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Hashing, Search Applications

› Hash tables
› Hash functions
› separate chaining
› linear probing

› Search applications
HASHING

- Hash functions
- Separate chaining
- Linear probing
### ST implementations: summary

<table>
<thead>
<tr>
<th>implementation</th>
<th>worst-case cost (after N inserts)</th>
<th>average-case cost (after N random inserts)</th>
<th>ordered iteration?</th>
<th>key interface</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>search</td>
<td>insert</td>
<td>delete</td>
<td>search hit</td>
</tr>
<tr>
<td>sequential search</td>
<td>N</td>
<td>N</td>
<td>N</td>
<td>N/2</td>
</tr>
<tr>
<td>(unordered list)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>binary search</td>
<td>lg N</td>
<td>N</td>
<td>N</td>
<td>lg N</td>
</tr>
<tr>
<td>(ordered array)</td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>BST</td>
<td>N</td>
<td>N</td>
<td>N</td>
<td>1.38 lg N</td>
</tr>
<tr>
<td>red-black BST</td>
<td>2 lg N</td>
<td>2 lg N</td>
<td>2 lg N</td>
<td>1.00 lg N</td>
</tr>
</tbody>
</table>

**Q.** Can we do better?  
**A.** Yes, but with different access to the data (if we don’t need ordered ops).
Hashing: basic plan

Save items in a key-indexed table (index is a function of the key).

Hash function. Method for computing array index from key.

Issues.

• Computing the hash function.
• Equality test: Method for checking whether two keys are equal.
• Collision resolution: Algorithm and data structure to handle two keys that hash to the same array index.

Classic space-time tradeoff.

• No space limitation: trivial hash function with key as index.
• No time limitation: trivial collision resolution with sequential search.
• Space and time limitations: hashing (the real world).
HASHING

- Hash functions
- Separate chaining
- Linear probing
Computing the hash function

Idealistic goal. Scramble the keys uniformly to produce a table index.

• Efficiently computable.
• Each table index equally likely for each key.

Ex 1. Phone numbers.
• Bad: first three digits.
• Better: last three digits.

Ex 2. Social Security numbers.
• Bad: first three digits.
• Better: last three digits.

Practical challenge. Need different approach for each key type.
Java’s hash code conventions

All Java classes inherit a method `hashCode()`, which returns a 32-bit `int`.

**Requirement.** If `x.equals(y)`, then `(x.hashCode() == y.hashCode())`.

**Highly desirable.** If `!x.equals(y)`, then `(x.hashCode() != y.hashCode())`.

**Default implementation.** Memory address of `x`.

**Legal (but poor) implementation.** Always return `17`.

**Customized implementations.** `Integer`, `Double`, `String`, `File`, `URL`, `Date`, ...

**User-defined types.** Users are on their own.
Implementing hash code: integers, booleans, and doubles

Java library implementations

```java
public final class Integer
{
    private final int value;
    ...

    public int hashCode()
    {  return value;  }
}
```

```java
public final class Double
{
    private final double value;
    ...

    public int hashCode()
    {  
       long bits = doubleToLongBits(value);
       return (int) (bits ^ (bits >>> 32));
    }
}
```

```java
public final class Boolean
{
    private final boolean value;
    ...

    public int hashCode()
    {
        if (value) return 1231;
        else       return 1237;
    }
}
```

- Convert to IEEE 64-bit representation;
- XOR most significant 32-bits with least significant 32-bits.
Implementing hash code: strings

Java library implementation

```java
public final class String {
   private final char[] s;
   ...
   public int hashCode()
   {
      int hash = 0;
      for (int i = 0; i < length(); i++)
         hash = s[i] + (31 * hash);
      return hash;
   }
}
```

- Horner's method to hash string of length \(L\): \(L\) multiplies/adds.
- Equivalent to \(h = s[0] \cdot 31^{L-1} + \ldots + s[L-3] \cdot 31^2 + s[L-2] \cdot 31^1 + s[L-1] \cdot 31^0\).

**Ex.** String \(s = "call"\);
```java
int code = s.hashCode();
```
\[
\begin{align*}
3045982 &= 99 \cdot 31^3 + 97 \cdot 31^2 + 108 \cdot 31^1 + 108 \cdot 31^0 \\
&= 108 + 31 \cdot (108 + 31 \cdot (97 + 31 \cdot (99)) \\
&\text{(Horner's method)}
\end{align*}
\]
Performance optimization.

• Cache the hash value in an instance variable.
• Return cached value.

```java
public final class String {
    private int hash = 0;
    private final char[] s;

    public int hashCode() {
        int h = hash;
        if (h != 0) return h;
        for (int i = 0; i < length(); i++)
            h = s[i] + (31 * hash);
        hash = h;
        return h;
    }
}
```
public final class Transaction implements Comparable<Transaction>
{
    private final String  who;
    private final Date    when;
    private final double  amount;

    public Transaction(String who, Date when, double amount)
    {  /* as before */  }

    ... 

    public boolean equals(Object y)
    {  /* as before */  }

    public int hashCode()
    {
        int hash = 17;
        hash = 31*hash + who.hashCode();
        hash = 31*hash + when.hashCode();
        hash = 31*hash + ((Double) amount).hashCode();
        return hash;
    }
}
Hash code design

"Standard" recipe for user-defined types.
• Combine each significant field using the $31x + y$ rule.
• If field is a primitive type, use wrapper type `hashCode()`.
• If field is null, return 0.
• If field is a reference type, use `hashCode()`.
• If field is an array, apply to each entry.

In practice. Recipe works reasonably well; used in Java libraries.
In theory. Keys are bitstring; "universal" hash functions exist.

Basic rule. Need to use the whole key to compute hash code; consult an expert for state-of-the-art hash codes.
**Modular hashing**

**Hash code.** An `int` between $-2^{31}$ and $2^{31}-1$.

**Hash function.** An `int` between 0 and $M-1$ (for use as array index).

```
private int hash(Key key)
{  return key.hashCode() % M;  }
```

- **bug**
  ```
  private int hash(Key key)
  {  return Math.abs(key.hashCode()) % M;  }
  ```

- **1-in-a-billion bug**
  ```
  private int hash(Key key)
  {  return (key.hashCode() & 0x7fffffff) % M;  }
  ```

- **correct**

  The `hashCode()` of "polygenelubricants" is $-2^{31}$
Uniform hashing assumption

Uniform hashing assumption. Each key is equally likely to hash to an integer between 0 and $M - 1$.

Bins and balls. Throw balls uniformly at random into $M$ bins.

Birthday problem. Expect two balls in the same bin after $\sim \sqrt{\pi M / 2}$ tosses.

Coupon collector. Expect every bin has $\geq 1$ ball after $\sim M \ln M$ tosses.

Load balancing. After $M$ tosses, expect most loaded bin has $\Theta(\log M / \log \log M)$ balls.
Uniform hashing assumption. Each key is equally likely to hash to an integer between 0 and $M - 1$.

Bins and balls. Throw balls uniformly at random into $M$ bins.

Hash value frequencies for words in Tale of Two Cities ($M = 97$)

Java's String data uniformly distribute the keys of Tale of Two Cities
HASHING

- Hash functions
- Separate chaining
- Linear probing
Collisions

**Collision.** Two distinct keys hashing to same index.
- Birthday problem $\Rightarrow$ can't avoid collisions unless you have a ridiculous (quadratic) amount of memory.
- Coupon collector + load balancing $\Rightarrow$ collisions will be evenly distributed.

**Challenge.** Deal with collisions efficiently.
Separate chaining symbol table

Use an array of $M < N$ linked lists. [H. P. Luhn, IBM 1953]

- **Hash:** map key to integer $i$ between 0 and $M - 1$.
- **Insert:** put at front of $i$th chain (if not already there).
- **Search:** need to search only $i$th chain.

<table>
<thead>
<tr>
<th>key</th>
<th>hash</th>
<th>value</th>
</tr>
</thead>
<tbody>
<tr>
<td>S</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>E</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>A</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>R</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>C</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>H</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>E</td>
<td>0</td>
<td>6</td>
</tr>
<tr>
<td>X</td>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td>A</td>
<td>0</td>
<td>8</td>
</tr>
<tr>
<td>M</td>
<td>4</td>
<td>9</td>
</tr>
<tr>
<td>P</td>
<td>3</td>
<td>10</td>
</tr>
<tr>
<td>L</td>
<td>3</td>
<td>11</td>
</tr>
<tr>
<td>E</td>
<td>0</td>
<td>12</td>
</tr>
</tbody>
</table>
public class SeparateChainingHashST<Key, Value> {
    private int M = 97; // number of chains
    private Node[] st = new Node[M]; // array of chains

    private static class Node {
        private Object key;
        private Object val;
        private Node next;
        ...
    }

    private int hash(Key key) {
        return (key.hashCode() & 0x7fffffff) % M;
    }

    public Value get(Key key) {
        int i = hash(key);
        for (Node x = st[i]; x != null; x = x.next)
            if (key.equals(x.key)) return (Value) x.val;
        return null;
    }
}
public class SeparateChainingHashST<Key, Value> {
    private int M = 97;               // number of chains
    private Node[] st = new Node[M]; // array of chains

    private static class Node {
        private Object key;
        private Object val;
        private Node next;
        ...
    }

    private int hash(Key key) {
        return (key.hashCode() & 0x7fffffff) % M;
    }

    public void put(Key key, Value val) {
        int i = hash(key);
        for (Node x = st[i]; x != null; x = x.next)
            if (key.equals(x.key)) { x.val = val; return; }
        st[i] = new Node(key, val, st[i]);
    }
}
Analysis of separate chaining

**Proposition.** Under uniform hashing assumption, probability that the number of keys in a list is within a constant factor of $\frac{N}{M}$ is extremely close to 1.

**Pf sketch.** Distribution of list size obeys a binomial distribution.

**Consequence.** Number of probes for search/insert is proportional to $\frac{N}{M}$.

- $M$ too large $\Rightarrow$ too many empty chains.
- $M$ too small $\Rightarrow$ chains too long.
- Typical choice: $M \sim \frac{N}{5} \Rightarrow$ constant-time ops.
### ST implementations: summary

<table>
<thead>
<tr>
<th>Implementation</th>
<th>Worst-case cost (after N inserts)</th>
<th>Average case (after N random inserts)</th>
<th>Ordered iteration?</th>
<th>Key interface</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>search</td>
<td>insert</td>
<td>delete</td>
<td>search</td>
</tr>
<tr>
<td>sequential search (unordered list)</td>
<td>N</td>
<td>N</td>
<td>N</td>
<td>N/2</td>
</tr>
<tr>
<td>binary search (ordered array)</td>
<td>lg N</td>
<td>N</td>
<td>N</td>
<td>lg N</td>
</tr>
<tr>
<td>BST</td>
<td>N</td>
<td>N</td>
<td>N</td>
<td>1.38 lg N</td>
</tr>
<tr>
<td>red-black tree</td>
<td>2 lg N</td>
<td>2 lg N</td>
<td>2 lg N</td>
<td>1.00 lg N</td>
</tr>
<tr>
<td>separate chaining</td>
<td>lg N *</td>
<td>lg N *</td>
<td>lg N *</td>
<td>3-5 *</td>
</tr>
</tbody>
</table>

* under uniform hashing assumption
HASHING

- Hash functions
- Separate chaining
- Linear probing
Collision resolution: open addressing

Open addressing. [Amdahl-Boehme-Rocherster-Samuel, IBM 1953]
When a new key collides, find next empty slot, and put it there.

linear probing (M = 30001, N = 15000)
**Linear probing hash table**

**Hash.** Map key to integer $i$ between 0 and $M - 1$.

**Insert.** Put at table index $i$ if free; if not try $i + 1$, $i + 2$, etc.

\[
\begin{array}{cccccccccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\
\text{st[]} & & & & & & & & & & & & & & & \\
\end{array}
\]

- \[\text{Insert } S\]
- \[\text{hash}(S) = 6\]

\[M = 16\]
Linear probing hash table

**Hash.** Map key to integer $i$ between 0 and $M - 1$.

**Insert.** Put at table index $i$ if free; if not try $i + 1$, $i + 2$, etc.

```
insert S
hash(S) = 6
```

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
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<tr>
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</tr>
</tbody>
</table>

$M = 16$
**Linear probing hash table**

**Hash.** Map key to integer $i$ between 0 and $M - 1$.

**Insert.** Put at table index $i$ if free; if not try $i + 1$, $i + 2$, etc.

```
insert E
hash(E) = 10
```

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
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<th>3</th>
<th>4</th>
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</tbody>
</table>

$M = 16$
Linear probing hash table

**Hash.** Map key to integer \( i \) between 0 and \( M - 1 \).

**Insert.** Put at table index \( i \) if free; if not try \( i + 1 \), \( i + 2 \), etc.

```
insert E
hash(E) = 10
```

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
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</thead>
<tbody>
<tr>
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</tr>
</tbody>
</table>

\( M = 16 \)
### Linear probing hash table

**Hash.** Map key to integer $i$ between 0 and $M - 1$.

**Insert.** Put at table index $i$ if free; if not try $i + 1$, $i + 2$, etc.

<table>
<thead>
<tr>
<th>i</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
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</tr>
</tbody>
</table>

Insert $A$

$\text{hash}(A) = 4$

$M = 16$
**Linear probing hash table**

**Hash.** Map key to integer \( i \) between 0 and \( M - 1 \).

**Insert.** Put at table index \( i \) if free; if not try \( i + 1 \), \( i + 2 \), etc.

```
insert A
hash(A) = 4
```

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
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</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
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<td>A</td>
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<td>S</td>
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</tr>
</tbody>
</table>

\( M = 16 \)
## Linear probing hash table

**Hash.** Map key to integer $i$ between 0 and $M - 1$.

**Insert.** Put at table index $i$ if free; if not try $i + 1$, $i + 2$, etc.

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
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<th>13</th>
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<th>15</th>
</tr>
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<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>A</td>
<td>S</td>
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<td></td>
<td>E</td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>

*M = 16*
Hash probing hash table

**Hash.** Map key to integer $i$ between 0 and $M - 1$.

**Insert.** Put at table index $i$ if free; if not try $i + 1$, $i + 2$, etc.

```
insert R
hash(R) = 14
```

```
+---+---+---+---+---+---+---+---+---+---+---+---+---+---+
|   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
+---+---+---+---+---+---+---+---+---+---+---+---+---+---+
  0  1  2  3  4  5  6  7  8  9 10 11 12 13 14 15
st[]
```

$M = 16$
**Linear probing hash table**

**Hash.** Map key to integer $i$ between 0 and $M - 1$.

**Insert.** Put at table index $i$ if free; if not try $i + 1$, $i + 2$, etc.

insert C
hash(C) = 5

<table>
<thead>
<tr>
<th>M = 16</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
</tr>
<tr>
<td>------</td>
</tr>
<tr>
<td></td>
</tr>
</tbody>
</table>
**Linear probing hash table**

**Hash.** Map key to integer \( i \) between 0 and \( M - 1 \).

**Insert.** Put at table index \( i \) if free; if not try \( i + 1, i + 2, \) etc.

```
insert C
hash(C) = 5
```

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
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<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>st[]</td>
<td></td>
<td></td>
<td></td>
<td>A</td>
<td>C</td>
<td>S</td>
<td></td>
<td></td>
<td></td>
<td>E</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>R</td>
</tr>
</tbody>
</table>

\( M = 16 \)
**Linear probing hash table**

**Hash.** Map key to integer \( i \) between 0 and \( M - 1 \).

**Insert.** Put at table index \( i \) if free; if not try \( i + 1 \), \( i + 2 \), etc.

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
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<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>st[]</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>A</td>
<td>C</td>
<td>S</td>
<td></td>
<td></td>
<td>E</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>R</td>
</tr>
</tbody>
</table>

\( M = 16 \)

**insert H**

hash(H) = 4
**Linear probing hash table**

**Hash.** Map key to integer $i$ between 0 and $M - 1$.

**Insert.** Put at table index $i$ if free; if not try $i + 1$, $i + 2$, etc.


tbl[]

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>A</td>
<td>C</td>
<td>S</td>
<td></td>
<td></td>
<td></td>
<td>E</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$M = 16$

![Table with elements A, C, S, E, R, and an empty space for H]
**Linear probing hash table**

**Hash.** Map key to integer $i$ between 0 and $M - 1$.

**Insert.** Put at table index $i$ if free; if not try $i + 1$, $i + 2$, etc.

---

```
insert H
hash(H) = 4
```

---

```
<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>A</td>
<td>C</td>
<td>S</td>
<td></td>
<td></td>
<td></td>
<td>E</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
```

$M = 16$
**Linear probing hash table**

**Hash.** Map key to integer $i$ between 0 and $M - 1$.

**Insert.** Put at table index $i$ if free; if not try $i + 1$, $i + 2$, etc.

```
insert H
hash(H) = 4
```

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>A</td>
<td>C</td>
<td>S</td>
<td></td>
<td></td>
<td></td>
<td>E</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>R</td>
</tr>
</tbody>
</table>

$M = 16$

H
Linear probing hash table

**Hash.** Map key to integer $i$ between 0 and $M - 1$.

**Insert.** Put at table index $i$ if free; if not try $i + 1$, $i + 2$, etc.

```plaintext
insert H
hash(H) = 4
```

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>A</td>
<td>C</td>
<td>S</td>
<td>H</td>
<td></td>
<td></td>
<td></td>
<td>E</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>R</td>
</tr>
</tbody>
</table>

$M = 16$
**Linear probing hash table**

**Hash.** Map key to integer $i$ between 0 and $M - 1$.

**Insert.** Put at table index $i$ if free; if not try $i + 1$, $i + 2$, etc.

insert $X$
hash($X$) = 15

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>st[]</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>A</td>
<td>C</td>
<td>S</td>
<td>H</td>
<td></td>
<td></td>
<td>E</td>
<td></td>
<td></td>
<td>R</td>
</tr>
</tbody>
</table>

$M = 16$
**Linear probing hash table**

**Hash.** Map key to integer $i$ between 0 and $M - 1$.

**Insert.** Put at table index $i$ if free; if not try $i + 1$, $i + 2$, etc.

```
insert X
hash(X) = 15
```

<table>
<thead>
<tr>
<th>st[]</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>A</td>
<td>C</td>
<td>S</td>
<td>H</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>X</td>
</tr>
</tbody>
</table>

$M = 16$
**Linear probing hash table**

**Hash.** Map key to integer $i$ between 0 and $M - 1$.

**Insert.** Put at table index $i$ if free; if not try $i + 1$, $i + 2$, etc.

```
insert M
hash(M) = 1
```

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>st[]</td>
<td></td>
<td></td>
<td></td>
<td>A</td>
<td>C</td>
<td>S</td>
<td>H</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>M = 16</td>
<td></td>
<td></td>
<td></td>
<td>R</td>
<td>X</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
**Linear probing hash table**

**Hash.** Map key to integer $i$ between 0 and $M - 1$.

**Insert.** Put at table index $i$ if free; if not try $i + 1$, $i + 2$, etc.

```
insert M
hash(M) = 1
```

<table>
<thead>
<tr>
<th>st[]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>4</td>
</tr>
<tr>
<td>5</td>
</tr>
<tr>
<td>6</td>
</tr>
<tr>
<td>7</td>
</tr>
<tr>
<td>8</td>
</tr>
<tr>
<td>9</td>
</tr>
<tr>
<td>10</td>
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<tr>
<td>11</td>
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<tr>
<td>12</td>
</tr>
<tr>
<td>13</td>
</tr>
<tr>
<td>14</td>
</tr>
<tr>
<td>15</td>
</tr>
<tr>
<td>M</td>
</tr>
<tr>
<td>A</td>
</tr>
<tr>
<td>C</td>
</tr>
<tr>
<td>S</td>
</tr>
<tr>
<td>H</td>
</tr>
<tr>
<td>E</td>
</tr>
<tr>
<td>R</td>
</tr>
<tr>
<td>X</td>
</tr>
</tbody>
</table>

$M = 16$
**Linear probing hash table**

**Hash.** Map key to integer $i$ between 0 and $M - 1$.

**Insert.** Put at table index $i$ if free; if not try $i + 1$, $i + 2$, etc.

```plaintext
insert P
hash(P) = 14
```

```
<p>| | | | | | | | | | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>M</td>
<td></td>
<td>A</td>
<td>C</td>
<td>S</td>
<td>H</td>
<td></td>
<td>E</td>
<td></td>
<td></td>
<td>R</td>
<td>X</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
```

$M = 16$
## Linear probing hash table

**Hash.** Map key to integer $i$ between 0 and $M - 1$.

**Insert.** Put at table index $i$ if free; if not try $i + 1$, $i + 2$, etc.

<table>
<thead>
<tr>
<th>st[]</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>M</td>
<td>A</td>
<td>C</td>
<td>S</td>
<td>H</td>
<td>E</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$M = 16$

- `P`
# Linear probing hash table

**Hash.** Map key to integer $i$ between 0 and $M - 1$.

**Insert.** Put at table index $i$ if free; if not try $i + 1$, $i + 2$, etc.

<table>
<thead>
<tr>
<th>i</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>st[]</td>
<td>P</td>
<td>M</td>
<td>A</td>
<td>C</td>
<td>S</td>
<td>H</td>
<td>E</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$M = 16$

insert $P$

hash($P$) = 14
**Linear probing hash table**

**Hash.** Map key to integer \( i \) between 0 and \( M - 1 \).

**Insert.** Put at table index \( i \) if free; if not try \( i + 1 \), \( i + 2 \), etc.

<table>
<thead>
<tr>
<th>( \text{st[]} )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{P} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \text{M} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \text{A} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \text{C} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \text{S} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \text{H} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \text{E} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \text{R} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \text{X} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\( M = 16 \)
**Linear probing hash table**

**Hash.** Map key to integer $i$ between 0 and $M - 1$.

**Insert.** Put at table index $i$ if free; if not try $i + 1$, $i + 2$, etc.

```
insert L
hash(L) = 6
```

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>M</td>
<td></td>
<td>A</td>
<td>C</td>
<td>S</td>
<td>H</td>
<td></td>
<td>E</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$M = 16$
### Linear probing hash table

**Hash.** Map key to integer \( i \) between 0 and \( M - 1 \).

**Insert.** Put at table index \( i \) if free; if not try \( i + 1, i + 2 \), etc.

```
insert L
hash(L) = 6
```

<table>
<thead>
<tr>
<th>st[]</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>P</td>
<td>M</td>
<td></td>
<td>A</td>
<td></td>
<td>C</td>
<td>S</td>
<td>H</td>
<td></td>
<td>E</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>R</td>
<td>X</td>
</tr>
</tbody>
</table>

\( M = 16 \)
Hash. Map key to integer $i$ between 0 and $M - 1$.

Insert. Put at table index $i$ if free; if not try $i + 1$, $i + 2$, etc.

```
insert L
hash(L) = 6
```

<table>
<thead>
<tr>
<th>st[]</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>P</td>
<td>M</td>
<td></td>
<td>A</td>
<td>C</td>
<td>S</td>
<td>H</td>
<td>L</td>
<td></td>
<td>E</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$M = 16$
Linear probing hash table

Hash. Map key to integer $i$ between 0 and $M - 1$.
Insert. Put at table index $i$ if free; if not try $i + 1$, $i + 2$, etc.
Search. Search table index $i$; if occupied but no match, try $i + 1$, $i + 2$, etc.

linear probing hash table

\[
\begin{array}{cccccccccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\
\text{st[]} & P & M & A & C & S & H & L & E & & & & & & R & X \\
\end{array}
\]

$M = 16$
**Linear probing hash table**

**Hash.** Map key to integer $i$ between 0 and $M - 1$.

**Insert.** Put at table index $i$ if free; if not try $i + 1, i + 2$, etc.

**Search.** Search table index $i$; if occupied but no match, try $i + 1, i + 2$, etc.

```
search E
hash(E) = 10
```

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>M</td>
<td></td>
<td></td>
<td>A</td>
<td>C</td>
<td>S</td>
<td>H</td>
<td>L</td>
<td>E</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>R</td>
<td>X</td>
</tr>
</tbody>
</table>

$M = 16$
**Linear probing hash table**

**Hash.** Map key to integer $i$ between 0 and $M - 1$.

**Insert.** Put at table index $i$ if free; if not try $i + 1$, $i + 2$, etc.

**Search.** Search table index $i$; if occupied but no match, try $i + 1$, $i + 2$, etc.

```plaintext
search E
hash(E) = 10
```

```
<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
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<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>M</td>
<td></td>
<td>A</td>
<td>C</td>
<td>S</td>
<td>H</td>
<td>L</td>
<td></td>
<td>E</td>
<td></td>
<td></td>
<td></td>
<td>R</td>
<td></td>
<td>X</td>
</tr>
</tbody>
</table>
```

$M = 16$

E

search hit
(return corresponding value)
Linear probing hash table

Hash. Map key to integer $i$ between 0 and $M - 1$.
Insert. Put at table index $i$ if free; if not try $i + 1$, $i + 2$, etc.
Search. Search table index $i$; if occupied but no match, try $i + 1$, $i + 2$, etc.

```
search L
hash(L) = 6
```

|M = 16|
|---|---|---|---|---|---|---|---|---|---|---|---|---|
|P |M |A |C |S |H |L |E |   |   |   |R |X |
Linear probing hash table

**Hash.** Map key to integer $i$ between 0 and $M - 1$.

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hash(L) = 6
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<table>
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<th>0</th>
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**Hash.** Map key to integer \( i \) between 0 and \( M - 1 \).

**Insert.** Put at table index \( i \) if free; if not try \( i + 1, i + 2, \) etc.

**Search.** Search table index \( i \); if occupied but no match, try \( i + 1, i + 2, \) etc.

```
search  L
hash(L) = 6
```

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
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<th>9</th>
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<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>st[]</td>
<td>P</td>
<td>M</td>
<td></td>
<td>A</td>
<td>C</td>
<td>S</td>
<td>H</td>
<td>L</td>
<td></td>
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</tr>
</tbody>
</table>

\( M = 16 \)
**Linear probing hash table**

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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
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</table>

$M = 16$

L

search hit
(return corresponding value)
**Linear probing hash table**

**Hash.** Map key to integer $i$ between 0 and $M - 1$.

**Insert.** Put at table index $i$ if free; if not try $i + 1$, $i + 2$, etc.

**Search.** Search table index $i$; if occupied but no match, try $i + 1$, $i + 2$, etc.

```
search K
hash(K) = 5
```

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
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$M = 16$
Linear probing hash table

Hash. Map key to integer $i$ between 0 and $M - 1$.

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Search. Search table index $i$; if occupied but no match, try $i + 1$, $i + 2$, etc.

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hash(K) = 5
```
Linear probing hash table

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</tr>
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</table>

$M = 16$
Linear probing hash table

**Hash.** Map key to integer \( i \) between 0 and \( M - 1 \).

**Insert.** Put at table index \( i \) if free; if not try \( i + 1, i + 2, \) etc.

**Search.** Search table index \( i \); if occupied but no match, try \( i + 1, i + 2, \) etc.

```plaintext
search  K
hash(K) = 5
```

\[
\begin{array}{cccccccccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\
\hline
\text{st[]} & P & M & | & A & C & S & H & L & E & | & R & X & | & \\
\end{array}
\]

\( M = 16 \)

\( K \)
Linear probing hash table

**Hash.** Map key to integer $i$ between 0 and $M - 1$.

**Insert.** Put at table index $i$ if free; if not try $i + 1$, $i + 2$, etc.

**Search.** Search table index $i$; if occupied but no match, try $i + 1$, $i + 2$, etc.

```
search K
hash(K) = 5
```

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<th>st[]</th>
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<td>L</td>
<td>E</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>K</td>
</tr>
</tbody>
</table>

$M = 16$

search miss
(return null)
Hash. Map key to integer $i$ between 0 and $M - 1$.

Insert. Put at table index $i$ if free; if not try $i + 1$, $i + 2$, etc.

Search. Search table index $i$; if occupied but no match, try $i + 1$, $i + 2$, etc.

Note. Array size $M$ must be greater than number of key-value pairs $N$.

M = 16

<table>
<thead>
<tr>
<th>st[]</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
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<td>C</td>
<td>S</td>
<td>H</td>
<td>L</td>
<td>E</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>R</td>
<td>X</td>
</tr>
</tbody>
</table>
public class LinearProbingHashST<Key, Value> {
    private int M = 30001;
    private Value[] vals = (Value[]) new Object[M];
    private Key[] keys = (Key[]) new Object[M];

    private int hash(Key key) {  /* as before */  }

    public void put(Key key, Value val) {
        int i;
        for (i = hash(key); keys[i] != null; i = (i+1) % M)
            if (keys[i].equals(key))
                break;
        keys[i] = key;
        vals[i] = val;
    }

    public Value get(Key key) {
        for (int i = hash(key); keys[i] != null; i = (i+1) % M)
            if (key.equals(keys[i]))
                return vals[i];
        return null;
    }
}
**Cluster.** A contiguous block of items.

**Observation.** New keys likely to hash into middle of big clusters.
**Knuth's parking problem**

**Model.** Cars arrive at one-way street with $M$ parking spaces. Each desires a random space $i$: if space $i$ is taken, try $i + 1$, $i + 2$, etc.

**Q.** What is mean displacement of a car?

**Half-full.** With $M/2$ cars, mean displacement is $\sim 3/2$.

**Full.** With $M$ cars, mean displacement is $\sim \sqrt{\pi M}/8$.
Analysis of linear probing

Proposition. Under uniform hashing assumption, the average number of probes in a linear probing hash table of size $M$ that contains $N = \alpha M$ keys is:

$$\sim \frac{1}{2} \left(1 + \frac{1}{1 - \alpha}\right) \quad \sim \frac{1}{2} \left(1 + \frac{1}{(1 - \alpha)^2}\right)$$

- search hit
- search miss / insert

Pf.

Parameters.

- $M$ too large $\Rightarrow$ too many empty array entries.
- $M$ too small $\Rightarrow$ search time blows up.
- Typical choice: $\alpha = N / M \sim 1/2$.  
  # probes for search hit is about 3/2
  # probes for search miss is about 5/2
## ST implementations: summary

<table>
<thead>
<tr>
<th>Implementation</th>
<th>Worst-case cost (after N inserts)</th>
<th>Average case (after N random inserts)</th>
<th>Ordered iteration?</th>
<th>Key interface</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>search</td>
<td>insert</td>
<td>delete</td>
<td>search hit</td>
</tr>
<tr>
<td>sequential search (unordered list)</td>
<td>N</td>
<td>N</td>
<td>N</td>
<td>N/2</td>
</tr>
<tr>
<td>binary search (ordered array)</td>
<td>lg N</td>
<td>N</td>
<td>N</td>
<td>lg N</td>
</tr>
<tr>
<td>BST</td>
<td>N</td>
<td>N</td>
<td>N</td>
<td>1.38 lg N</td>
</tr>
<tr>
<td>red-black tree</td>
<td>2 lg N</td>
<td>2 lg N</td>
<td>2 lg N</td>
<td>1.00 lg N</td>
</tr>
<tr>
<td>separate chaining</td>
<td>lg N</td>
<td>lg N</td>
<td>lg N</td>
<td>3-5</td>
</tr>
<tr>
<td>linear probing</td>
<td>lg N</td>
<td>lg N</td>
<td>lg N</td>
<td>3-5</td>
</tr>
</tbody>
</table>

* under uniform hashing assumption
War story: String hashing in Java

String hashCode() in Java 1.1.

- For long strings: only examine 8-9 evenly spaced characters.
- Benefit: saves time in performing arithmetic.

```java
public int hashCode()
{
    int hash = 0;
    int skip = Math.max(1, length() / 8);
    for (int i = 0; i < length(); i += skip)
        hash = s[i] + (37 * hash);
    return hash;
}
```

- Downside: great potential for bad collision patterns.

http://www.cs.princeton.edu/introcs/13loop/Hello.java
http://www.cs.princeton.edu/introcs/13loop/Hello.class
http://www.cs.princeton.edu/introcs/12type/index.html
War story: algorithmic complexity attacks

Q. Is the uniform hashing assumption important in practice?
A. Obvious situations: aircraft control, nuclear reactor, pacemaker.
A. Surprising situations: denial-of-service attacks.

Real-world exploits. [Crosby-Wallach 2003]
• Bro server: send carefully chosen packets to DOS the server, using less bandwidth than a dial-up modem.
• Perl 5.8.0: insert carefully chosen strings into associative array.
• Linux 2.4.20 kernel: save files with carefully chosen names.
Algorithmic complexity attack on Java

**Goal.** Find family of strings with the same hash code.

**Solution.** The base 31 hash code is part of Java's string API.

<table>
<thead>
<tr>
<th>key</th>
<th>hashCode ()</th>
</tr>
</thead>
<tbody>
<tr>
<td>&quot;Aa&quot;</td>
<td>2112</td>
</tr>
<tr>
<td>&quot;BB&quot;</td>
<td>2112</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>key</th>
<th>hashCode ()</th>
</tr>
</thead>
<tbody>
<tr>
<td>&quot;AaAaAaAa&quot;</td>
<td>-540425984</td>
</tr>
<tr>
<td>&quot;AaAaAaBB&quot;</td>
<td>-540425984</td>
</tr>
<tr>
<td>&quot;AaAaBBAa&quot;</td>
<td>-540425984</td>
</tr>
<tr>
<td>&quot;AaAaBBBB&quot;</td>
<td>-540425984</td>
</tr>
<tr>
<td>&quot;AaBBAaAa&quot;</td>
<td>-540425984</td>
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</tr>
</tbody>
</table>

2^N strings of length 2N that hash to same value!
Diversion: one-way hash functions

One-way hash function. "Hard" to find a key that will hash to a desired value (or two keys that hash to same value).

Ex. MD4, MD5, SHA-0, SHA-1, SHA-2, WHIRLPOOL, RIPEMD-160, ….

known to be insecure

String password = args[0];
MessageDigest sha1 =
MessageDigest.getInstance("SHA1");
byte[] bytes = sha1.digest(password);

/* prints bytes as hex string */

Applications. Digital fingerprint, message digest, storing passwords.
Caveat. Too expensive for use in ST implementations.
Separate chaining vs. linear probing

Separate chaining.
- Easier to implement delete.
- Performance degrades gracefully.
- Clustering less sensitive to poorly-designed hash function.

Linear probing.
- Less wasted space.
- Better cache performance.

Q. How to delete?
Q. How to resize?
Hashing: variations on the theme

Many improved versions have been studied.

Two-probe hashing.  (separate-chaining variant)
• Hash to two positions, insert key in shorter of the two chains.
• Reduces expected length of the longest chain to $\log \log N$.

Double hashing.  (linear-probing variant)
• Use linear probing, but skip a variable amount, not just 1 each time.
• Effectively eliminates clustering.
• Can allow table to become nearly full.
• More difficult to implement delete.

Cuckoo hashing.  (linear-probing variant)
• Hash key to two positions; insert key into either position; if occupied, reinsert displaced key into its alternative position (and recur).
• Constant worst case time for search.
Hash tables vs. balanced search trees

Hash tables.
• Simpler to code.
• No effective alternative for unordered keys.
• Faster for simple keys (a few arithmetic ops versus $\log N$ compares).
• Better system support in Java for strings (e.g., cached hash code).

Balanced search trees.
• Stronger performance guarantee.
• Support for ordered ST operations.
• Easier to implement compareTo() correctly than equals() and hashCode().

Java system includes both.
• Red-black BSTs: java.util.TreeMap, java.util.TreeSet.
• Hash tables: java.util.HashMap, java.util.IdentityHashMap.
Hashing, Search Applications

- Hash tables
- Search applications
  - Sets
  - Dictionary clients
  - Indexing clients
  - Sparse vectors
Search Applications

- Sets
- Dictionary clients
- Indexing clients
- Sparse vectors
**Set API**

**Mathematical set.** A collection of distinct keys.

```java
public class SET<Key extends Comparable<Key>> {
    SET() // create an empty set
    void add(Key key) // add the key to the set
    boolean contains(Key key) // is the key in the set?
    void remove(Key key) // remove the key from the set
    int size() // return the number of keys in the set
    Iterator<Key> iterator() // iterator through keys in the set
}
```

**Q.** How to implement?

**A.** Remove “value” from any ST implementation
Exception filter

- Read in a list of words from one file.
- Print out all words from standard input that are { in, not in } the list.

```
% more list.txt
was it the of

% java WhiteList list.txt < tinyTale.txt
it was the of it was the of
it was the of it was the of
it was the of it was the of
it was the of it was the of
it was the of it was the of
it was the of it was the of

% java BlackList list.txt < tinyTale.txt
best times worst times
age wisdom age foolishness
epoch belief epoch incredulity
season light season darkness
spring hope winter despair
```
Exception filter applications

- Read in a list of words from one file.
- Print out all words from standard input that are \{ in, not in \} the list.

<table>
<thead>
<tr>
<th>application</th>
<th>purpose</th>
<th>key</th>
<th>in list</th>
</tr>
</thead>
<tbody>
<tr>
<td>spell checker</td>
<td>identify misspelled words</td>
<td>word</td>
<td>dictionary words</td>
</tr>
<tr>
<td>browser</td>
<td>mark visited pages</td>
<td>URL</td>
<td>visited pages</td>
</tr>
<tr>
<td>parental controls</td>
<td>block sites</td>
<td>URL</td>
<td>bad sites</td>
</tr>
<tr>
<td>chess</td>
<td>detect draw</td>
<td>board</td>
<td>positions</td>
</tr>
<tr>
<td>spam filter</td>
<td>eliminate spam</td>
<td>IP address</td>
<td>spam addresses</td>
</tr>
<tr>
<td>credit cards</td>
<td>check for stolen cards</td>
<td>number</td>
<td>stolen cards</td>
</tr>
</tbody>
</table>
Exception filter: Java implementation

- Read in a list of words from one file.
- Print out all words from standard input that are { in, not in } the list.

```java
public class WhiteList
{
   public static void main(String[] args)
   {
      SET<String> set = new SET<String>();

      In in = new In(args[0]);
      while (!in.isEmpty())
         set.add(in.readString());

      while (!StdIn.isEmpty())
      {
         String word = StdIn.readString();
         if (set.contains(word))
            StdOut.println(word);
      }
   }
}
```
• Read in a list of words from one file.
• Print out all words from standard input that are \{ in, not in \} the list.

```java
public class BlackList
{
   public static void main(String[] args)
   {
      SET<String> set = new SET<String>();
      In in = new In(args[0]);
      while (!in.isEmpty())
         set.add(in.readString());
      while (!StdIn.isEmpty())
      {
         String word = StdIn.readString();
         if (!set.contains(word))
            StdOut.println(word);
      }
   }
}
```
Search Applications

- Sets
- Dictionary clients
- Indexing clients
- Sparse vectors
Dictionary lookup

Command-line arguments.
• A comma-separated value (CSV) file.
• Key field.
• Value field.

Ex 1. DNS lookup.

% more ip.csv
www.princeton.edu,128.112.128.15
www.cs.princeton.edu,128.112.136.35
www.math.princeton.edu,128.112.18.11
www.cs.harvard.edu,140.247.50.127
www.harvard.edu,128.103.60.24
www.yale.edu,130.132.51.8
www.econ.yale.edu,128.36.236.74
www.cs.yale.edu,128.36.229.30
espn.com,199.181.135.201
yahoo.com,66.94.234.13
msn.com,207.68.172.246
google.com,64.233.167.99
baidu.com,202.108.22.33
yahoo.co.jp,202.93.91.141
sina.com.cn,202.108.33.32
ebay.com,66.135.192.87
adobe.com,192.150.18.60
163.com,220.181.29.154
passport.net,65.54.179.226
tom.com,61.135.158.237
nate.com,203.226.253.11
cnn.com,64.236.16.20
daum.net,211.115.77.211
blogger.com,66.102.15.100
fastclick.com,205.180.86.4
wikipedia.org,66.230.200.100
rakuten.co.jp,202.72.51.22
...

% java LookupCSV ip.csv 0 1
adobe.com
192.150.18.60
www.princeton.edu
128.112.128.15
ebay.edu
Not found

% java LookupCSV ip.csv 1 0
128.112.128.15
www.princeton.edu
999.999.999.99
Not found

URL is key
IP is value

IP is key
URL is value
Dictionary lookup

Command-line arguments.
- A comma-separated value (CSV) file.
- Key field.
- Value field.

Ex 2. Amino acids.

```
% java LookupCSV amino.csv 0 3
ACT
  Threonine
TAG
  Stop
CAT
  Histidine
```

% more amino.csv
TTT,Phe,F,Phenylalanine
TTC,Phe,F,Phenylalanine
TTA,Leu,L,Leucine
TTG,Leu,L,Leucine
TCT,Ser,S,Serine
TCC,Ser,S,Serine
TCA,Ser,S,Serine
TCG,Ser,S,Serine
TAT,Tyr,Y,Tyrosine
TAC,Tyr,Y,Tyrosine
TAA,Stop,Stop,Stop
TAG,Stop,Stop,Stop
TGT,Cys,C,Cysteine
TGC,Cys,C,Cysteine
TGA,Stop,Stop,Stop
TGG,Trp,W,Tryptophan
CTT,Leu,L,Leucine
CTC,Leu,L,Leucine
CTA,Leu,L,Leucine
CTG,Leu,L,Leucine
CCT,Pro,P,Proline
CCC,Pro,P,Proline
CCA,Pro,P,Proline
CCG,Pro,P,Proline
CAT,His,H,Histidine
CAC,His,H,Histidine
CAA,Gln,Q,Glutamine
CAG,Gln,Q,Glutamine
CGT,Arg,R,Arginine
CGC,Arg,R,Arginine
...
Dictionary lookup

Command-line arguments.

- A comma-separated value (CSV) file.
- Key field.
- Value field.

Ex 3. Class list.

```
% java LookupCSV classlist.csv 4 1
eberl
Ethan
% java LookupCSV classlist.csv 4 3
dpan
P01
```

```
... more classlist.csv
13,Berl,Ethan Michael,P01,eberl
11,Bourque,Alexander Joseph,P01,abourque
12,Cao,Phillips Minghua,P01,pcao
11,Chen,Xin,P01,ccchen
10,Douglas,Malia Morioka,P01,malia
12,Haddock,Sara Lynn,P01,shaddock
12,Hantman,Nicole Samantha,P01,nhantman
11,Hesterberg,Adam Classen,P01,ahesterb
13,Hwang,Roland Lee,P01,rhwang
13,Hyde,Gregory Thomas,P01,ghyde
13,Kim,Hyunmoon,P01,hktwo
11,Kleinfeld,Ivan Maximilian,P01,ikleinfeld
12,Korac,Damjan,P01,dkorac
12,MacDonald,Graham David,P01,gmacdona
10,Michal,Brian Thomas,P01,bmichal
12,Nam,Seung Hyeon,P01,seungnam
11,Nastasescu,Maria Monica,P01,mnastase
11,Pan,Di,P01,dpan
12,Partridge,Brenton Alan,P01,bpartridge
13,Rilee,Alexander,P01,arilee
13,Roopakalu,Ajay,P01,aroopaka
11,Sheng,Ben C,P01,bsheng
12,Webb,Natalie Sue,P01,nwebb
...```
public class LookupCSV
{
   public static void main(String[] args)
   {
      In in = new In(args[0]);
      int keyField = Integer.parseInt(args[1]);
      int valField = Integer.parseInt(args[2]);

      ST<String, String> st = new ST<String, String>();
      while (!in.isEmpty())
      {
         String line = in.readLine();
         String[] tokens = database[i].split(",");
         String key = tokens[keyField];
         String val = tokens[valField];
         st.put(key, val);
      }

      while (!StdIn.isEmpty())
      {
         String s = StdIn.readString();
         if (!st.contains(s)) StdOut.println("Not found");
         else StdOut.println(st.get(s));
      }
   }
}
Search Applications

- Sets
- Dictionary clients
- Indexing clients
- Sparse vectors
Goal. Index a PC (or the web).
Goal. Given a list of files specified, create an index so that you can efficiently find all files containing a given query string.

```
% ls *.txt
aesop.txt magna.txt moby.txt
sawyer.txt tale.txt

% java FileIndex *.txt
freedom
magna.txt moby.txt tale.txt
whale
moby.txt
lamb
sawyer.txt aesop.txt

% ls *.java
BlackList.java Concordance.java
DeDup.java FileIndex.java ST.java
SET.java WhiteList.java

import
FileIndex.java SET.java ST.java

Comparator
null
```
File indexing

Goal. Given a list of files specified, create an index so that you can efficiently find all files containing a given query string.

Solution. Key = query string; value = set of files containing that string.
public class FileIndex {
    public static void main(String[] args) {
        ST<String, SET<File>> st = new ST<String, SET<File>>() {
            @Override
            public SET<File> get(String key) {
                return super.get(key); // Use the base class's implementation
            }
        };

        for (String filename : args) {
            File file = new File(filename);
            In in = new In(file);
            while (!in.isEmpty()) {
                String word = in.readString();
                if (!st.contains(word))
                    st.put(s, new SET<File>());
                SET<File> set = st.get(key);
                set.add(file);
            }
        }

        while (!StdIn.isEmpty()) {
            String query = StdIn.readString();
            StdOut.println(st.get(query));
        }
    }
}
Book index

Goal. Index for an e-book.

Index

Abstract data type (ADT), 127-195
abstract classes, 163
classes, 129-136
collections of items, 137-139
creating, 157-164
defined, 128
duplicate items, 173-176
equivalence-relations, 159-162
FIFO queues, 163-171
first-class, 177-186
generic operations, 273
index items, 177
insert/remove operations, 138-139
modular programming, 135
polynomial, 188-192
priority queues, 375-376
pushdown stack, 138-156
stubs, 135
symbol table, 497-506
ADT interfaces
array (myArray), 274
complex number (Complex), 181
collection table (ET), 663
full priority queue (PQfull), 397
indirect priority queue (PQi), 403
item (myItem), 273, 498
key (myKey), 498
polynomial (Poly), 189
point (Point), 134
priority queue (PQ), 375
queue of int (intQueue), 166
stack of int (intStack), 140
symbol table (ST), 503
text index (TI), 525
union-find (UF), 159
Abstract in-place merging, 351-353
Abstract operation, 10
Access control state, 131
Actual data, 31
Adapter class, 155-157
Adaptive sort, 268
Address, 84-85
Adjacency list, 120-123
depth-first search, 251-256
Adjacency matrix, 120-122
Ajtai, M., 464
Algorithm, 4-6, 27-64
abstract operations, 10, 31, 34-35
analysis of, 6
average/worst-case performance, 35, 60-62
big-Oh notation, 44-47
binary search, 56-59
computational complexity, 62-64
efficiency, 6, 30, 32
empirical analysis, 30-32, 58
exponential-time, 219
implementation, 28-30
logarithm function, 40-43
mathematical analysis, 33-36, 58
primary parameter, 36
probabilistic, 331
recurrences, 49-52, 57
recursive, 198
running time, 34-40
search, 53-56, 498
steps in, 22-23
See also Randomized algorithm
Amortization approach, 557, 627
Arithmetic operator, 177-179, 188, 191
Array, 12, 83
binary search, 57
dynamic allocation, 87
and linked lists, 92, 94-95
merging, 349-350
multidimensional, 117-118
references, 86-87, 89
sorting, 265-267, 273-276
and strings, 119
two-dimensional, 117-118, 120-124
vectors, 87
visualizations, 295
See also Index, array
Array representation
binary tree, 381
FIFO queue, 168-169
linked lists, 110
polynomial ADT, 191-192
priority queue, 377-378, 403, 406
pushdown stack, 148-150
random search, 170
symbol table, 508, 511-512, 521
Asymptotic expression, 45-46
Average deviation, 80-81
Average-case performance, 35, 60-61
AVL tree, 583
B tree, 584, 692-704
external/external pages, 695
4-5-6-7-8 tree, 693-704
Markov chain, 701
remove, 701-703
search/insert, 697-701
select/short, 701
Balanced tree, 238, 555-598
B tree, 584
bottom-up, 576, 584-585
height-balanced, 583
indexed sequential access, 690-692
performance, 575-576, 581-582, 595-598
randomized, 559-564
red-black, 577-585
skip lists, 587-594
splay, 566-571
Concordance

**Goal.** Preprocess a text corpus to support concordance queries: given a word, find all occurrences with their immediate contexts.

```bash
% java Concordance tale.txt
cities
tongues of the two *cities* that were blended in

majesty
t heir turnkeys and the *majesty* of the law fired
me treason against the *majesty* of the people in
of his most gracious *majesty* king george the third

princeton
no matches
```
public class Concordance
{
   public static void main(String[] args)
   {
      In in = new In(args[0]);
      String[] words = StdIn.readAll().split("\\s+");  
      ST<String, SET<Integer>> st = new ST<String, SET<Integer>>()
      for (int i = 0; i < words.length; i++)
      {
         String s = words[i];
         if (!st.contains(s))
            st.put(s, new SET<Integer>());
         SET<Integer> pages = st.get(s);
         set.put(i);
      }
      while (!StdIn.isEmpty())
      {
         String query = StdIn.readString();
         SET<Integer> set = st.get(query);
         for (int k : set)
            // print words[k-5] to words[k+5]
      }
   }
}
Search Applications

- Sets
- Dictionary clients
- Indexing clients
- Sparse vectors
Matrix-vector multiplication (standard implementation)

\[
\begin{bmatrix}
0 & .90 & 0 & 0 & 0 \\
0 & 0 & .36 & .36 & .18 \\
0 & 0 & 0 & .90 & 0 \\
.90 & 0 & 0 & 0 & 0 \\
.47 & 0 & .47 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
.05 \\
.04 \\
.36 \\
.37 \\
.19 \\
\end{bmatrix}
= 
\begin{bmatrix}
.036 \\
.297 \\
.333 \\
.045 \\
.1927 \\
\end{bmatrix}
\]

... double[][] a = new double[N][N];
double[] x = new double[N];
double[] b = new double[N];
...
// initialize a[][] and x[]
...
for (int i = 0; i < N; i++)
{
    sum = 0.0;
    for (int j = 0; j < N; j++)
        sum += a[i][j]*x[j];
    b[i] = sum;
}
Problem. Sparse matrix-vector multiplication.

Assumptions. Matrix dimension is 10,000; average nonzeros per row $\sim 10$. 

$$A \times x = b$$
Vector representations

ID array (standard) representation.
• Constant time access to elements.
• Space proportional to N.

Symbol table representation.
• Key = index, value = entry.
• Efficient iterator.
• Space proportional to number of nonzeros.
Sparse vector data type

```java
public class SparseVector {
    private HashST<Integer, Double> v;

    public SparseVector() {
        v = new HashST<Integer, Double>();
    }

    public void put(int i, double x) {
        v.put(i, x);
    }

    public double get(int i) {
        if (!v.contains(i)) return 0.0;
        else return v.get(i);
    }

    public Iterable<Integer> indices() {
        return v.keys();
    }

    public double dot(double[] that) {
        double sum = 0.0;
        for (int i : indices())
            sum += that[i] * this.get(i);
        return sum;
    }
}
```

- HashST because order not important
- Empty ST represents all 0s vector
- `a[i] = value`
- Return `a[i]`
- Dot product is constant time for sparse vectors
Matrix representations

2D array (standard) matrix representation: Each row of matrix is an array.
- Constant time access to elements.
- Space proportional to $N^2$.

Sparse matrix representation: Each row of matrix is a sparse vector.
- Efficient access to elements.
- Space proportional to number of nonzeros (plus $N$).
Sparse matrix–vector multiplication

.. SparseVector[] a = new SparseVector[N];
   double[] x = new double[N];
   double[] b = new double[N];
   ...
   // Initialize a[] and x[]
   ...
   for (int i = 0; i < N; i++)
       b[i] = a[i].dot(x);

Linear running time for sparse matrix

\[
\begin{bmatrix}
0 & .90 & 0 & 0 & 0 \\
0 & 0 & .36 & .36 & .18 \\
0 & 0 & 0 & .90 & 0 \\
.90 & 0 & 0 & 0 & 0 \\
.47 & 0 & .47 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
.05 \\
.04 \\
.36 \\
.37 \\
.19 \\
\end{bmatrix}
\
= \\
\begin{bmatrix}
.036 \\
.297 \\
.333 \\
.045 \\
.1927 \\
\end{bmatrix}
\]