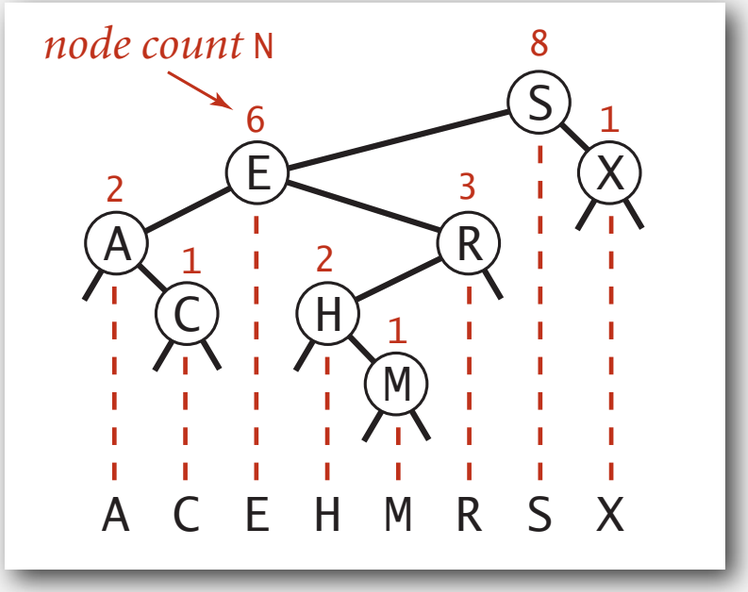
| data structure | insert | range count | range search |
|---|---|---|---|
| unordered array | 1 | N | N |
| ordered array | N | log N | R + log N |
| goal | log N | log N | R + log N |

Parameters.

- $N$ = number of keys.
- $R$ = number of keys that match.

# 1d range count: BST implementation

1d range count. How many keys between `lo` and `hi` ?




```
public int size(Key lo, Key hi)
{
   if (contains(hi)) return rank(hi) - rank(lo) + 1;
   else              return rank(hi) - rank(lo);
}
```

number of keys < hi

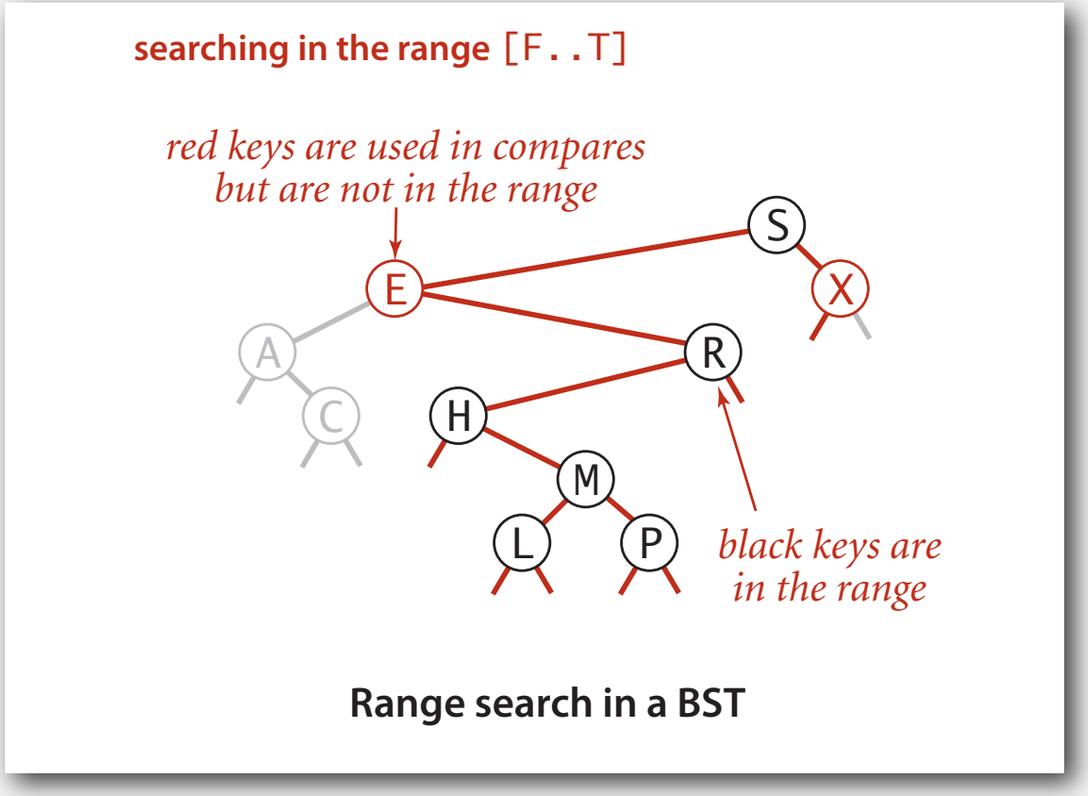
Proposition. Running time is proportional to $\log N$ (assuming BST is balanced).

Pf. Nodes examined = search path to `lo` + search path to `hi`.

# 1d range search: BST implementation

1d range search. Find all keys between `lo` and `hi`.

- Recursively find all keys in left subtree (if any could fall in range).
- Check key in current node.
- Recursively find all keys in right subtree (if any could fall in range).




Proposition. Running time is proportional to $R + \log N$ (assuming BST is balanced).

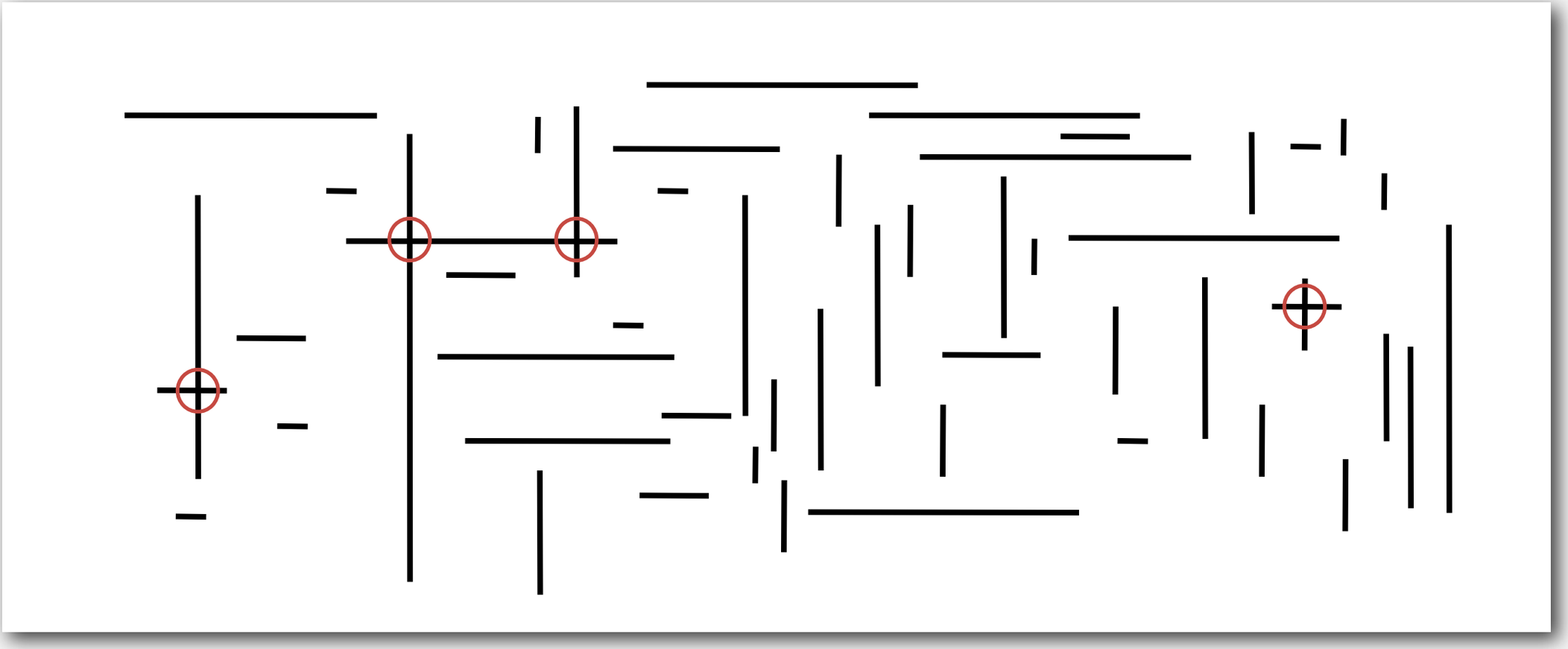
Pf. Nodes examined = search path to `lo` + search path to `hi` + matching keys.

# Geometric applications of BSTs

- 1D range search
- Line segment intersection
- kd trees

# Orthogonal line segment intersection search

Given $N$ horizontal and vertical line segments, find all intersections.



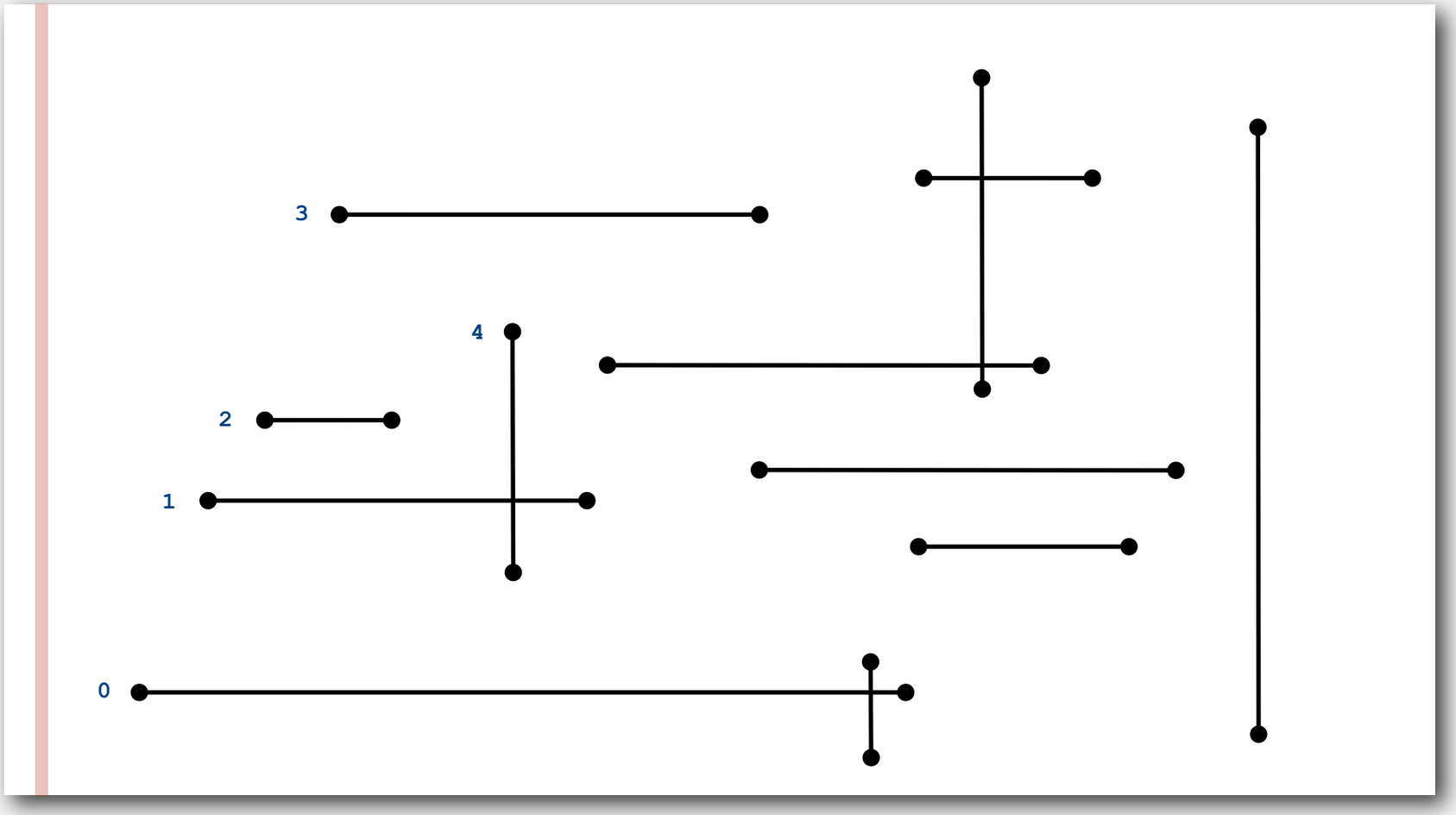
Quadratic algorithm. Check all pairs of line segments for intersection.

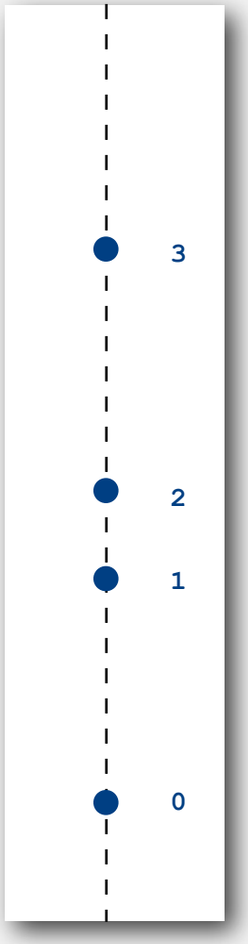
Nondegeneracy assumption. All $x$- and $y$-coordinates are distinct.

# Orthogonal line segment intersection search: sweep-line algorithm

Sweep vertical line from left to right.

- $x$-coordinates define events.
- $h$-segment (left endpoint): insert $y$-coordinate into BST.



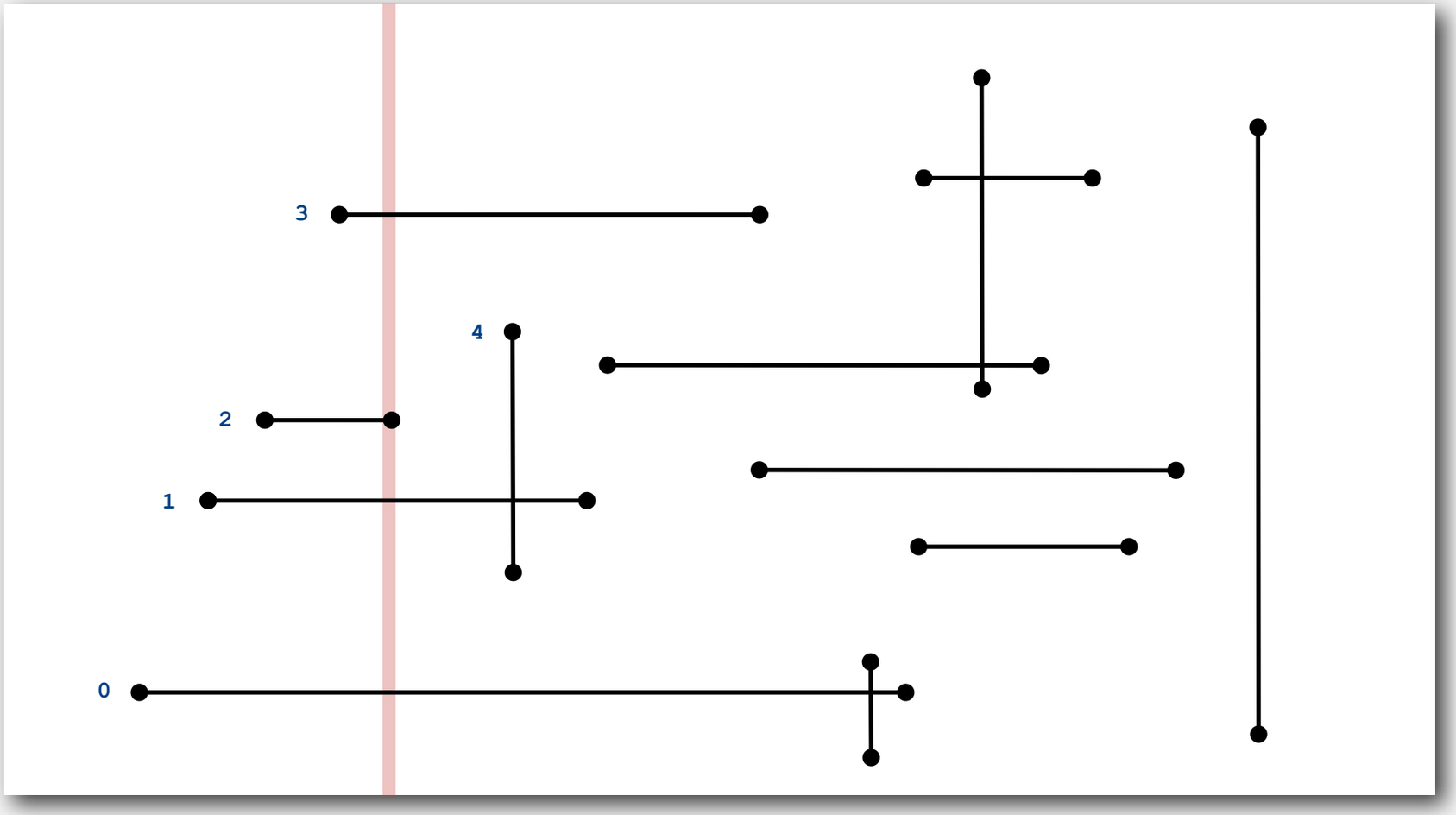



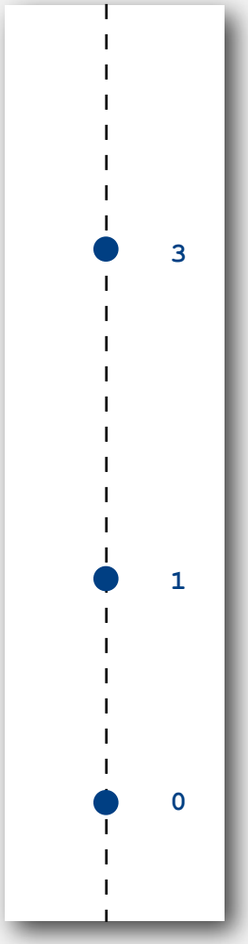

y-coordinates

# Orthogonal line segment intersection search: sweep-line algorithm

Sweep vertical line from left to right.

- $x$-coordinates define events.
- $h$-segment (left endpoint): insert $y$-coordinate into BST.
- $h$-segment (right endpoint): remove $y$-coordinate from BST.



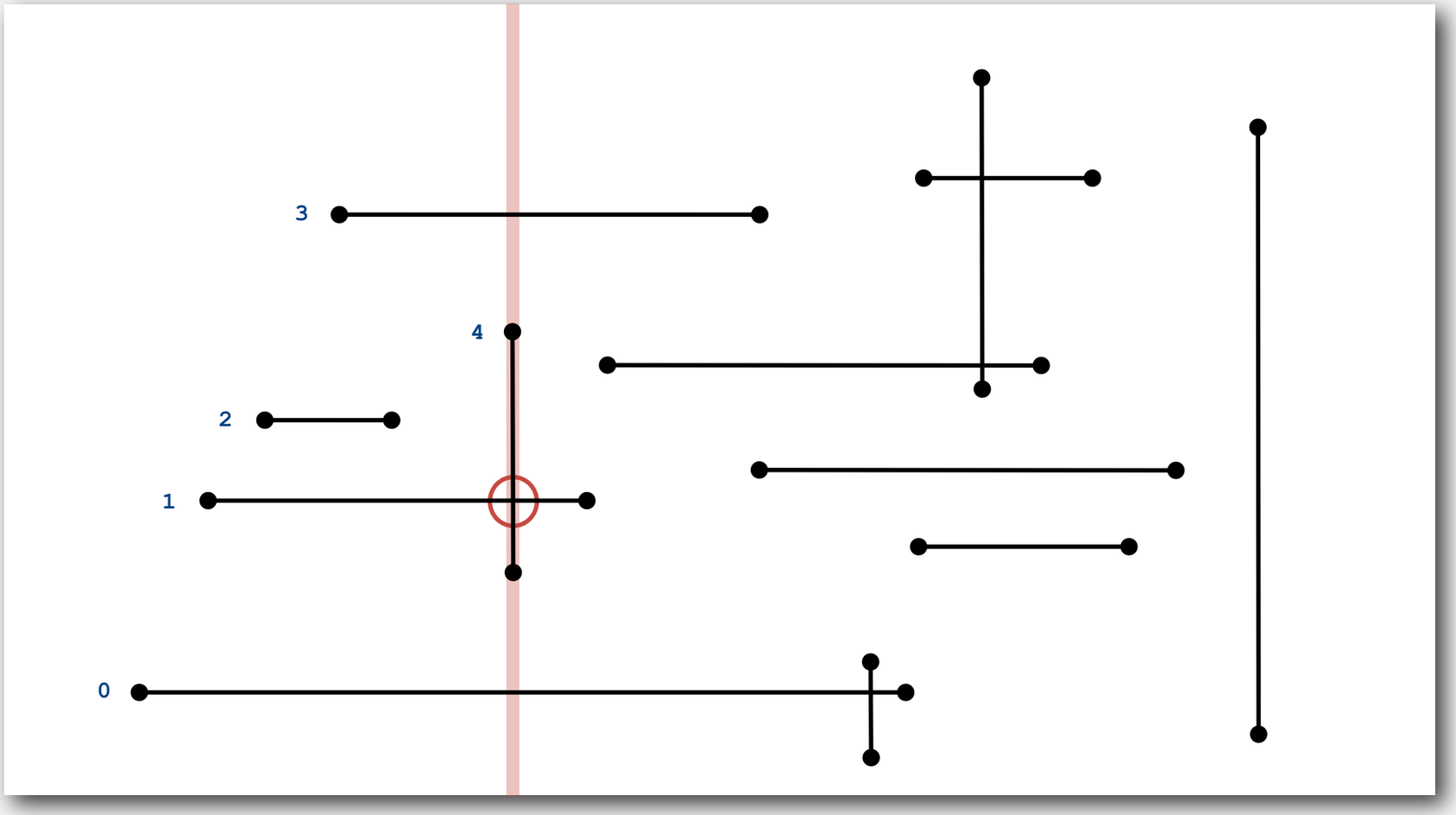



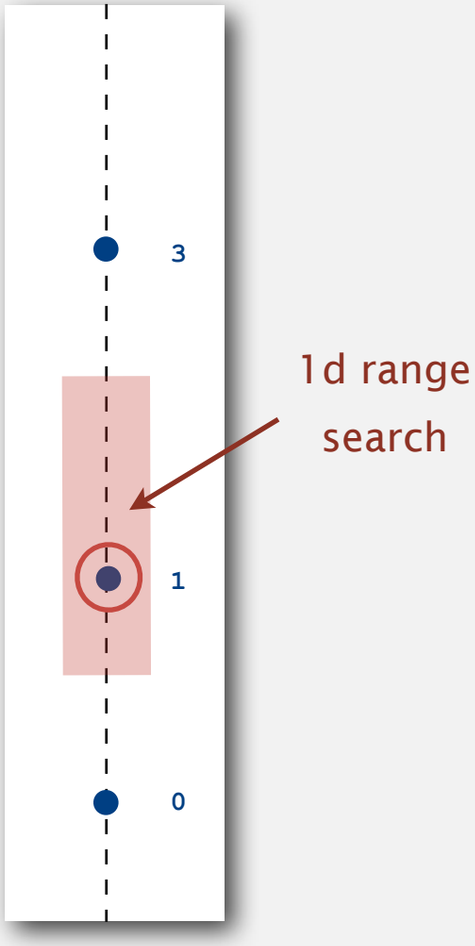

y-coordinates

# Orthogonal line segment intersection search: sweep-line algorithm

Sweep vertical line from left to right.

- $x$-coordinates define events.
- $h$-segment (left endpoint): insert $y$-coordinate into BST.
- $h$-segment (right endpoint): remove $y$-coordinate from BST.
- $v$-segment: range search for interval of $y$-endpoints.



# Orthogonal line segment intersection search: sweep-line algorithm analysis

Proposition. The sweep-line algorithm takes time proportional to $N \log N + R$ to find all $R$ intersections among $N$ orthogonal line segments.

Pf.

- Put $x$-coordinates on a PQ (or sort). ← N log N
- Insert $y$-coordinates into BST. ← N log N
- Delete $y$-coordinates from BST. ← N log N
- Range searches in BST. ← N log N + R

Bottom line. Sweep line reduces 2d orthogonal line segment intersection search to 1d range search.

# Geometric applications of BSTs

- 1D range search
- Line segment intersection
- kd trees

# 2-d orthogonal range search

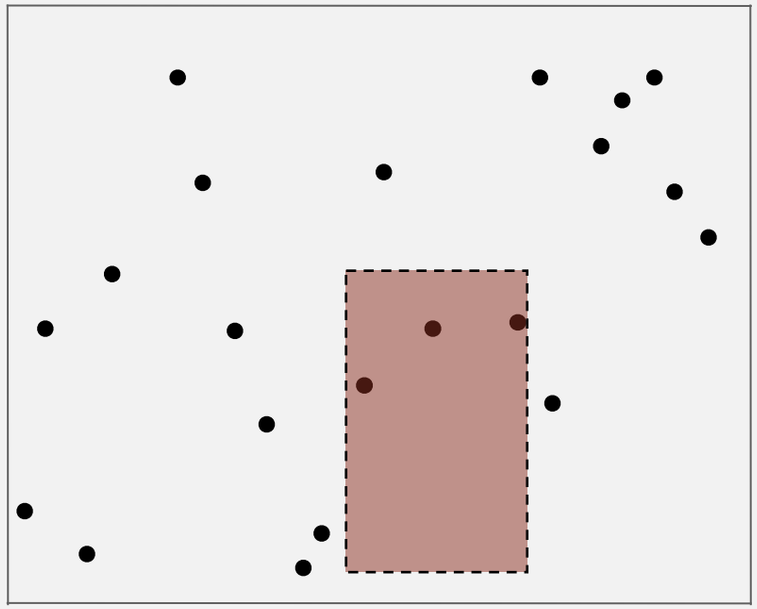
Extension of ordered symbol-table to 2d keys.

- Insert a 2d key.
- Delete a 2d key.
- Search for a 2d key.
- Range search: find all keys that lie in a 2d range.
- Range count: number of keys that lie in a 2d range.

Geometric interpretation.

- Keys are point in the plane.
- Find/count points in a given $h$–$v$ rectangle.

rectangle is axis-aligned

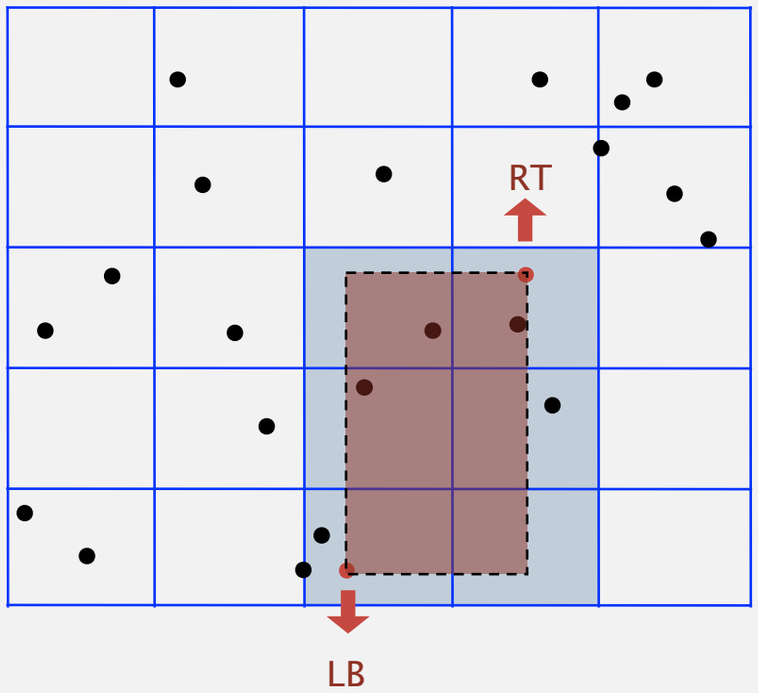


Applications. Networking, circuit design, databases,...

# 2d orthogonal range search: grid implementation

## Grid implementation.

- Divide space into $M$-by-$M$ grid of squares.
- Create list of points contained in each square.
- Use 2d array to directly index relevant square.
- Insert: add $(x, y)$ to list for corresponding square.
- Range search: examine only those squares that intersect 2d range query.



# 2d orthogonal range search: grid implementation costs

## Space-time tradeoff.

- Space: $M^2 + N$.
- Time: $1 + N / M^2$ per square examined, on average.

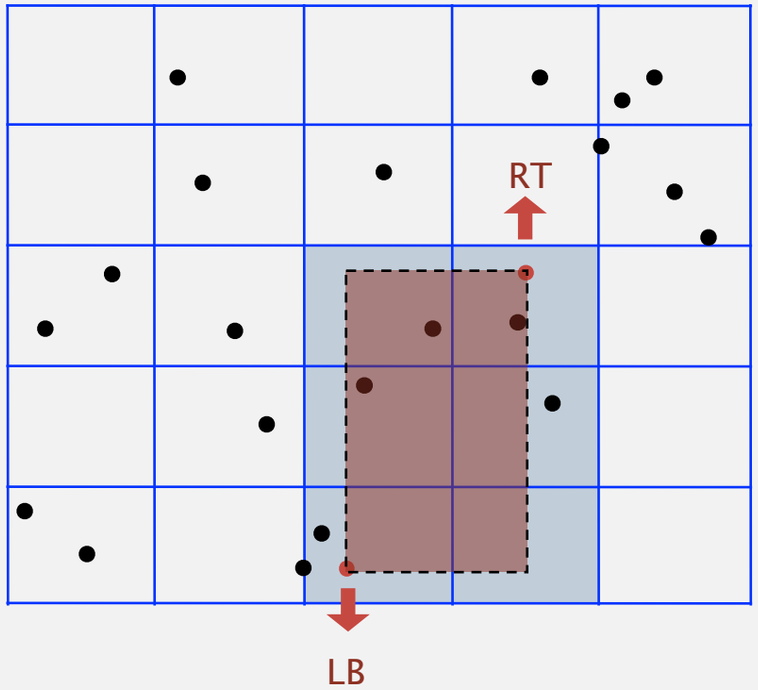
## Choose grid square size to tune performance.

- Too small: wastes space.
- Too large: too many points per square.
- Rule of thumb: $\sqrt{N}$-by-$\sqrt{N}$ grid.

## Running time. [if points are evenly distributed]

- Initialize data structure: $N$.
- Insert point: $1$.
- Range search: $1$ per point in range.

choose M ~ √N

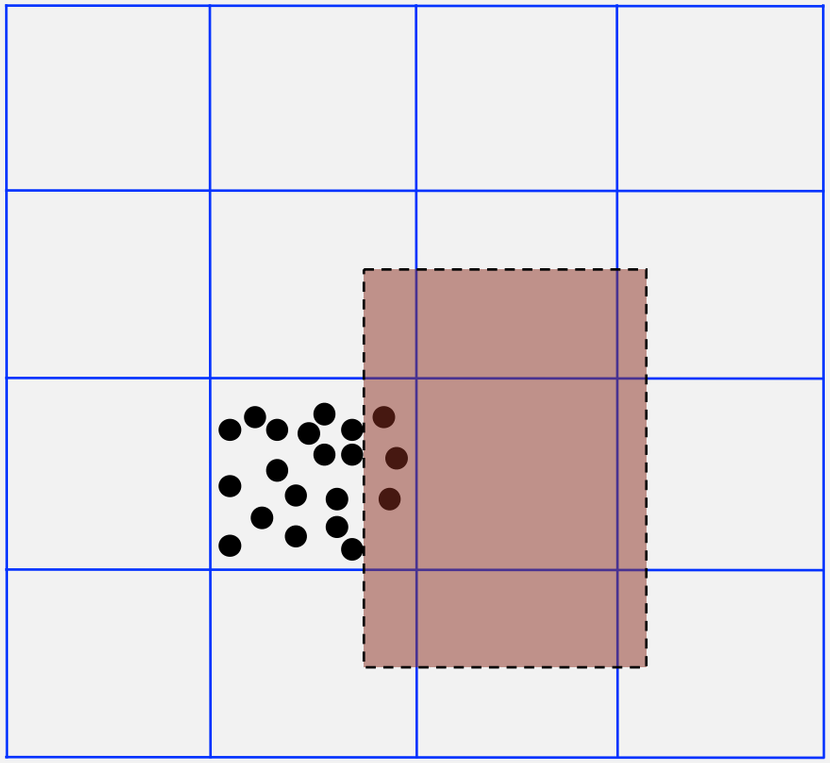


# Clustering

Grid implementation. Fast and simple solution for evenly-distributed points.

Problem. Clustering a well-known phenomenon in geometric data.

- Lists are too long, even though average length is short.
- Need data structure that gracefully adapts to data.



# Clustering

Grid implementation. Fast and simple solution for evenly-distributed points.

Problem. Clustering a well-known phenomenon in geometric data.

Ex. USA map data.

**13,000 points, 1000 grid squares**

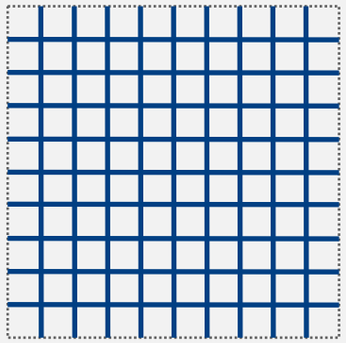
half the squares are empty

half the points are in 10% of the squares

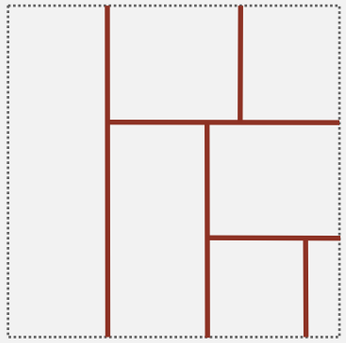
# Space-partitioning trees

Use a tree to represent a recursive subdivision of 2d space.

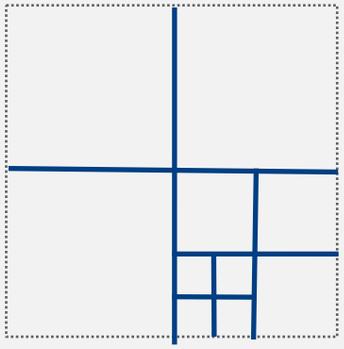
Grid. Divide space uniformly into squares.
2d tree. Recursively divide space into two halfplanes.
Quadtree. Recursively divide space into four quadrants.
BSP tree. Recursively divide space into two regions.



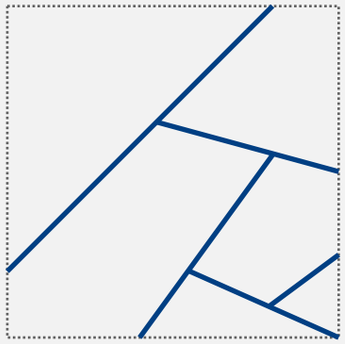
Grid



2d tree



Quadtree

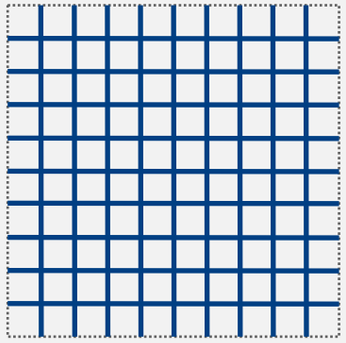


BSP tree

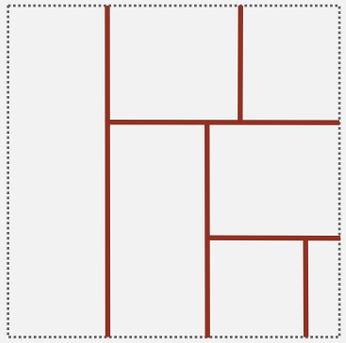
# Space-partitioning trees: applications

## Applications.

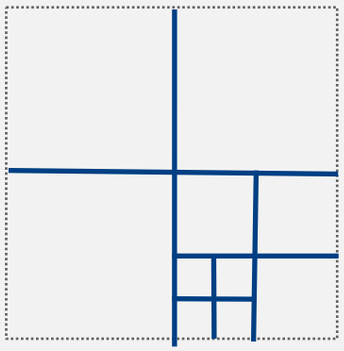
- Ray tracing.
- 2d range search.
- Flight simulators.
- N-body simulation.
- Collision detection.
- Astronomical databases.
- Nearest neighbor search.
- Adaptive mesh generation.
- Accelerate rendering in Doom.
- Hidden surface removal and shadow casting.

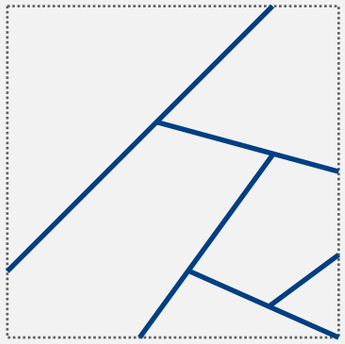
Grid



2d tree



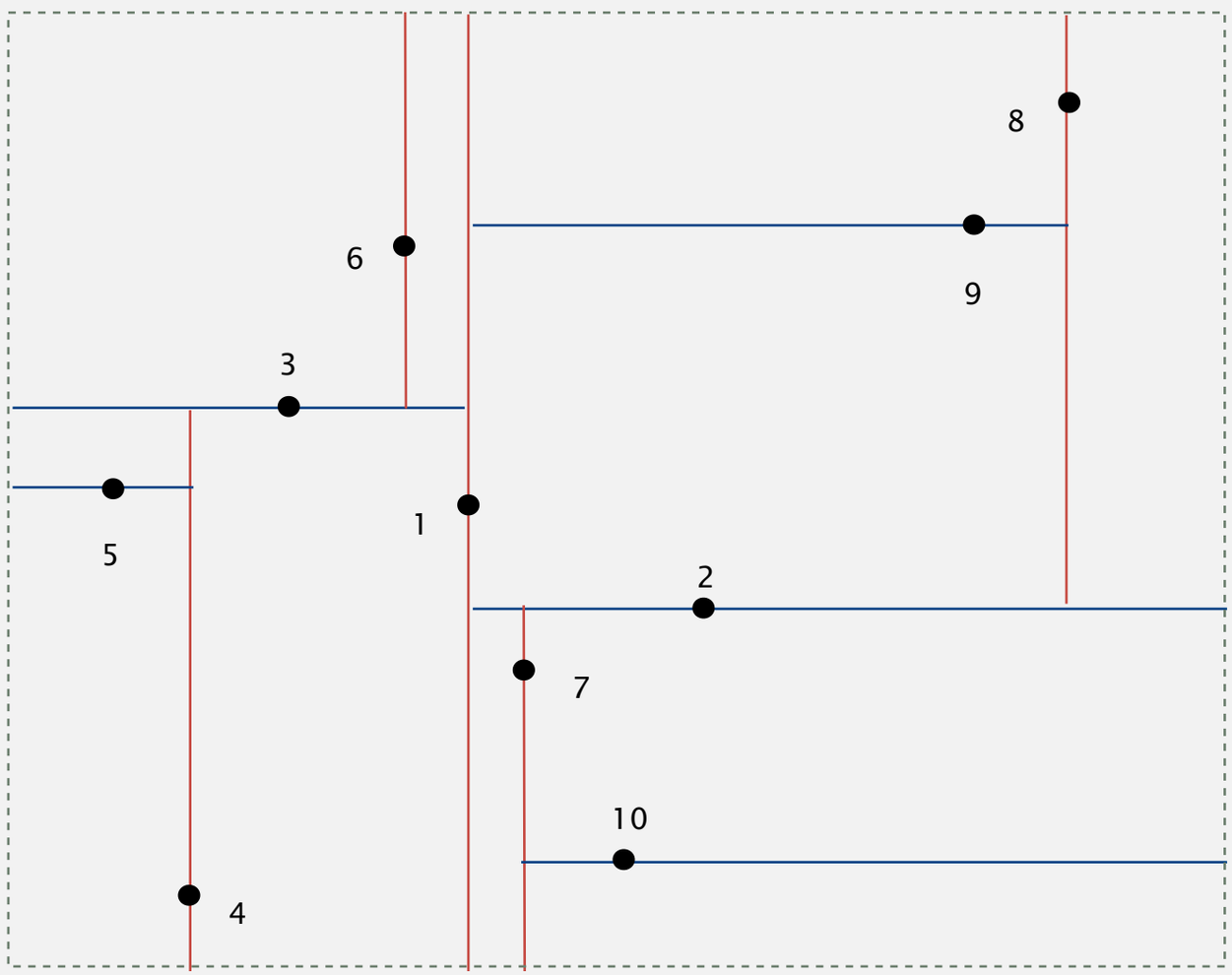
Quadtree

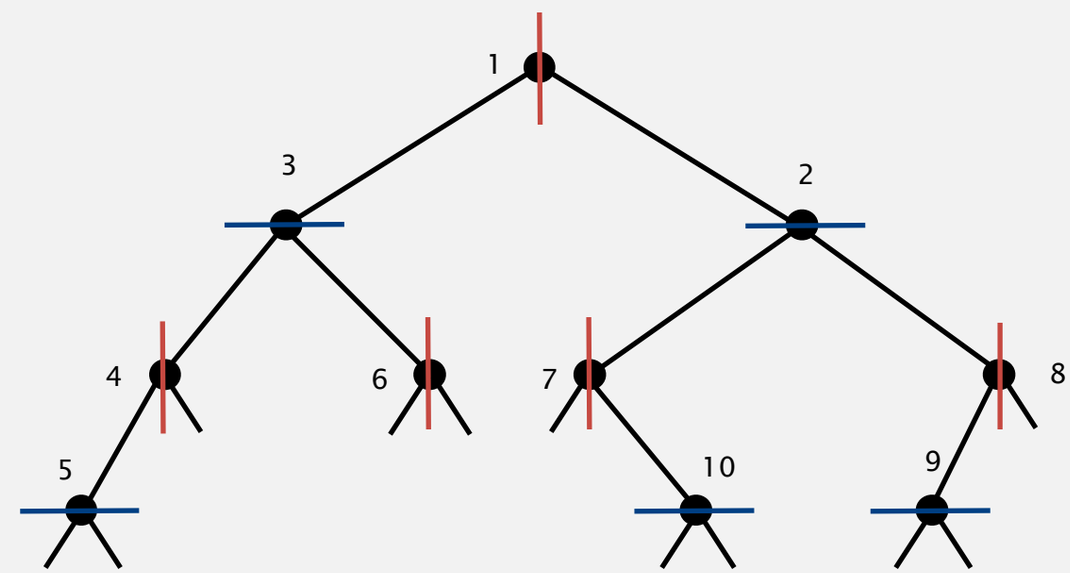


BSP tree

# Insertion in a 2d tree

Recursively partition plane into two halfplanes.

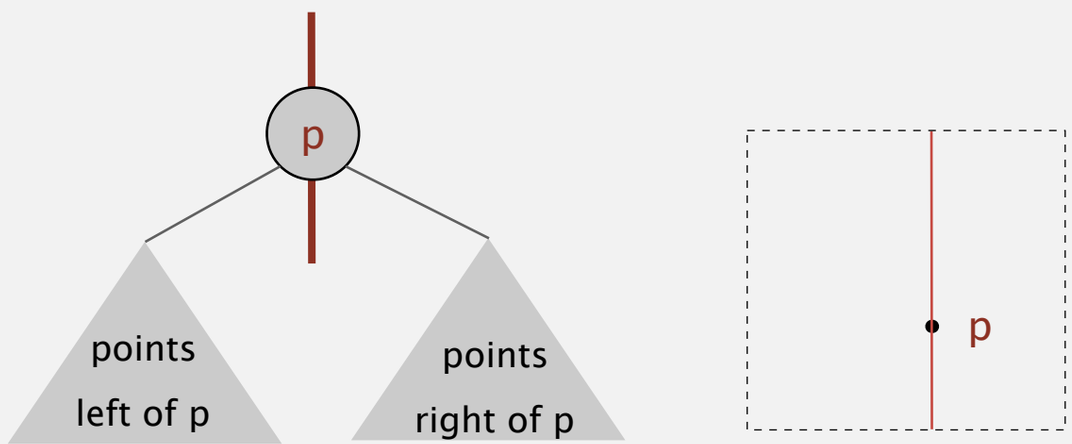


# 2d tree implementation

Data structure. BST, but alternate using $x$- and $y$-coordinates as key.

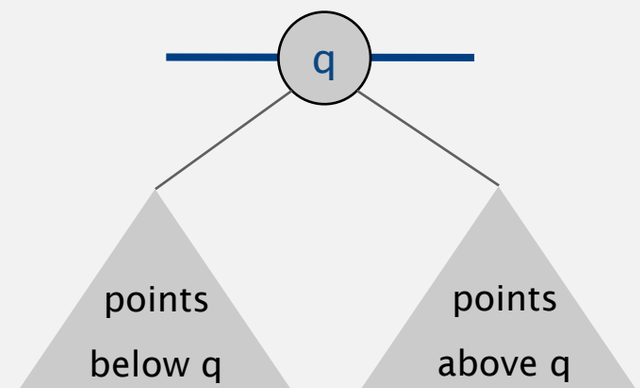
- Search gives rectangle containing point.
- Insert further subdivides the plane.

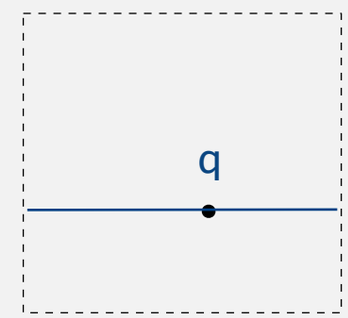




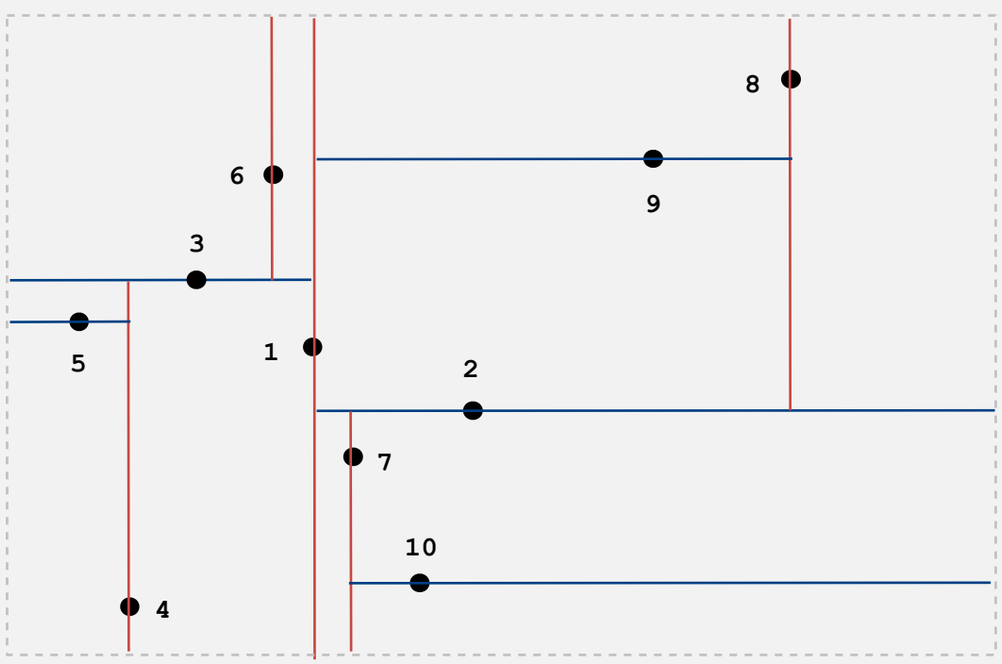


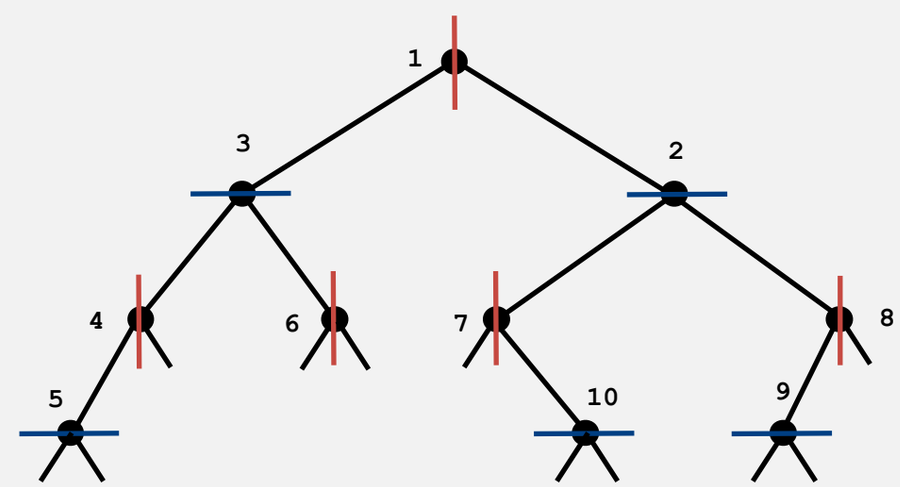
even levels



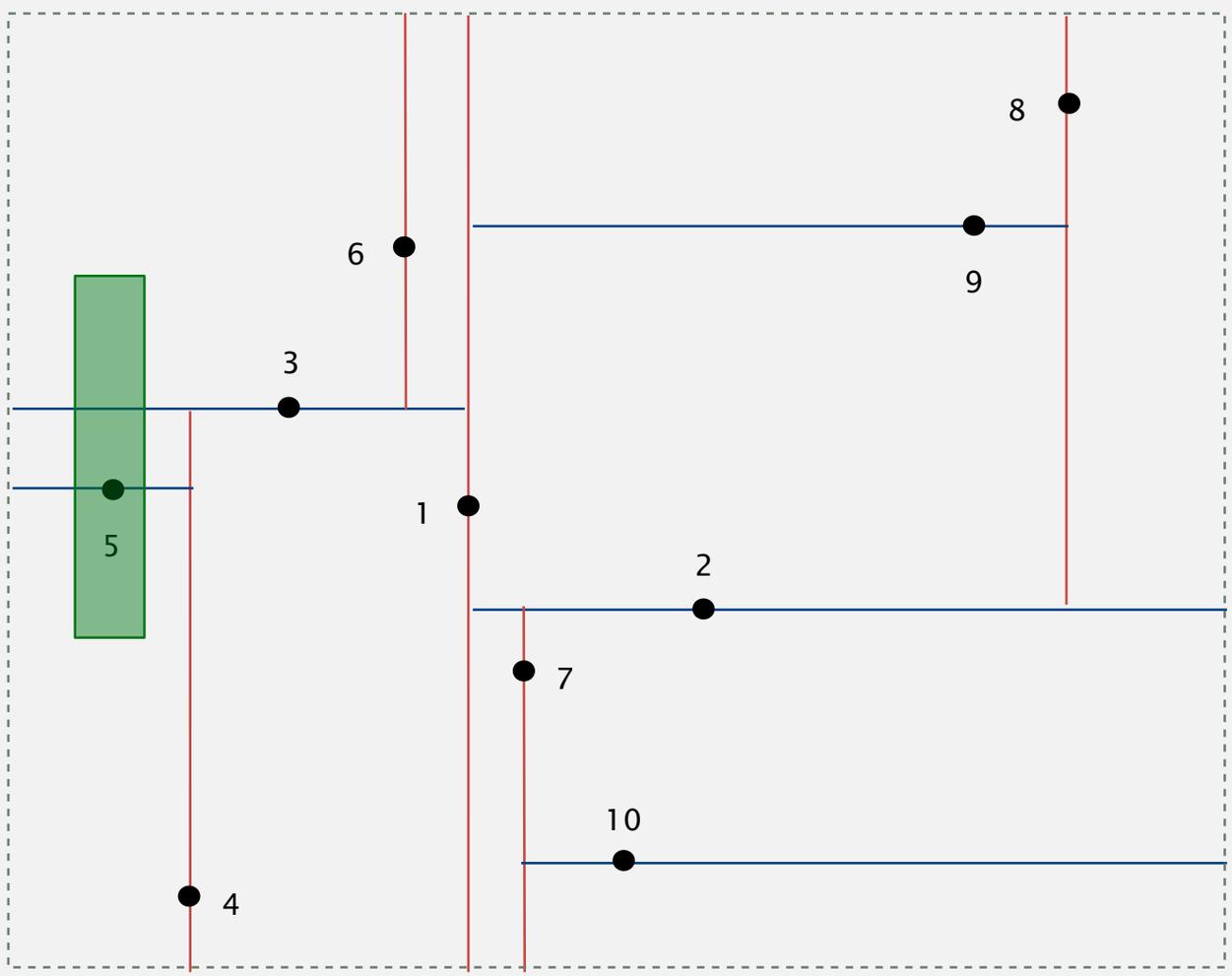

odd levels

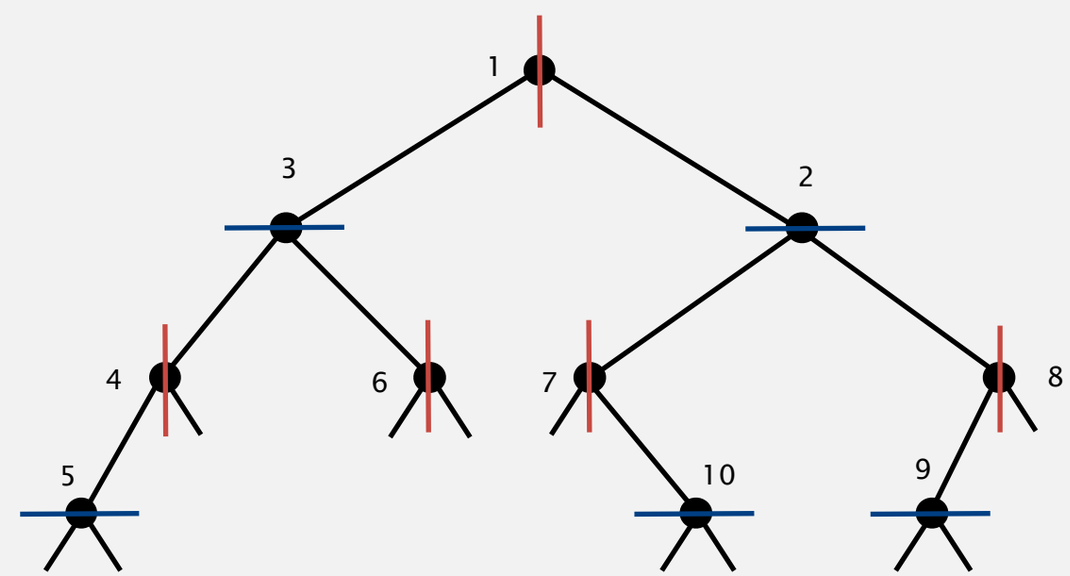


# Range search in a 2d tree

- Check if point in node lies in given rectangle.
- Recursively search left/bottom subdivision (if any could fall in rectangle).
- Recursively search right/top subdivision (if any could fall in rectangle).

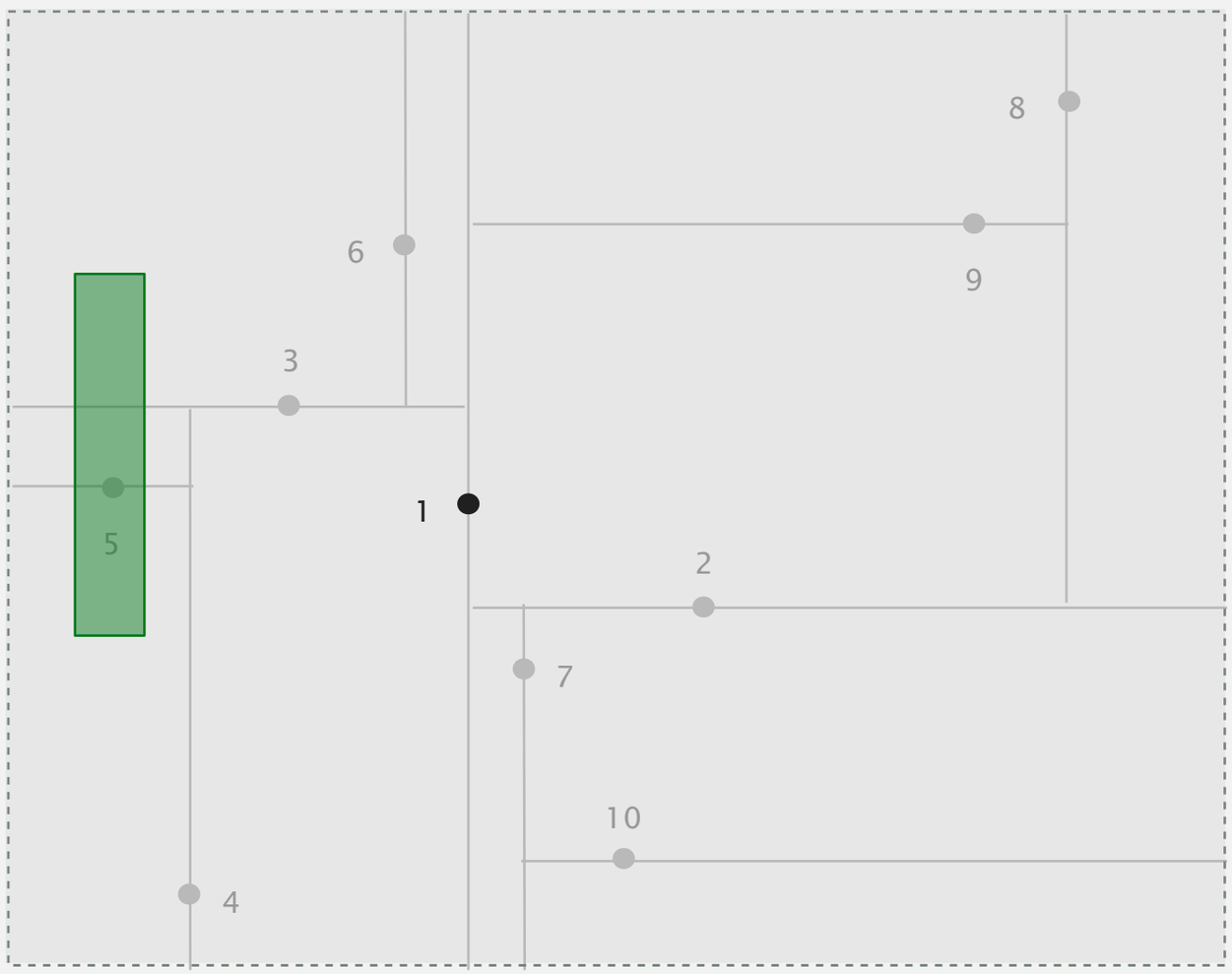


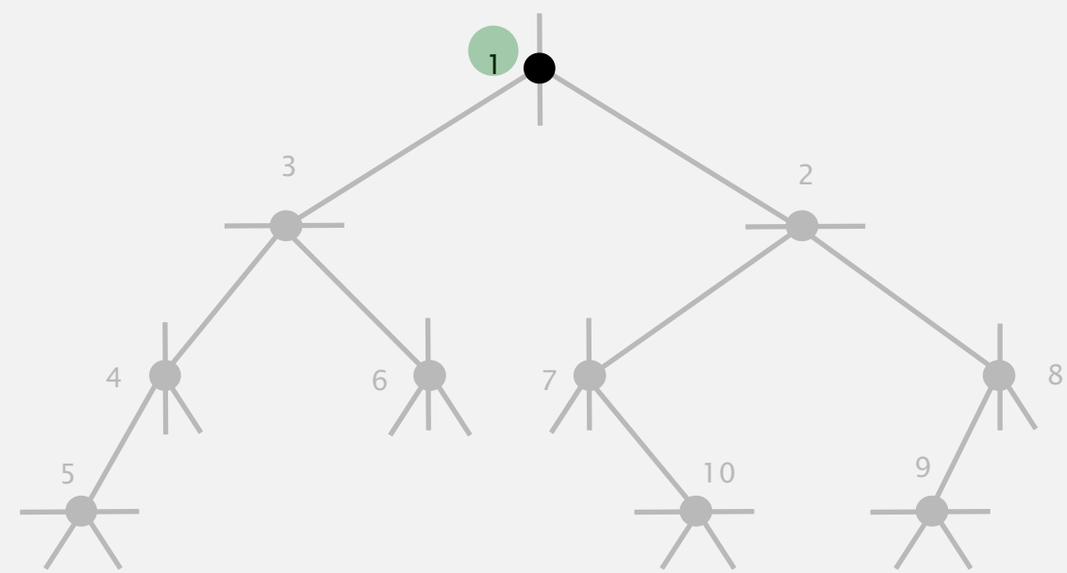




**find all points in 2d tree that are contained in green query rectangle**

# Range search in a 2d tree

- Check if point in node lies in given rectangle.
- Recursively search left/bottom subdivision (if any could fall in rectangle).
- Recursively search right/top subdivision (if any could fall in rectangle).



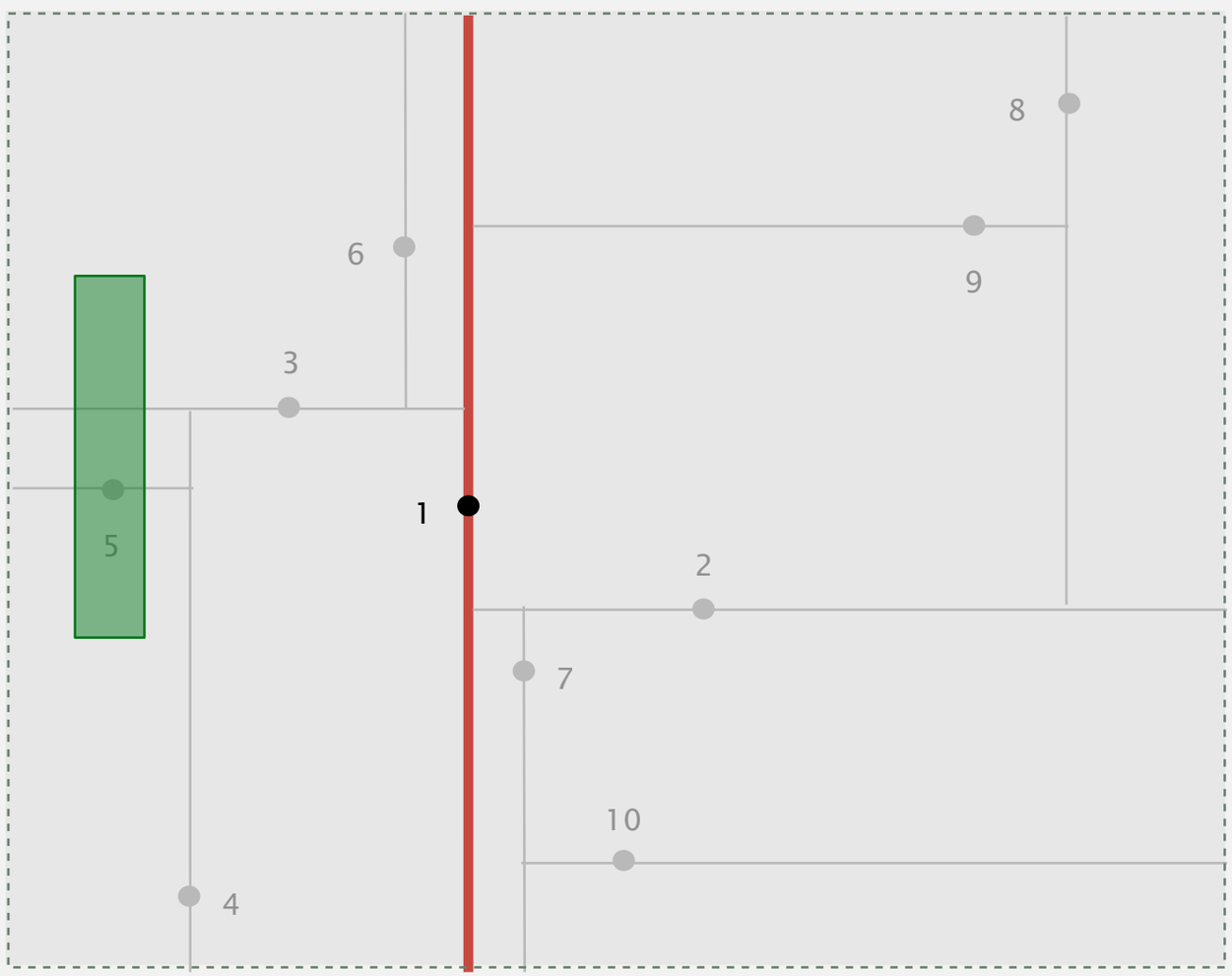



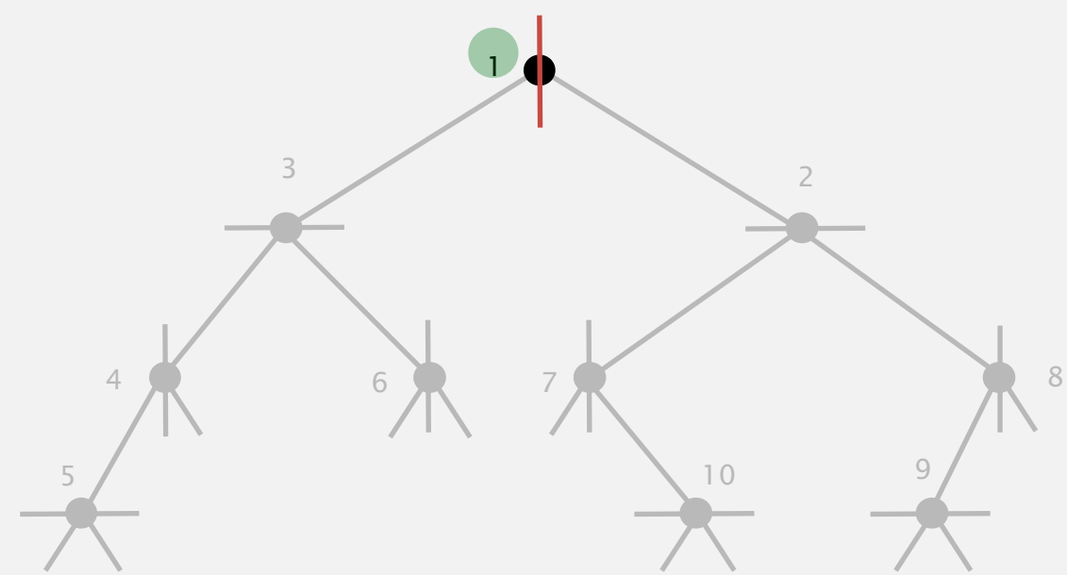

**search root node**

**check if query rectangle contains point 1**

# Range search in a 2d tree

- Check if point in node lies in given rectangle.
- Recursively search left/bottom subdivision (if any could fall in rectangle).
- Recursively search right/top subdivision (if any could fall in rectangle).

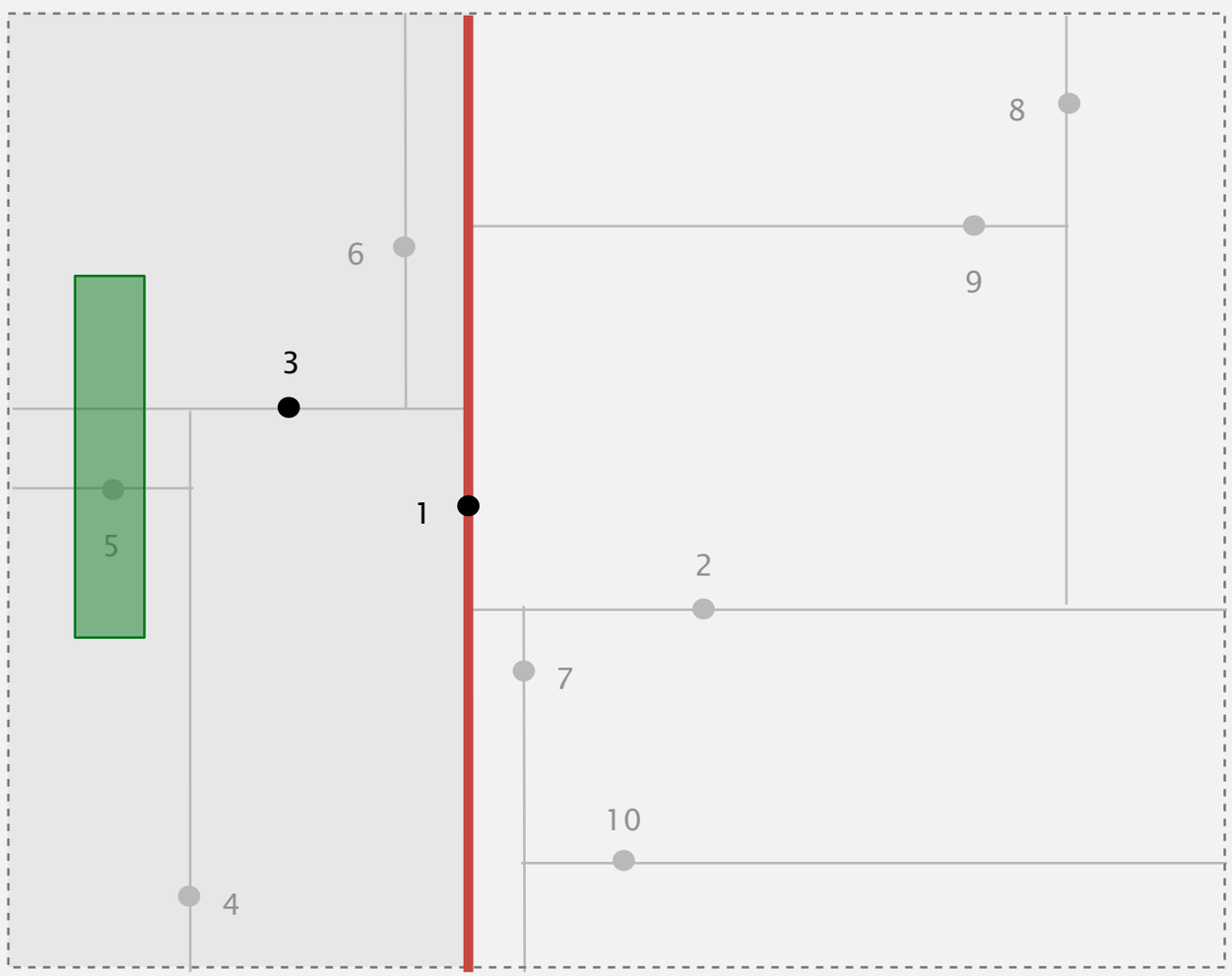


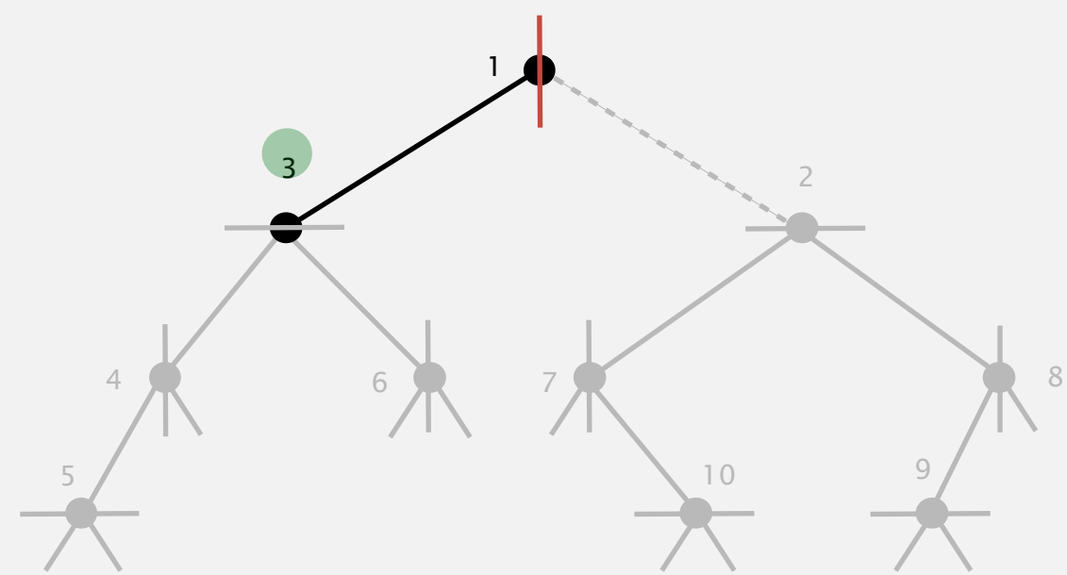




**query rectangle to left of splitting line**
**search only in left subtree**

# Range search in a 2d tree

- Check if point in node lies in given rectangle.
- Recursively search left/bottom subdivision (if any could fall in rectangle).
- Recursively search right/top subdivision (if any could fall in rectangle).



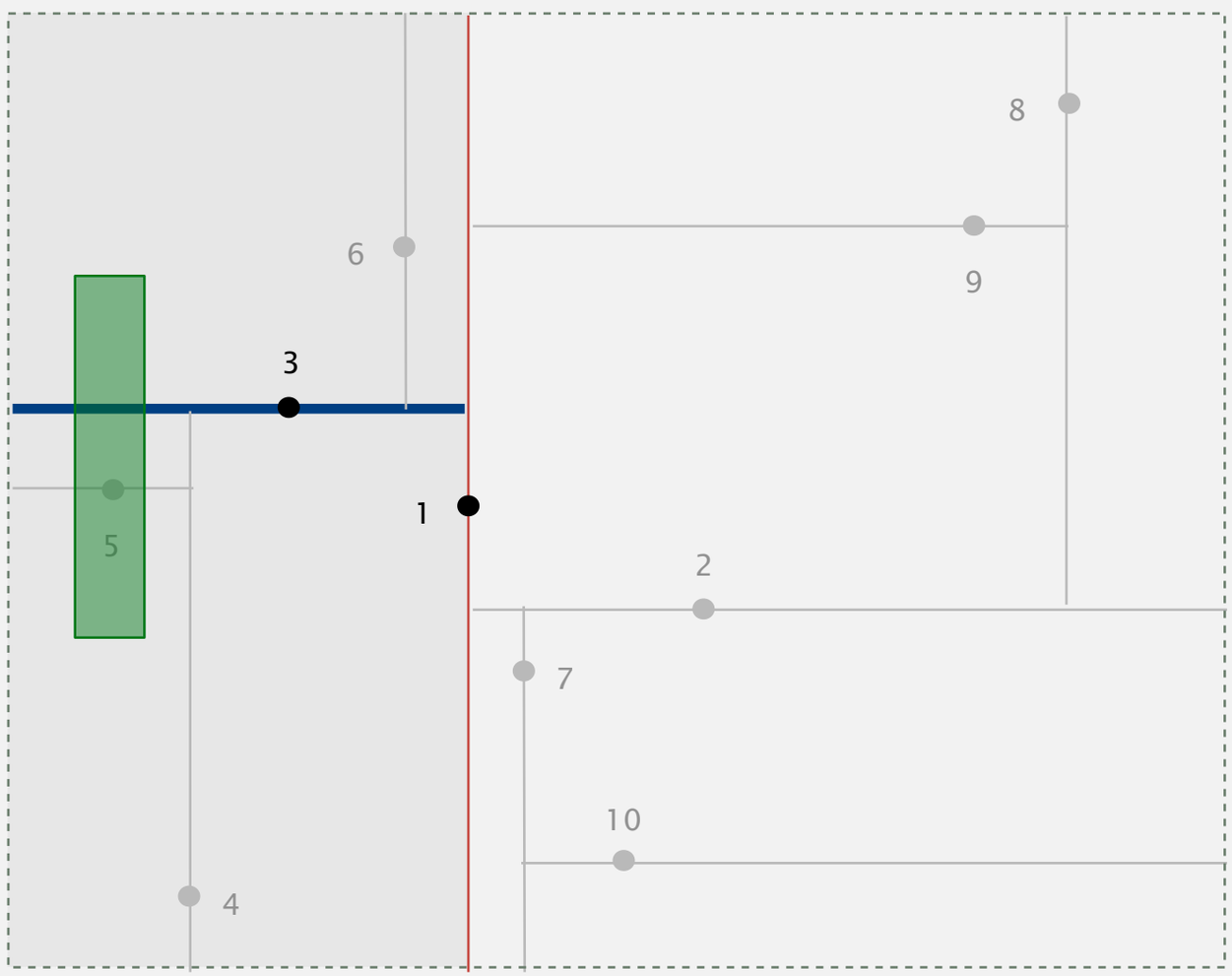



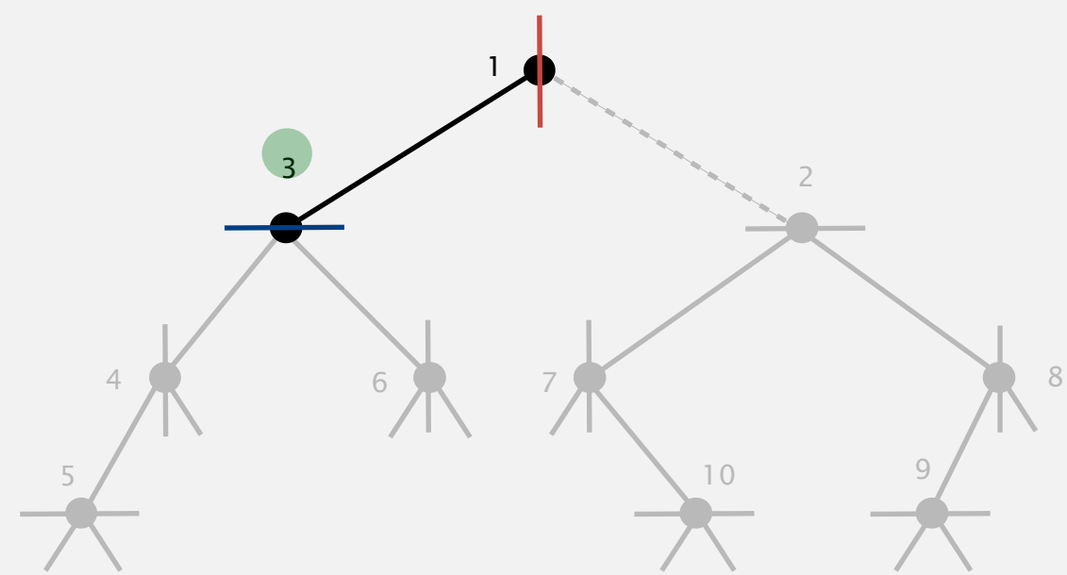

**search left subtree**

**check if query rectangle contains point 3**

# Range search in a 2d tree

- Check if point in node lies in given rectangle.
- Recursively search left/bottom subdivision (if any could fall in rectangle).
- Recursively search right/top subdivision (if any could fall in rectangle).

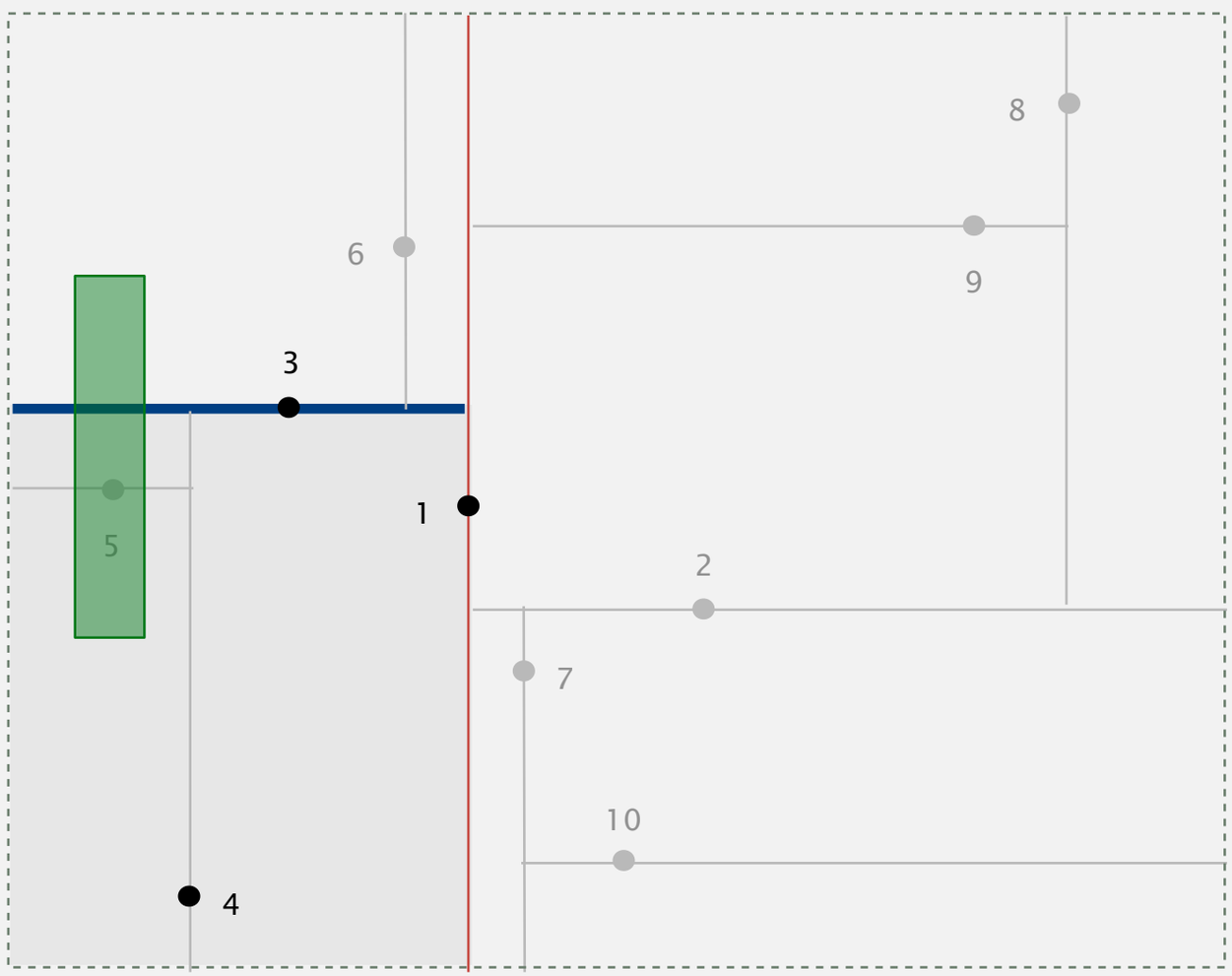


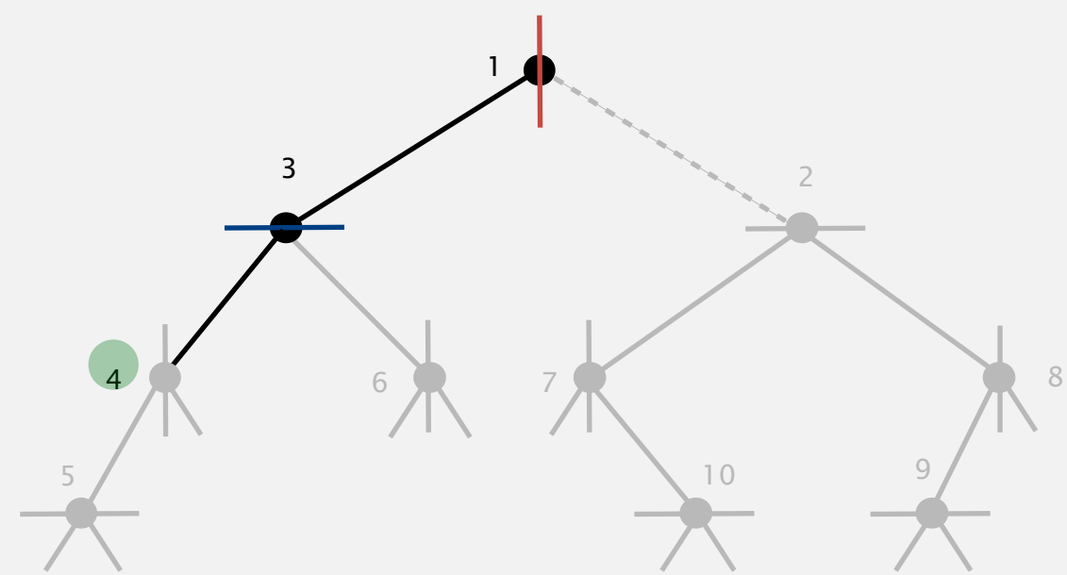




**query rectangle intersects splitting line**
**search bottom and top subtrees**

# Range search in a 2d tree

- Check if point in node lies in given rectangle.
- Recursively search left/bottom subdivision (if any could fall in rectangle).
- Recursively search right/top subdivision (if any could fall in rectangle).



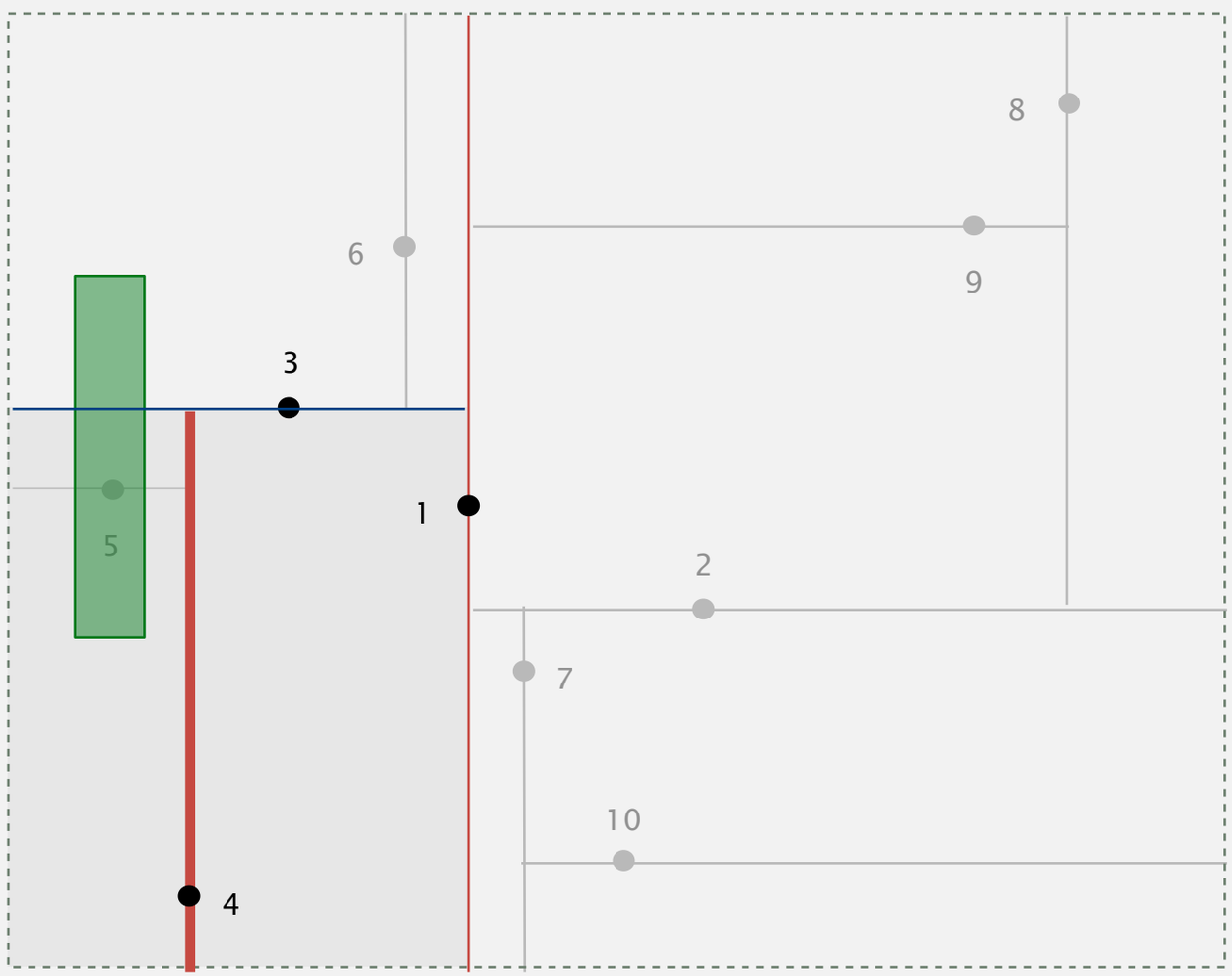



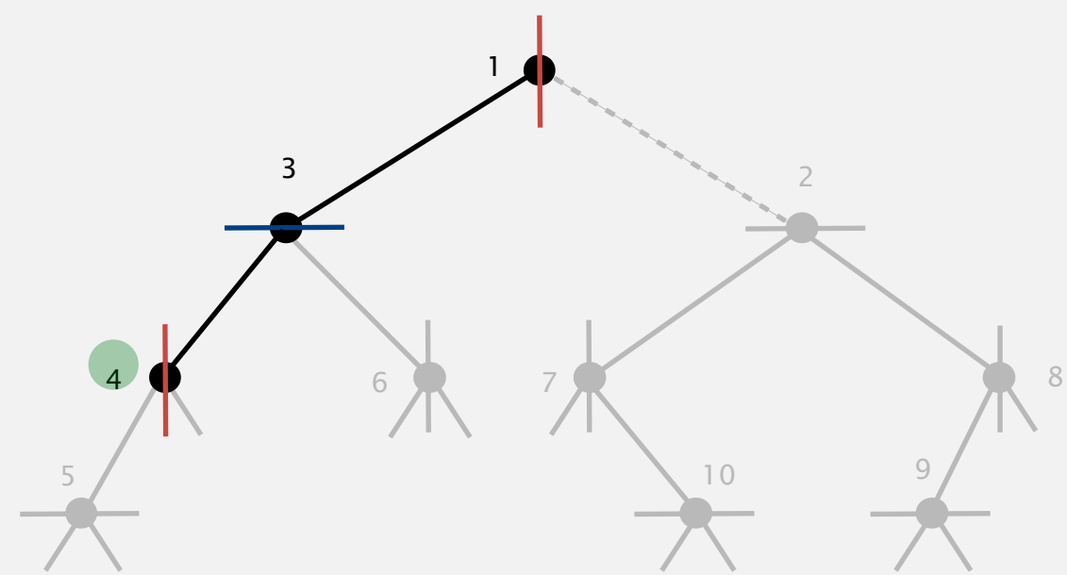

**search left subtree**

**check if query rectangle contains point 4**

# Range search in a 2d tree

- Check if point in node lies in given rectangle.
- Recursively search left/bottom subdivision (if any could fall in rectangle).
- Recursively search right/top subdivision (if any could fall in rectangle).

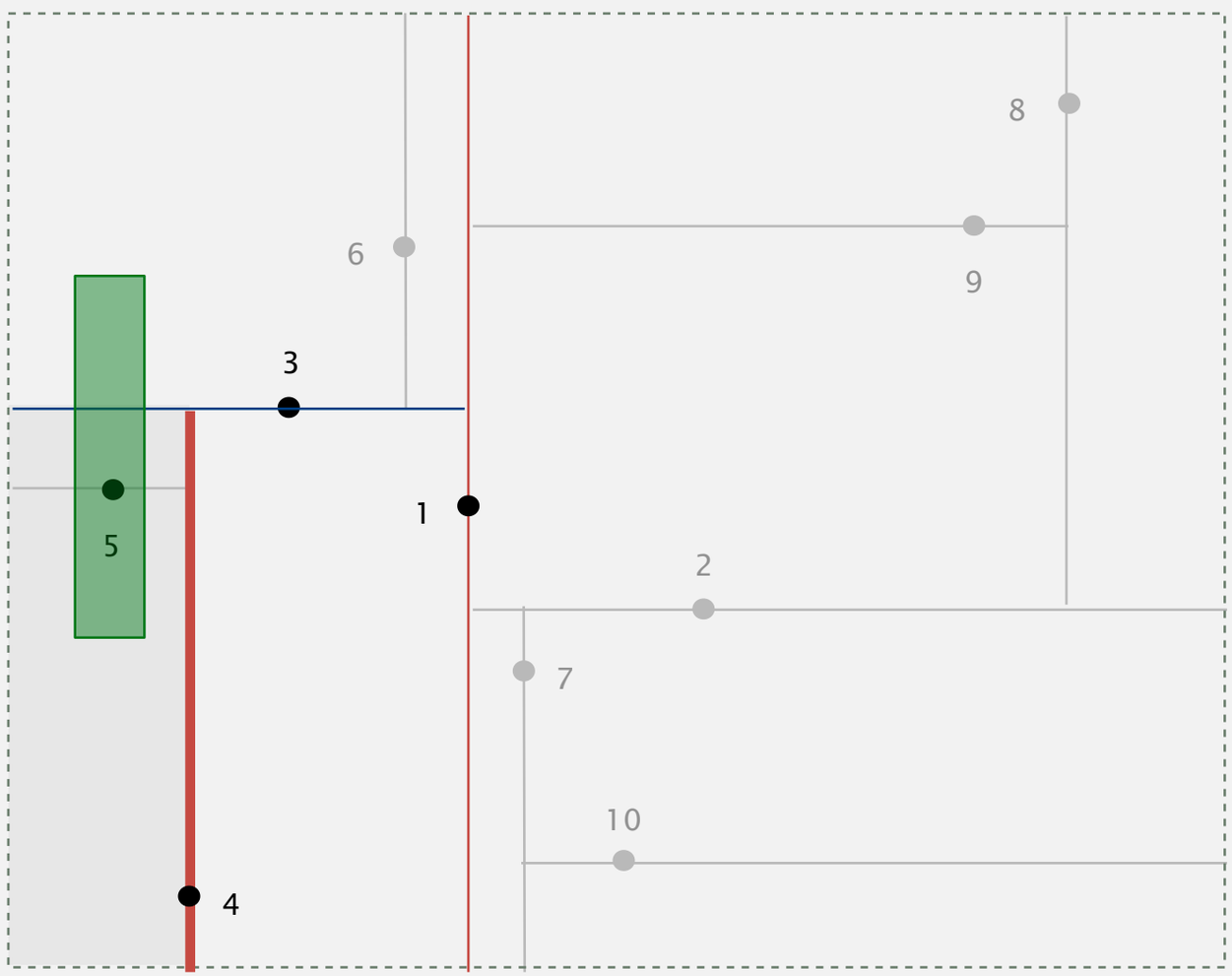


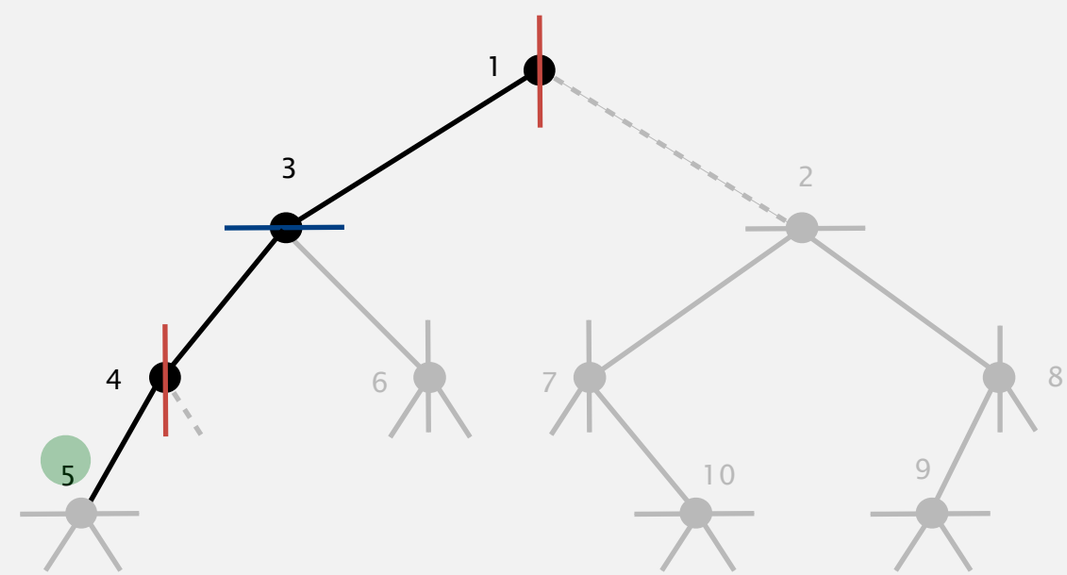




**query rectangle to left of splitting line**
**search only in left subtree**

# Range search in a 2d tree

- Check if point in node lies in given rectangle.
- Recursively search left/bottom subdivision (if any could fall in rectangle).
- Recursively search right/top subdivision (if any could fall in rectangle).

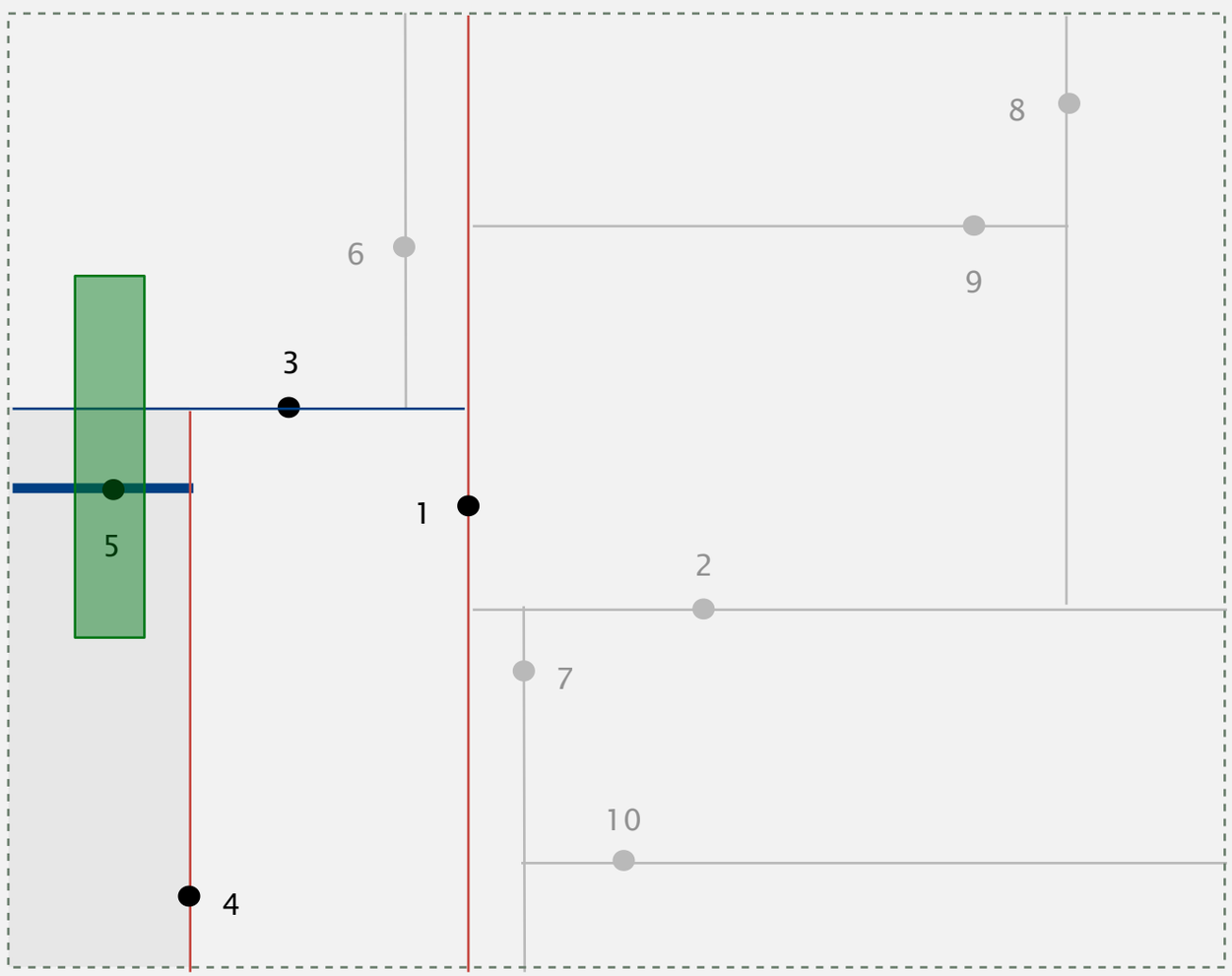


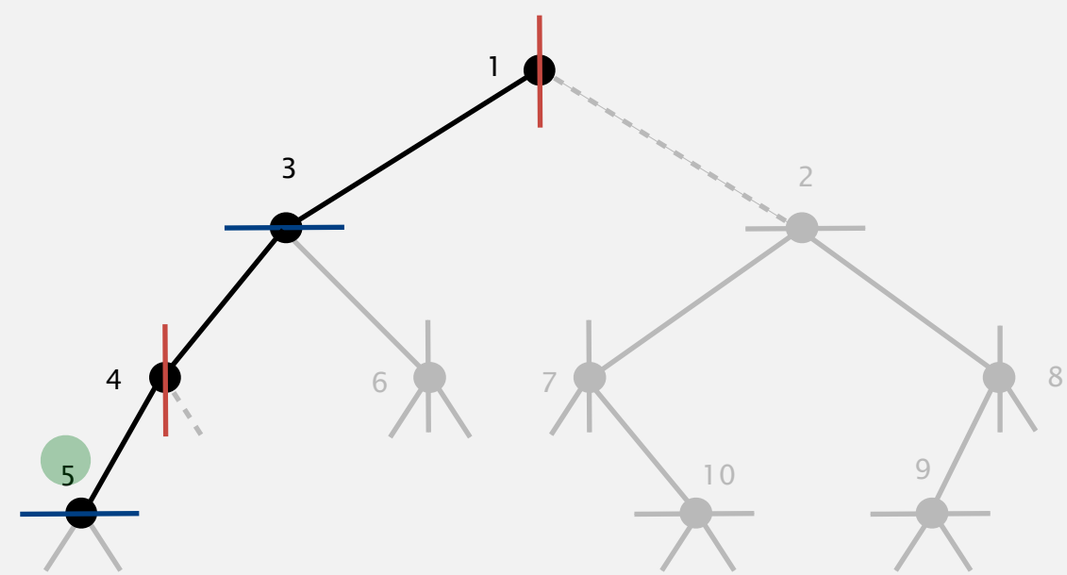




**search left subtree**
**check if query rectangle contains point 5**
**(search hit)**

# Range search in a 2d tree

- Check if point in node lies in given rectangle.
- Recursively search left/bottom subdivision (if any could fall in rectangle).
- Recursively search right/top subdivision (if any could fall in rectangle).

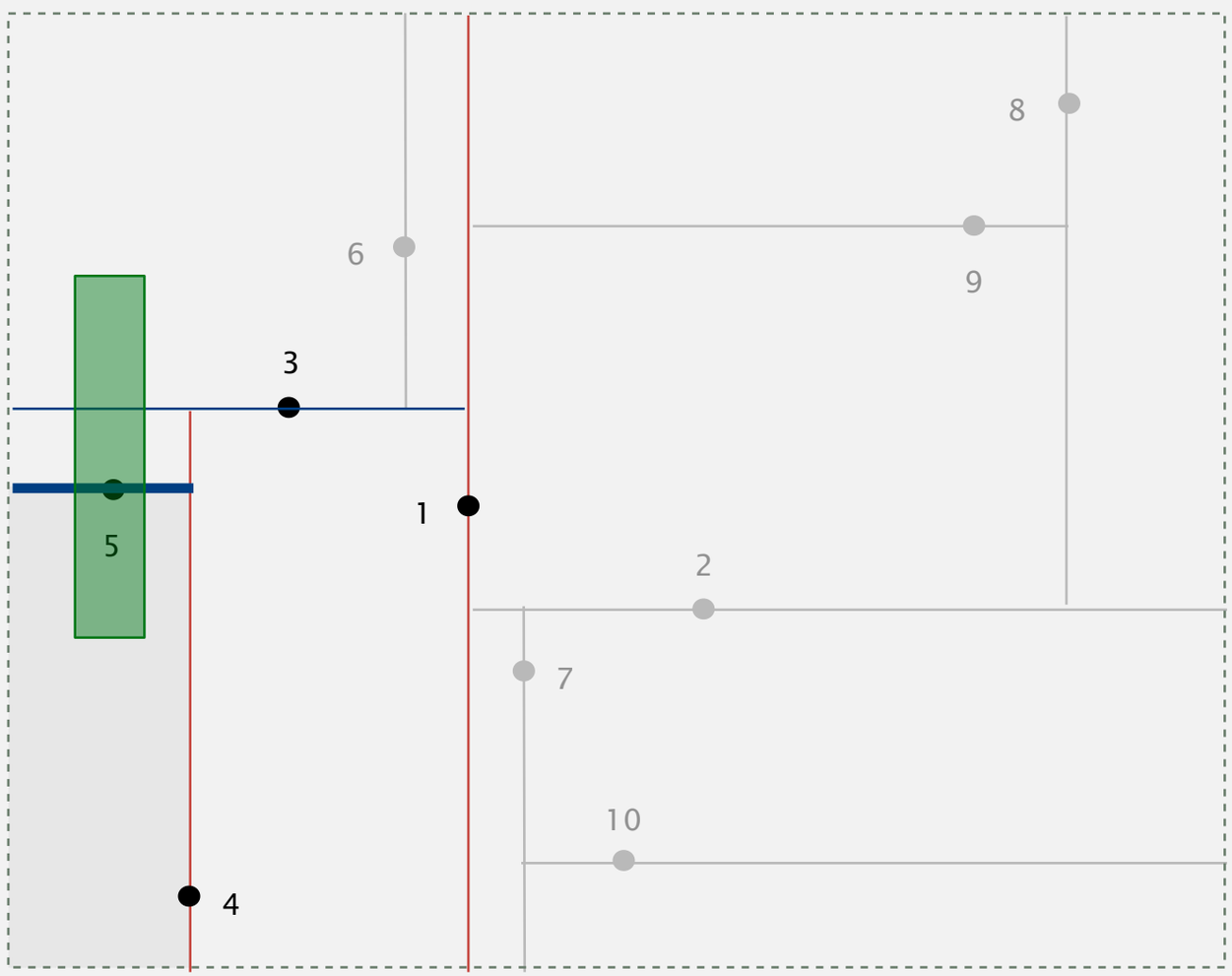


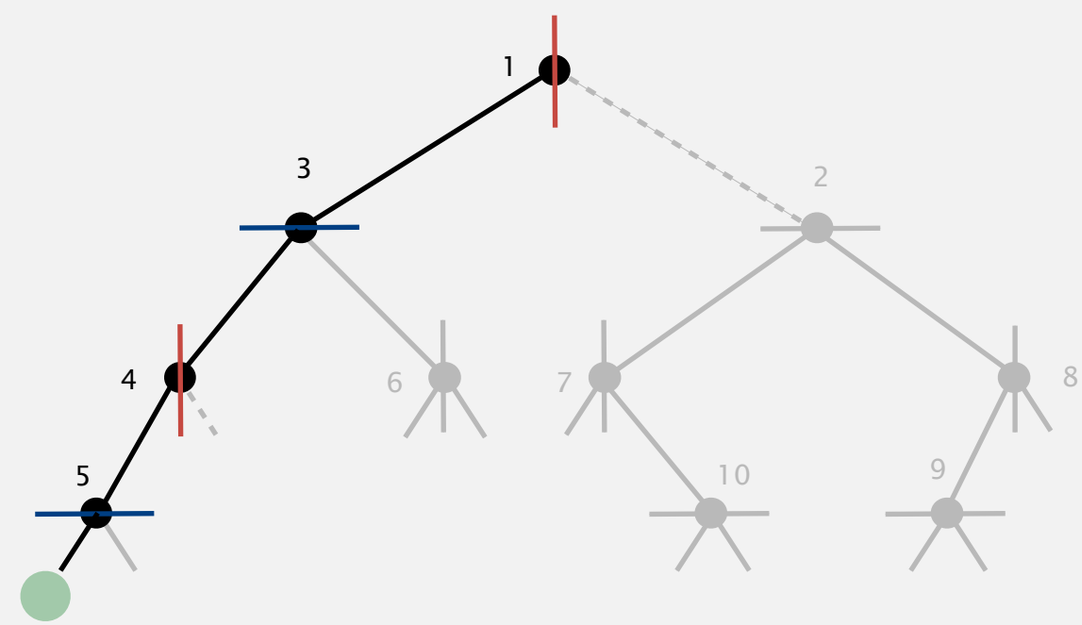




**query rectangle intersects splitting line**
**search bottom and top subtrees**

# Range search in a 2d tree

- Check if point in node lies in given rectangle.
- Recursively search left/bottom subdivision (if any could fall in rectangle).
- Recursively search right/top subdivision (if any could fall in rectangle).



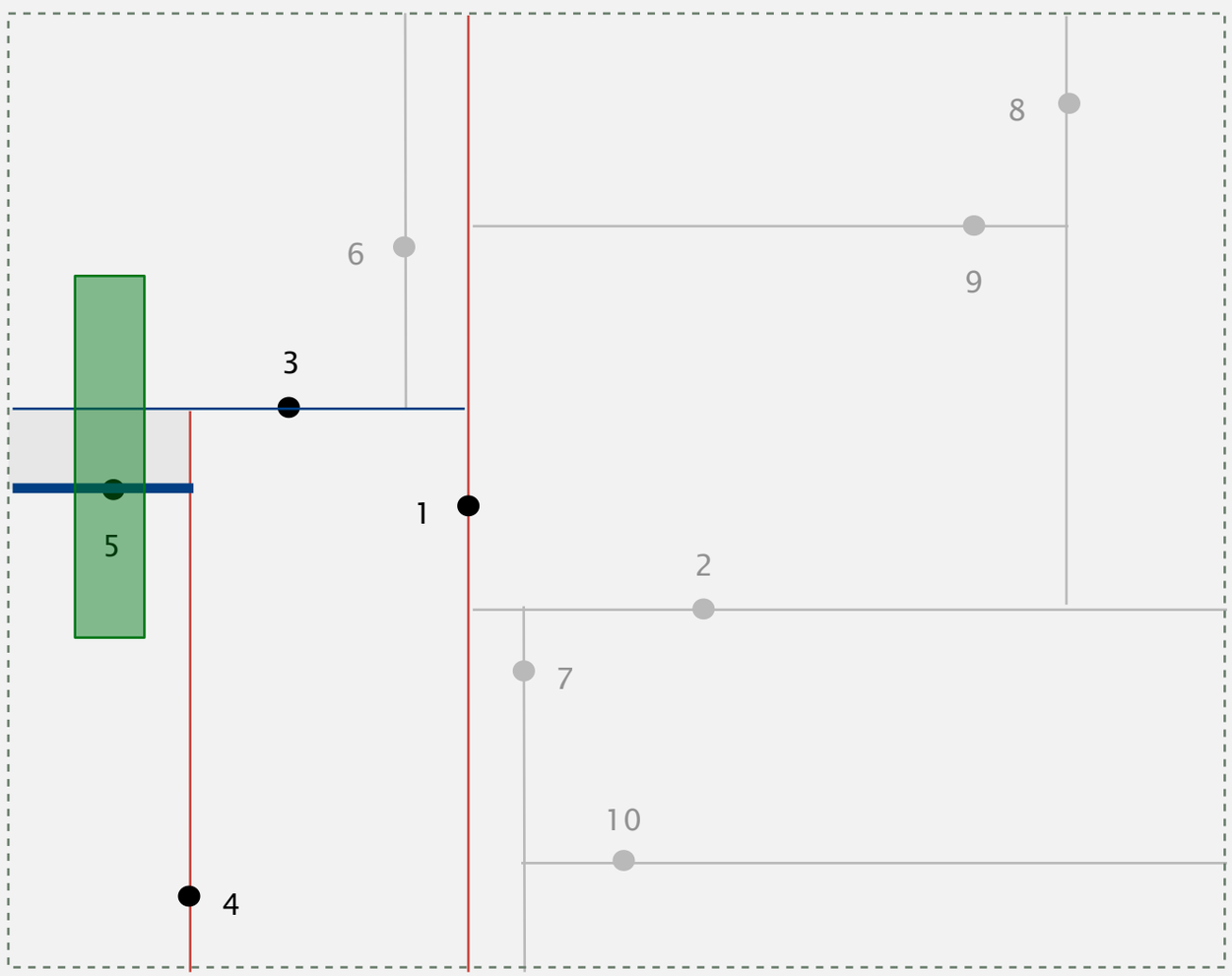



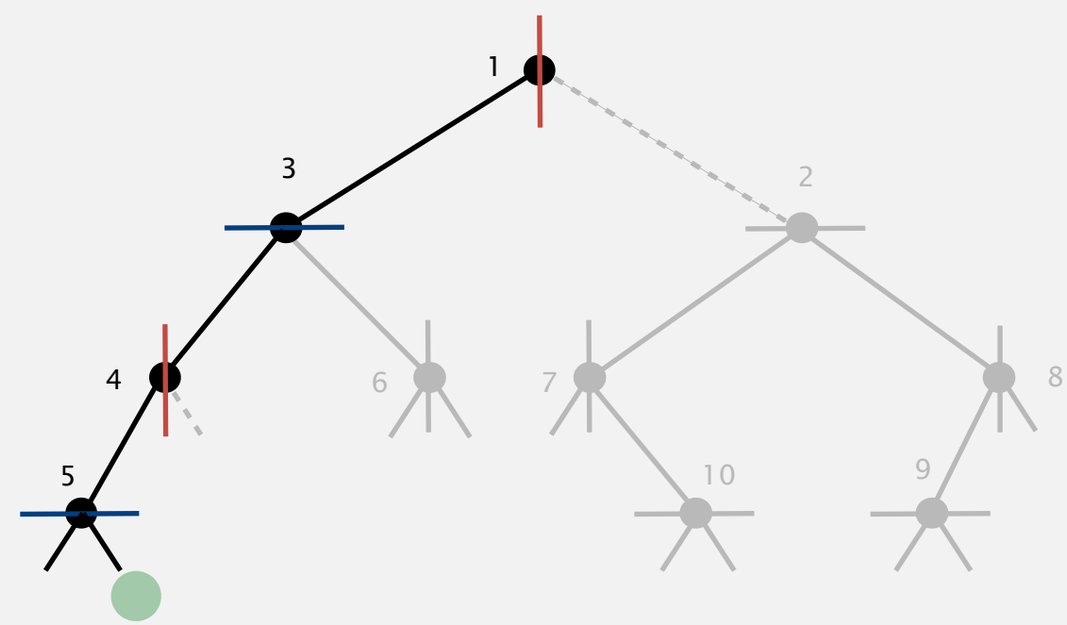

**search bottom subtree**

**stop since empty**

# Range search in a 2d tree

- Check if point in node lies in given rectangle.
- Recursively search left/bottom subdivision (if any could fall in rectangle).
- Recursively search right/top subdivision (if any could fall in rectangle).



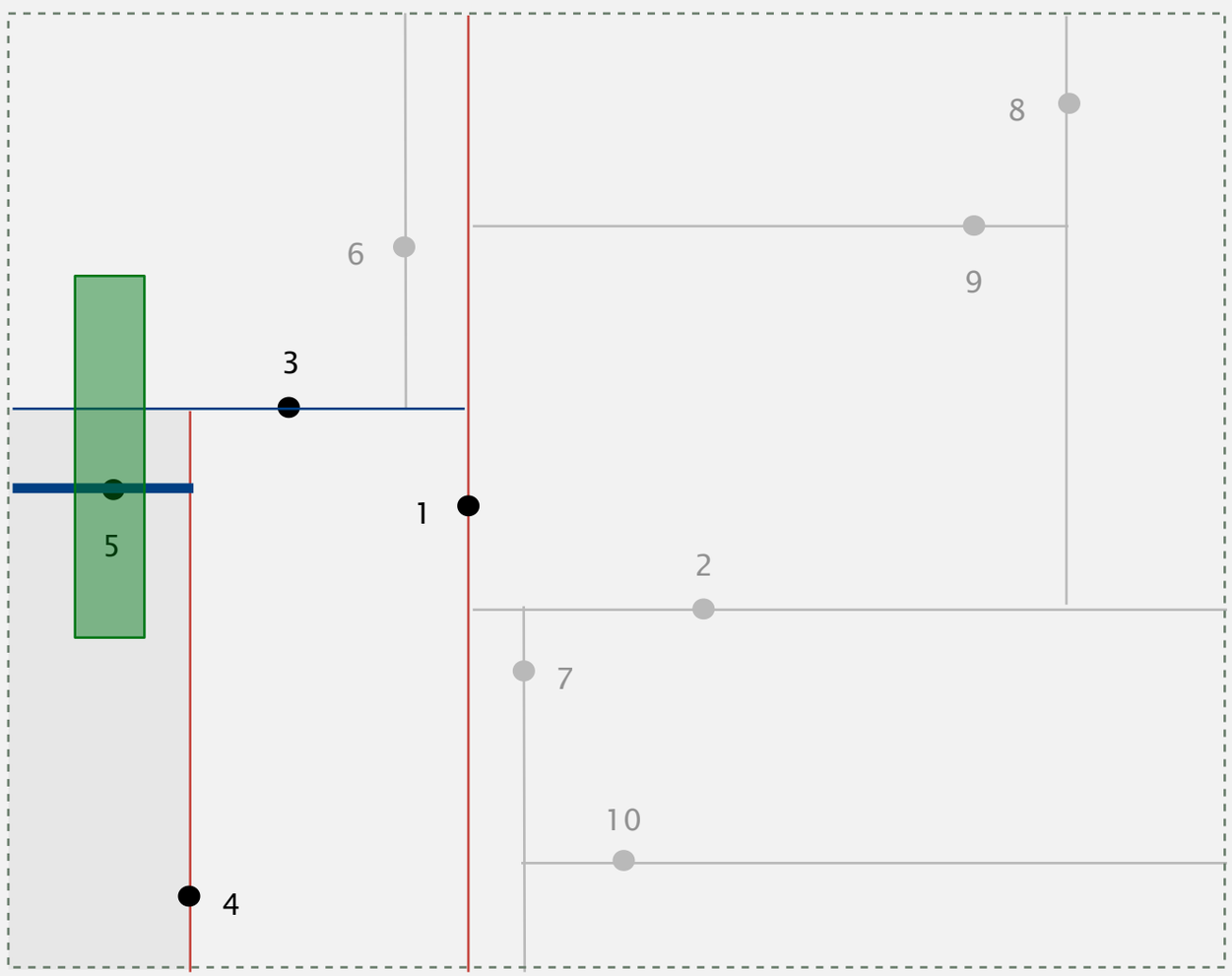



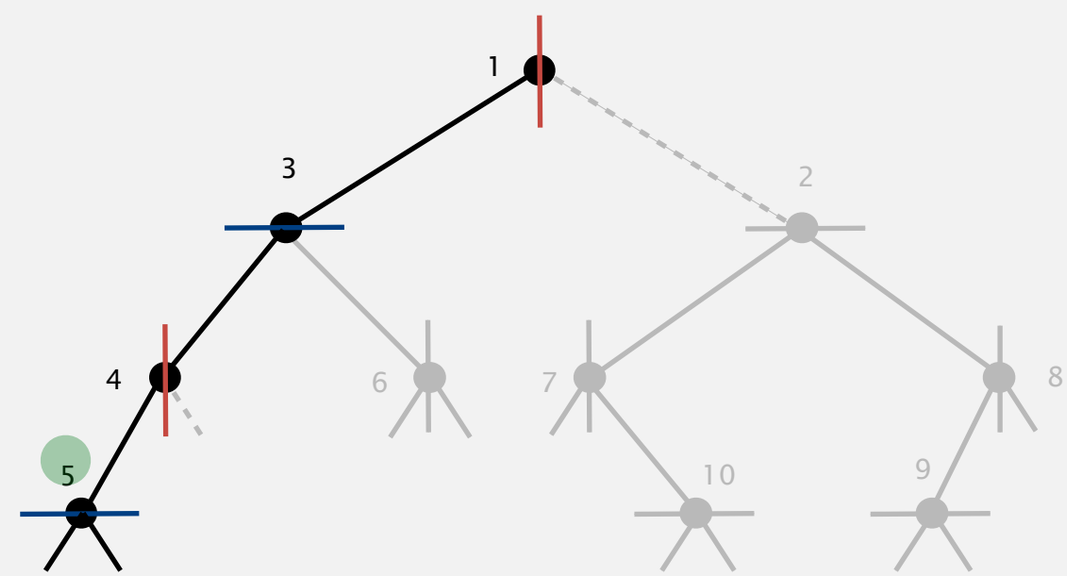

**search top subtree**

**stop since empty**

# Range search in a 2d tree

- Check if point in node lies in given rectangle.
- Recursively search left/bottom subdivision (if any could fall in rectangle).
- Recursively search right/top subdivision (if any could fall in rectangle).

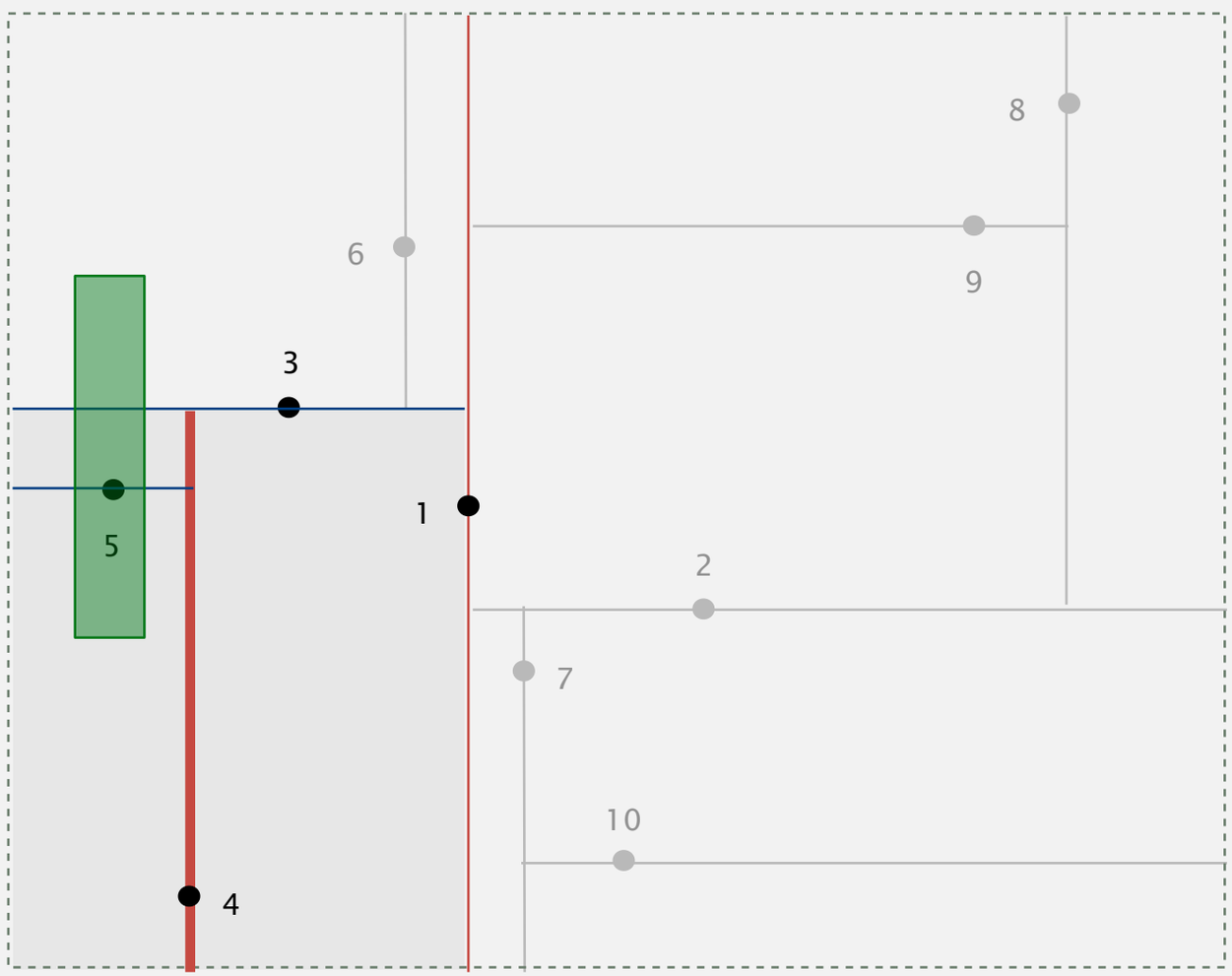


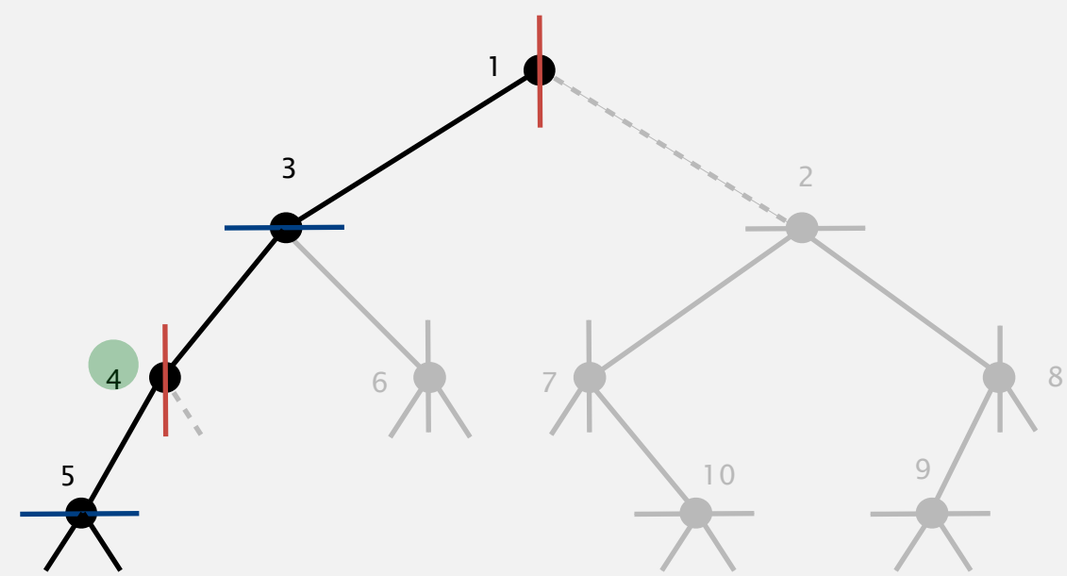




return from function call

# Range search in a 2d tree

- Check if point in node lies in given rectangle.
- Recursively search left/bottom subdivision (if any could fall in rectangle).
- Recursively search right/top subdivision (if any could fall in rectangle).

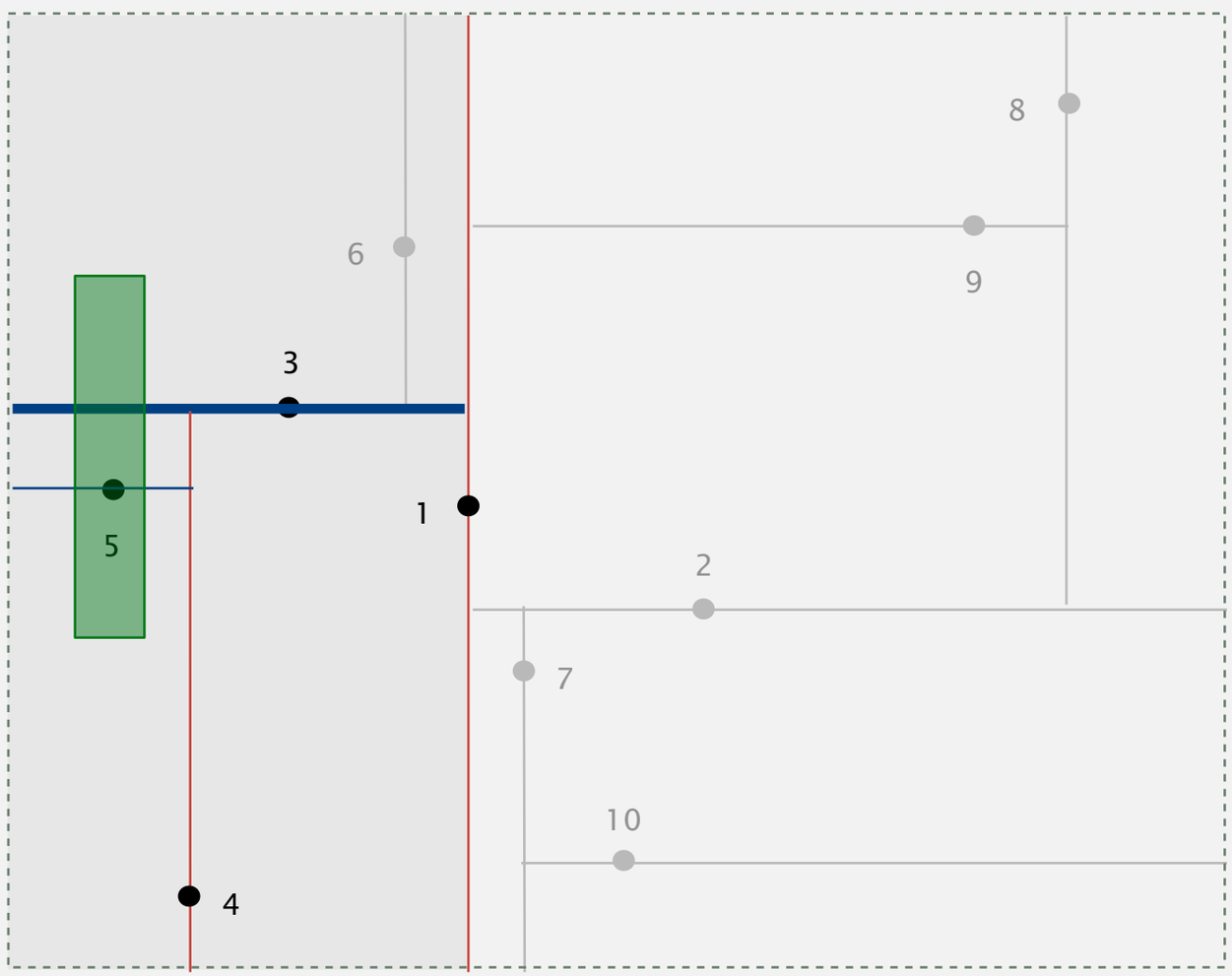


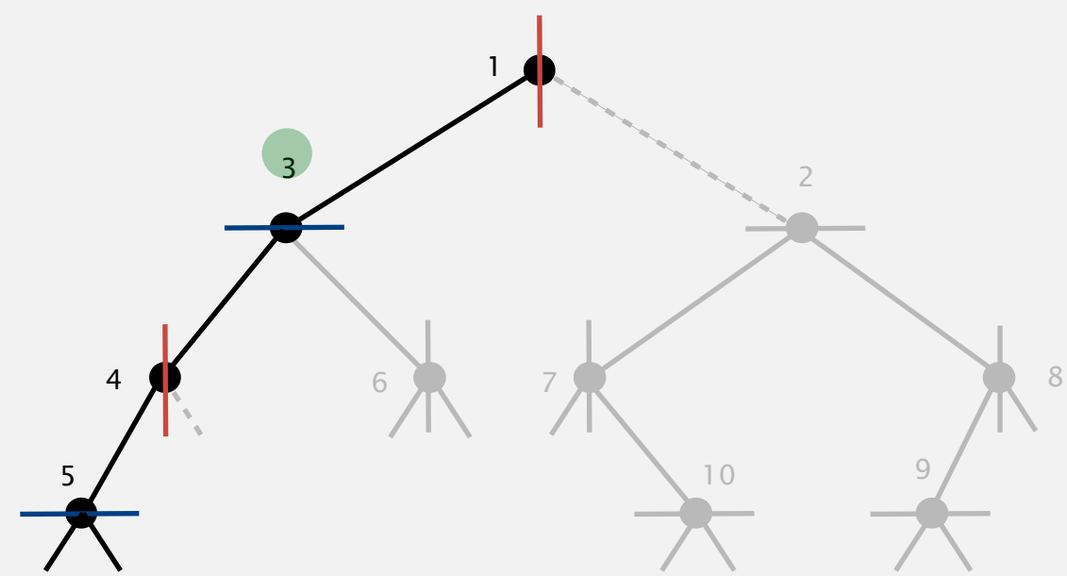




return from function call

# Range search in a 2d tree

- Check if point in node lies in given rectangle.
- Recursively search left/bottom subdivision (if any could fall in rectangle).
- Recursively search right/top subdivision (if any could fall in rectangle).

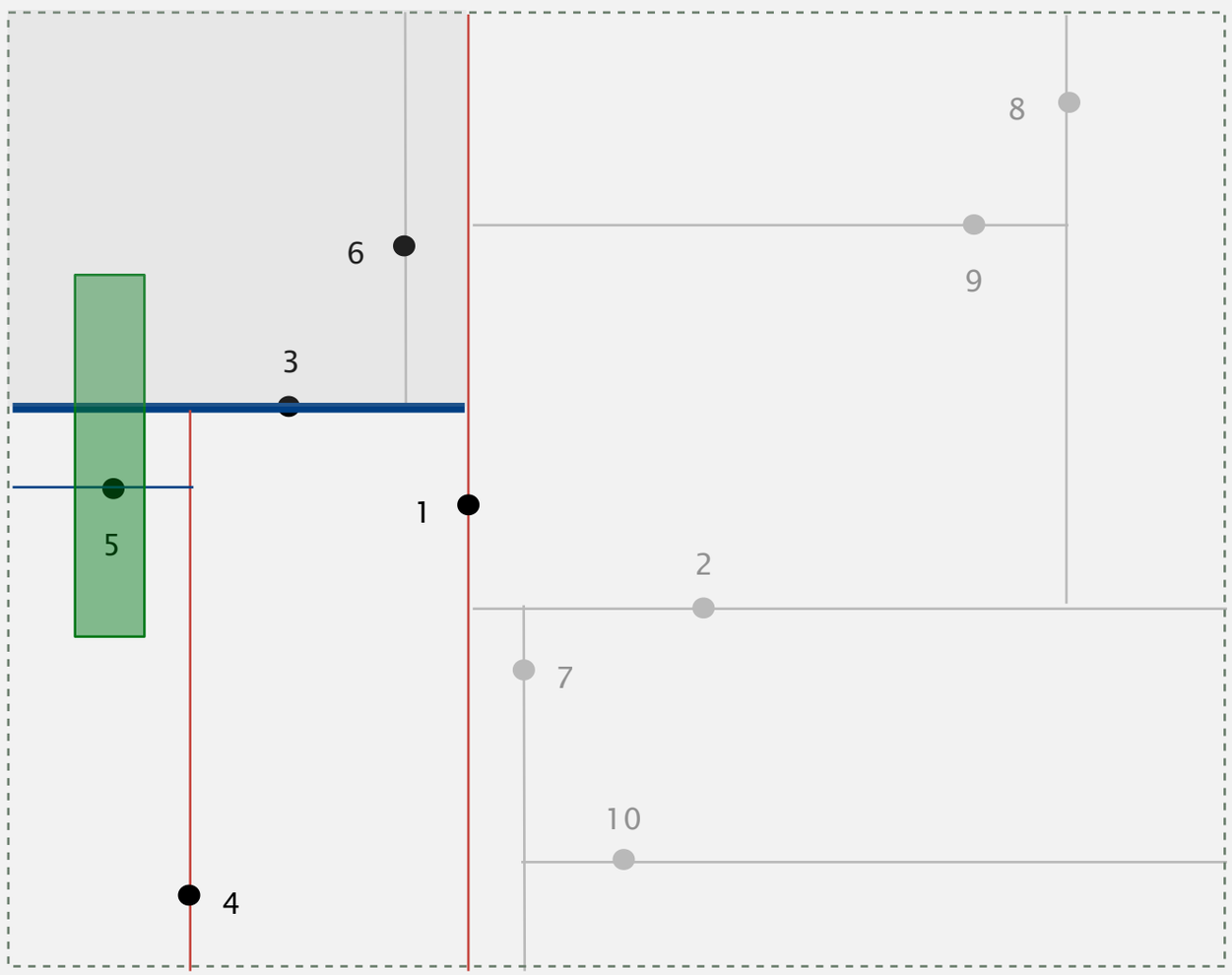


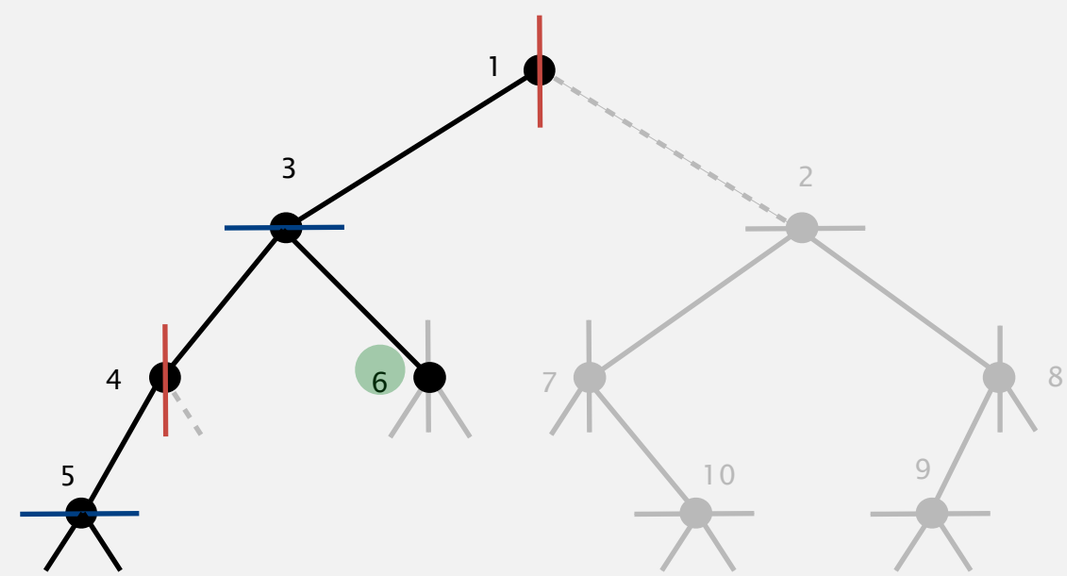




**return from function call**

# Range search in a 2d tree

- Check if point in node lies in given rectangle.
- Recursively search left/bottom subdivision (if any could fall in rectangle).
- Recursively search right/top subdivision (if any could fall in rectangle).



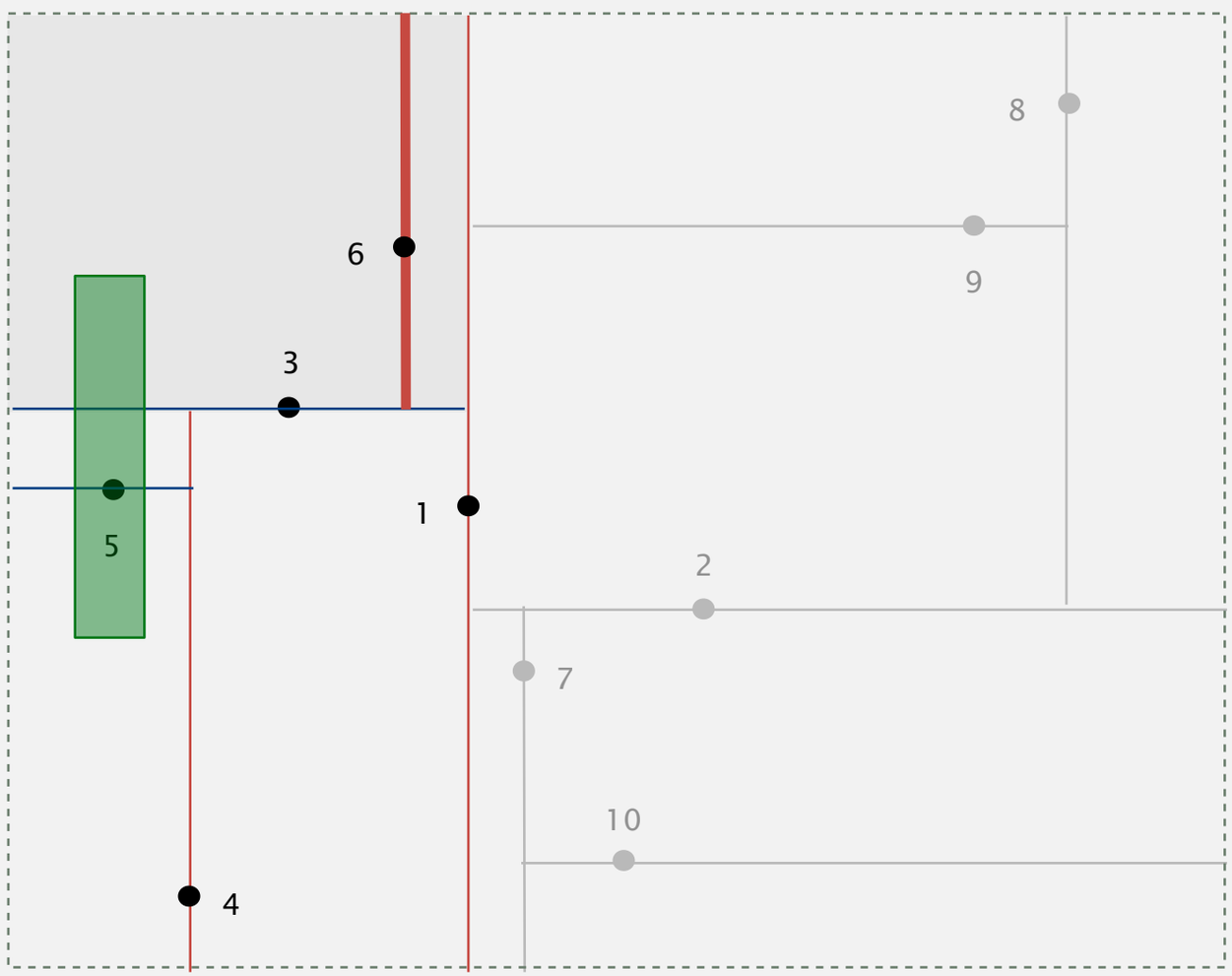



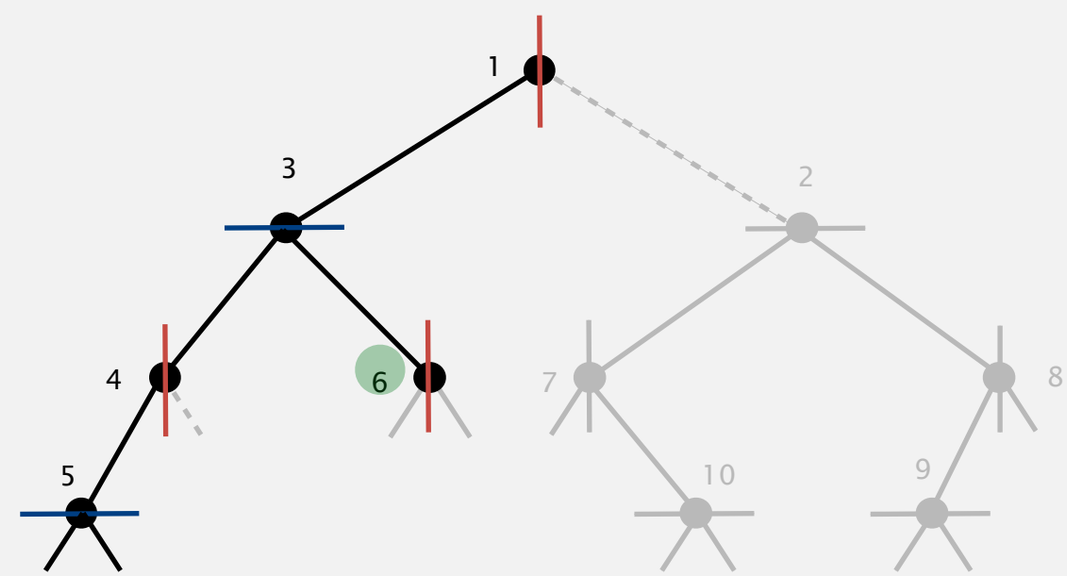

**search top subtree**

**check if query rectangle contains point 6**

# Range search in a 2d tree

- Check if point in node lies in given rectangle.
- Recursively search left/bottom subdivision (if any could fall in rectangle).
- Recursively search right/top subdivision (if any could fall in rectangle).

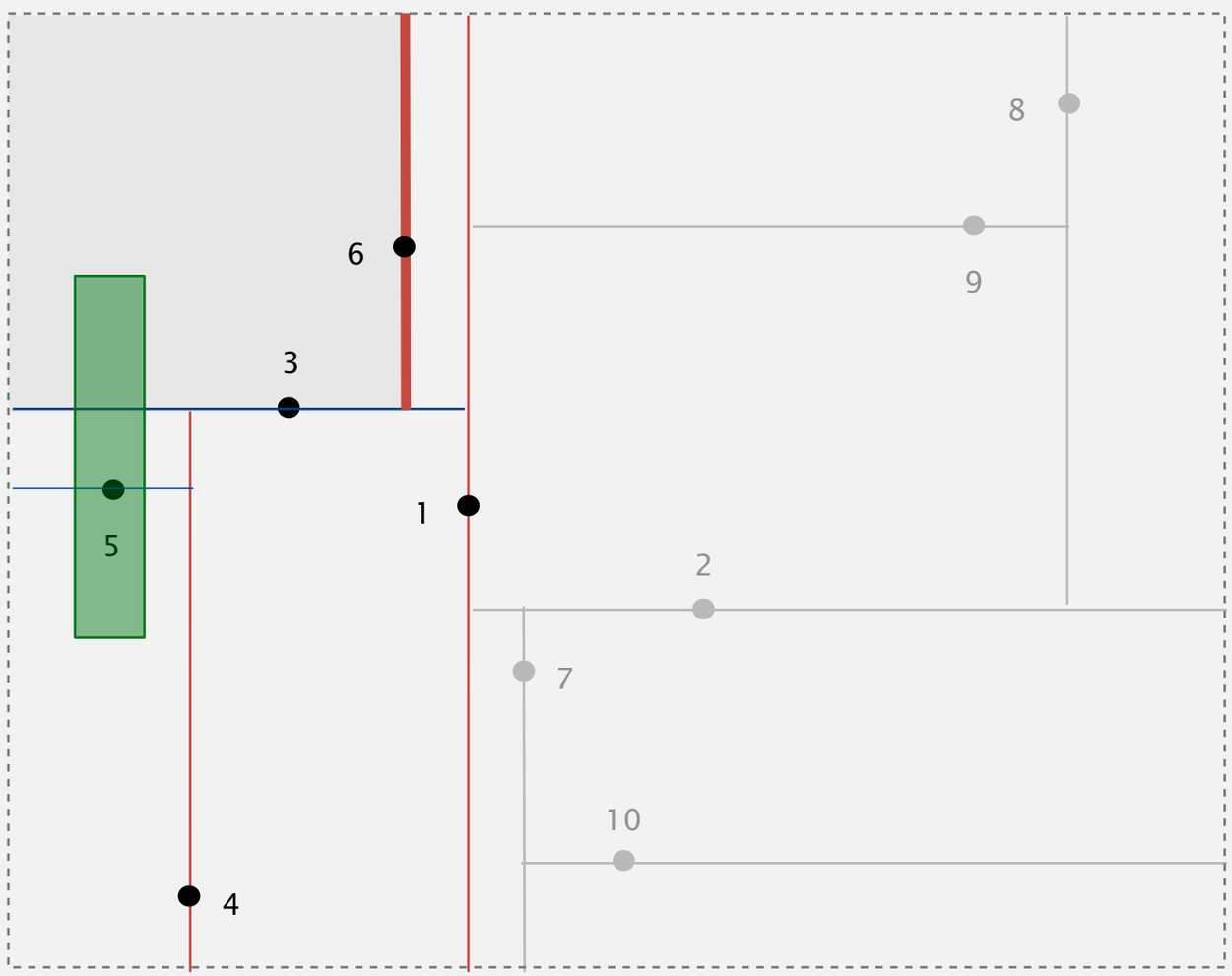


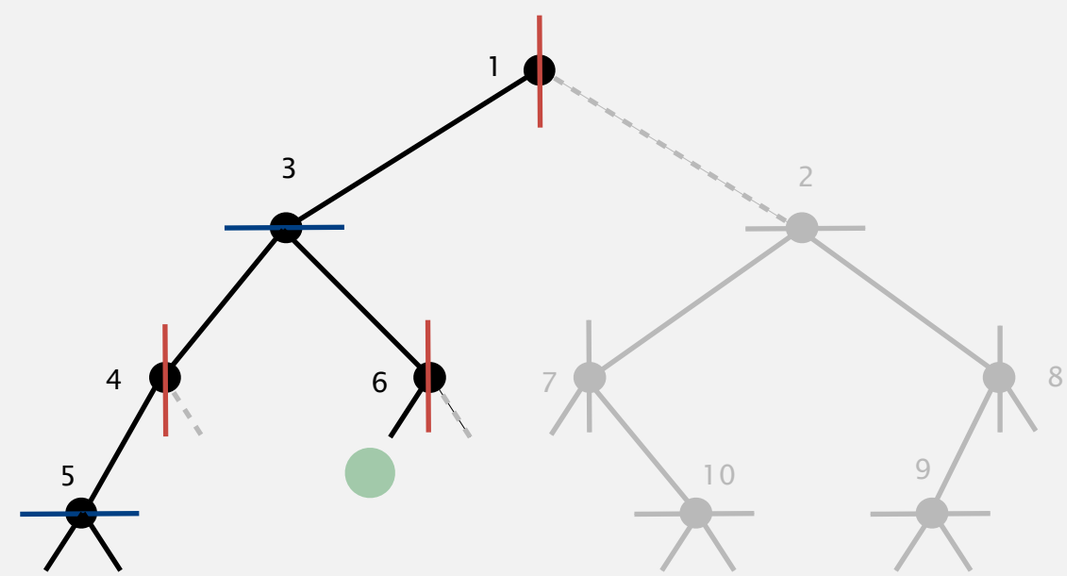




**query rectangle to left of splitting line**
**search only in left subtree**

# Range search in a 2d tree

- Check if point in node lies in given rectangle.
- Recursively search left/bottom subdivision (if any could fall in rectangle).
- Recursively search right/top subdivision (if any could fall in rectangle).



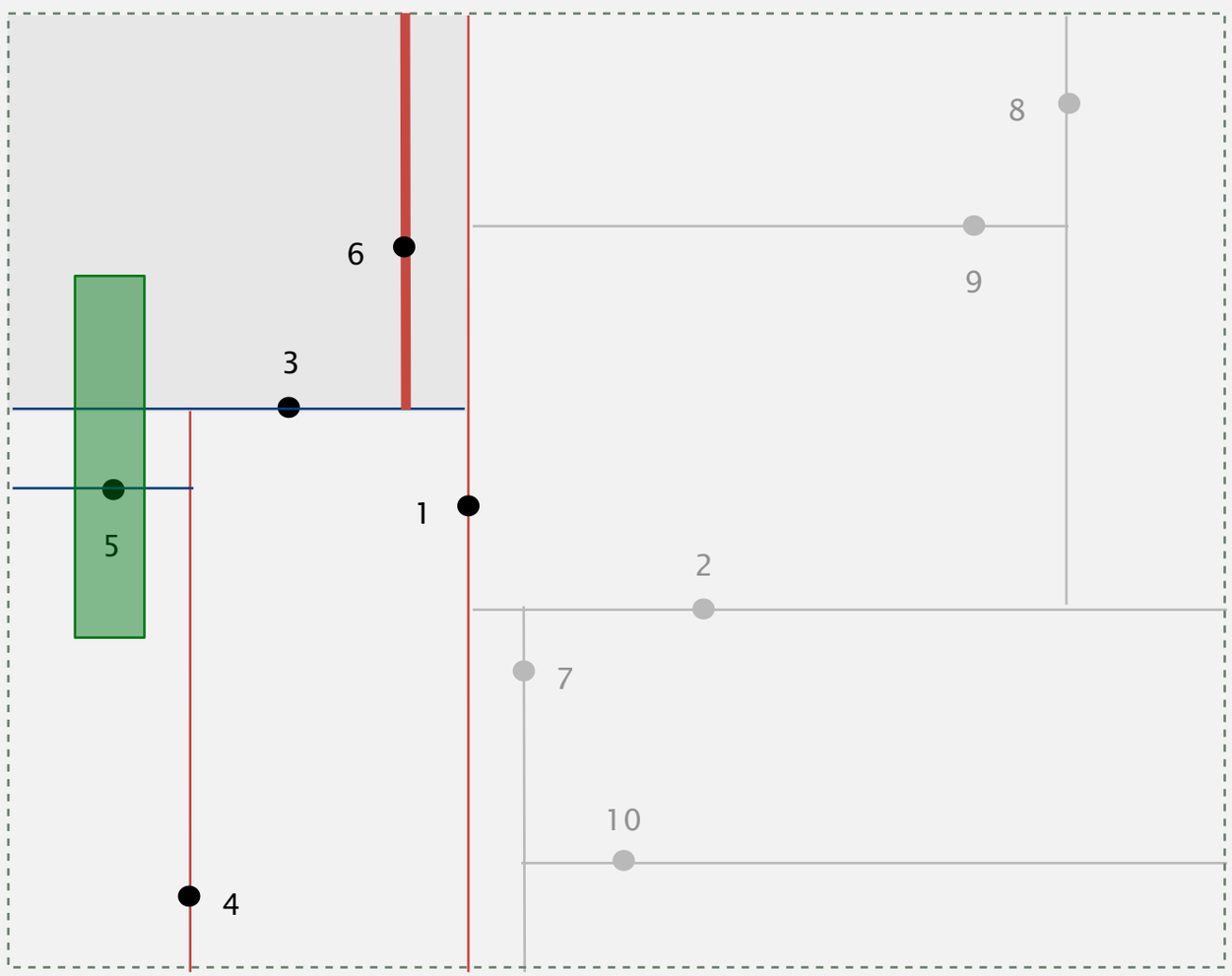



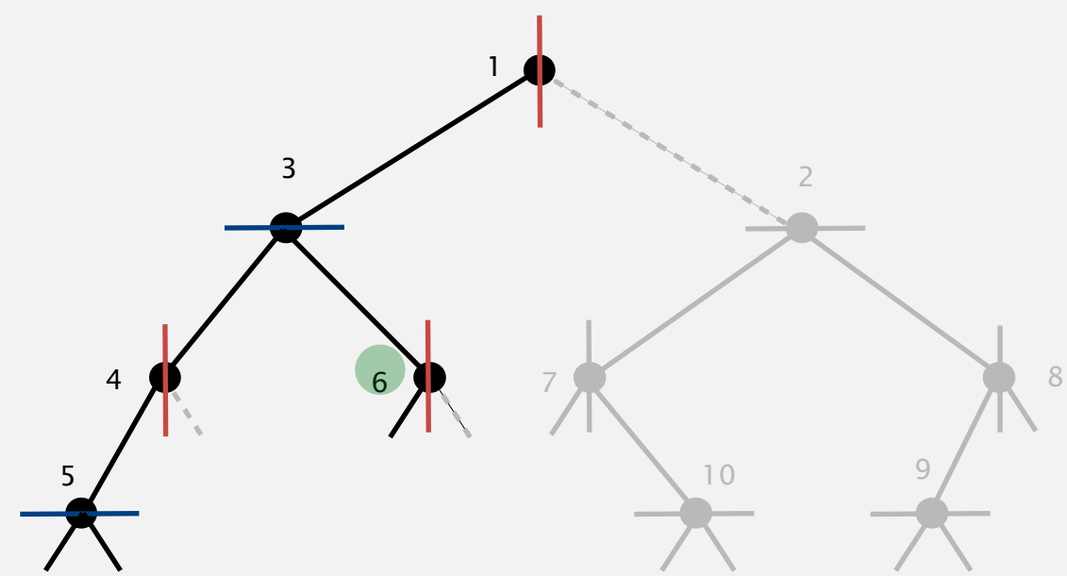

**search left subtree**

**stop since empty**

# Range search in a 2d tree

- Check if point in node lies in given rectangle.
- Recursively search left/bottom subdivision (if any could fall in rectangle).
- Recursively search right/top subdivision (if any could fall in rectangle).

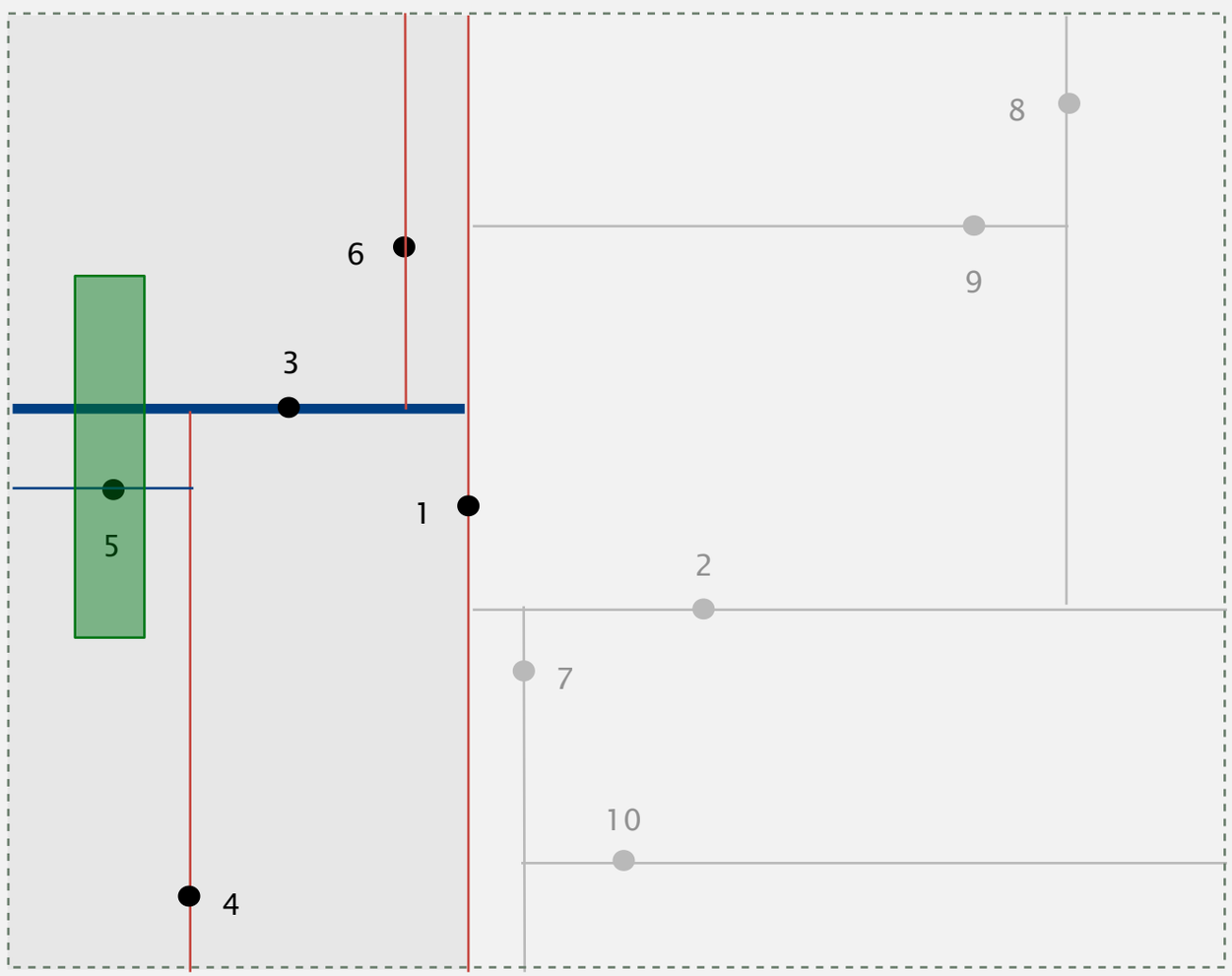


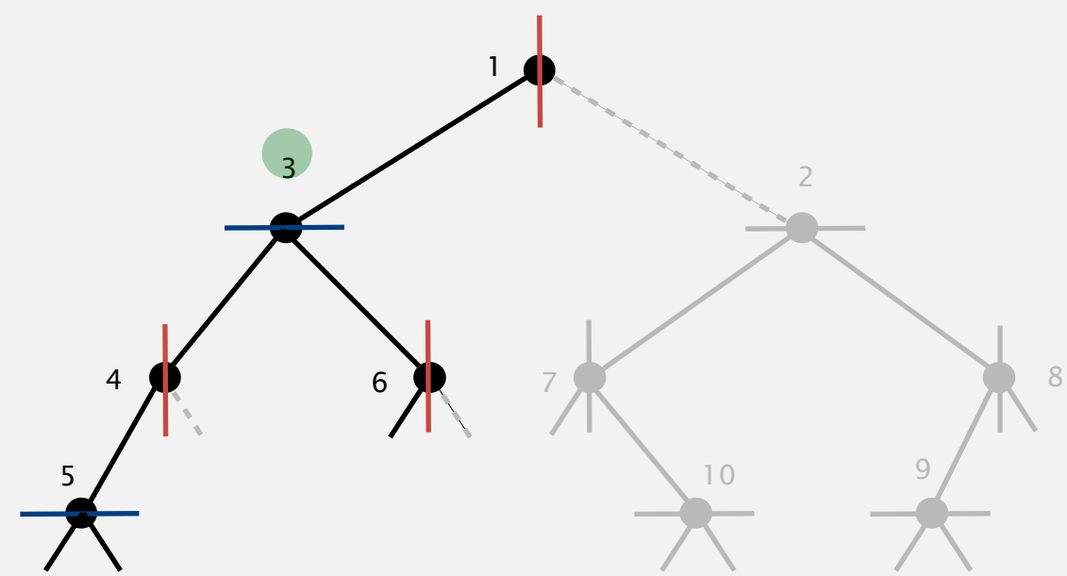




return from function call

# Range search in a 2d tree

- Check if point in node lies in given rectangle.
- Recursively search left/bottom subdivision (if any could fall in rectangle).
- Recursively search right/top subdivision (if any could fall in rectangle).

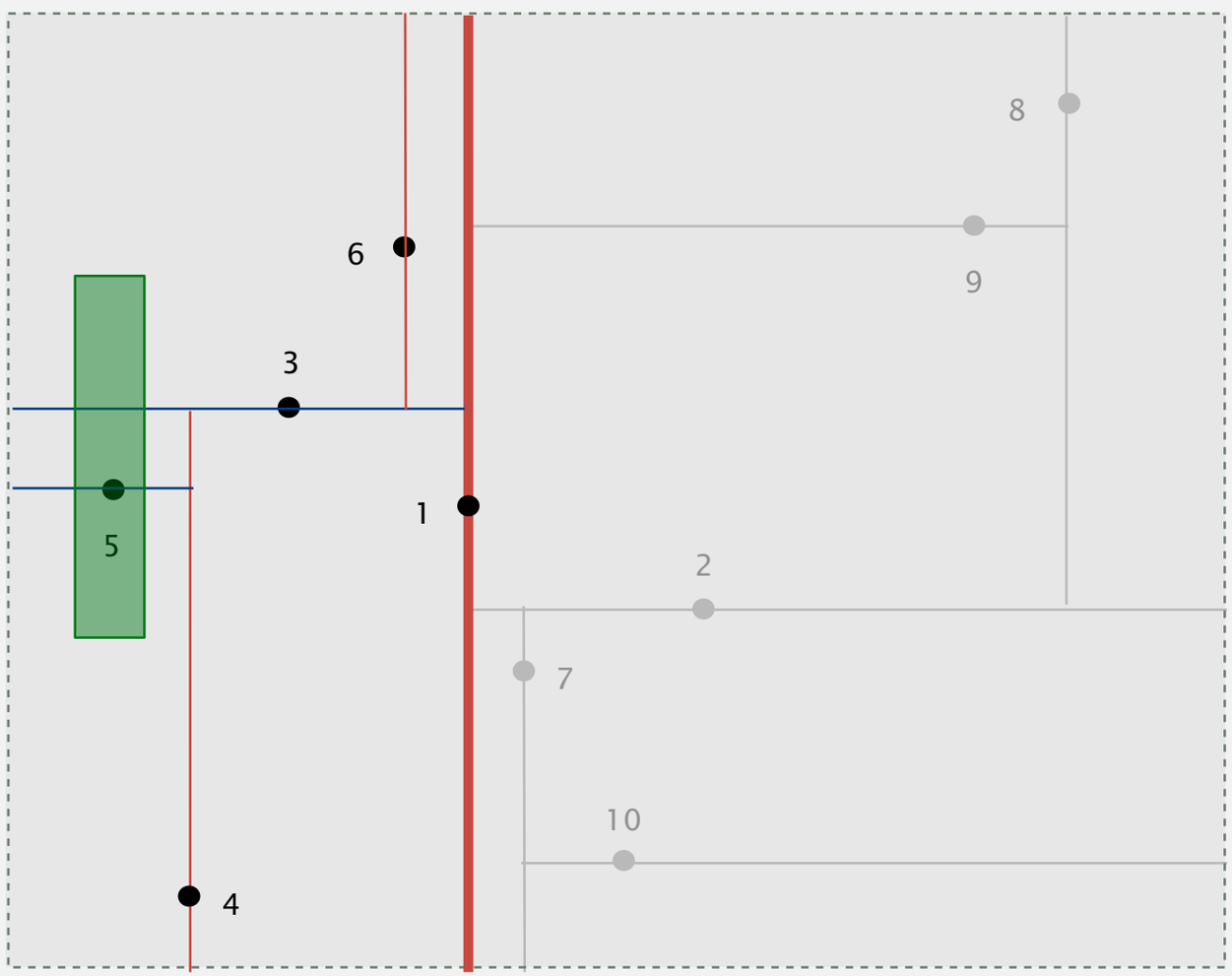


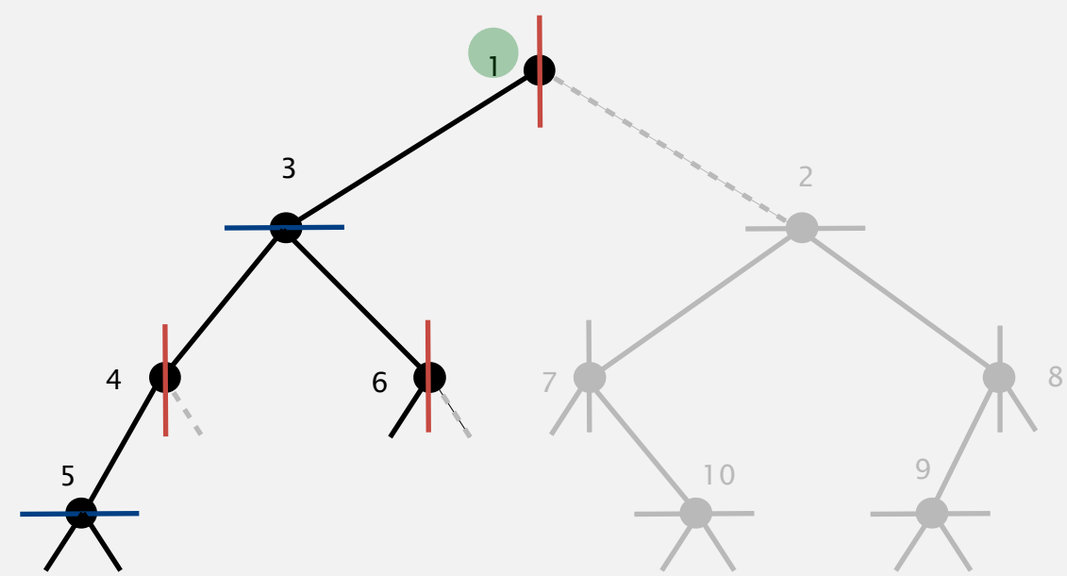




return from function call

# Range search in a 2d tree

- Check if point in node lies in given rectangle.
- Recursively search left/bottom subdivision (if any could fall in rectangle).
- Recursively search right/top subdivision (if any could fall in rectangle).

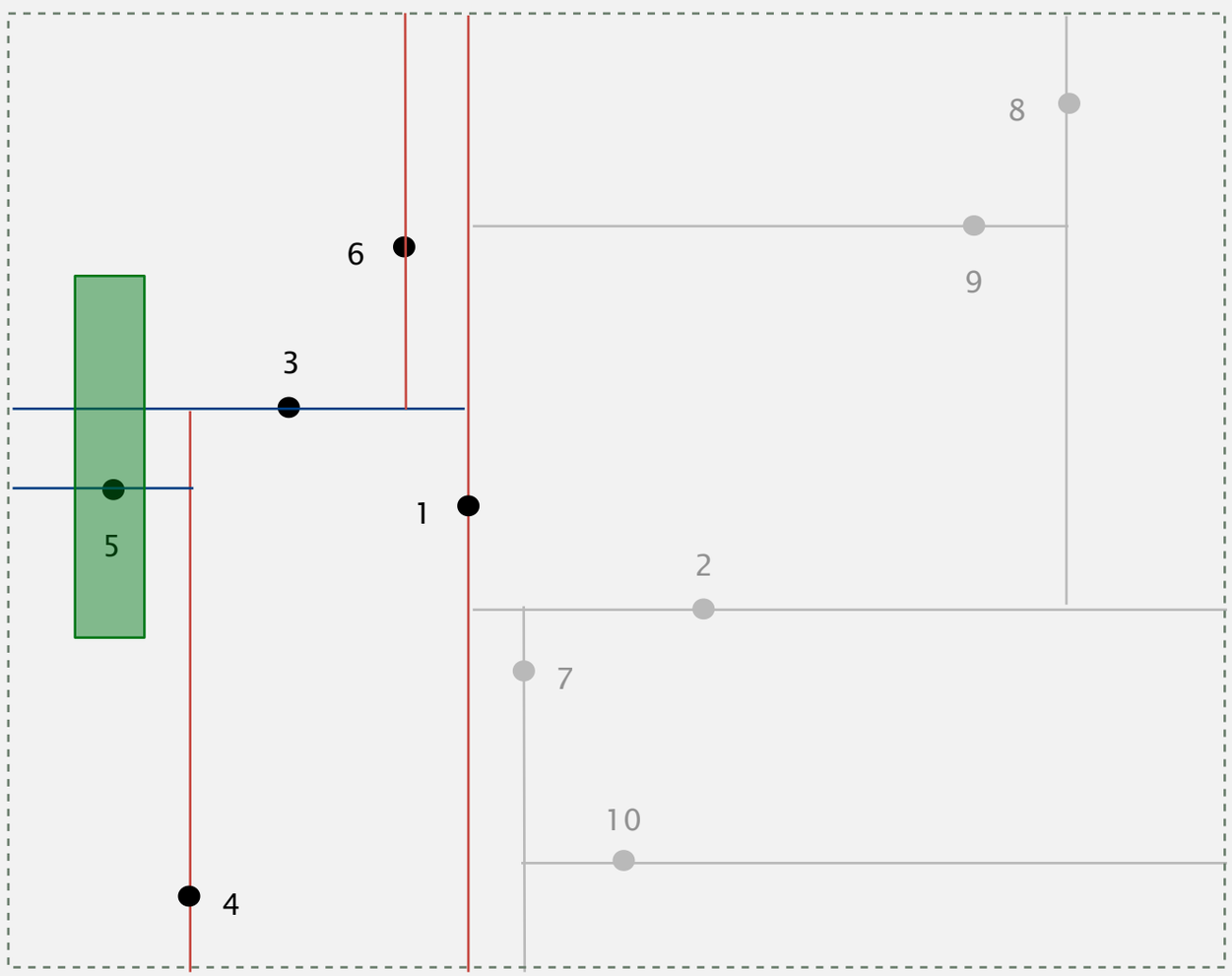


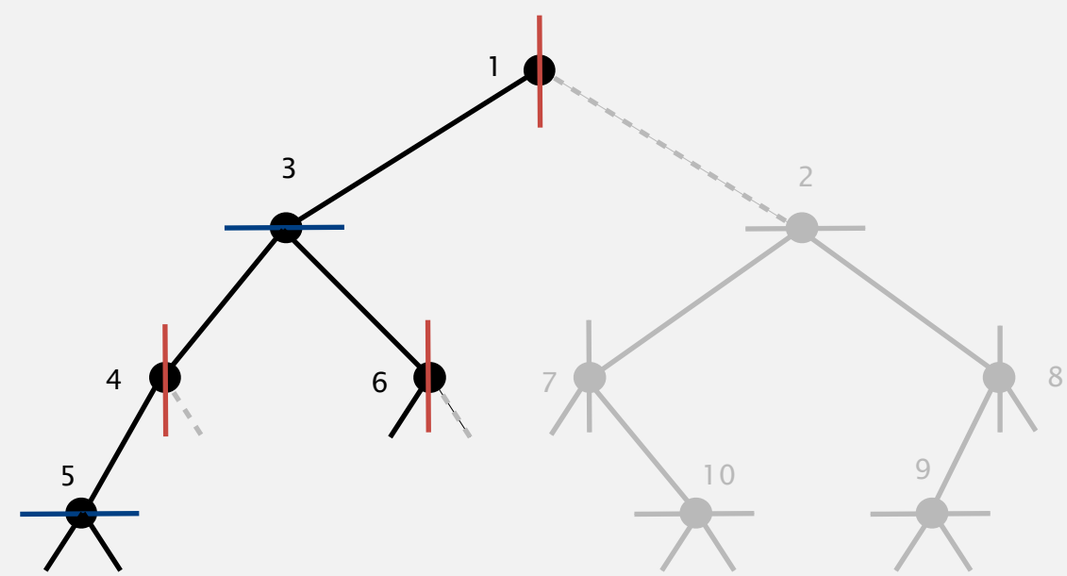




return from function call

# Range search in a 2d tree

- Check if point in node lies in given rectangle.
- Recursively search left/bottom subdivision (if any could fall in rectangle).
- Recursively search right/top subdivision (if any could fall in rectangle).




done

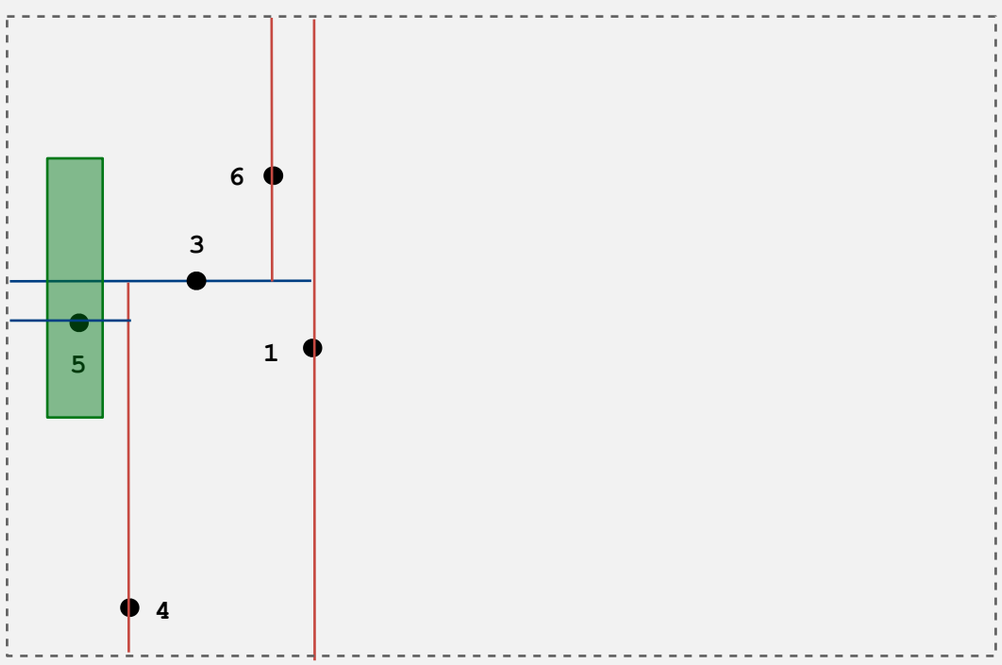
# Range search in a 2d tree

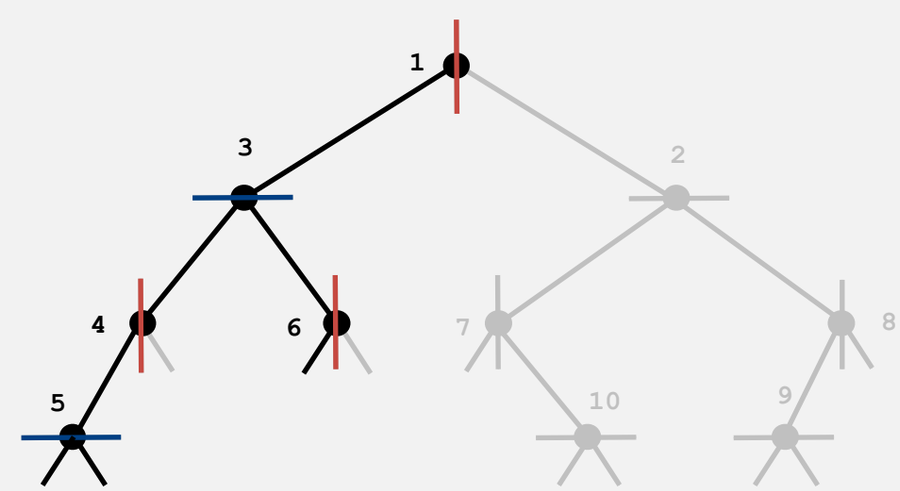
Range search. Find all points in a query axis-aligned rectangle.

- Check if point in node lies in given rectangle.
- Recursively search left/bottom subdivision (if any could fall in rectangle).
- Recursively search right/top subdivision (if any could fall in rectangle).

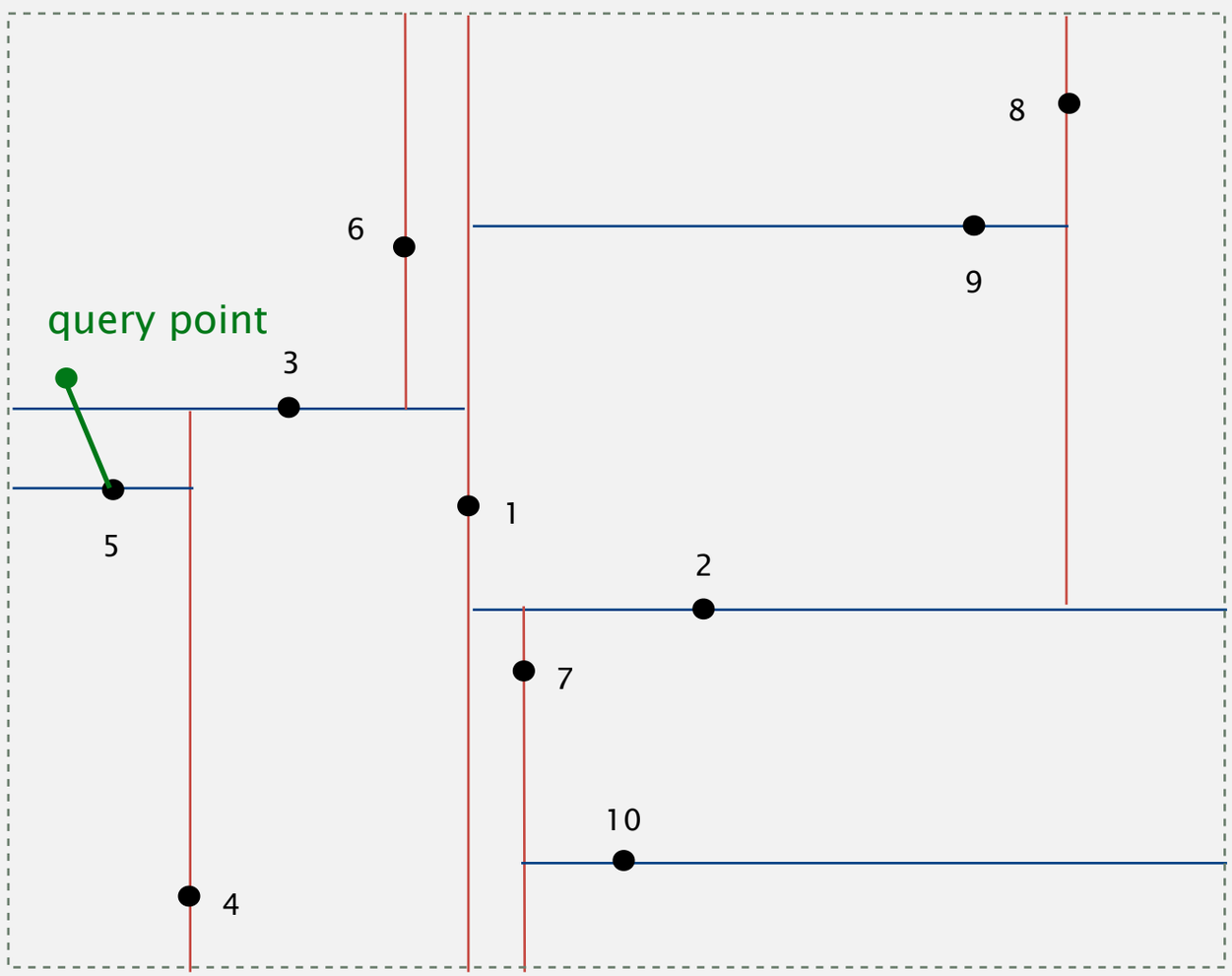
Typical case. $R + \log N$.

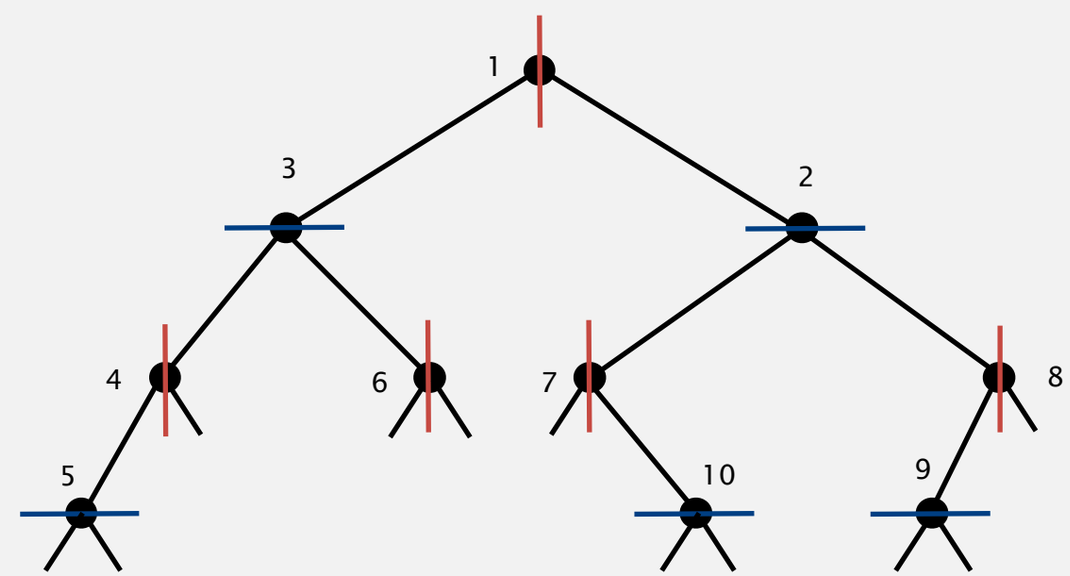
Worst case (assuming tree is balanced). $R + \sqrt{N}$.



# Nearest neighbor search in a 2d tree

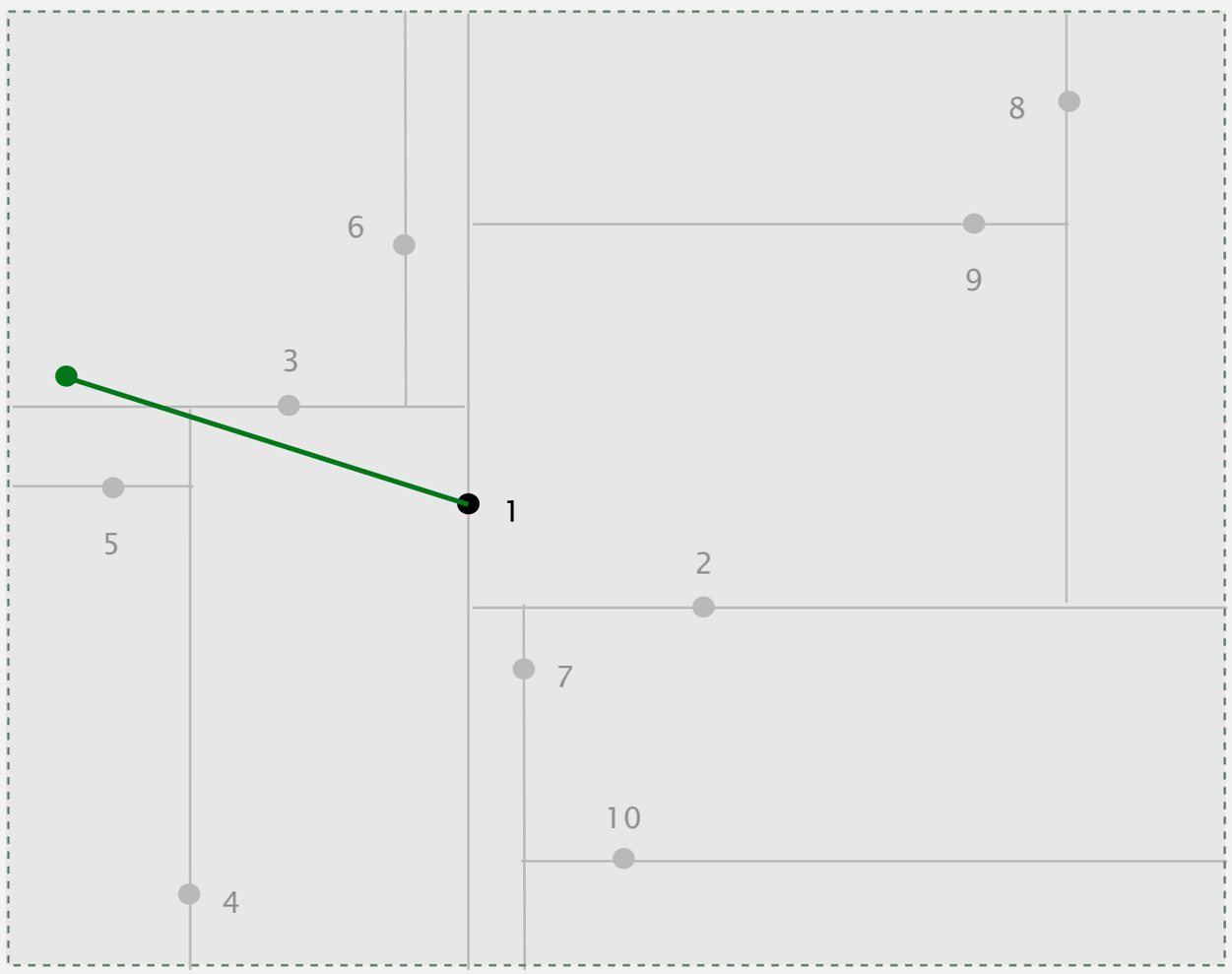


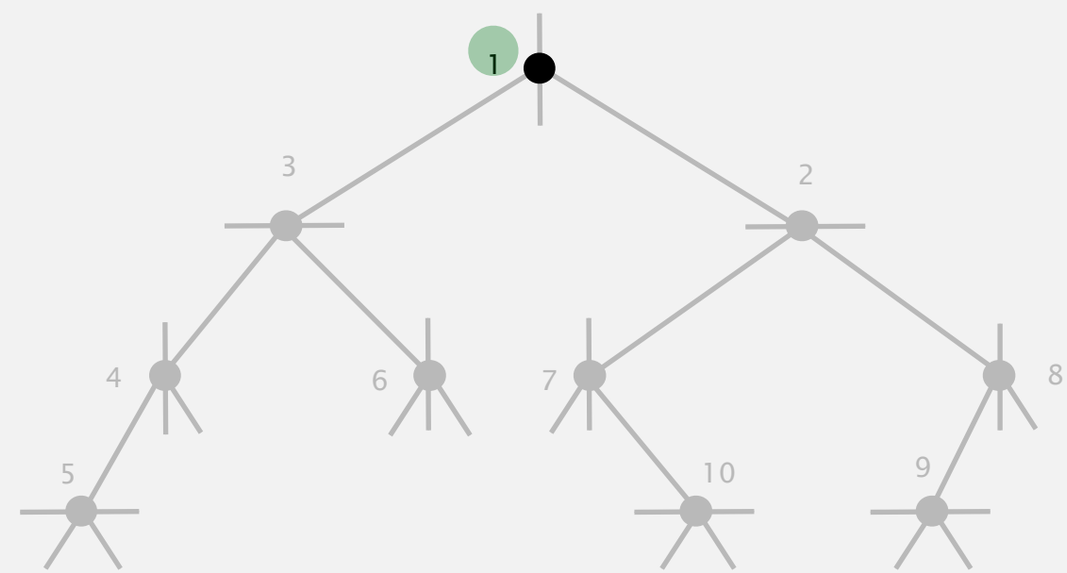




**find closest points in 2d tree to green query point**

# Nearest neighbor search in a 2d tree

- Check distance from point in node to query point.
- Recursively search left/bottom subdivision (if it could contain a closer point).
- Recursively search right/top subdivision (if it could contain a closer point).
- Organize recursive method so that it begins by searching for query point.



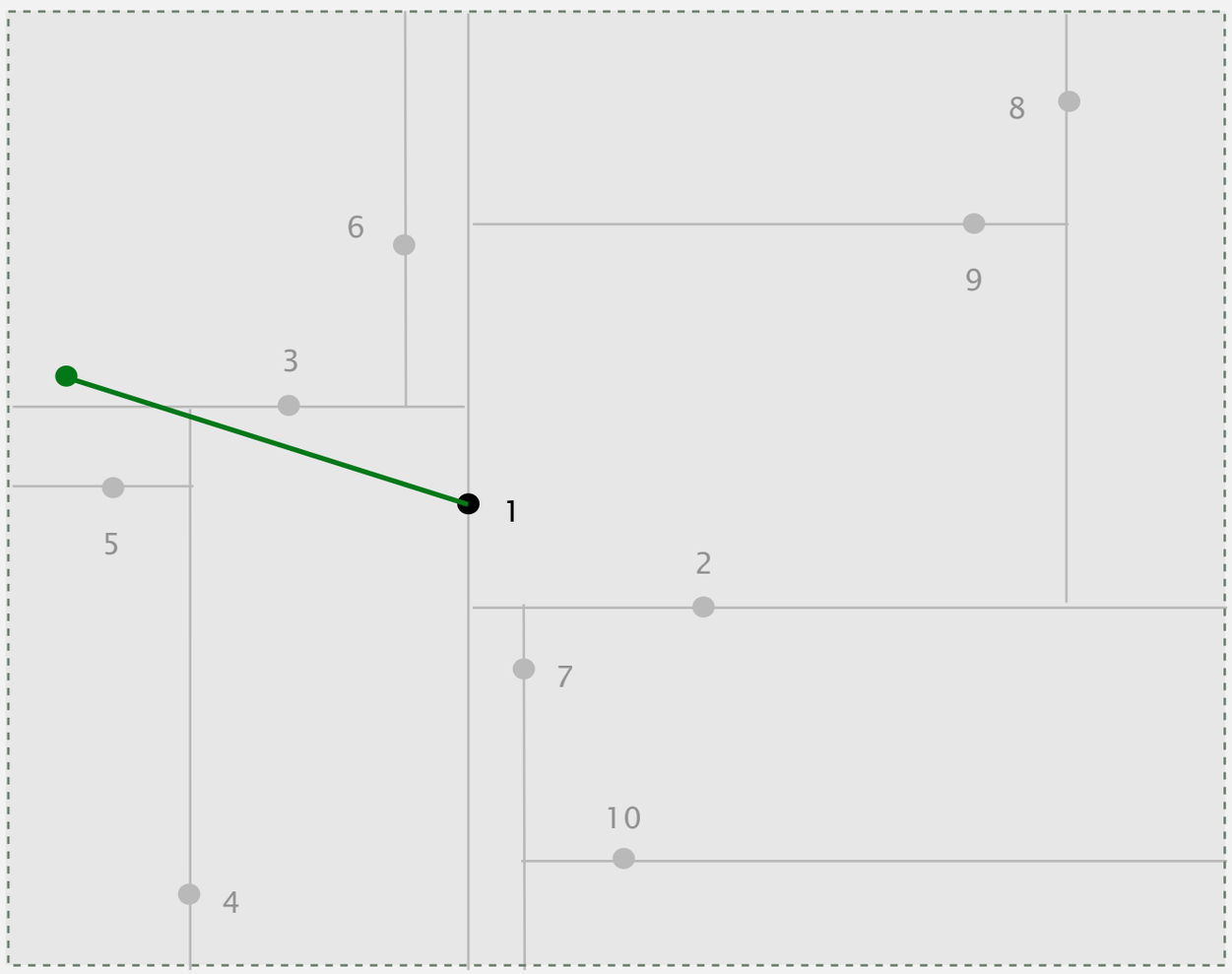

**search root node**

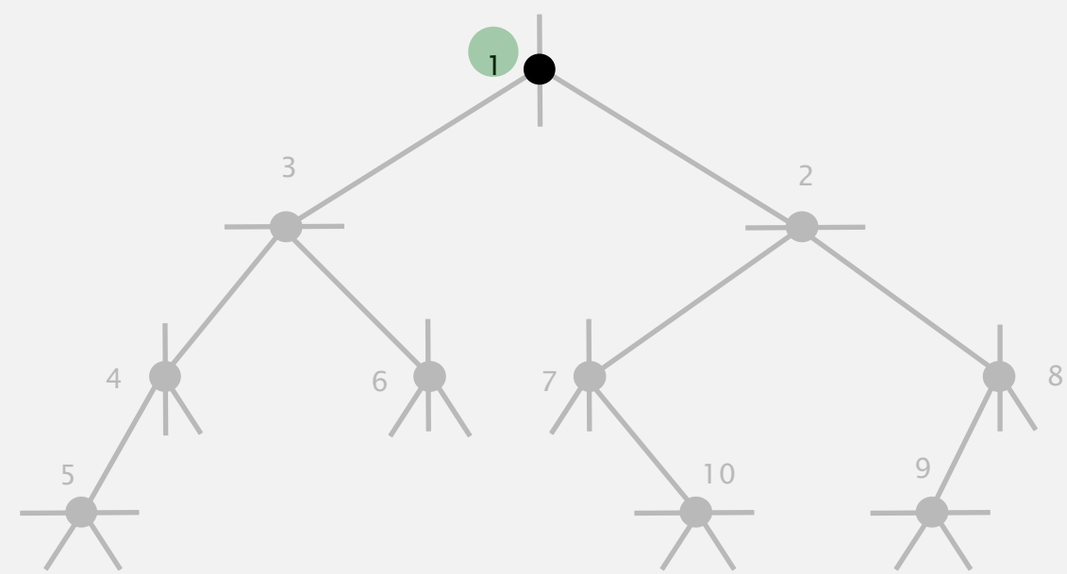
**compute distance from query point to 1**

**(update champion nearest neighbor)**

# Nearest neighbor search in a 2d tree

- Check distance from point in node to query point.
- Recursively search left/bottom subdivision (if it could contain a closer point).
- Recursively search right/top subdivision (if it could contain a closer point).
- Organize recursive method so that it begins by searching for query point.

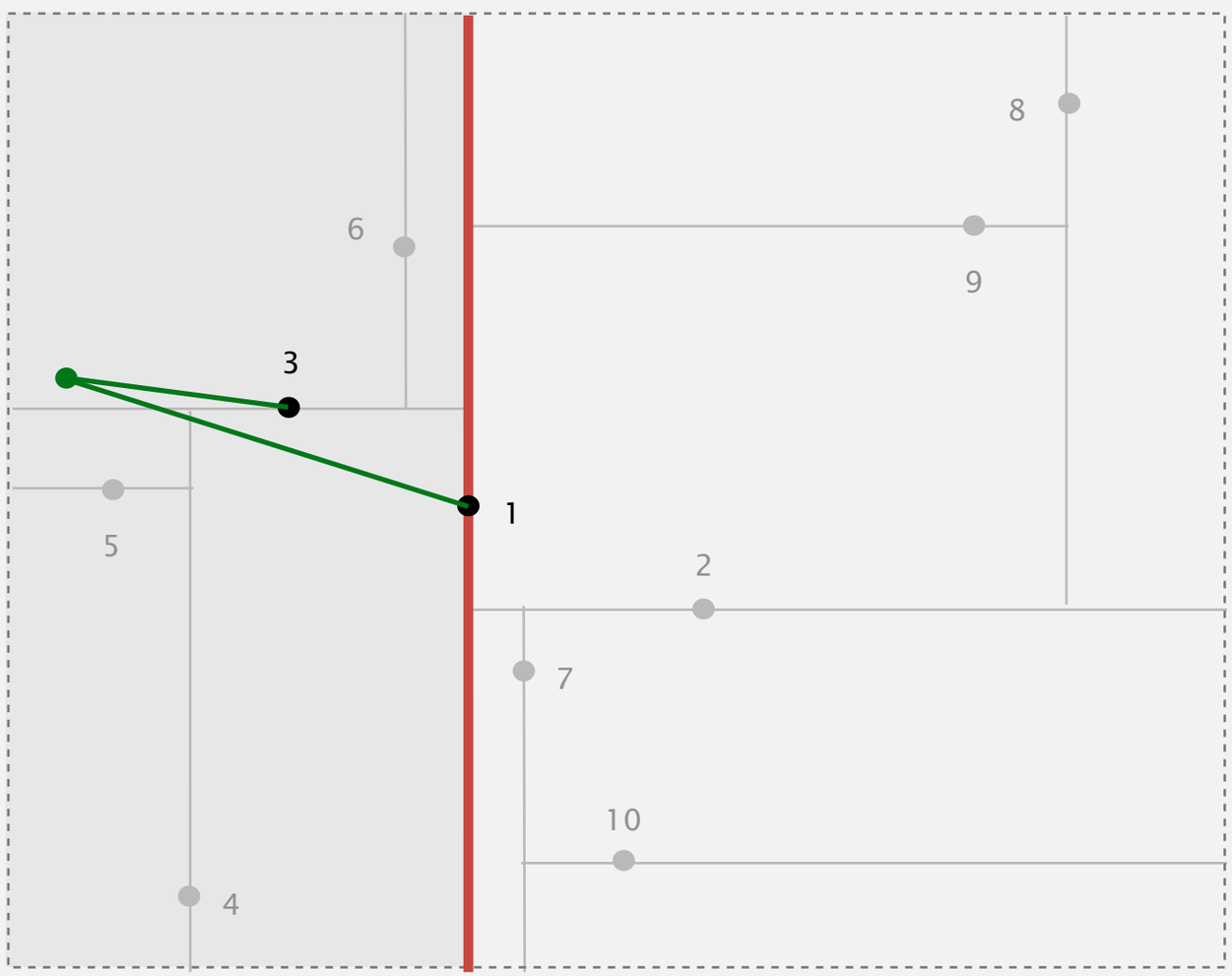


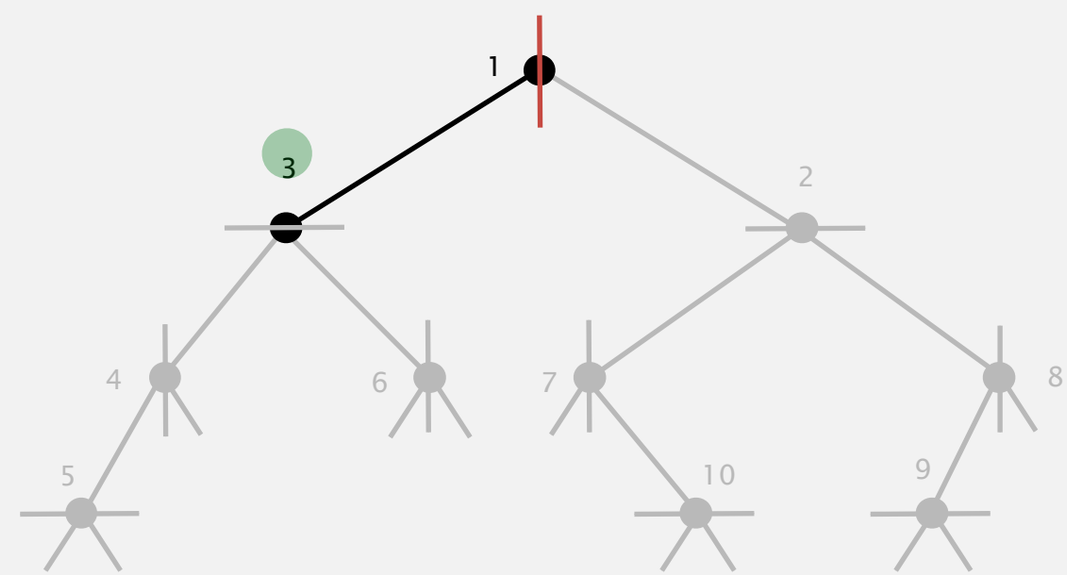




**query point is to the left of splitting line**
**search left subtree first**

# Nearest neighbor search in a 2d tree

- Check distance from point in node to query point.
- Recursively search left/bottom subdivision (if it could contain a closer point).
- Recursively search right/top subdivision (if it could contain a closer point).
- Organize recursive method so that it begins by searching for query point.

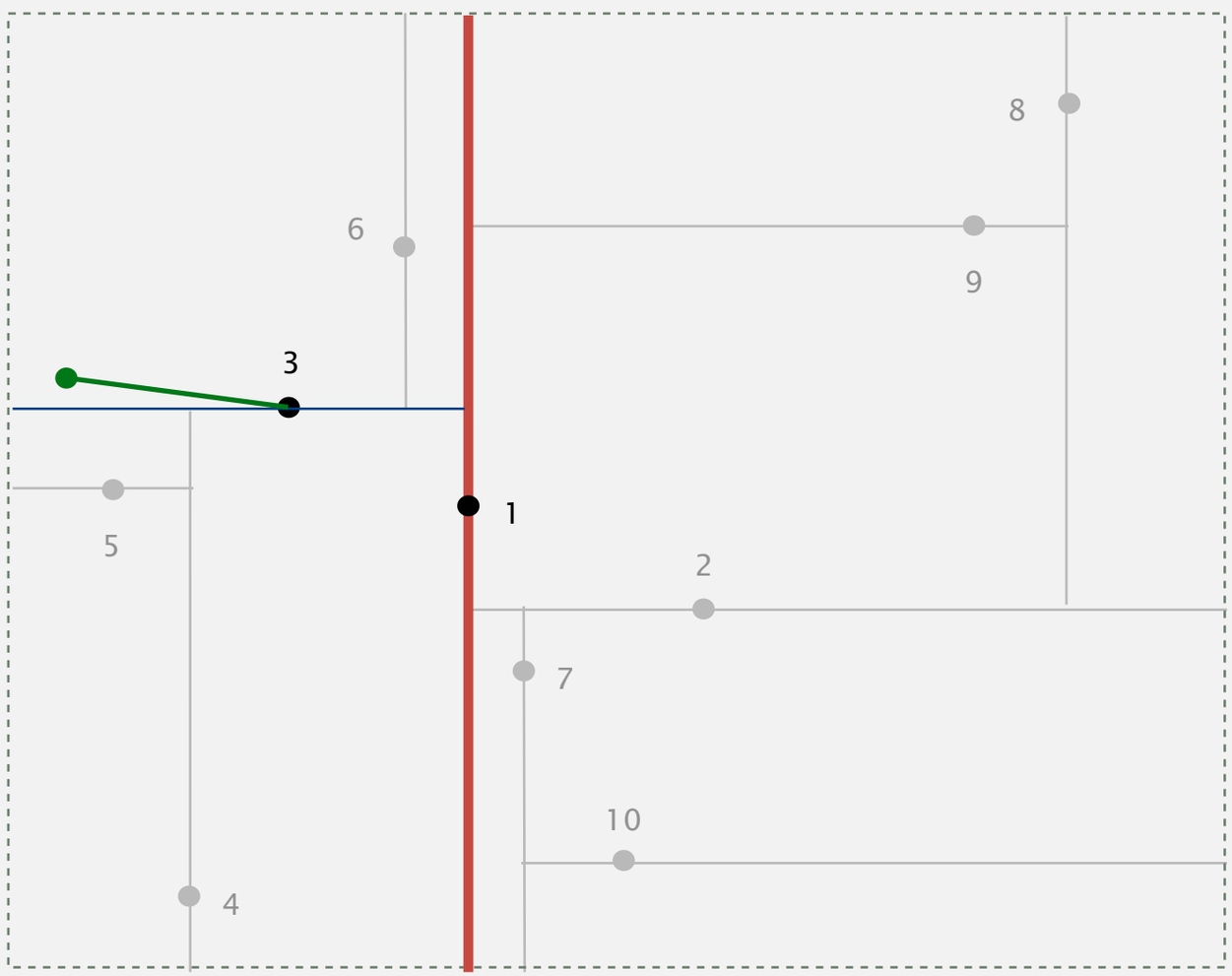


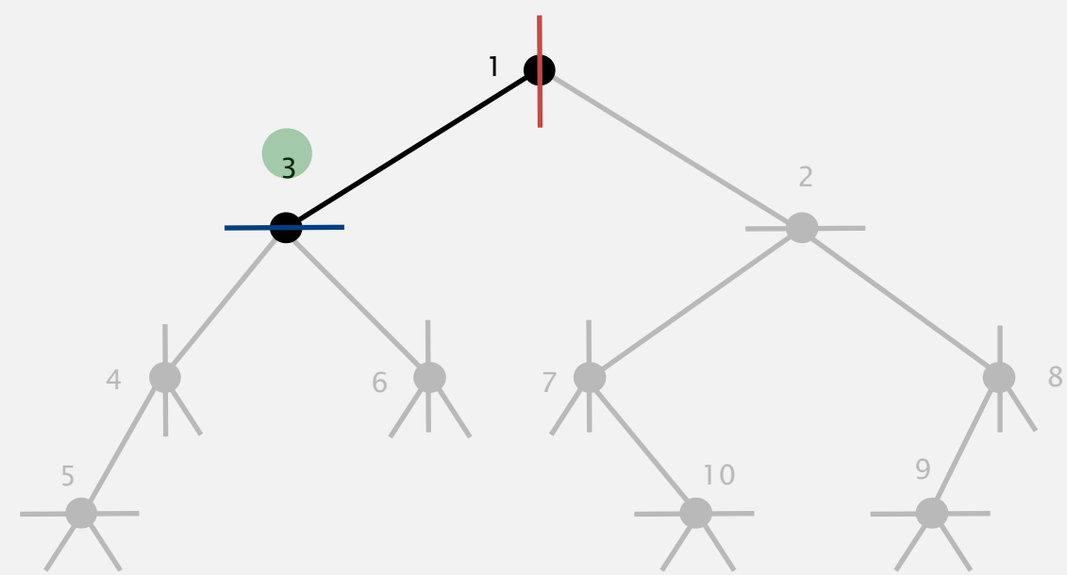




**search left subtree**
**compute distance from query point to 3**
**(update champion)**

# Nearest neighbor search in a 2d tree

- Check distance from point in node to query point.
- Recursively search left/bottom subdivision (if it could contain a closer point).
- Recursively search right/top subdivision (if it could contain a closer point).
- Organize recursive method so that it begins by searching for query point.

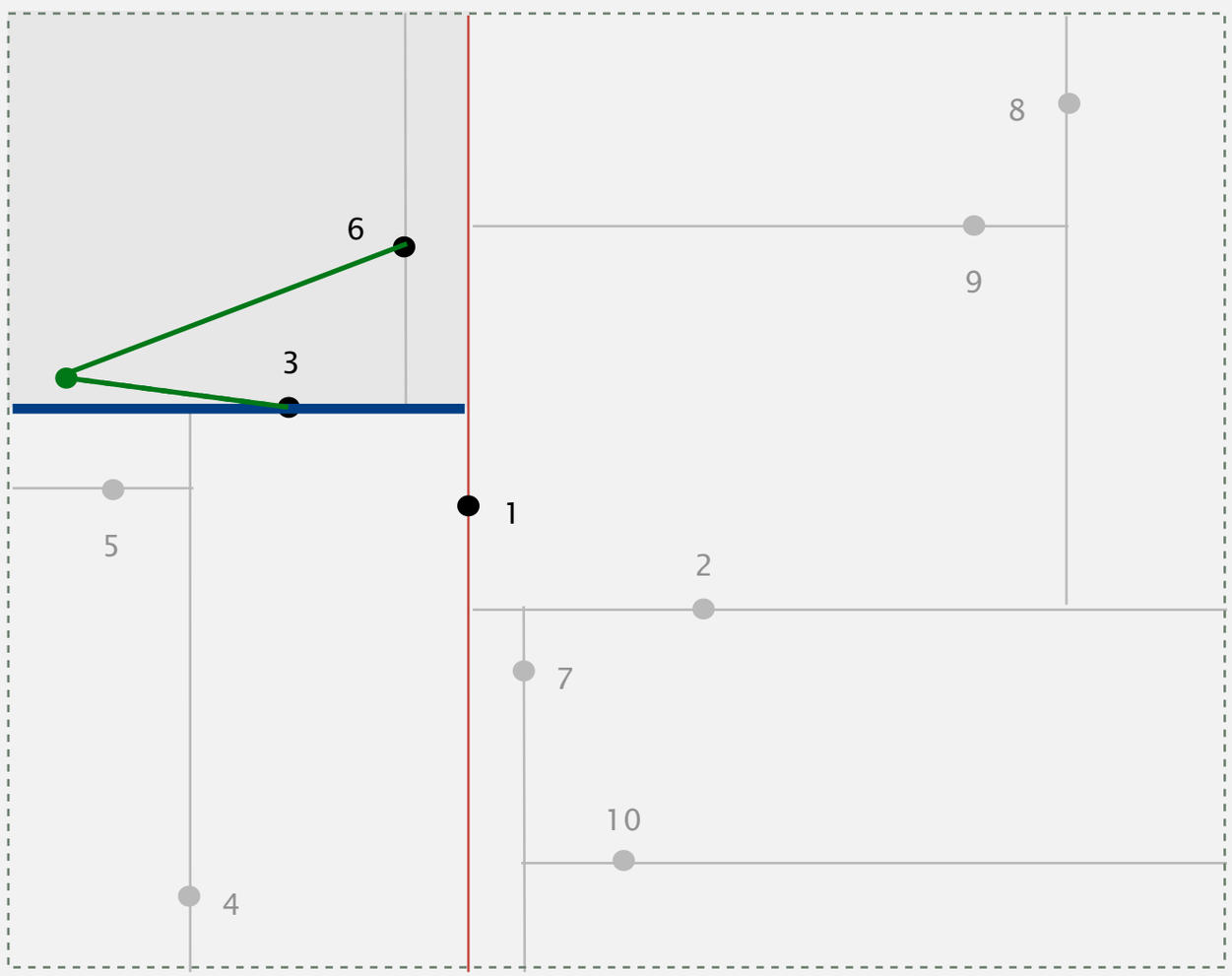


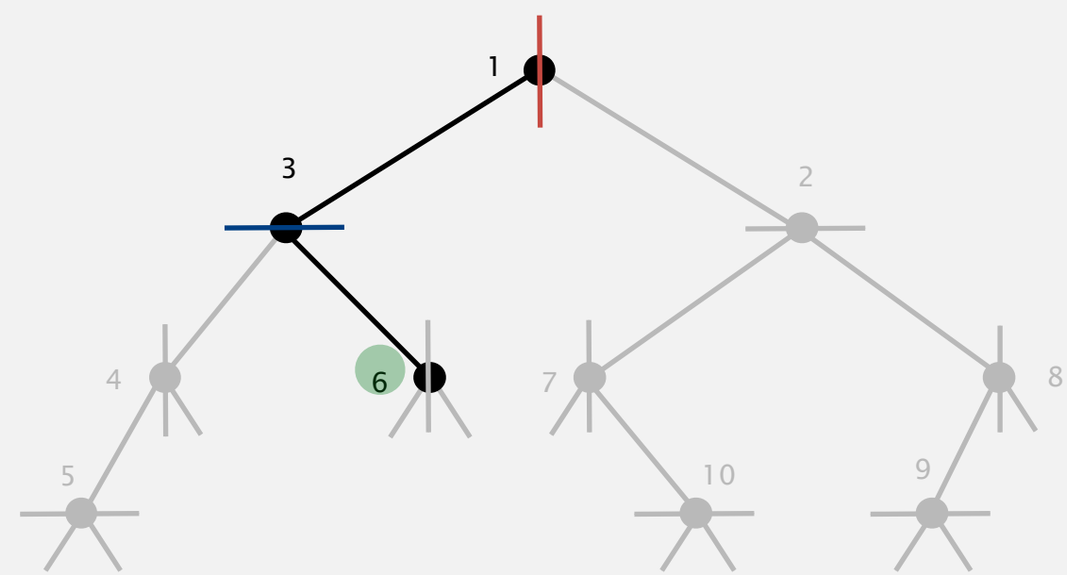




**query point is above splitting line**
**search top subtree first**

# Nearest neighbor search in a 2d tree

- Check distance from point in node to query point.
- Recursively search left/bottom subdivision (if it could contain a closer point).
- Recursively search right/top subdivision (if it could contain a closer point).
- Organize recursive method so that it begins by searching for query point.



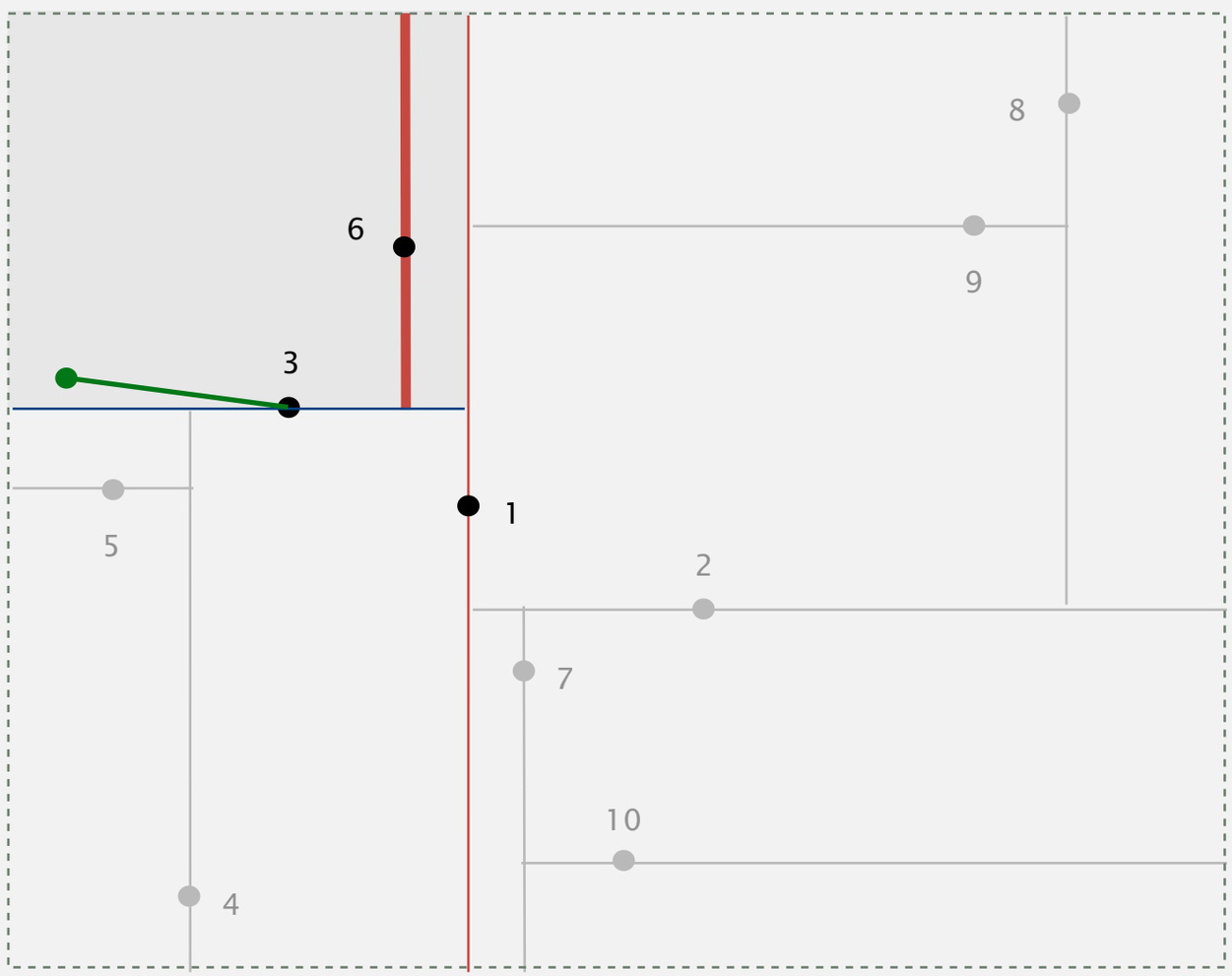



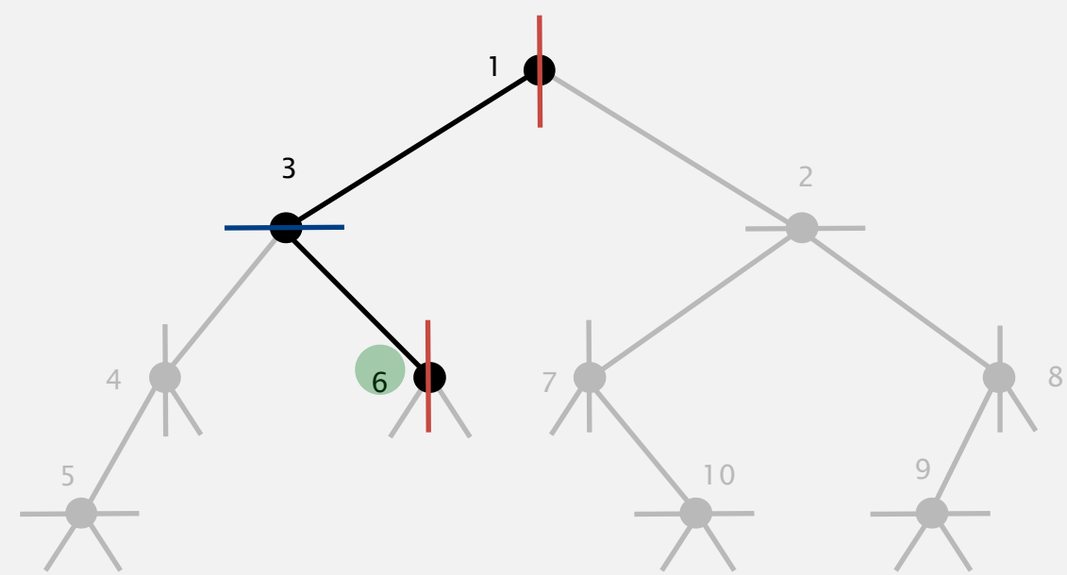

**search top subtree**

**compute distance from query point to 6**

# Nearest neighbor search in a 2d tree

- Check distance from point in node to query point.
- Recursively search left/bottom subdivision (if it could contain a closer point).
- Recursively search right/top subdivision (if it could contain a closer point).
- Organize recursive method so that it begins by searching for query point.

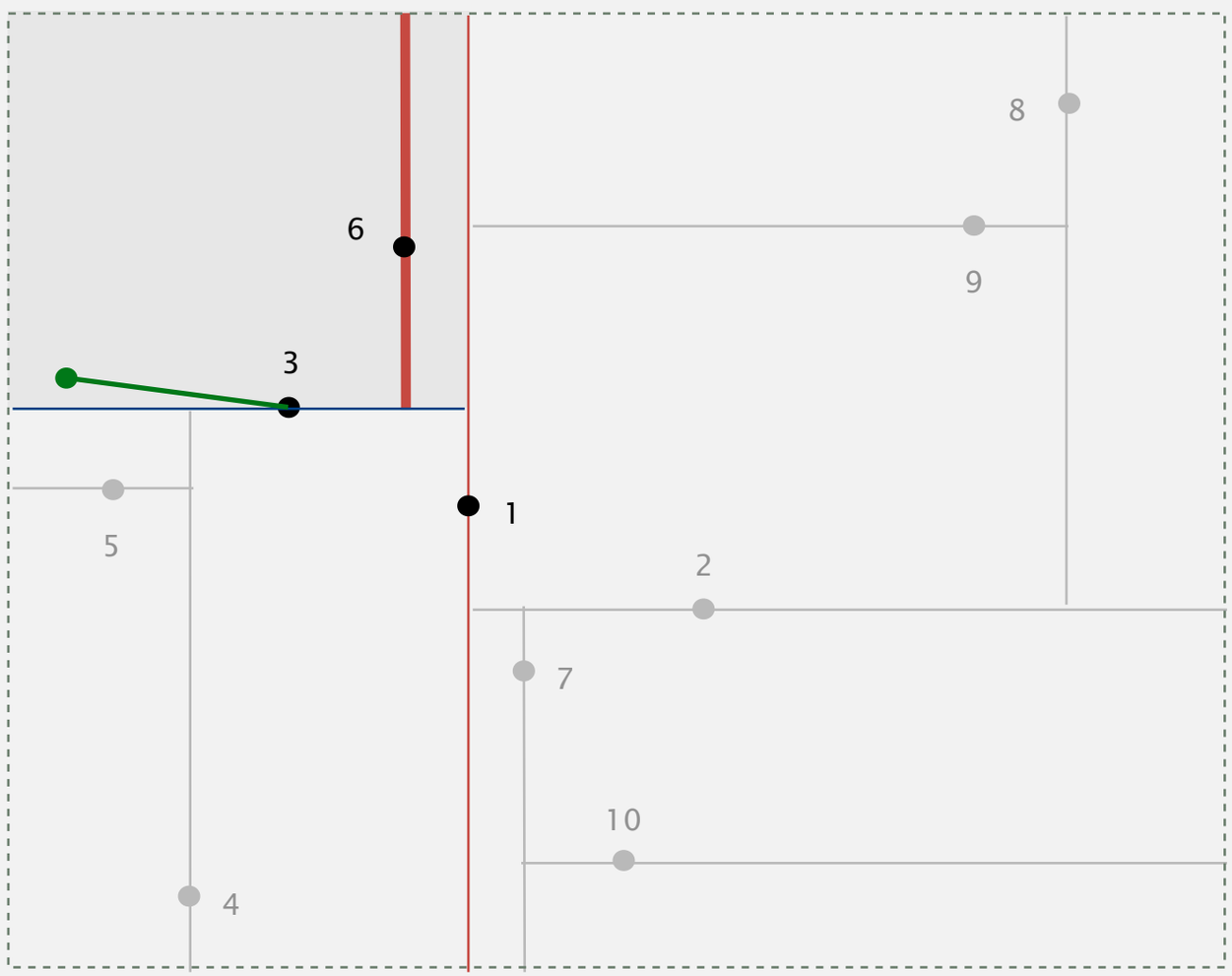


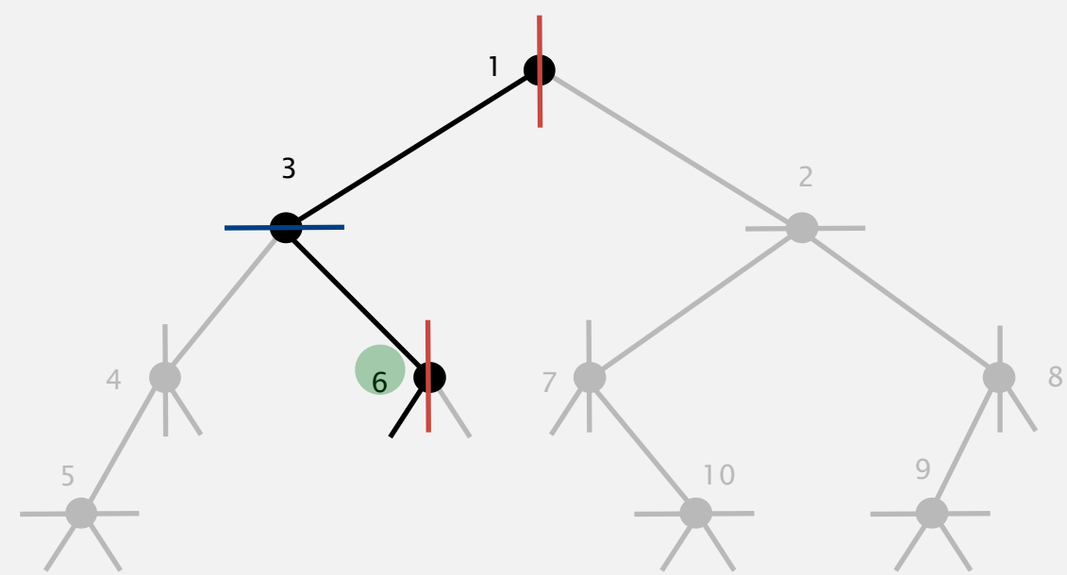




**query point is to left of splitting line**
**search left subtree first**

# Nearest neighbor search in a 2d tree

- Check distance from point in node to query point.
- Recursively search left/bottom subdivision (if it could contain a closer point).
- Recursively search right/top subdivision (if it could contain a closer point).
- Organize recursive method so that it begins by searching for query point.



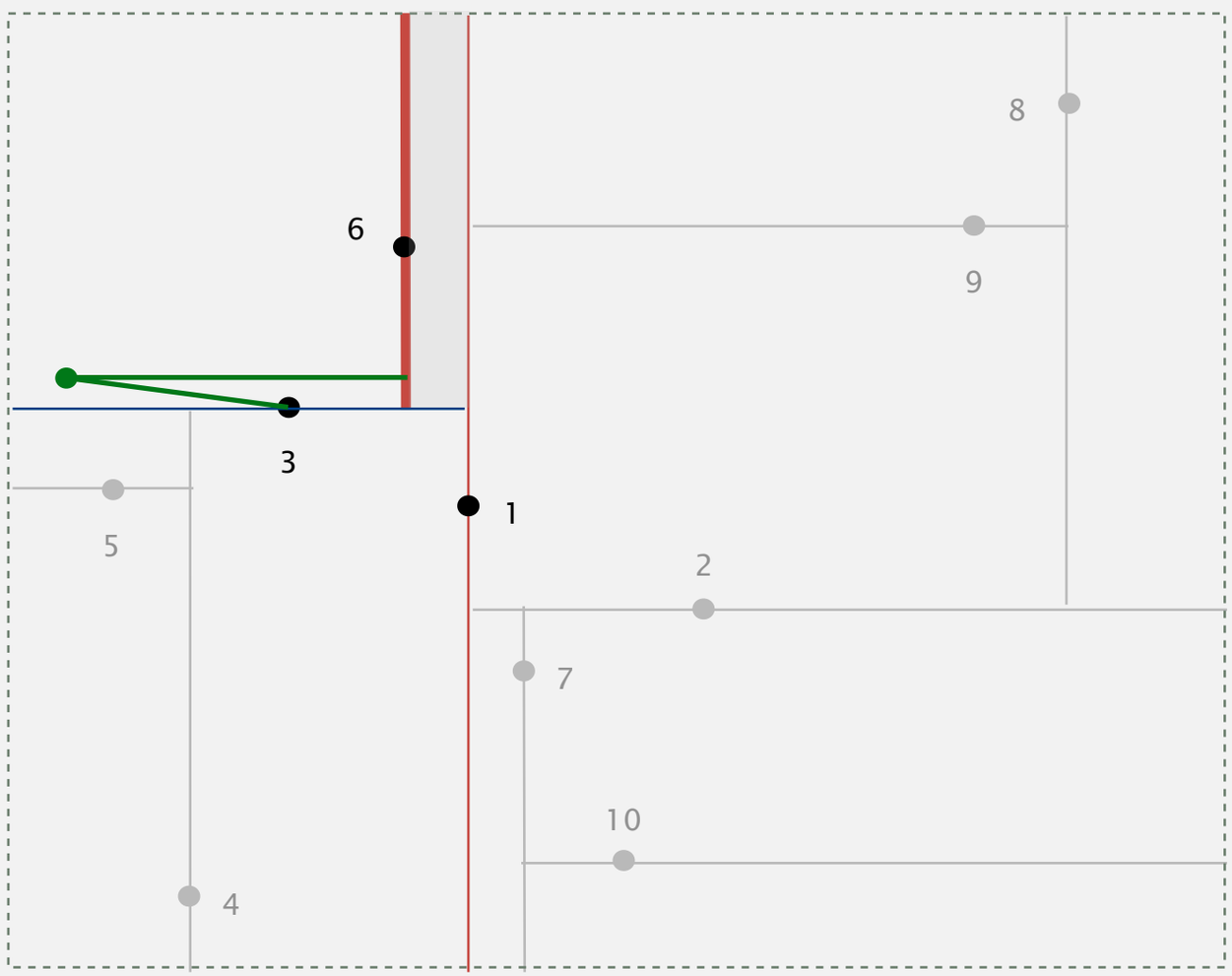



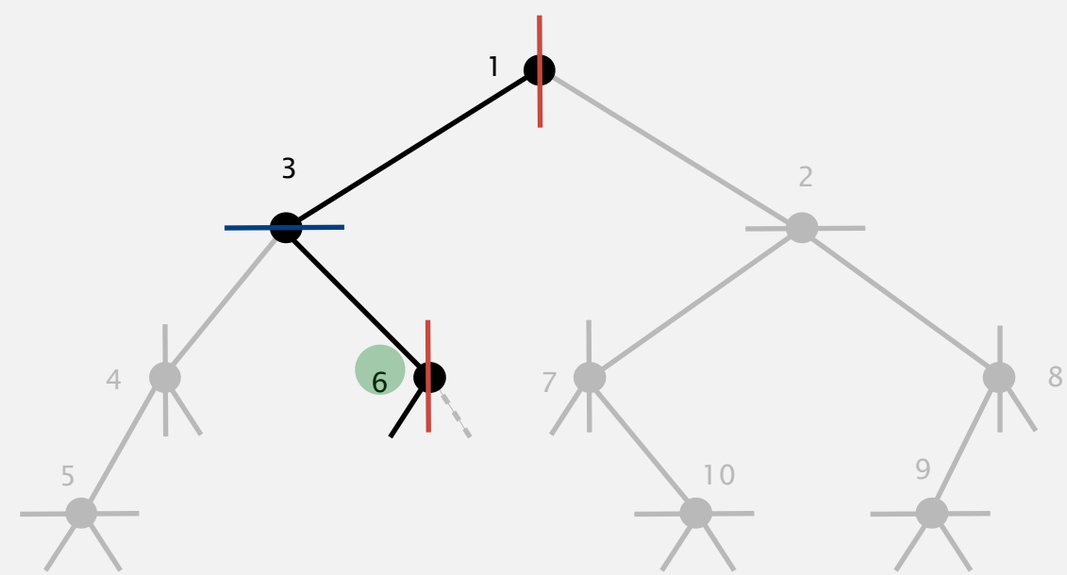

**search left subtree**

**return since empty**

# Nearest neighbor search in a 2d tree

- Check distance from point in node to query point.
- Recursively search left/bottom subdivision (if it could contain a closer point).
- Recursively search right/top subdivision (if it could contain a closer point).
- Organize recursive method so that it begins by searching for query point.

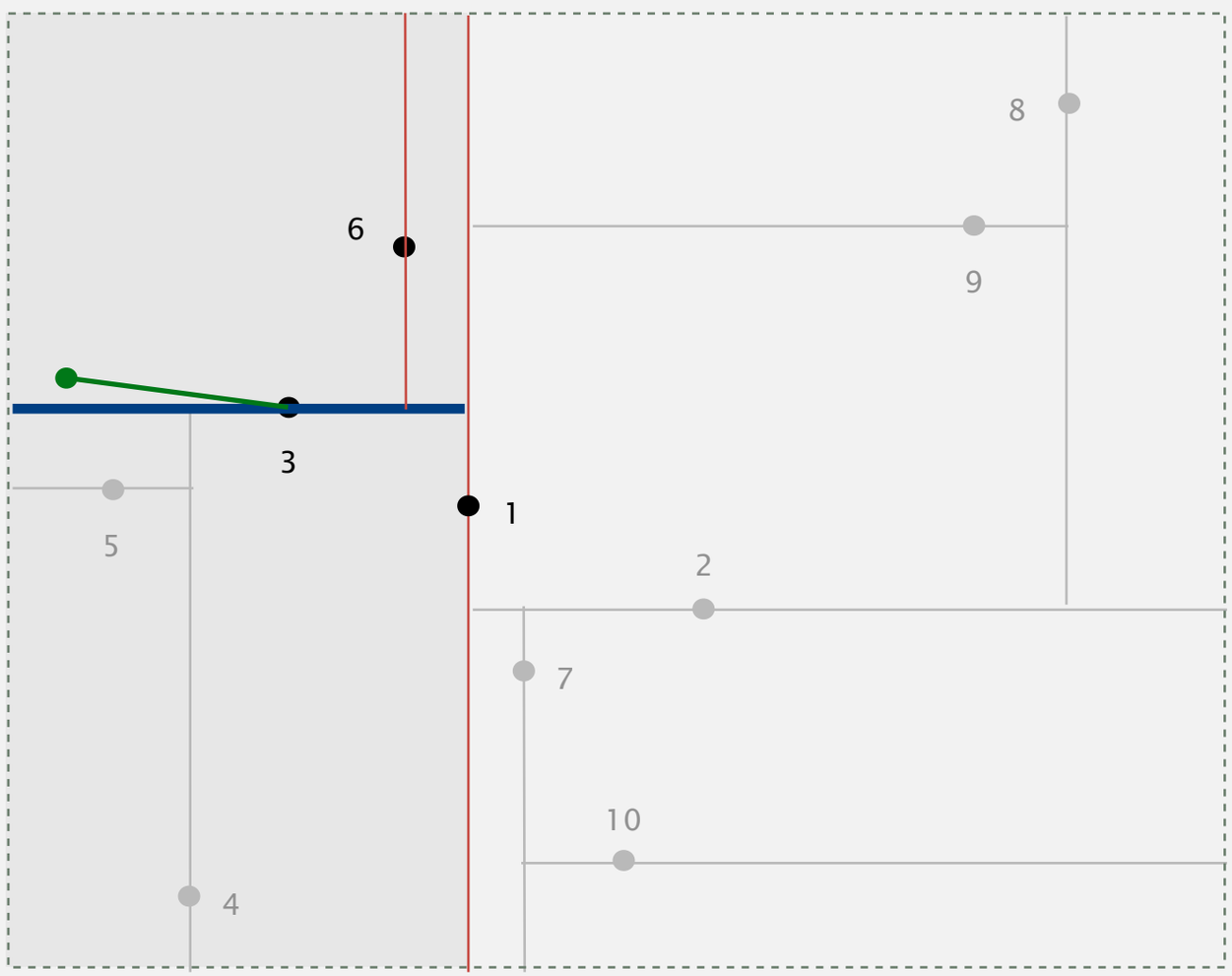


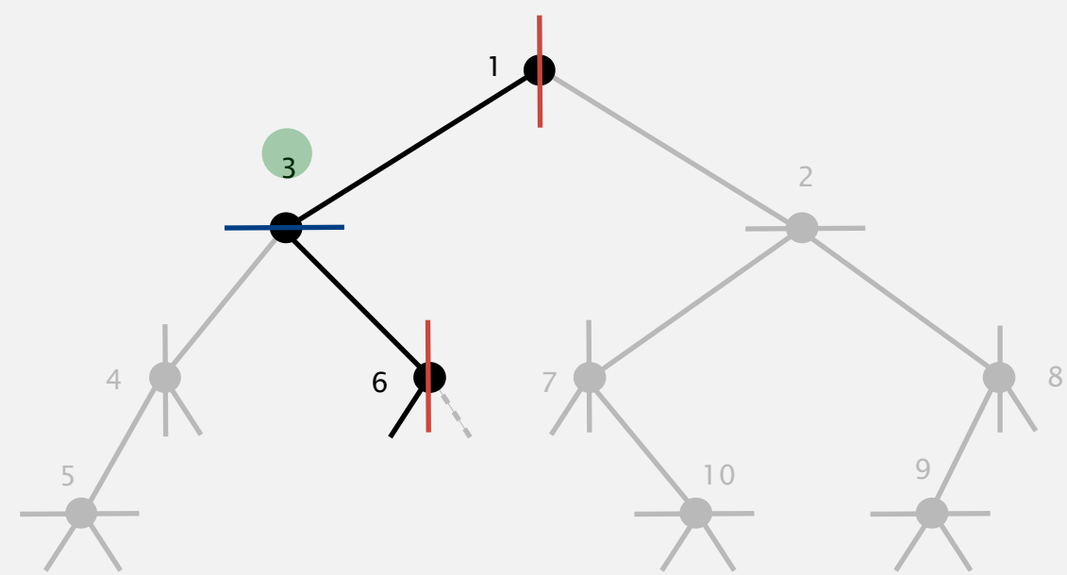




**search right subtree**
**prune since nearest neighbor**
**can't be in subdivision**

# Nearest neighbor search in a 2d tree

- Check distance from point in node to query point.
- Recursively search left/bottom subdivision (if it could contain a closer point).
- Recursively search right/top subdivision (if it could contain a closer point).
- Organize recursive method so that it begins by searching for query point.

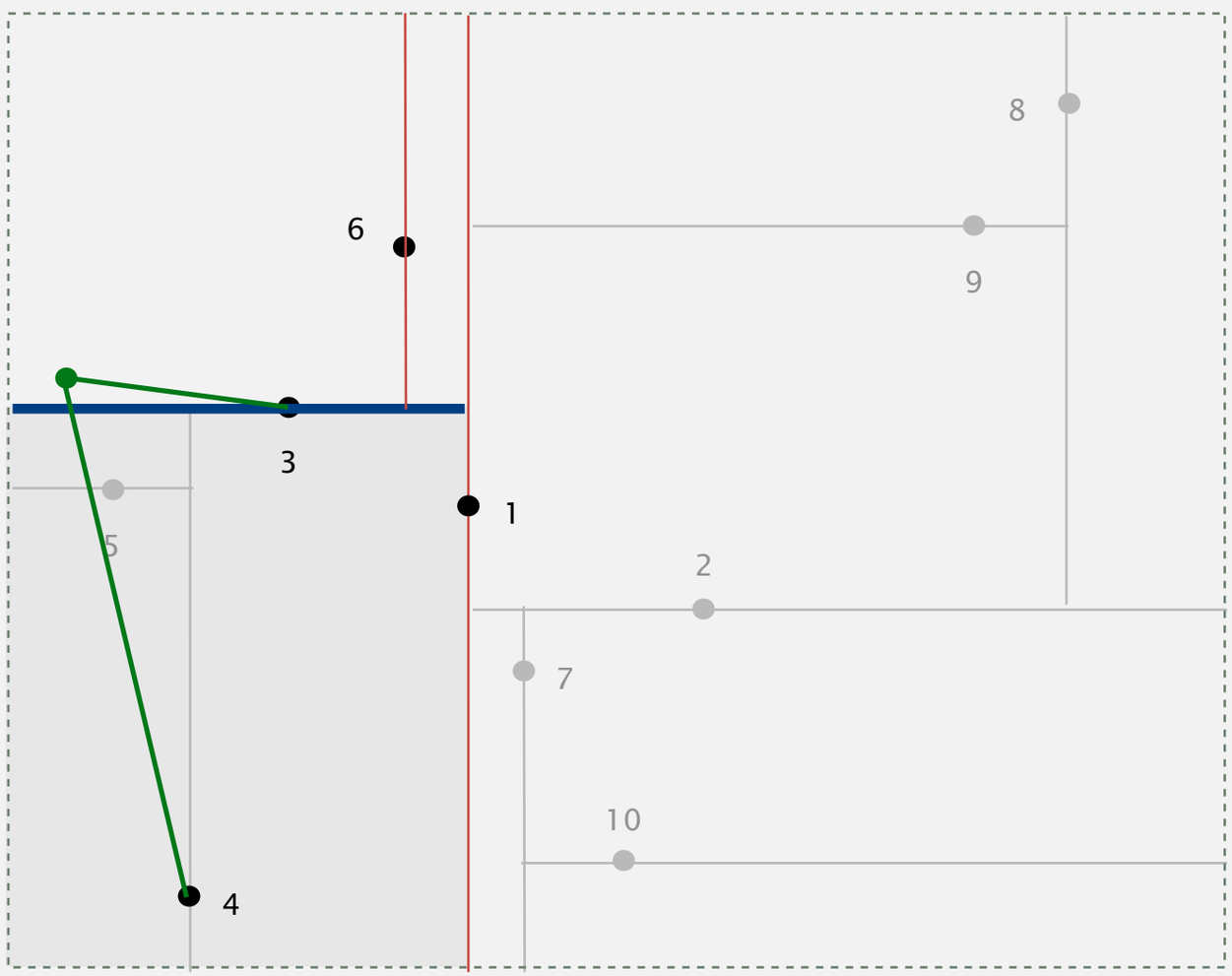


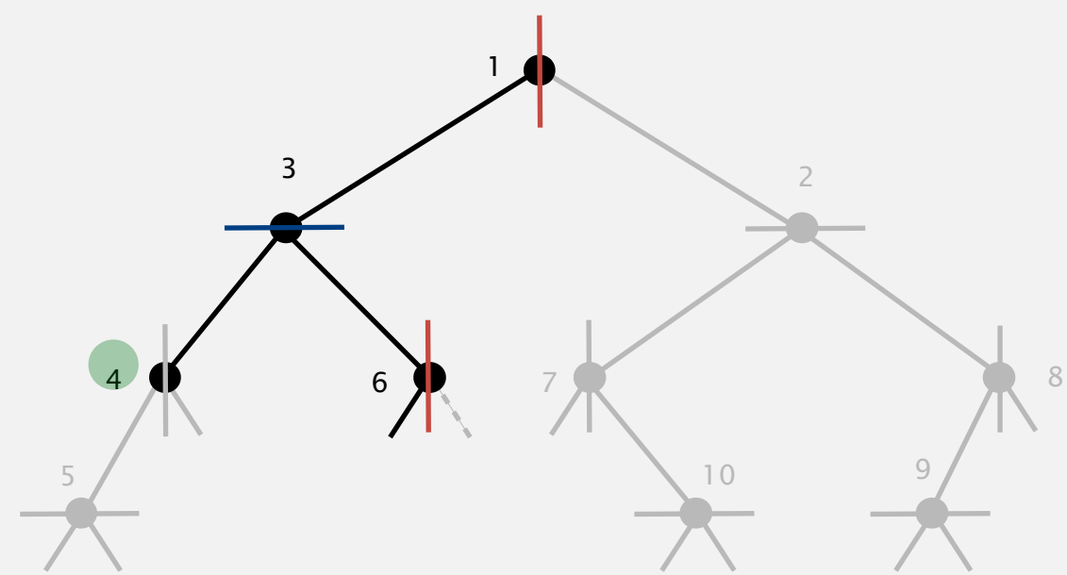




**return from function call**
**search bottom subtree next**

# Nearest neighbor search in a 2d tree

- Check distance from point in node to query point.
- Recursively search left/bottom subdivision (if it could contain a closer point).
- Recursively search right/top subdivision (if it could contain a closer point).
- Organize recursive method so that it begins by searching for query point.



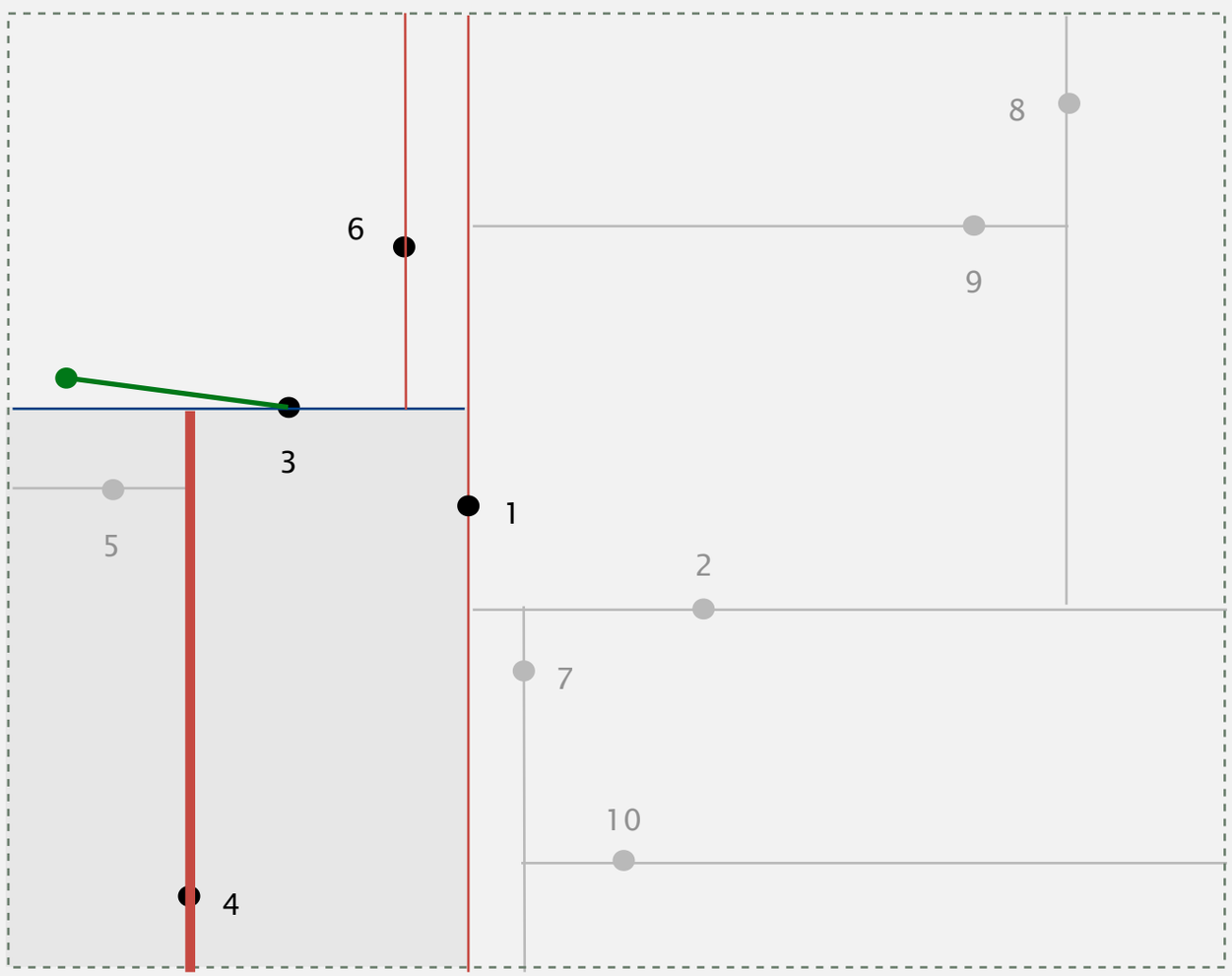



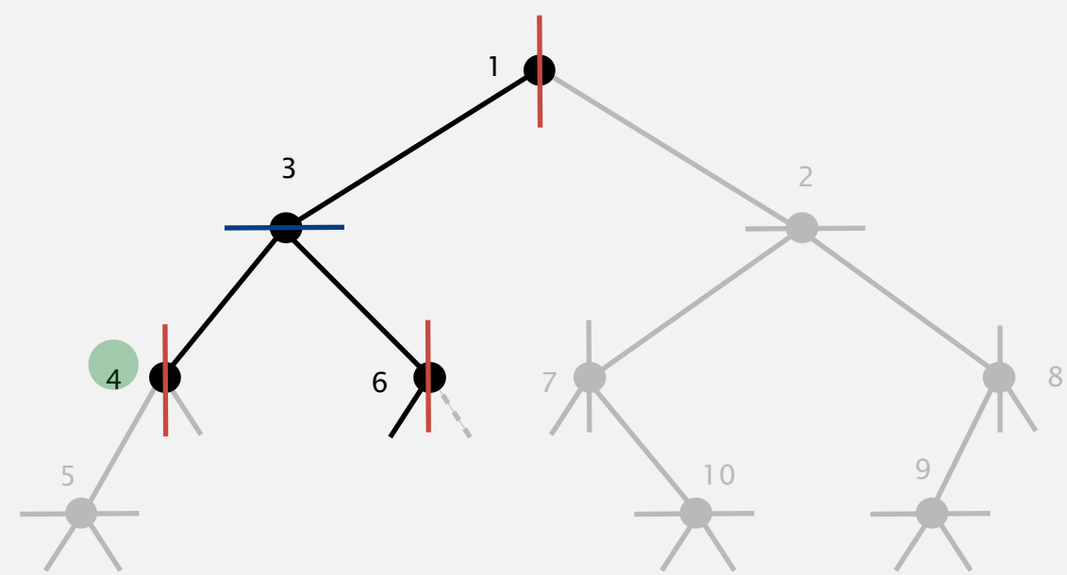

**search bottom subtree**

**compute distance from query point to 4**

# Nearest neighbor search in a 2d tree

- Check distance from point in node to query point.
- Recursively search left/bottom subdivision (if it could contain a closer point).
- Recursively search right/top subdivision (if it could contain a closer point).
- Organize recursive method so that it begins by searching for query point.

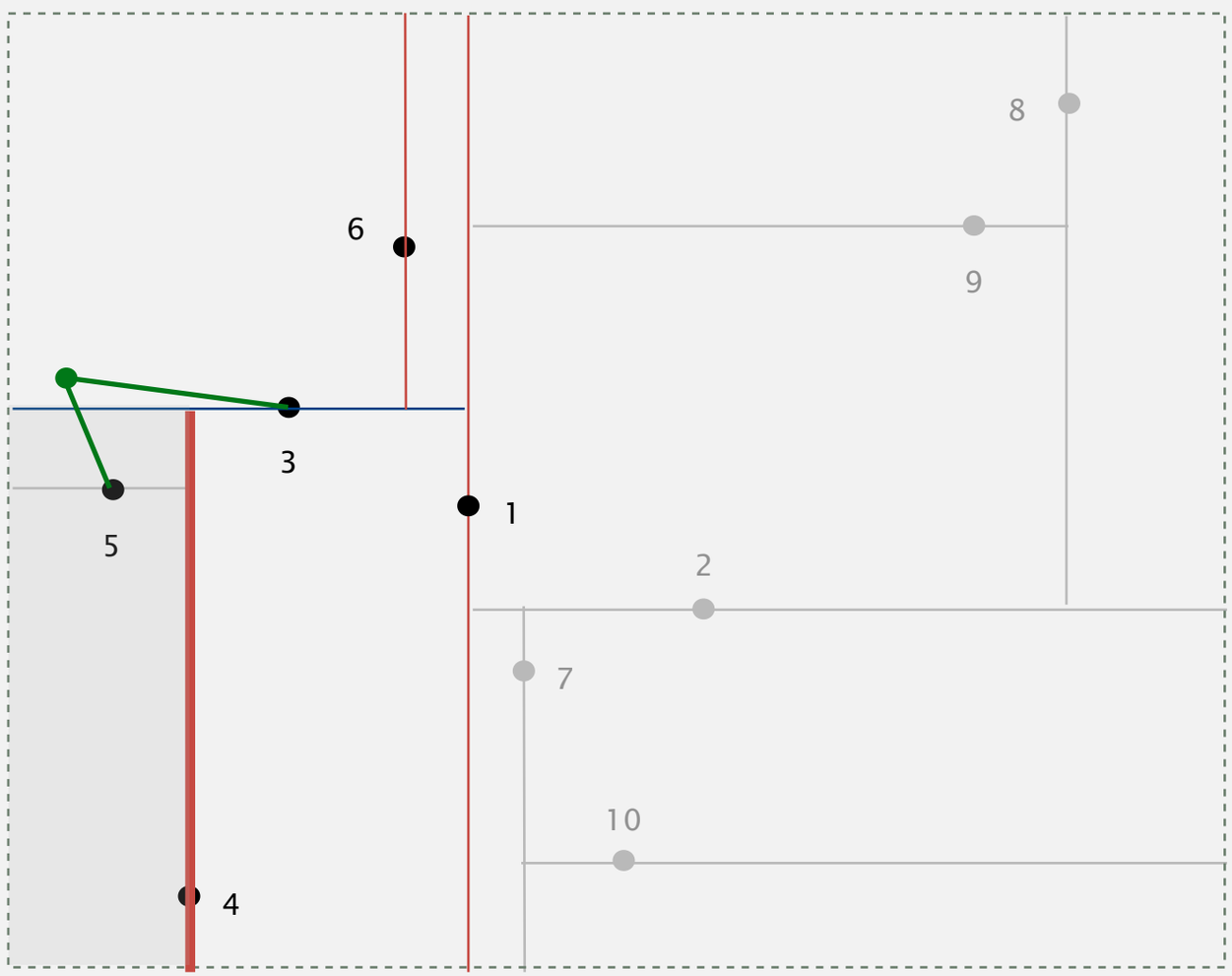


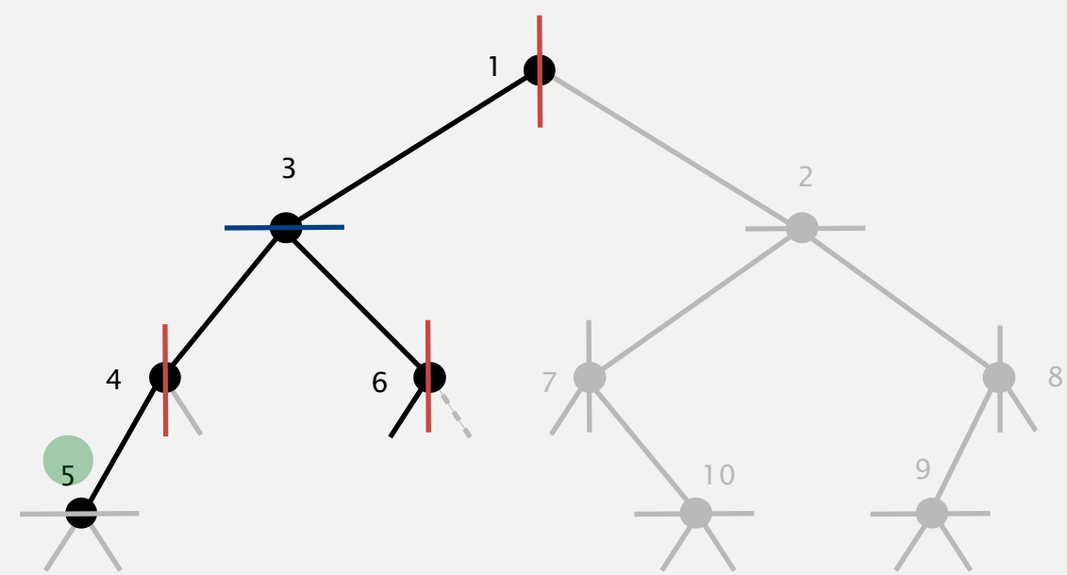




**query point is to left of splitting line**
**search left subtree first**

# Nearest neighbor search in a 2d tree

- Check distance from point in node to query point.
- Recursively search left/bottom subdivision (if it could contain a closer point).
- Recursively search right/top subdivision (if it could contain a closer point).
- Organize recursive method so that it begins by searching for query point.

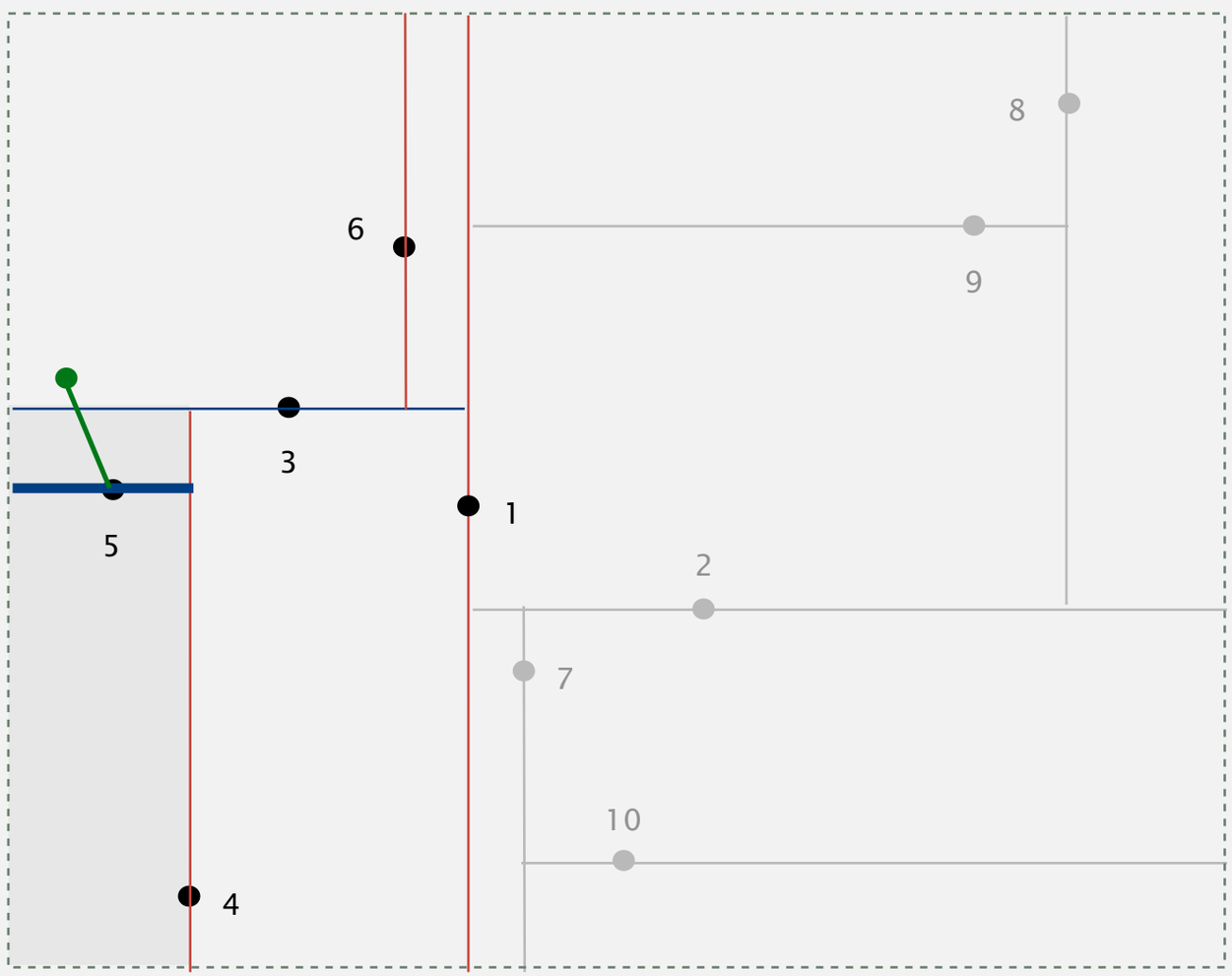


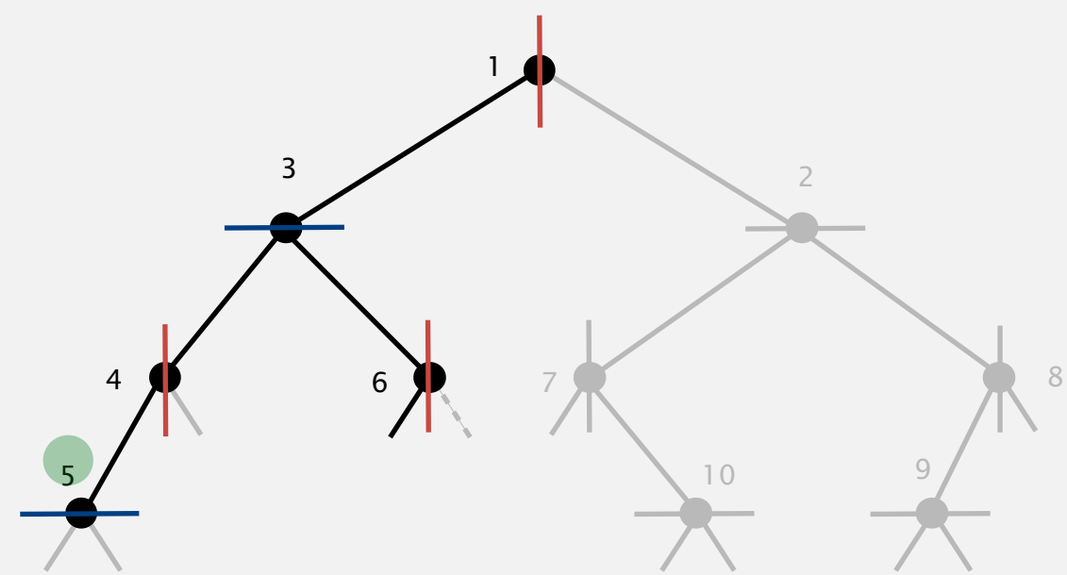




**search left subtree**
**compute distance from query point to 5**
**(update champion)**

# Nearest neighbor search in a 2d tree

- Check distance from point in node to query point.
- Recursively search left/bottom subdivision (if it could contain a closer point).
- Recursively search right/top subdivision (if it could contain a closer point).
- Organize recursive method so that it begins by searching for query point.

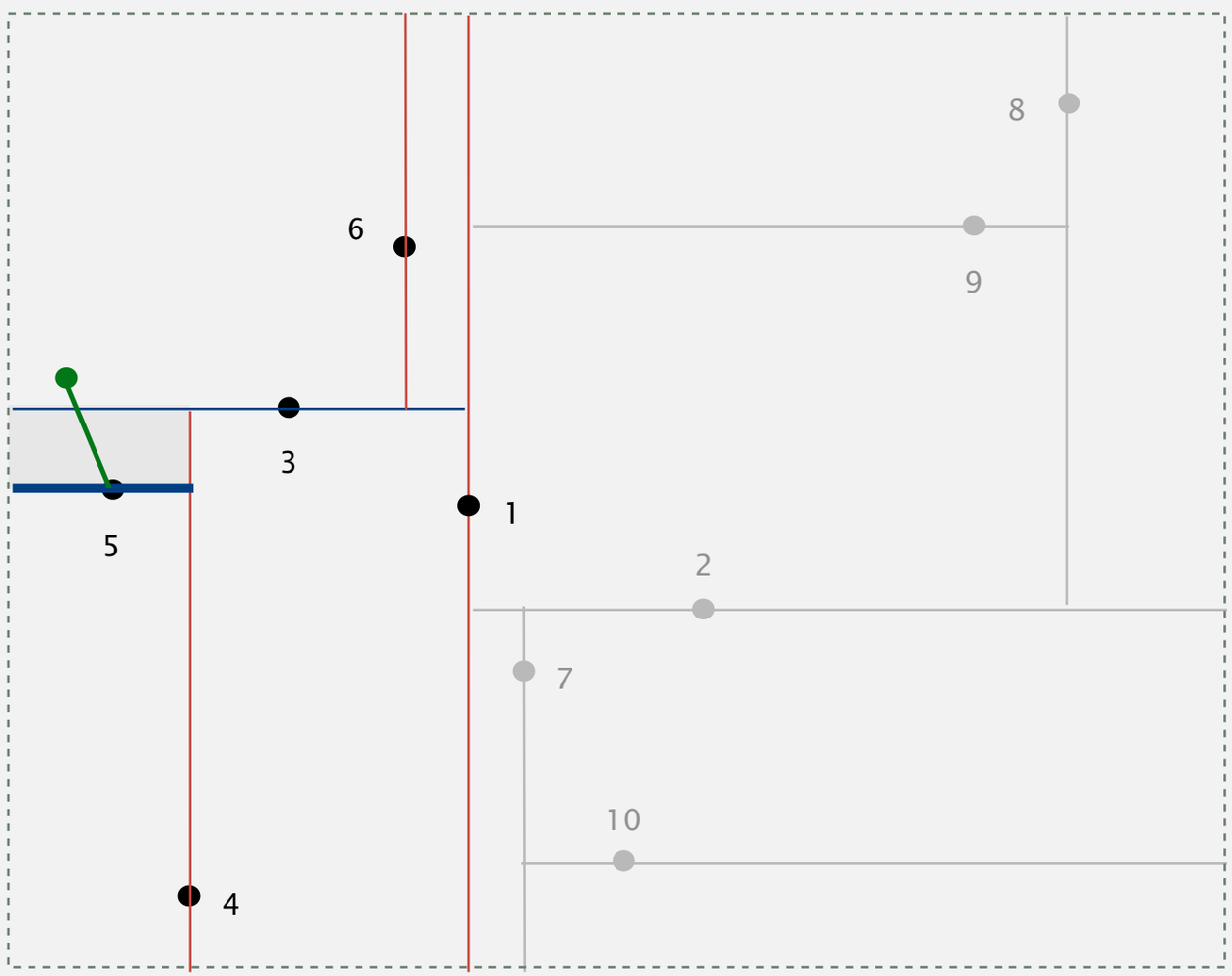


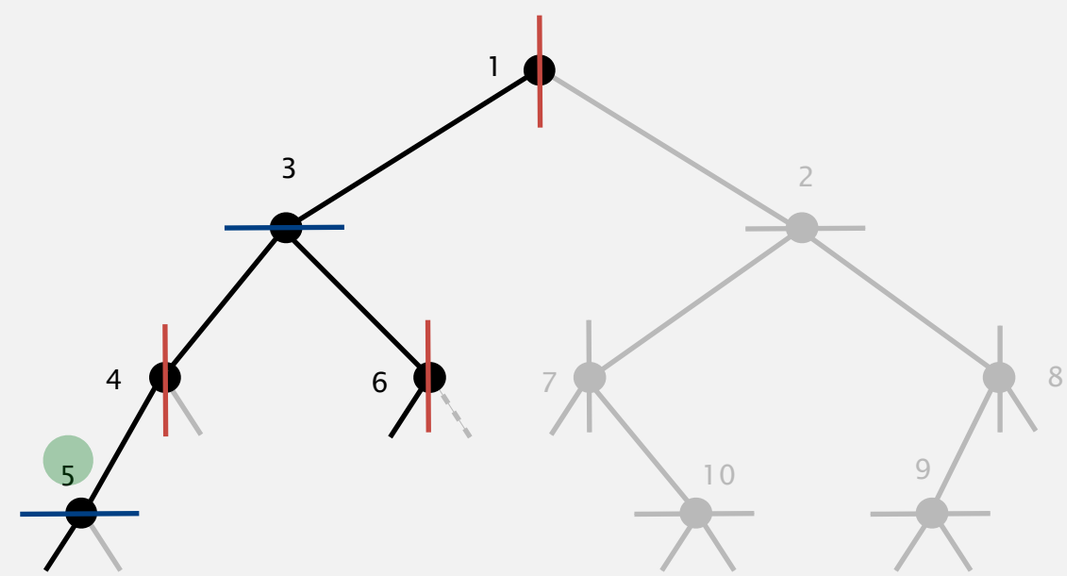




**query point is above splitting line**
**search top subtree first**

# Nearest neighbor search in a 2d tree

- Check distance from point in node to query point.
- Recursively search left/bottom subdivision (if it could contain a closer point).
- Recursively search right/top subdivision (if it could contain a closer point).
- Organize recursive method so that it begins by searching for query point.



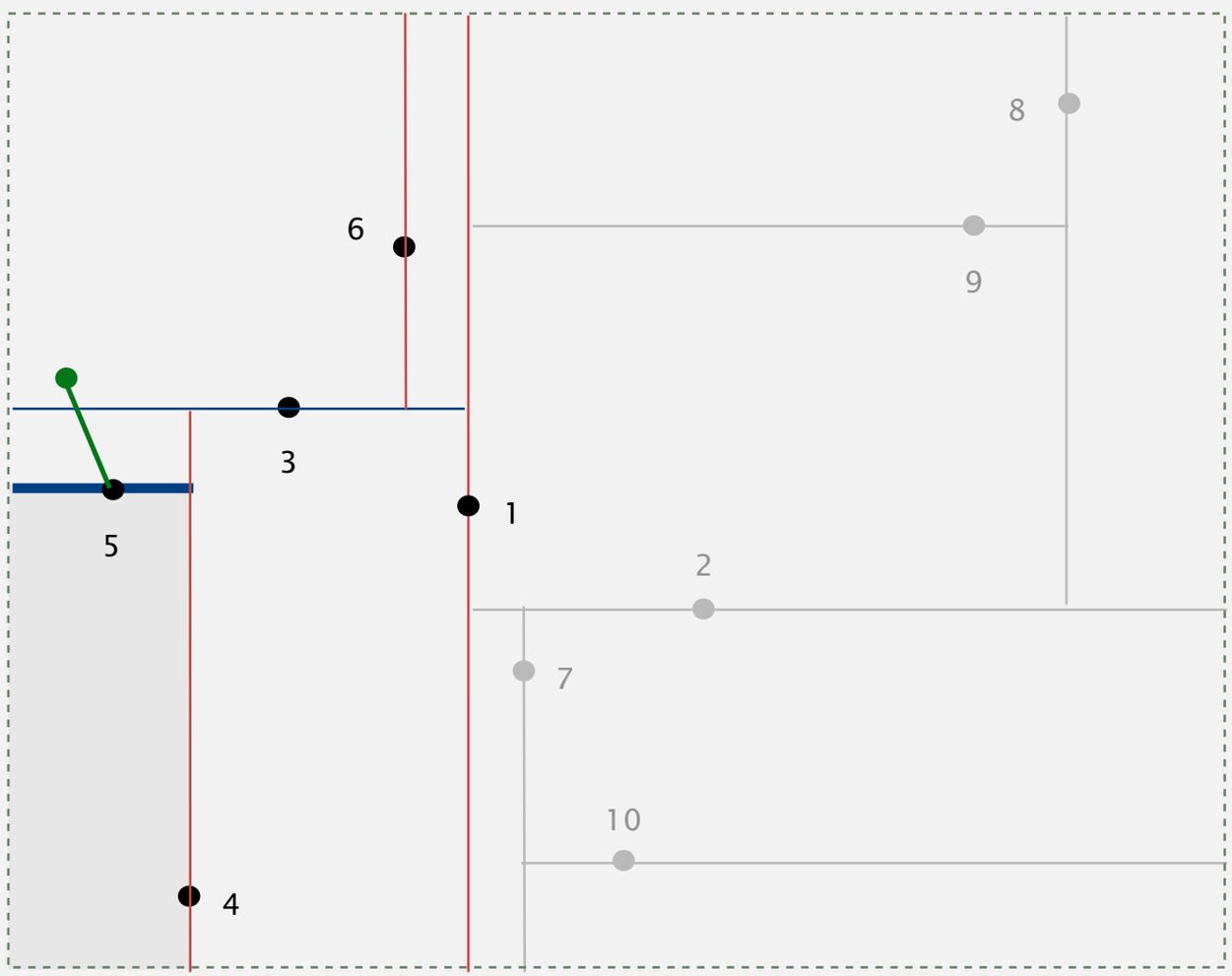



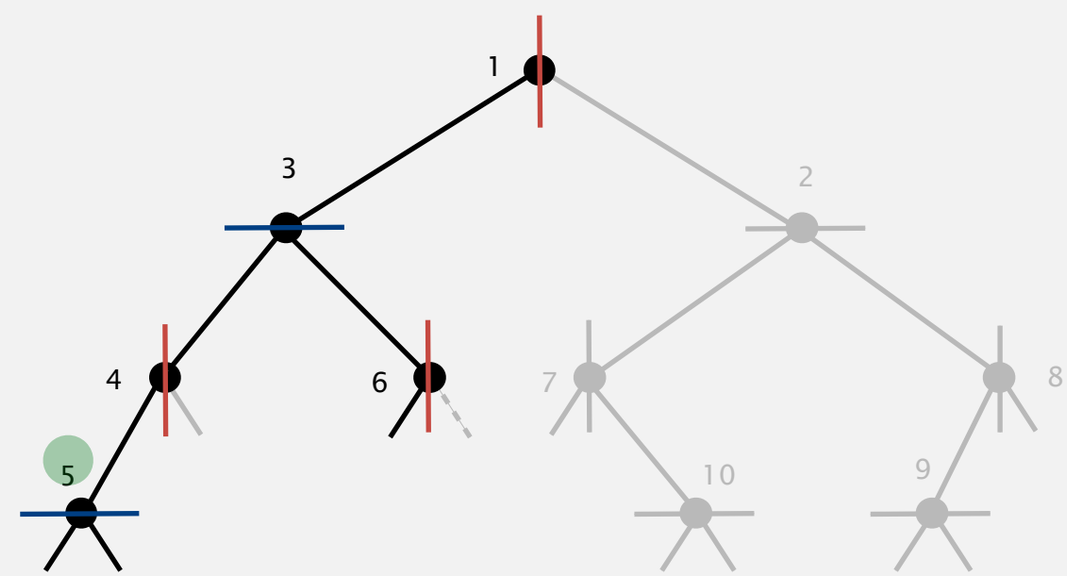

**search top subtree**

**return since empty**

# Nearest neighbor search in a 2d tree

- Check distance from point in node to query point.
- Recursively search left/bottom subdivision (if it could contain a closer point).
- Recursively search right/top subdivision (if it could contain a closer point).
- Organize recursive method so that it begins by searching for query point.



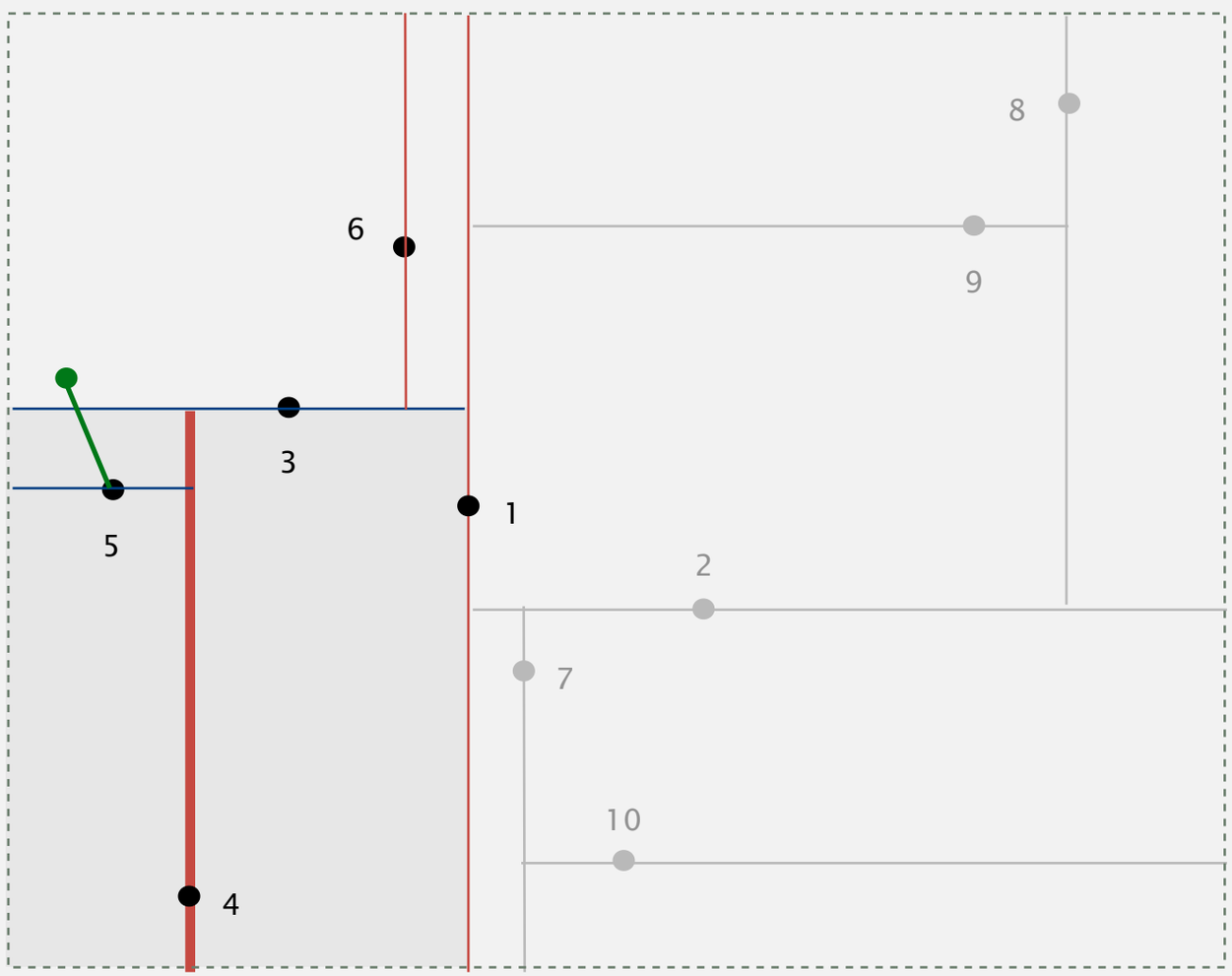



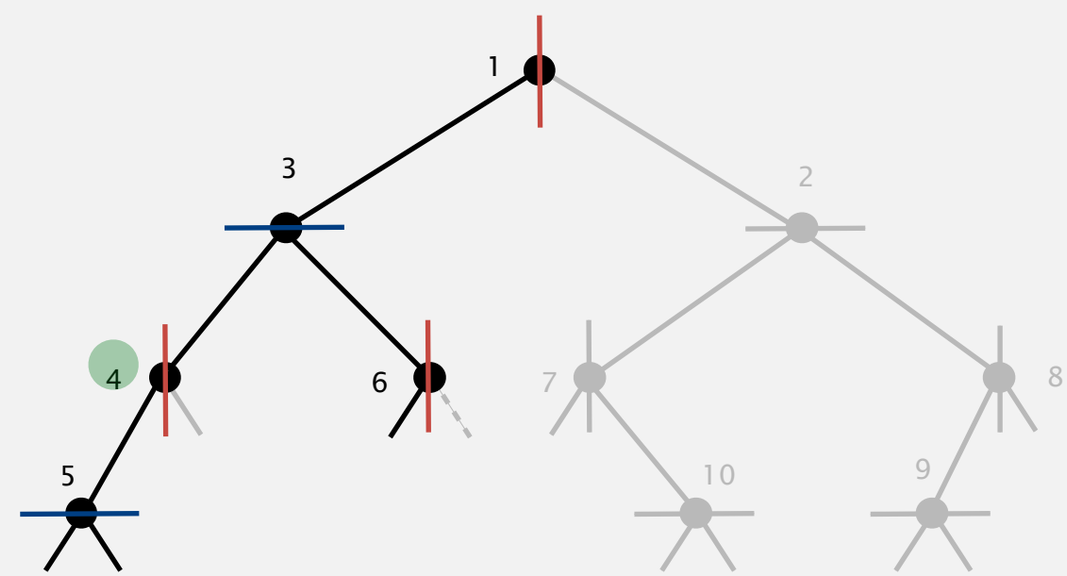

**search bottom subtree**

**return since empty**

# Nearest neighbor search in a 2d tree

- Check distance from point in node to query point.
- Recursively search left/bottom subdivision (if it could contain a closer point).
- Recursively search right/top subdivision (if it could contain a closer point).
- Organize recursive method so that it begins by searching for query point.

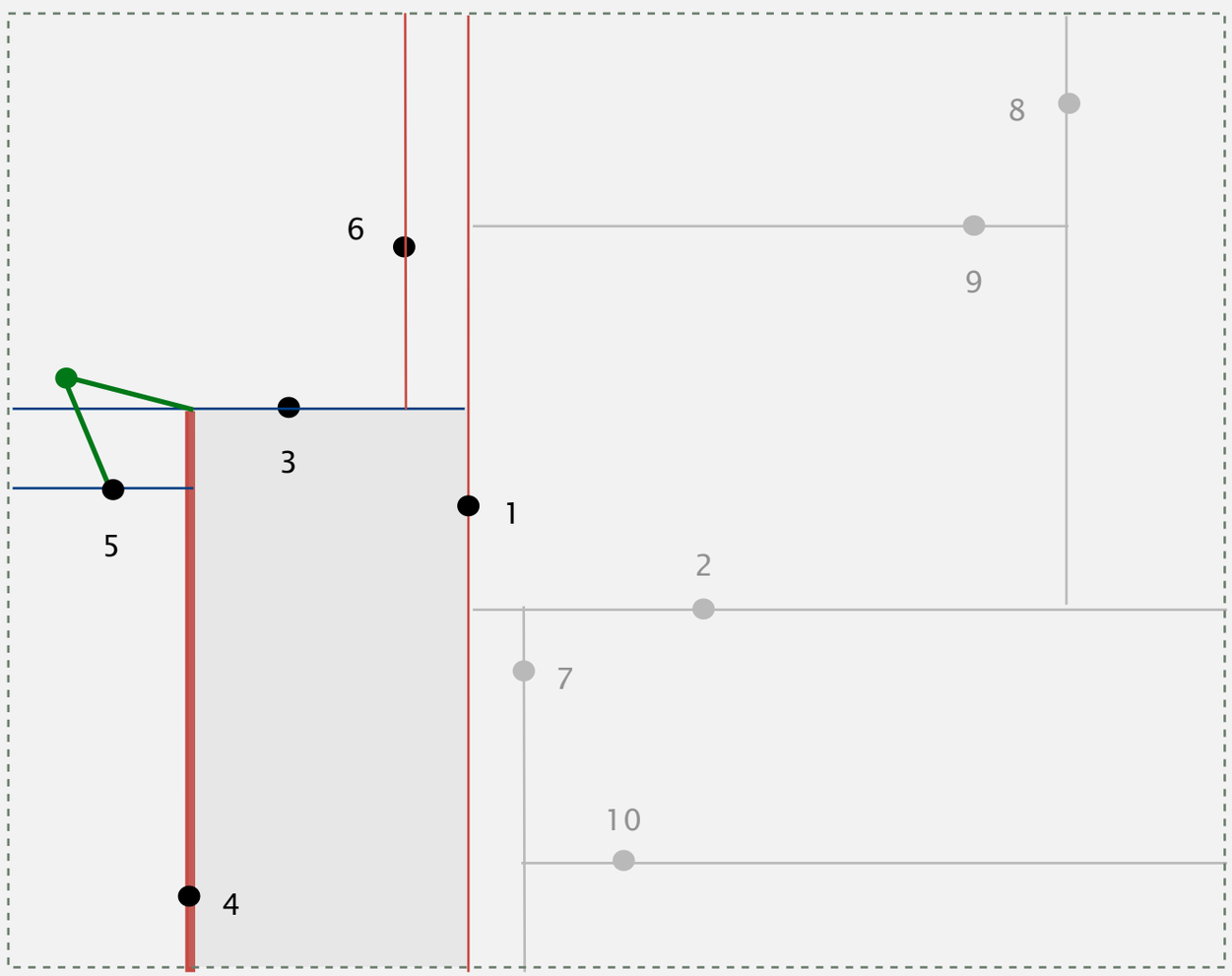


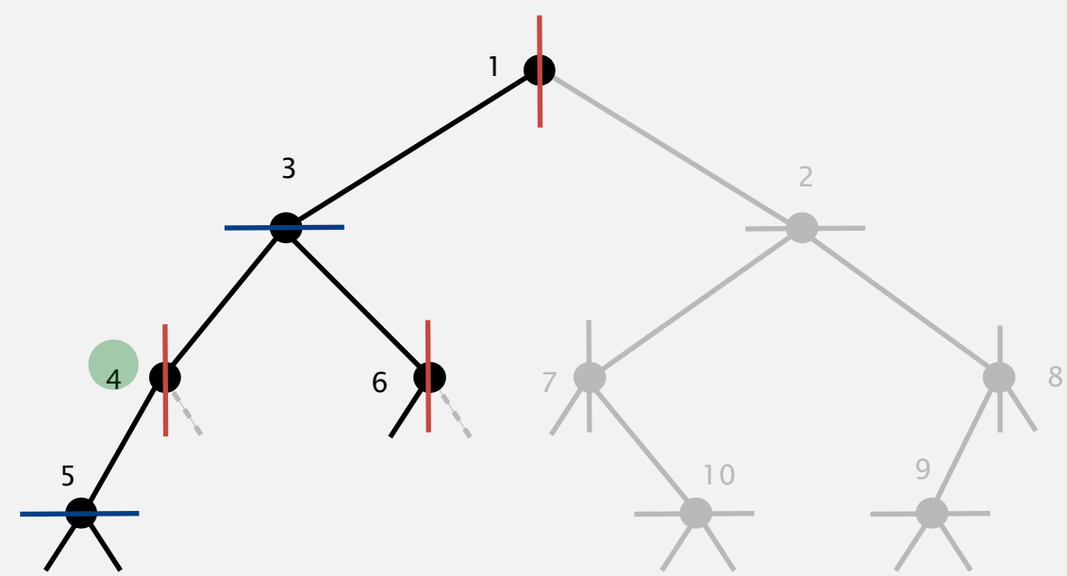




**return from function call**
**search right subtree next**

# Nearest neighbor search in a 2d tree

- Check distance from point in node to query point.
- Recursively search left/bottom subdivision (if it could contain a closer point).
- Recursively search right/top subdivision (if it could contain a closer point).
- Organize recursive method so that it begins by searching for query point.

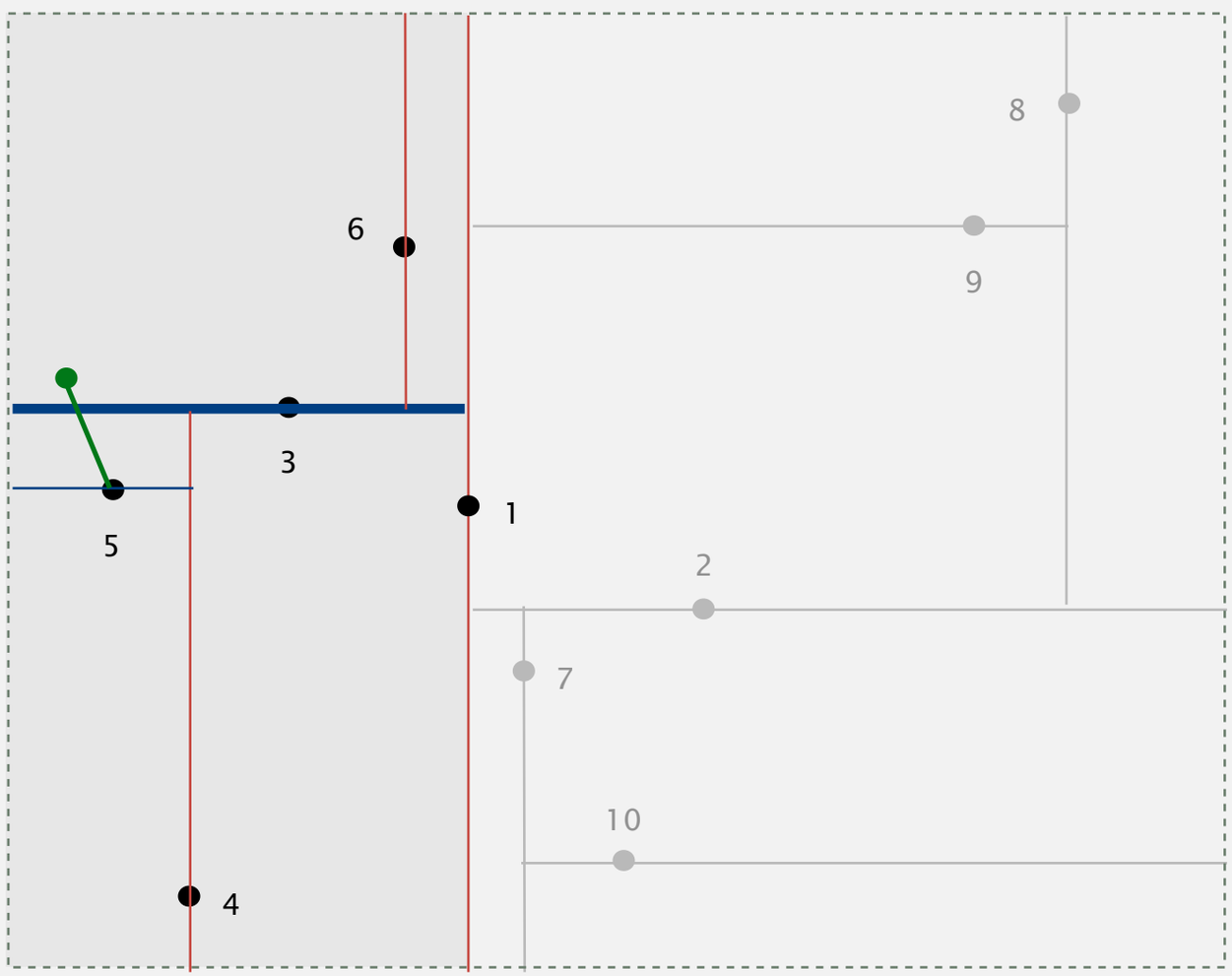


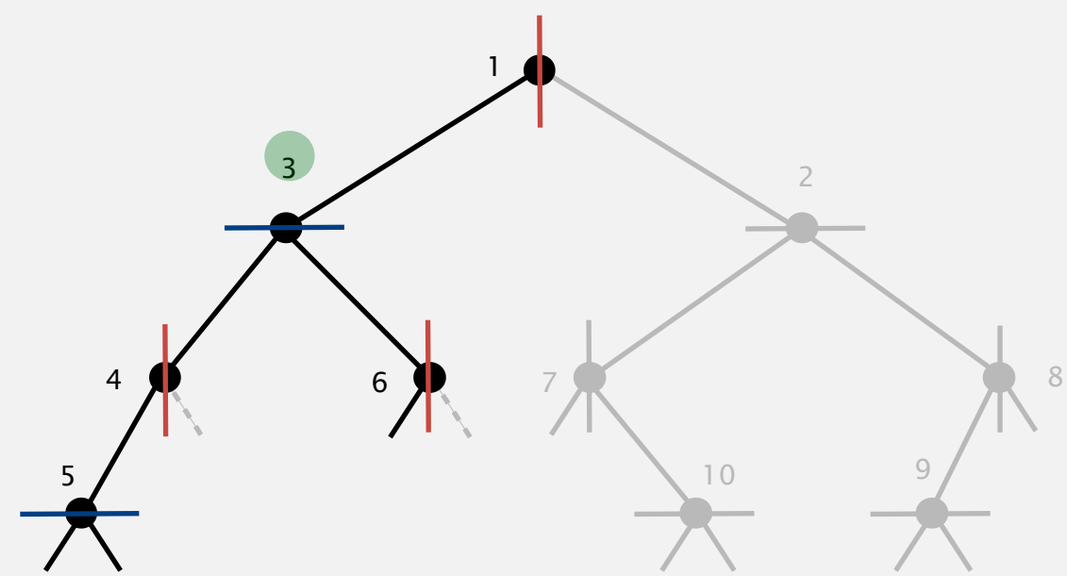




**search right subtree**
**prune since nearest neighbor**
**can't be in subdivision**

# Nearest neighbor search in a 2d tree

- Check distance from point in node to query point.
- Recursively search left/bottom subdivision (if it could contain a closer point).
- Recursively search right/top subdivision (if it could contain a closer point).
- Organize recursive method so that it begins by searching for query point.

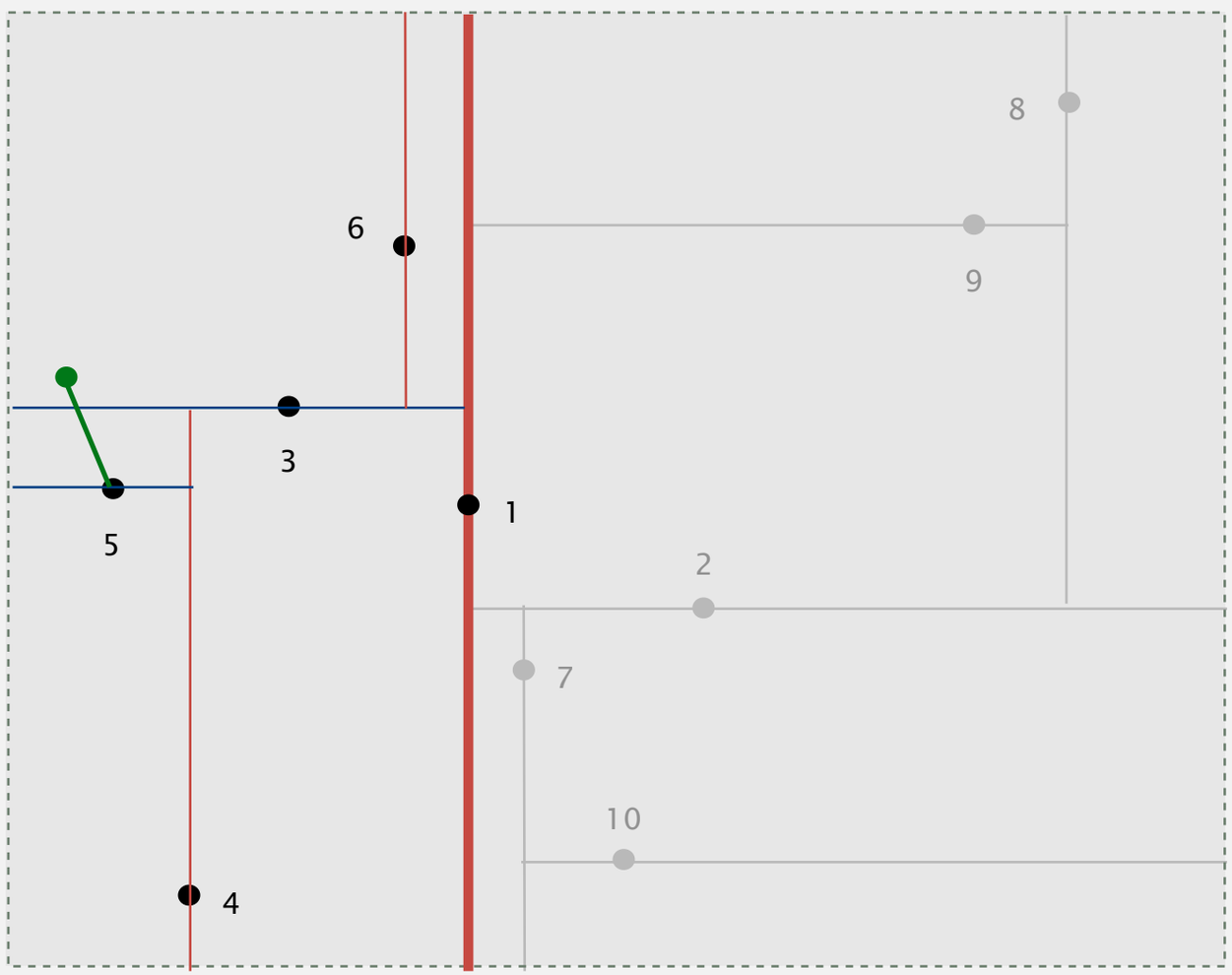


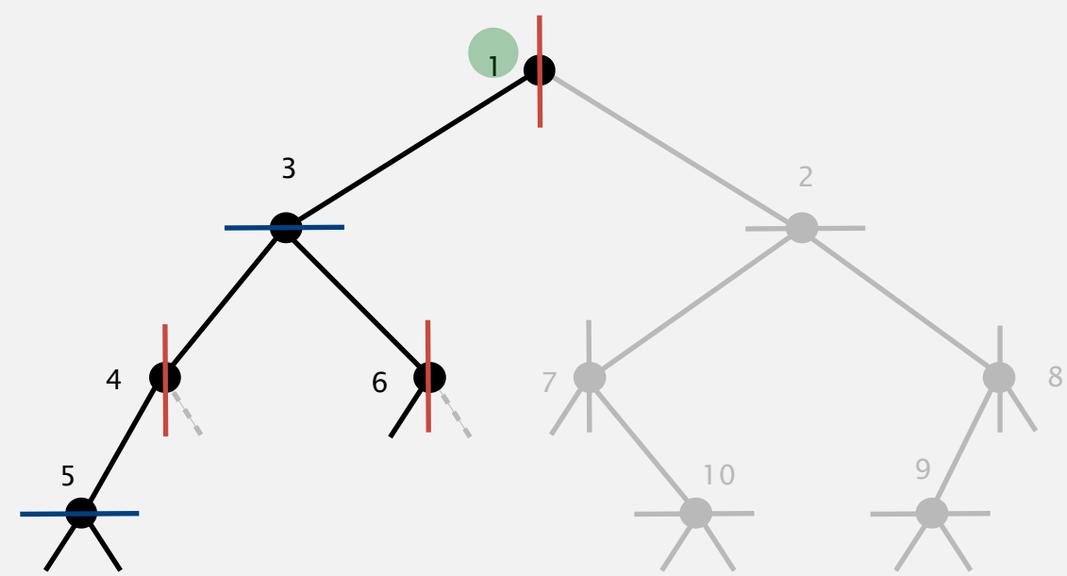




**return from function call**

# Nearest neighbor search in a 2d tree

- Check distance from point in node to query point.
- Recursively search left/bottom subdivision (if it could contain a closer point).
- Recursively search right/top subdivision (if it could contain a closer point).
- Organize recursive method so that it begins by searching for query point.



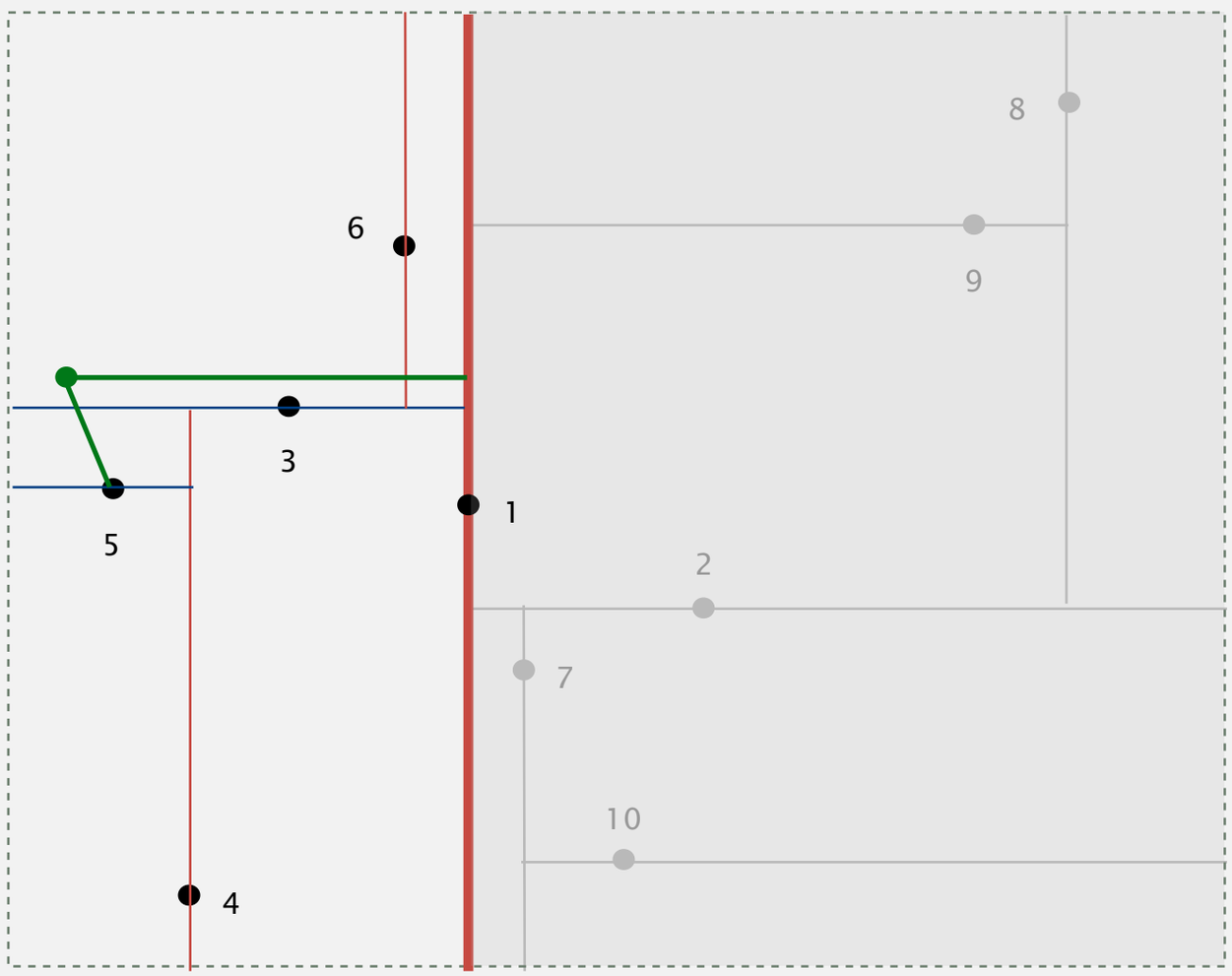



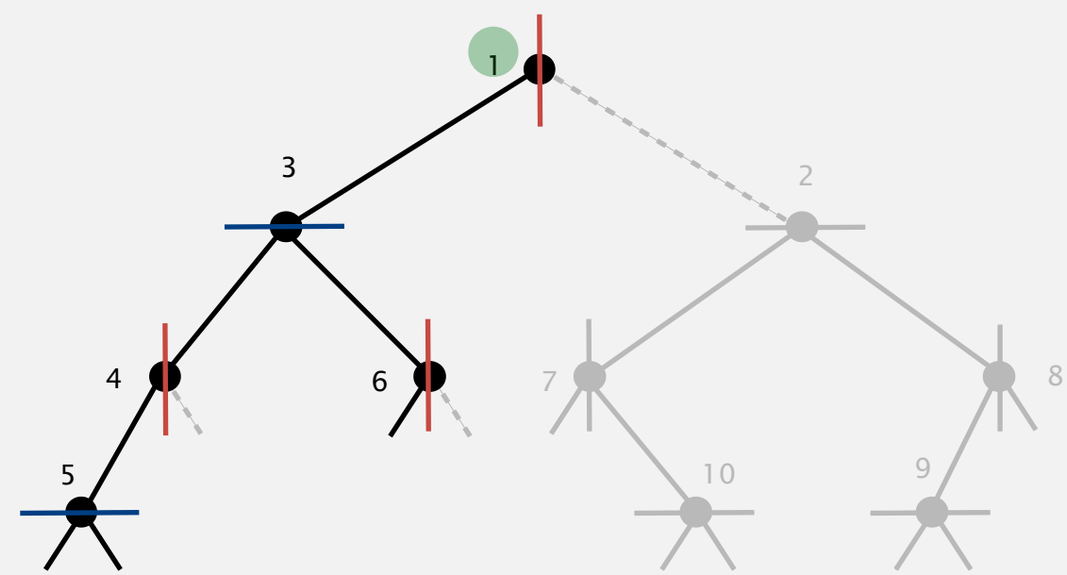

**return from function call**

**search right subtree next**

# Nearest neighbor search in a 2d tree

- Check distance from point in node to query point.
- Recursively search left/bottom subdivision (if it could contain a closer point).
- Recursively search right/top subdivision (if it could contain a closer point).
- Organize recursive method so that it begins by searching for query point.

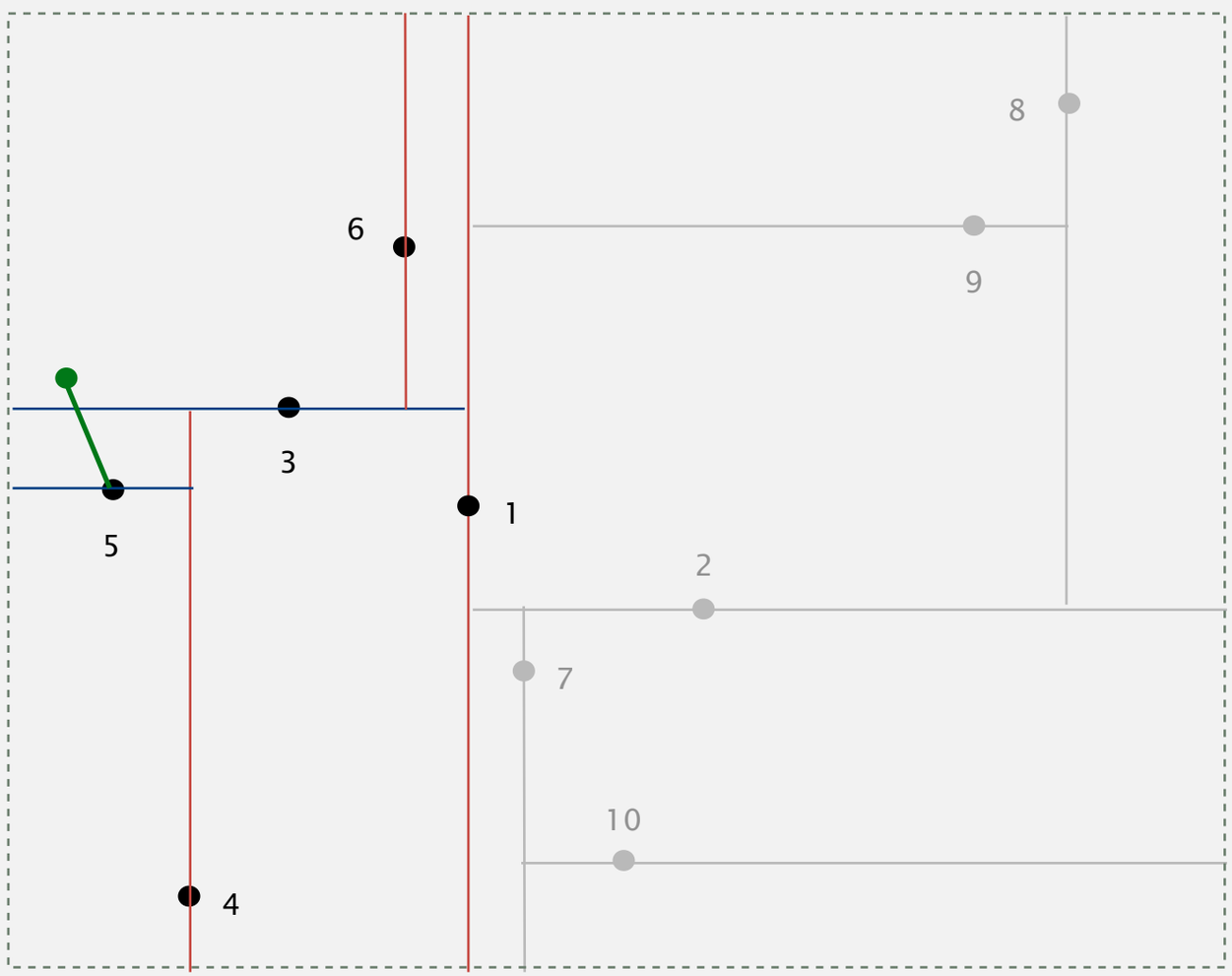


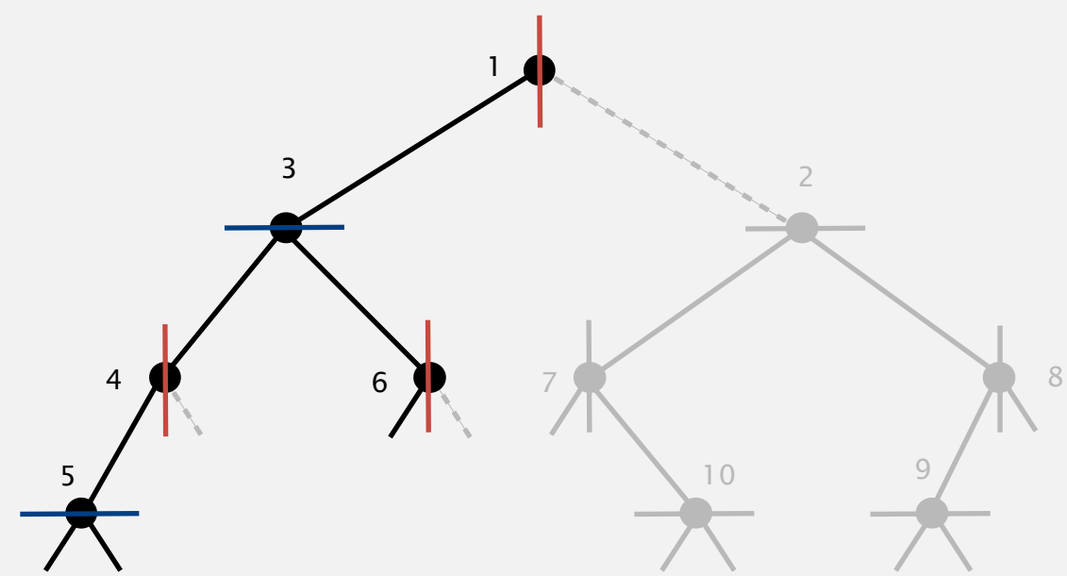




**search right subtree**
**prune since nearest neighbor**
**can't be in subdivision**

# Nearest neighbor search in a 2d tree

- Check distance from point in node to query point.
- Recursively search left/bottom subdivision (if it could contain a closer point).
- Recursively search right/top subdivision (if it could contain a closer point).
- Organize recursive method so that it begins by searching for query point.




nearest neighbor = 5

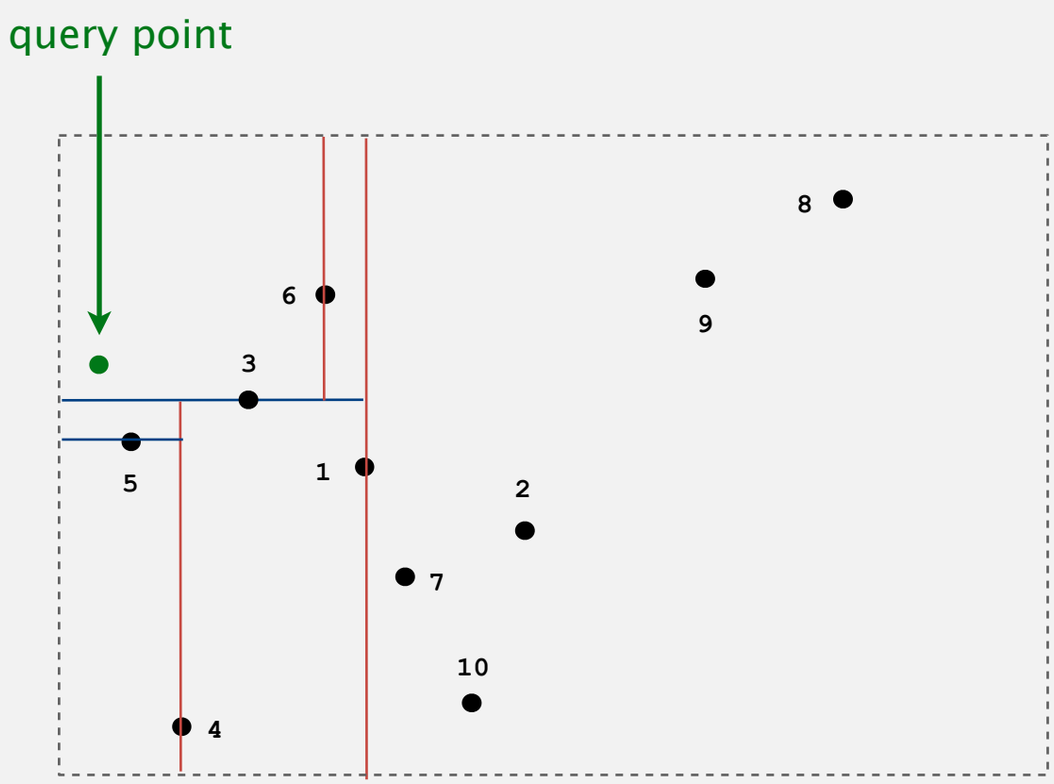
# Nearest neighbor search in a 2d tree

Nearest neighbor search. Given a query point, find the closest point.

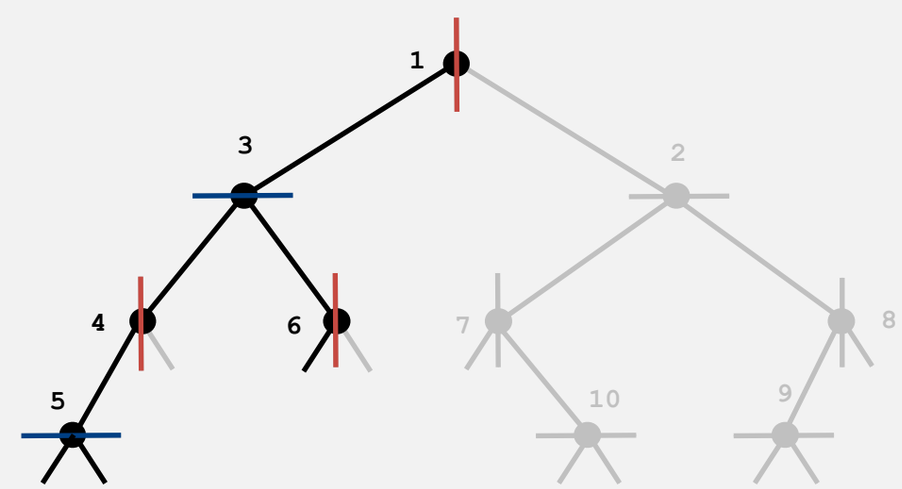
- Check distance from point in node to query point.
- Recursively search left/bottom subdivision (if it could contain a closer point).
- Recursively search right/top subdivision (if it could contain a closer point).
- Organize recursive method so that it begins by searching for query point.

Typical case. $\log N$.

Worst case (even if tree is balanced). $N$.



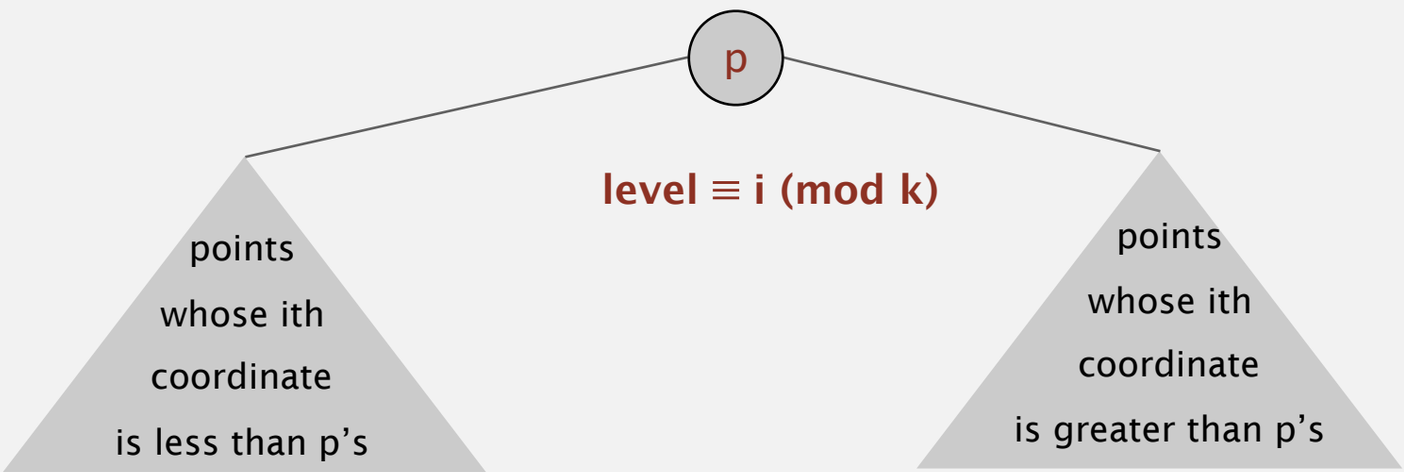

**closest point = 5**

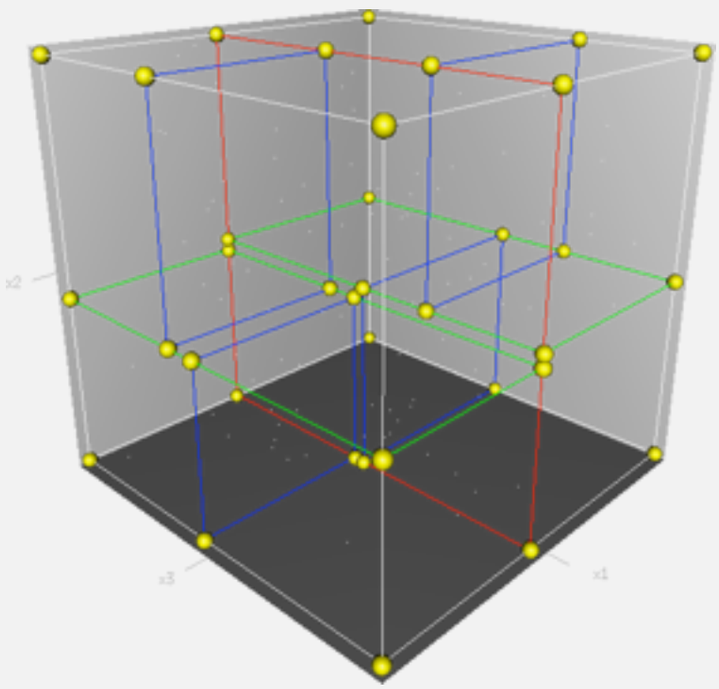


# Kd tree

Kd tree. Recursively partition $k$-dimensional space into 2 halfspaces.

Implementation. BST, but cycle through dimensions ala 2d trees.



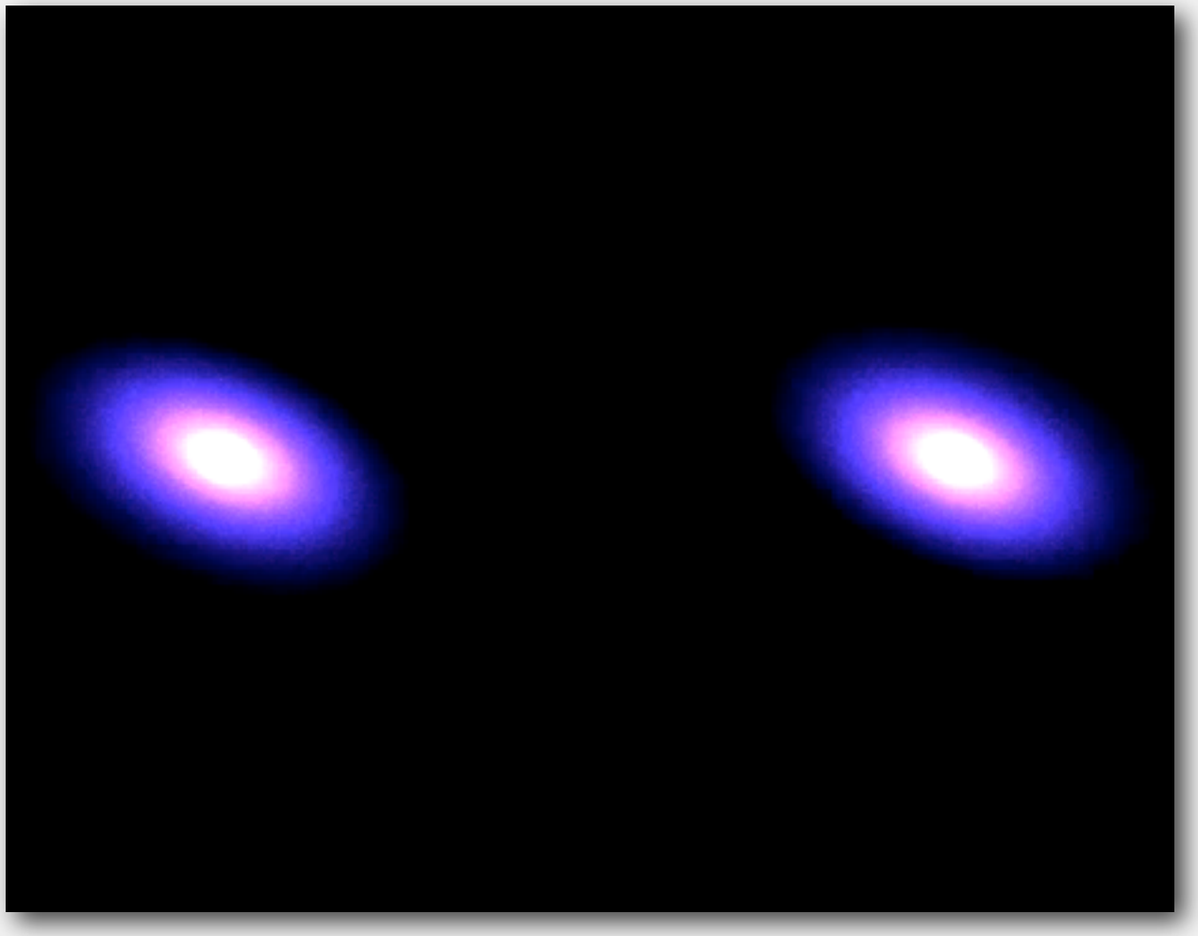
Efficient, simple data structure for processing $k$-dimensional data.

- Widely used.
- Adapts well to high-dimensional and clustered data.
- Discovered by an undergrad in an algorithms class!

Jon Bentley

# N-body simulation

Goal. Simulate the motion of $N$ particles, mutually affected by gravity.



http://www.youtube.com/watch?v=ua7YlN4eL_w

Brute force. For each pair of particles, compute force. $F = \frac{G\, m_1\, m_2}{r^2}$

# Appel algorithm for N-body simulation

Key idea. Suppose particle is far, far away from cluster of particles.

- Treat cluster of particles as a single aggregate particle.
- Compute force between particle and center of mass of aggregate particle.

# Appel algorithm for N-body simulation

- Build 3d-tree with $N$ particles as nodes.
- Store center-of-mass of subtree in each node.
- To compute total force acting on a particle, traverse tree, but stop as soon as distance from particle to subdivision is sufficiently large.

SIAM J. SCI. STAT. COMPUT.
Vol. 6, No. 1, January 1985



**AN EFFICIENT PROGRAM FOR MANY-BODY SIMULATION***

ANDREW W. APPEL†

**Abstract.** The simulation of $N$ particles interacting in a gravitational force field is useful in astrophysics, but such simulations become costly for large $N$. Representing the universe as a tree structure with the particles at the leaves and internal nodes labeled with the centers of mass of their descendants allows several simultaneous attacks on the computation time required by the problem. These approaches range from algorithmic changes (replacing an $O(N^2)$ algorithm with an algorithm whose time-complexity is believed to be $O(N \log N)$) to data structure modifications, code-tuning, and hardware modifications. The changes reduced the running time of a large problem ($N = 10{,}000$) by a factor of four hundred. This paper describes both the particular program and the methodology underlying such speedups.

Impact. Running time per step is $N \log N$ instead of $N^2 \Rightarrow$ enables new research.