Minimum Spanning Trees, Shortest Path

May. 2, 2013

Acknowledgement: The course slides are adapted from the slides prepared by R. Sedgewick and K. Wayne of Princeton University.
Graphs

- Minimum Spanning Trees
- Greedy algorithm
- Edge-weighted graph API
- Kruskal's algorithm
- Prim's algorithm
- Context

- Shortest Paths
Given. Undirected graph $G$ with positive edge weights (connected).

Def. A spanning tree of $G$ is a subgraph $T$ that is connected and acyclic.

Goal. Find a min weight spanning tree.
Given. Undirected graph $G$ with positive edge weights (connected).
Def. A spanning tree of $G$ is a subgraph $T$ that is connected and acyclic.
Goal. Find a min weight spanning tree.
Minimum spanning tree

**Given.** Undirected graph $G$ with positive edge weights (connected).

**Def.** A **spanning tree** of $G$ is a subgraph $T$ that is connected and acyclic.

**Goal.** Find a min weight spanning tree.
**Minimum spanning tree**

**Given.** Undirected graph $G$ with positive edge weights (connected).

**Def.** A spanning tree of $G$ is a subgraph $T$ that is connected and acyclic.

**Goal.** Find a min weight spanning tree.

---

The given graph $G$ is shown with its edge weights.

**spanning tree $T$:** $\text{cost} = 50 = 4 + 6 + 8 + 5 + 11 + 9 + 7$

---

**Brute force.** Try all spanning trees?
Network design

MST of bicycle routes in North Seattle

http://www.flickr.com/photos/ewedistrict/21980840
Models of nature

MST of random graph

http://algo.inria.fr/broutin/gallery.html
MST describes arrangement of nuclei in the epithelium for cancer research

http://www.bccrc.ca/ci/ta01_archlevel.html
Medical image processing

MST dithering

http://www.flickr.com/photos/quasimondo/2695389651
MST is fundamental problem with diverse applications.

- Dithering.
- Cluster analysis.
- Max bottleneck paths.
- Real-time face verification.
- LDPC codes for error correction.
- Image registration with Renyi entropy.
- Find road networks in satellite and aerial imagery.
- Reducing data storage in sequencing amino acids in a protein.
- Model locality of particle interactions in turbulent fluid flows.
- Autoconfig protocol for Ethernet bridging to avoid cycles in a network.
- Approximation algorithms for NP-hard problems (e.g., TSP, Steiner tree).
- Network design (communication, electrical, hydraulic, cable, computer, road).

Minimum Spanning Trees

- Greedy algorithm
- Edge-weighted graph API
- Kruskal's algorithm
- Prim's algorithm
- Context
Cut property

Simplifying assumptions. Edge weights are distinct; graph is connected.

Def. A cut in a graph is a partition of its vertices into two (nonempty) sets. A crossing edge connects a vertex in one set with a vertex in the other.

Cut property. Given any cut, the crossing edge of min weight is in the MST.
Cut property: correctness proof

Simplifying assumptions. Edge weights are distinct; graph is connected.

Def. A cut in a graph is a partition of its vertices into two (nonempty) sets. A crossing edge connects a vertex in one set with a vertex in the other.

Cut property. Given any cut, the crossing edge of min weight is in the MST.

Pf. Let $e$ be the min-weight crossing edge in cut.
- Suppose $e$ is not in the MST.
- Adding $e$ to the MST creates a cycle.
- Some other edge $f$ in cycle must be a crossing edge.
- Removing $f$ and adding $e$ is also a spanning tree.
- Since weight of $e$ is less than the weight of $f$, that spanning tree is lower weight.
- Contradiction. □
Greedy MST algorithm

- Start with all edges colored gray.
- Find a cut with no black crossing edges, and color its min-weight edge black.
- Repeat until $V - 1$ edges are colored black.
Greedy MST algorithm

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**MST edges**

0–2
Greedy MST algorithm

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MST edges

0–2  5–7
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**MST edges**

$0-2$, $5-7$, $6-2$

**Crossing edges (sorted by weight)**

- $0-7$ 0.16
- $2-3$ 0.17
- $2-7$ 0.34
- $4-5$ 0.35
- $1-2$ 0.36
- $4-7$ 0.37
- $3-6$ 0.52
Greedy MST algorithm

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MST edges

0–2  5–7  6–2  0–7
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MST edges

0-2  5-7  6-2  0-7  2-3
### Greedy MST algorithm

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**MST edges**

- 0–2
- 5–7
- 6–2
- 0–7
- 2–3
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- 0–7
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- 1–7
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MST edges

0–2  5–7  6–2  0–7  2–3  1–7  4–5
Proposition. The greedy algorithm computes the MST.

Pf.

• Any edge colored black is in the MST (via cut property).
• If fewer than $V - 1$ black edges, there exists a cut with no black crossing edges. (consider cut whose vertices are one connected component)
Proposition. The greedy algorithm computes the MST:

Efficient implementations. Choose cut? Find min-weight edge?
Ex 1. Kruskal's algorithm. [stay tuned]
Ex 2. Prim's algorithm. [stay tuned]
Ex 3. Borůvka's algorithm.
Removing two simplifying assumptions

Q. What if edge weights are not all distinct?
A. Greedy MST algorithm still correct if equal weights are present! (our correctness proof fails, but that can be fixed)

Q. What if graph is not connected?
A. Compute minimum spanning forest = MST of each component.

Weights can be 0 or negative

MST may not be unique when weights have equal values
Minimum Spanning Trees

- Greedy algorithm
- Edge-weighted graph API
- Kruskal's algorithm
- Prim's algorithm
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# Weighted edge API

Edge abstraction needed for weighted edges.

```java
public class Edge implements Comparable<Edge>

    Edge(int v, int w, double weight) create a weighted edge v-w

    int either() either endpoint

    int other(int v) the endpoint that's not v

    int compareTo(Edge that) compare this edge to that edge

    double weight() the weight

    String toString() string representation
```

Idiom for processing an edge `e`: `int v = e.either(), w = e.other(v);`
public class Edge implements Comparable<Edge>
{
    private final int v, w;
    private final double weight;

    public Edge(int v, int w, double weight)
    {
        this.v = v;
        this.w = w;
        this.weight = weight;
    }

    public int either()
    {
        return v;
    }

    public int other(int vertex)
    {
        if (vertex == v) return w;
        else return v;
    }

    public int compareTo(Edge that)
    {
        if      (this.weight < that.weight) return -1;
        else if (this.weight > that.weight) return +1;
        else                                return  0;
    }
}
### Edge-weighted graph API

**public class** `EdgeWeightedGraph`

<table>
<thead>
<tr>
<th>Method</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>EdgeWeightedGraph(int V)</code></td>
<td>create an empty graph with V vertices</td>
</tr>
<tr>
<td><code>EdgeWeightedGraph(In in)</code></td>
<td>create a graph from input stream</td>
</tr>
<tr>
<td><code>void addEdge(Edge e)</code></td>
<td>add weighted edge e to this graph</td>
</tr>
<tr>
<td><code>Iterable&lt;Edge&gt; adj(int v)</code></td>
<td>edges incident to v</td>
</tr>
<tr>
<td><code>Iterable&lt;Edge&gt; edges()</code></td>
<td>all edges in this graph</td>
</tr>
<tr>
<td><code>int V()</code></td>
<td>number of vertices</td>
</tr>
<tr>
<td><code>int E()</code></td>
<td>number of edges</td>
</tr>
<tr>
<td><code>String toString()</code></td>
<td>string representation</td>
</tr>
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</table>

**Conventions.** Allow self-loops and parallel edges.
Edge-weighted graph: adjacency-lists representation

Maintain vertex-indexed array of $\text{Edge}$ lists.

```
adj[]
8
0 6 0.58 0 2.26 0 4.38 0 7.16
1 1 3.29 1 2.36 1 7.19 1 5.32
2 6 2.40 2 7.34 1 2.36 0 2.26 2 3.17
3 3 6.52 1 3.29 2 3.17
4 6 4.93 0 4.38 4 7.37 4 5.35
5 1 5.32 5 7.28 4 5.35
6 6 4.93 6 0.58 3 6.52 6 2.40
7 2 7.34 1 7.19 0 7.16 5 7.28 5 7.28
```

tinyEWG.txt

References to the same $\text{Edge}$ object

Bag objects
Edge-weighted graph: adjacency-lists implementation

```java
class EdgeWeightedGraph
{
    private final int V;
    private final Bag<Edge>[] adj;

    public EdgeWeightedGraph(int V)
    {
        this.V = V;
        adj = (Bag<Edge>[]) new Bag[V];
        for (int v = 0; v < V; v++)
            adj[v] = new Bag<Edge>();
    }

    public void addEdge(Edge e)
    {
        int v = e.either(), w = e.other(v);
        adj[v].add(e);
        adj[w].add(e);
    }

    public Iterable<Edge> adj(int v)
    {  return adj[v];  }
}
```

- **constructor**: same as Graph, but adjacency lists of Edges instead of integers
- **add edge to both adjacency lists**: add edge to both adjacency lists
Q. How to represent the MST?

public class MST

MST(EdgeWeightedGraph G)  constructor

Iterable<Edge> edges()  edges in MST

double weight()  weight of MST

tinyEWG.txt

% java MST tinyEWG.txt
0-7 0.16
1-7 0.19
0-2 0.26
2-3 0.17
5-7 0.28
4-5 0.35
6-2 0.40
1.81
Q. How to represent the MST?

```
public class MST {
    MST(EdgeWeightedGraph G)
    Iterable<Edge> edges()
    double weight()
}
```

```
public static void main(String[] args) {
    In in = new In(args[0]);
    EdgeWeightedGraph G = new EdgeWeightedGraph(in);
    MST mst = new MST(G);
    for (Edge e : mst.edges())
        StdOut.println(e);
    StdOut.printf("%.2f\n", mst.weight());
}
```

% java MST tinyEWG.txt
0-7 0.16
1-7 0.19
0-2 0.26
2-3 0.17
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1.81
Minimum Spanning Trees

- Greedy algorithm
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- Context
Kruskal's algorithm

- Consider edges in ascending order of weight.
- Add next edge to tree $T$ unless doing so would create a cycle.

an edge-weighted graph

<table>
<thead>
<tr>
<th>Graph Edges</th>
<th>Weight</th>
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<tr>
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Kraskal's algorithm

- Consider edges in ascending order of weight.
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![Diagram of a graph with edges and nodes labeled 0, 1, 2, 3, 4, 5, 6, 7. Some edges are highlighted in red, indicating they do not create a cycle.](image)

in MST $\rightarrow$ 0-7 0.16

does not create a cycle
Kruskal's algorithm

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Kruskal's algorithm

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In MST:

- $0-7$ 0.16
- $2-3$ 0.17
- $1-7$ 0.19

The edge $0-7$ does not create a cycle.
Kruskal's algorithm

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Kruskal's algorithm

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4-7  0.37
0-4  0.38
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Kruskal's algorithm

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![Graph](image.png)
Kruskal's algorithm

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creates a cycle

not in MST
Kruskal's algorithm

- Consider edges in ascending order of weight.
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![Complete graph with edge weights]

a minimum spanning tree

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</table>
Kruskal's algorithm: visualization
Proposition. [Kruskal 1956] Kruskal's algorithm computes the MST.

**Pf.** Kruskal's algorithm is a special case of the greedy MST algorithm.
- Suppose Kruskal's algorithm colors the edge $e = v \rightarrow w$ black.
- Cut = set of vertices connected to $v$ in tree $T$.
- No crossing edge is black.
- No crossing edge has lower weight. Why?

![Diagram of a graph with edges colored to demonstrate the correctness of Kruskal's algorithm.](image)
Challenge. Would adding edge $v\rightarrow w$ to tree $T$ create a cycle? If not, add it.

How difficult?

- $E + V$
- $V$
- $\log V$
- $\log^* V$
- 1

run DFS from $v$, check if $w$ is reachable
(T has at most $V - 1$ edges)

use the union-find data structure!
Challenge. Would adding edge $v \rightarrow w$ to tree $T$ create a cycle? If not, add it.

Efficient solution. Use the union-find data structure.

- Maintain a set for each connected component in $T$.
- If $v$ and $w$ are in the same set, then adding $v \rightarrow w$ would create a cycle.
- To add $v \rightarrow w$ to $T$, merge sets containing $v$ and $w$.

Case 1: adding $v \rightarrow w$ creates a cycle
Case 2: add $v \rightarrow w$ to $T$ and merge sets containing $v$ and $w$
**Kruskal's algorithm: Java implementation**

```
public class KruskalMST
{
    private Queue<Edge> mst = new Queue<Edge>();

    public KruskalMST(EdgeWeightedGraph G)
    {
        MinPQ<Edge> pq = new MinPQ<Edge>();
        for (Edge e : G.edges())
            pq.insert(e);

        UF uf = new UF(G.V());
        while (!pq.isEmpty() && mst.size() < G.V()-1)
        {
            Edge e = pq.delMin();
            int v = e.either(), w = e.other(v);
            if (!uf.connected(v, w))
            {
                uf.union(v, w);
                mst.enqueue(e);
            }
        }
    }

    public Iterable<Edge> edges()
    {  return mst;  }
}
```

- **build priority queue**
- **greedily add edges to MST**
- **edge v–w does not create cycle**
- **merge sets**
- **add edge to MST**
Kruskal's algorithm: running time

Proposition. Kruskal's algorithm computes MST in time proportional to $E \log E$ (in the worst case).

Pf.

<table>
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<tr>
<th>operation</th>
<th>frequency</th>
<th>time per op</th>
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<tr>
<td>build pq</td>
<td>1</td>
<td>$E$</td>
</tr>
<tr>
<td>delete-min</td>
<td>$E$</td>
<td>$\log E$</td>
</tr>
<tr>
<td>union</td>
<td>$V$</td>
<td>$\log^* V$</td>
</tr>
<tr>
<td>connected</td>
<td>$E$</td>
<td>$\log^* V$</td>
</tr>
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</table>

↑ amortized bound using weighted quick union with path compression

recall: $\log^* V \leq 5$ in this universe

Remark. If edges are already sorted, order of growth is $E \log^* V$. 
Minimum Spanning Trees

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Prim's algorithm

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
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an edge–weighted graph

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Prim's algorithm

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Prim's algorithm

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MST edges

0–7
Prim's algorithm

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MST edges

$0-7$
Prim's algorithm

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MST edges

0-7 1-7
Prim's algorithm

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Prism's algorithm

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- Repeat until $V-1$ edges.

**MST edges**

0–7  1–7  0–2
Prim's algorithm

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.
Prim's algorithm

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

MST edges

$0-7$  $1-7$  $0-2$  $2-3$
Prim's algorithm

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

MST edges

- $0-7$
- $1-7$
- $0-2$
- $2-3$

Edges with exactly one endpoint in $T$ (sorted by weight):

- $5-7: 0.28$
- $1-5: 0.32$
- $4-7: 0.37$
- $0-4: 0.38$
- $6-2: 0.40$
- $3-6: 0.52$
- $6-0: 0.58$

Min weight edge with exactly one endpoint in $T$: $0-7$
Prim's algorithm

• Start with vertex 0 and greedily grow tree $T$.
• Add to $T$ the min weight edge with exactly one endpoint in $T$.
• Repeat until $V-1$ edges.

MST edges

$0-7$  $1-7$  $0-2$  $2-3$  $5-7$
• Start with vertex 0 and greedily grow tree $T$.
• Add to $T$ the min weight edge with exactly one endpoint in $T$.
• Repeat until $V-1$ edges.

**MST edges**

0–7  1–7  0–2  2–3  5–7

**Edges with exactly one endpoint in $T$**

(sorted by weight)

- 4–5  0.35
- 4–7  0.37
- 0–4  0.38
- 6–2  0.40
- 3–6  0.52
- 6–0  0.58
**Prim's algorithm**

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

![MST edges](MST.png)

MST edges

<table>
<thead>
<tr>
<th>0–7</th>
<th>1–7</th>
<th>0–2</th>
<th>2–3</th>
<th>5–7</th>
<th>4–5</th>
</tr>
</thead>
</table>
Prim's algorithm

• Start with vertex 0 and greedily grow tree $T$.
• Add to $T$ the min weight edge with exactly one endpoint in $T$.
• Repeat until $V-1$ edges.

MST edges

0–7 1–7 0–2 2–3 5–7 4–5
Prim's algorithm

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

**MST edges**

0–7  1–7  0–2  2–3  5–7  4–5  6–2
Prim’s algorithm: visualization
Proposition. [Jarník 1930, Dijkstra 1957, Prim 1959]
Prim's algorithm computes the MST.

Pf. Prim's algorithm is a special case of the greedy MST algorithm.
- Suppose edge $e = \text{min weight edge connecting a vertex on the tree to a vertex not on the tree}$.
- Cut = set of vertices connected on tree.
- No crossing edge is black.
- No crossing edge has lower weight.

\[
edge \ e = 7-5 \text{ added to tree}
\]
Challenge. Find the min weight edge with exactly one endpoint in $T$.

How difficult?

- $E$  
  try all edges
- $V$
- $\log E$
  use a priority queue!
- $\log^* E$
- 1

1-7 is min weight edge with exactly one endpoint in $T$
**Challenge.** Find the min weight edge with exactly one endpoint in $T$.

**Lazy solution.** Maintain a PQ of edges with (at least) one endpoint in $T$.

- Key = edge; priority = weight of edge.
- Delete-min to determine next edge $e = v\rightarrow w$ to add to $T$.
- Disregard if both endpoints $v$ and $w$ are in $T$.
- Otherwise, let $v$ be vertex not in $T$:
  - add to PQ any edge incident to $v$ (assuming other endpoint not in $T$)
  - add $v$ to $T$

1-7 is min weight edge with exactly one endpoint in $T$
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.
Prim's algorithm - Lazy implementation

• Start with vertex 0 and greedily grow tree $T$.
• Add to $T$ the min weight edge with exactly one endpoint in $T$.
• Repeat until $V-1$ edges.

add to PQ all edges incident to 0

edges on PQ (sorted by weight)

<table>
<thead>
<tr>
<th>*</th>
<th>0–7</th>
<th>0.16</th>
</tr>
</thead>
<tbody>
<tr>
<td>*</td>
<td>0–2</td>
<td>0.26</td>
</tr>
<tr>
<td>*</td>
<td>0–4</td>
<td>0.38</td>
</tr>
<tr>
<td>*</td>
<td>6–0</td>
<td>0.58</td>
</tr>
</tbody>
</table>
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree \( T \).
- Add to \( T \) the min weight edge with exactly one endpoint in \( T \).
- Repeat until \( V-1 \) edges.

delete 0–7 and add to MST
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

### MST edges

0–7
Prim's algorithm - Lazy implementation

• Start with vertex 0 and greedily grow tree $T$.
• Add to $T$ the min weight edge with exactly one endpoint in $T$.
• Repeat until $V-1$ edges.

add to PQ all edges incident to 7

MST edges

0–7

edges on PQ
(sorted by weight)

* 1–7 0.19
0–2 0.26
* 5–7 0.28
* 2–7 0.34
* 4–7 0.37
0–4 0.38
6–0 0.58
• Start with vertex 0 and greedily grow tree $T$.
• Add to $T$ the min weight edge with exactly one endpoint in $T$.
• Repeat until $V-1$ edges.

delete 1–7 and add to MST

<table>
<thead>
<tr>
<th>edges on PQ</th>
<th>(sorted by weight)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1–7</td>
<td>0.19</td>
</tr>
<tr>
<td>0–2</td>
<td>0.26</td>
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<tr>
<td>5–7</td>
<td>0.28</td>
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<tr>
<td>2–7</td>
<td>0.34</td>
</tr>
<tr>
<td>4–7</td>
<td>0.37</td>
</tr>
<tr>
<td>0–4</td>
<td>0.38</td>
</tr>
<tr>
<td>6–0</td>
<td>0.58</td>
</tr>
</tbody>
</table>

MST edges

0–7
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

**MST edges**

- 0–7
- 1–7

**edges on PQ (sorted by weight)**

<table>
<thead>
<tr>
<th>Edge</th>
<th>Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>0–2</td>
<td>0.26</td>
</tr>
<tr>
<td>5–7</td>
<td>0.28</td>
</tr>
<tr>
<td>2–7</td>
<td>0.34</td>
</tr>
<tr>
<td>4–7</td>
<td>0.37</td>
</tr>
<tr>
<td>0–4</td>
<td>0.38</td>
</tr>
<tr>
<td>6–0</td>
<td>0.58</td>
</tr>
</tbody>
</table>
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

add to PQ all edges incident to 1

MST edges

0–7  1–7
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree \( T \).
- Add to \( T \) the min weight edge with exactly one endpoint in \( T \).
- Repeat until \( V-1 \) edges.

delete edge 0–2 and add to MST

MST edges

| 0–7 | 1–7 |

edges on PQ (sorted by weight)

<table>
<thead>
<tr>
<th>Edge</th>
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</tr>
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<tbody>
<tr>
<td>0–2</td>
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<td>5–7</td>
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<tr>
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<td>0.38</td>
</tr>
<tr>
<td>6–0</td>
<td>0.58</td>
</tr>
</tbody>
</table>
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

add to PQ all edges incident to 2

edges on PQ (sorted by weight):

<table>
<thead>
<tr>
<th></th>
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</tr>
</thead>
<tbody>
<tr>
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<td></td>
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<td></td>
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<tr>
<td></td>
<td>1–2</td>
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<tr>
<td></td>
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<td>*</td>
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<td>0.40</td>
</tr>
<tr>
<td></td>
<td>6–0</td>
<td>0.58</td>
</tr>
</tbody>
</table>

MST edges

0–7  1–7  0–2
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

delete 2–3 and add to MST

MST edges

0–7 1–7 0–2

edges on PQ (sorted by weight)

| 2–3 | 0.17 |
| 5–7 | 0.28 |
| 1–3 | 0.29 |
| 1–5 | 0.32 |
| 2–7 | 0.34 |
| 1–2 | 0.36 |
| 4–7 | 0.37 |
| 0–4 | 0.38 |
| 6–2 | 0.40 |
| 6–0 | 0.58 |
• Start with vertex 0 and greedily grow tree $T$.
• Add to $T$ the min weight edge with exactly one endpoint in $T$.
• Repeat until $V-1$ edges.

**Prim's algorithm - Lazy implementation**

MST edges

0–7 1–7 0–2 2–3

<table>
<thead>
<tr>
<th>edges on PQ (sorted by weight)</th>
<th>weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>5–7</td>
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</tr>
<tr>
<td>1–3</td>
<td>0.29</td>
</tr>
<tr>
<td>1–5</td>
<td>0.32</td>
</tr>
<tr>
<td>2–7</td>
<td>0.34</td>
</tr>
<tr>
<td>1–2</td>
<td>0.36</td>
</tr>
<tr>
<td>4–7</td>
<td>0.37</td>
</tr>
<tr>
<td>0–4</td>
<td>0.38</td>
</tr>
<tr>
<td>6–2</td>
<td>0.40</td>
</tr>
<tr>
<td>6–0</td>
<td>0.58</td>
</tr>
</tbody>
</table>
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

add to PQ all edges incident to 3

edges on PQ (sorted by weight)

<table>
<thead>
<tr>
<th>Edge</th>
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</thead>
<tbody>
<tr>
<td>5–7</td>
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<td>0.29</td>
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<td>1–5</td>
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</tr>
<tr>
<td>1–2</td>
<td>0.36</td>
</tr>
<tr>
<td>4–7</td>
<td>0.37</td>
</tr>
<tr>
<td>0–4</td>
<td>0.38</td>
</tr>
<tr>
<td>6–2</td>
<td>0.40</td>
</tr>
<tr>
<td>*3–6</td>
<td>0.52</td>
</tr>
<tr>
<td>6–0</td>
<td>0.58</td>
</tr>
</tbody>
</table>

MST edges

0–7  1–7  0–2  2–3
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

**MST edges**

0–7  1–7  0–2  2–3

**edges on PQ**

<table>
<thead>
<tr>
<th>Edge</th>
<th>Weight</th>
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<tbody>
<tr>
<td>5–7</td>
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<tr>
<td>1–3</td>
<td>0.29</td>
</tr>
<tr>
<td>1–5</td>
<td>0.32</td>
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<tr>
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<td>1–2</td>
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<td>4–7</td>
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<tr>
<td>0–4</td>
<td>0.38</td>
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<tr>
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<tr>
<td>3–6</td>
<td>0.52</td>
</tr>
<tr>
<td>6–0</td>
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</tr>
</tbody>
</table>

delete 5–7 and add to MST
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$
- Add to $T$ the min weight edge with exactly one endpoint in $T$
- Repeat until $V-1$ edges

edges on PQ (sorted by weight)

<table>
<thead>
<tr>
<th>Edge</th>
<th>Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>1–3</td>
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<tr>
<td>1–5</td>
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<tr>
<td>2–7</td>
<td>0.34</td>
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<tr>
<td>1–2</td>
<td>0.36</td>
</tr>
<tr>
<td>4–7</td>
<td>0.37</td>
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<tr>
<td>0–4</td>
<td>0.38</td>
</tr>
<tr>
<td>6–2</td>
<td>0.40</td>
</tr>
<tr>
<td>3–6</td>
<td>0.52</td>
</tr>
<tr>
<td>6–0</td>
<td>0.58</td>
</tr>
</tbody>
</table>

MST edges

0–7  1–7  0–2  2–3  5–7
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

add to PQ all edges incident to 5

edges on PQ
(sorted by weight)

<table>
<thead>
<tr>
<th>Edge</th>
<th>Weight</th>
</tr>
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<tbody>
<tr>
<td>1–3</td>
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<td>4–7</td>
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<td>0–4</td>
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<td>6–2</td>
<td>0.40</td>
</tr>
<tr>
<td>3–6</td>
<td>0.52</td>
</tr>
<tr>
<td>6–0</td>
<td>0.58</td>
</tr>
</tbody>
</table>

MST edges
0–7   1–7   0–2   2–3   5–7
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

**MST edges**

0–7  1–7  0–2  2–3  5–7

**edges on PQ**

<table>
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<tr>
<th>Edge</th>
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<tbody>
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<td>4–7</td>
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<td>0.52</td>
</tr>
<tr>
<td>6–0</td>
<td>0.58</td>
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</tbody>
</table>

delete 1–3 and discard obsolete edge
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

delete 1–5 and discard obsolete edge

MST edges

0–7  1–7  0–2  2–3  5–7

edges on PQ (sorted by weight)

<table>
<thead>
<tr>
<th>Edge</th>
<th>Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>1–5</td>
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<tr>
<td>2–7</td>
<td>0.34</td>
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<tr>
<td>4–5</td>
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<td>6–0</td>
<td>0.58</td>
</tr>
</tbody>
</table>
• Start with vertex 0 and greedily grow tree $T$.
• Add to $T$ the min weight edge with exactly one endpoint in $T$.
• Repeat until $V-1$ edges.

delete 2–7 and discard obsolete edge

MST edges
0–7 1–7 0–2 2–3 5–7

<table>
<thead>
<tr>
<th>edges on PQ (sorted by weight)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2–7 0.34</td>
</tr>
<tr>
<td>4–5 0.35</td>
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<tr>
<td>1–2 0.36</td>
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<td>3–6 0.52</td>
</tr>
<tr>
<td>6–0 0.58</td>
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</table>
**Prim's algorithm - Lazy implementation**

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

**delete 4–5 and add to MST**

**edges on PQ (sorted by weight)**

<table>
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<tbody>
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<tr>
<td>6–0</td>
<td>0.58</td>
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**MST edges**

- 0–7
- 1–7
- 0–2
- 2–3
- 5–7
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

MST edges:

- $0-7$
- $1-7$
- $0-2$
- $2-3$
- $5-7$
- $4-5$

Edges on PQ (sorted by weight):

<table>
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<tr>
<td>1-2</td>
<td>0.36</td>
</tr>
<tr>
<td>4-7</td>
<td>0.37</td>
</tr>
<tr>
<td>0-4</td>
<td>0.38</td>
</tr>
<tr>
<td>6-2</td>
<td>0.40</td>
</tr>
<tr>
<td>3-6</td>
<td>0.52</td>
</tr>
<tr>
<td>6-0</td>
<td>0.58</td>
</tr>
</tbody>
</table>
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

add to PQ all edges incident to 4

MST edges

0–7  1–7  0–2  2–3  5–7  4–5

edges on PQ
(sorted by weight)

<table>
<thead>
<tr>
<th>Edge</th>
<th>Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>1–2</td>
<td>0.36</td>
</tr>
<tr>
<td>4–7</td>
<td>0.37</td>
</tr>
<tr>
<td>0–4</td>
<td>0.38</td>
</tr>
<tr>
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<tr>
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<td>0.93</td>
</tr>
</tbody>
</table>
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

delete 1–2 and discard obsolete edge

MST edges

<p>| | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0–7</td>
<td>1–7</td>
<td>0–2</td>
<td>2–3</td>
<td>5–7</td>
<td>4–5</td>
</tr>
</tbody>
</table>

edges on PQ (sorted by weight)

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
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<td>0.93</td>
</tr>
</tbody>
</table>
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

**delete 4–7 and discard obsolete edge**

**MST edges**

<table>
<thead>
<tr>
<th>Edge</th>
<th>Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>0–7</td>
<td>1.00</td>
</tr>
<tr>
<td>1–7</td>
<td>1.00</td>
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<tr>
<td>0–2</td>
<td>1.00</td>
</tr>
<tr>
<td>2–3</td>
<td>1.00</td>
</tr>
<tr>
<td>5–7</td>
<td>1.00</td>
</tr>
<tr>
<td>4–5</td>
<td>1.00</td>
</tr>
</tbody>
</table>

**edges on PQ (sorted by weight)**

<table>
<thead>
<tr>
<th>Edge</th>
<th>Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>4–7</td>
<td>0.37</td>
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Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

delete 0–4 and discard obsolete edge

MST edges

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
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<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0-7</td>
<td>1-7</td>
<td>0-2</td>
<td>2-3</td>
<td>5-7</td>
</tr>
</tbody>
</table>

edges on PQ (sorted by weight)

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
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Prim's algorithm - Lazy implementation

• Start with vertex 0 and greedily grow tree $T$.
• Add to $T$ the min weight edge with exactly one endpoint in $T$.
• Repeat until $V-1$ edges.

delete 6–2 and add to MST

MST edges
0–7 1–7 0–2 2–3 5–7 4–5

generates edges on PQ (sorted by weight)

<table>
<thead>
<tr>
<th>Edge</th>
<th>Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>6–2</td>
<td>0.40</td>
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Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

delete 6–2 and add to MST

MST edges

0–7  1–7  0–2  2–3  5–7  4–5  6–2

edges on PQ
(sorted by weight)

<table>
<thead>
<tr>
<th>Weight</th>
<th>Edge</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.52</td>
<td>3–6</td>
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<td>6–4</td>
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</tbody>
</table>
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

stop since $V-1$ edges

MST edges

0–7  1–7  0–2  2–3  5–7  4–5  6–2

edges on PQ
(sorted by weight)

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<th>Edge</th>
<th>Weight</th>
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<tbody>
<tr>
<td>3–6</td>
<td>0.52</td>
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<tr>
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<td>0.93</td>
</tr>
</tbody>
</table>

stop since $V-1$ edges
• Start with vertex 0 and greedily grow tree $T$.
• Add to $T$ the min weight edge with exactly one endpoint in $T$.
• Repeat until $V-1$ edges.

MST edges

0–7  1–7  0–2  2–3  5–7  4–5  6–2
Prim's algorithm: lazy implementation

```java
public class LazyPrimMST
{
    private boolean[] marked; // MST vertices
    private Queue<Edge> mst; // MST edges
    private MinPQ<Edge> pq; // PQ of edges

    public LazyPrimMST(WeightedGraph G)
    {
        pq = new MinPQ<Edge>();
        mst = new Queue<Edge>();
        marked = new boolean[G.V()];
        visit(G, 0);
        while (!pq.isEmpty())
        {
            Edge e = pq.delMin();
            int v = e.either(), w = e.other(v);
            if (marked[v] && marked[w]) continue;
            mst.enqueue(e);
            if (!marked[v]) visit(G, v);
            if (!marked[w]) visit(G, w);
        }
    }
}
```

- Assume G is connected
- Repeatedly delete the min weight edge $e = v-w$ from PQ
- Ignore if both endpoints in T
- Add edge $e$ to tree
- Add $v$ or $w$ to tree
Prim's algorithm: lazy implementation

```java
private void visit(WeightedGraph G, int v)
{
    marked[v] = true;
    for (Edge e : G.adj(v))
        if (!marked[e.other(v)])
            pq.insert(e);
}

public Iterable<Edge> mst()
{  return mst;  }
```

- For each edge \( e = v \rightarrow w \), add to \( PQ \) if \( w \) not already in \( T \)
- Add \( v \) to \( T \)
Lazy Prim's algorithm: running time

**Proposition.** Lazy Prim's algorithm computes the MST in time proportional to $E \log E$ and extra space proportional to $E$ (in the worst case).

**Pf.**

<table>
<thead>
<tr>
<th>operation</th>
<th>frequency</th>
<th>binary heap</th>
</tr>
</thead>
<tbody>
<tr>
<td>delete min</td>
<td>$E$</td>
<td>$\log E$</td>
</tr>
<tr>
<td>insert</td>
<td>$E$</td>
<td>$\log E$</td>
</tr>
</tbody>
</table>
**Prim's algorithm: eager implementation**

**Challenge.** Find min weight edge with exactly one endpoint in $T$.

**Eager solution.** Maintain a PQ of vertices connected by an edge to $T$, where priority of vertex $v = \text{weight of shortest edge connecting } v \text{ to } T$.

- Delete min vertex $v$ and add its associated edge $e = v-w$ to $T$.
- Update PQ by considering all edges $e = v-x$ incident to $v$
  - ignore if $x$ is already in $T$
  - add $x$ to PQ if not already on it
  - decrease priority of $x$ if $v-x$ becomes shortest edge connecting $x$ to $T$
Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

an edge-weighted graph
Prim's algorithm - Eager implementation

• Start with vertex 0 and greedily grow tree $T$.
• Add to $T$ the min weight edge with exactly one endpoint in $T$.
• Repeat until $V-1$ edges.
Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree \( T \).
- Add to \( T \) the min weight edge with exactly one endpoint in \( T \).
- Repeat until \( V-1 \) edges.

<table>
<thead>
<tr>
<th>vertex</th>
<th>edgeTo[]</th>
<th>distTo[]</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>0–7</td>
<td>0.16</td>
</tr>
<tr>
<td>2</td>
<td>0–2</td>
<td>0.26</td>
</tr>
<tr>
<td>4</td>
<td>0–4</td>
<td>0.38</td>
</tr>
<tr>
<td>6</td>
<td>6–0</td>
<td>0.58</td>
</tr>
</tbody>
</table>

vertices on PQ
(sorted by weight)

add vertices 7, 2, 4, and 6 to PQ
Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V - 1$ edges.

![Diagram showing Prim's algorithm]

<table>
<thead>
<tr>
<th>vertex</th>
<th>edgeTo[]</th>
<th>distTo[]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>7</td>
<td>0–7</td>
<td>0.16</td>
</tr>
<tr>
<td>2</td>
<td>0–2</td>
<td>0.26</td>
</tr>
<tr>
<td>4</td>
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<td>0.38</td>
</tr>
<tr>
<td>6</td>
<td>6–0</td>
<td>0.58</td>
</tr>
</tbody>
</table>

Vertices on PQ (sorted by weight)
Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

<table>
<thead>
<tr>
<th>v</th>
<th>edgeTo[]</th>
<th>distTo[]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>7</td>
<td>0–7</td>
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<td>6–0</td>
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</tr>
</tbody>
</table>

vertices on PQ
(sorted by weight)

MST edges

0–7
Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

```
<table>
<thead>
<tr>
<th>v</th>
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</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>7</td>
<td>0–7</td>
<td>0.16</td>
</tr>
<tr>
<td>1</td>
<td>1–7</td>
<td>0.19</td>
</tr>
<tr>
<td>2</td>
<td>0–2</td>
<td>0.26</td>
</tr>
<tr>
<td>5</td>
<td>5–7</td>
<td>0.28</td>
</tr>
<tr>
<td>4</td>
<td>0–4</td>
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</tr>
<tr>
<td>6</td>
<td>6–0</td>
<td>0.58</td>
</tr>
</tbody>
</table>
```

- Add vertex 1 to PQ
- Add vertex 5 to PQ
- Vertices on PQ (sorted by weight):
  - 1–7
  - 0–2
  - 5–7
  - 0–4
  - 6–0

**MST edges**

- 0–7

Already a better connection to 2 and 4 (discard)
Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

![Graph](image)

<table>
<thead>
<tr>
<th>v</th>
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</thead>
<tbody>
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</tr>
</tbody>
</table>

vertices on PQ (sorted by weight)

**MST edges**

0–7 1–7
Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree \( T \).
- Add to \( T \) the min weight edge with exactly one endpoint in \( T \).
- Repeat until \( V-1 \) edges.

### Vertices on PQ (sorted by weight)

<table>
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<tr>
<th>v</th>
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### MST edges

- 0–7
- 1–7
Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

![Diagram of Prim's algorithm with MST edges: 0–7, 1–7.]

<table>
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<tr>
<th>v</th>
<th>edgeTo[]</th>
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</tr>
</thead>
<tbody>
<tr>
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already a better connection to 5 and 7 (discard)
Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

### MST edges
- 0–7
- 1–7
Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

**Table: Prim's algorithm**

<table>
<thead>
<tr>
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</table>

**Diagram: MST edges**

- 0–7
- 1–7
- 0–2
Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree \( T \).
- Add to \( T \) the min weight edge with exactly one endpoint in \( T \).
- Repeat until \( V-1 \) edges.

\[
\begin{array}{c|c|c}
\text{v} & \text{edgeTo[]} & \text{distTo[]} \\
\hline
0 & - & - \\
7 & 0-7 & 0.16 \\
1 & 1-7 & 0.19 \\
2 & 0-2 & 0.26 \\
3 & 1-3 & 0.29 \\
5 & 5-7 & 0.28 \\
4 & 0-4 & 0.38 \\
6 & 6-0 & 0.58 \\
\end{array}
\]

- decrease key of vertex 3 from 0.29 to 0.17
- decrease key of vertex 6 from 0.58 to 0.40

MST edges

0-7  1-7  0-2
• Start with vertex 0 and greedily grow tree $T$.
• Add to $T$ the min weight edge with exactly one endpoint in $T$.
• Repeat until $V-1$ edges.

**MST edges**

0–7  1–7  0–2  2–3
Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

### MST edges
- 0–7
- 1–7
- 0–2
- 2–3

### Table

<table>
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<tr>
<th>v</th>
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<th>distTo[]</th>
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Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

**MST edges**

0–7 1–7 0–2 2–3

already a better connection to 6 (discard)
**Prim's algorithm - Eager implementation**

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</tr>
<tr>
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<td>0.16</td>
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**Diagram:**

MST edges

0–7 1–7 0–2 2–3
Prim's algorithm - Eager implementation

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MST edges

$0-7$  $1-7$  $0-2$  $2-3$  $5-7$
• Start with vertex 0 and greedily grow tree $T$.
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• Repeat until $V-1$ edges.

Prim's algorithm - Eager implementation

MST edges
0–7 1–7 0–2 2–3 5–7
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### MST edges

0–7  1–7  0–2  2–3  5–7

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MST edges

0–7 1–7 0–2 2–3 5–7 4–5

already a better connection to 6 (discard)
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### Indexed priority queue

Associate an index between 0 and $N - 1$ with each key in a priority queue.

- Client can insert and delete-the-minimum.
- Client can change the key by specifying the index.

```java
public class IndexMinPQ<Key extends Comparable<Key>> {
    IndexMinPQ(int N) {
        create indexed priority queue with indices 0, 1, ..., N-1
    }
    void insert(int k, Key key) {
        associate key with index k
    }
    void decreaseKey(int k, Key key) {
        decrease the key associated with index k
    }
    boolean contains() {
        is k an index on the priority queue?
    }
    int delMin() {
        remove a minimal key and return its associated index
    }
    boolean isEmpty() {
        is the priority queue empty?
    }
    int size() {
        number of entries in the priority queue
    }
```
Indexed priority queue implementation

Implementation.

• Start with same code as MinPQ.
• Maintain parallel arrays `keys[]`, `pq[]`, and `qp[]` so that:
  - `keys[i]` is the priority of `i`
  - `pq[i]` is the index of the key in heap position `i`
  - `qp[i]` is the heap position of the key with index `i`
• Use `swim(qp[k])` implement `decreaseKey(k, key)`.
## Prim's algorithm: running time

Depending on PQ implementation: \( V \) insert, \( V \) delete-min, \( E \) decrease-key.

<table>
<thead>
<tr>
<th>PQ implementation</th>
<th>insert</th>
<th>delete-min</th>
<th>decrease-key</th>
<th>total</th>
</tr>
</thead>
<tbody>
<tr>
<td>array</td>
<td>1</td>
<td>( V )</td>
<td>1</td>
<td>( V^2 )</td>
</tr>
<tr>
<td>binary heap</td>
<td>( \log V )</td>
<td>( \log V )</td>
<td>( \log V )</td>
<td>( E \log V )</td>
</tr>
<tr>
<td>d-way heap (Johnson 1975)</td>
<td>( d \log d V )</td>
<td>( d \log d V )</td>
<td>( \log d V )</td>
<td>( E \log_{E/V} V )</td>
</tr>
<tr>
<td>Fibonacci heap (Fredman-Tarjan 1984)</td>
<td>( 1 ) ( \dagger )</td>
<td>( \log V ) ( \dagger )</td>
<td>( 1 ) ( \dagger )</td>
<td>( E + V \log V )</td>
</tr>
</tbody>
</table>

\( \dagger \) amortized

**Bottom line.**
- Array implementation optimal for dense graphs.
- Binary heap much faster for sparse graphs.
- 4-way heap worth the trouble in performance-critical situations.
- Fibonacci heap best in theory, but not worth implementing.
Minimum Spanning Trees

- Greedy algorithm
- Edge-weighted graph API
- Kruskal's algorithm
- Prim's algorithm
- Context
Given $N$ points in the plane, find MST connecting them, where the distances between point pairs are their Euclidean distances.

**Brute force.** Compute $\sim N^2 / 2$ distances and run Prim's algorithm.

**Ingenuity.** Exploit geometry and do it in $\sim c N \log N$. 

**Euclidean MST**
Scientific application: clustering

k-clustering. Divide a set of objects classified into k coherent groups.

Distance function. Numeric value specifying "closeness" of two objects.

Goal. Divide into clusters so that objects in different clusters are far apart.

Applications.
- Routing in mobile ad hoc networks.
- Document categorization for web search.
- Similarity searching in medical image databases.
- Skycat: cluster $10^9$ sky objects into stars, quasars, galaxies.

outbreak of cholera deaths in London in 1850s (Nina Mishra)
Single-link clustering

k-clustering. Divide a set of objects classify into k coherent groups.

Distance function. Numeric value specifying "closeness" of two objects.

Single link. Distance between two clusters equals the distance between the two closest objects (one in each cluster).

Single-link clustering. Given an integer k, find a k-clustering that maximizes the distance between two closest clusters.
Single-link clustering algorithm

“Well-known” algorithm for single-link clustering:

- Form $V$ clusters of one object each.
- Find the closest pair of objects such that each object is in a different cluster, and merge the two clusters.
- Repeat until there are exactly $k$ clusters.

**Observation.** This is Kruskal's algorithm (stop when $k$ connected components).

Alternate solution. Run Prim's algorithm and delete $k-1$ max weight edges.
Tumors in similar tissues cluster together.

Reference: Botstein & Brown group
Minimum Spanning Trees

Shortest Paths
- Edge-weighted digraph API
- Shortest-paths properties
- Dijkstra's algorithm
- Edge-weighted DAGs
- Negative weights
Shortest Paths

- Edge-weighted digraph API
- Shortest-paths properties
- Dijkstra's algorithm
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- Negative weights
Shortest paths in a weighted digraph

Given an edge-weighted digraph, find the shortest (directed) path from $s$ to $t$. 

**edge-weighted digraph**

- $4\rightarrow 5 \ 0.35$
- $5\rightarrow 4 \ 0.35$
- $4\rightarrow 7 \ 0.37$
- $5\rightarrow 7 \ 0.28$
- $7\rightarrow 5 \ 0.28$
- $5\rightarrow 1 \ 0.32$
- $0\rightarrow 4 \ 0.38$
- $0\rightarrow 2 \ 0.26$
- $7\rightarrow 3 \ 0.39$
- $1\rightarrow 3 \ 0.29$
- $2\rightarrow 7 \ 0.34$
- $6\rightarrow 2 \ 0.40$
- $3\rightarrow 6 \ 0.52$
- $6\rightarrow 0 \ 0.58$
- $6\rightarrow 4 \ 0.93$

**shortest path from 0 to 6**

- $0\rightarrow 2 \ 0.26$
- $2\rightarrow 7 \ 0.34$
- $7\rightarrow 3 \ 0.39$
- $3\rightarrow 6 \ 0.52$
Google maps
Car navigation
Shortest path applications

- PERT/CPM.
- Map routing.
- Seam carving.
- Robot navigation.
- Texture mapping.
- Typesetting in TeX.
- Urban traffic planning.
- Optimal pipelining of VLSI chip.
- Telemarketer operator scheduling.
- Routing of telecommunications messages.
- Network routing protocols (OSPF, BGP, RIP).
- Exploiting arbitrage opportunities in currency exchange.
- Optimal truck routing through given traffic congestion pattern.

Shortest path variants

Which vertices?
• Source-sink: from one vertex to another.
• **Single source:** from one vertex to every other.
• All pairs: between all pairs of vertices.

Restrictions on edge weights?
• Nonnegative weights.
• Arbitrary weights.
• Euclidean weights.

Cycles?
• No directed cycles.
• No "negative cycles."

Simplifying assumption. Shortest paths from $s$ to each vertex $v$ exist.
Shortest Paths

- Edge-weighted digraph API
- Shortest-paths properties
- Dijkstra's algorithm
- Edge-weighted DAGs
- Negative weights
Weighted directed edge API

public class DirectedEdge

    DirectedEdge(int v, int w, double weight)  
        weighted edge v→w

    int from()  
        vertex v

    int to()  
        vertex w

    double weight()  
        weight of this edge

    String toString()  
        string representation

Idiom for processing an edge e: int v = e.from(), w = e.to();
Similar to Edge for undirected graphs, but a bit simpler.

```java
public class DirectedEdge {
    private final int v, w;
    private final double weight;

    public DirectedEdge(int v, int w, double weight) {
        this.v = v;
        this.w = w;
        this.weight = weight;
    }

    public int from() {
        return v;
    }

    public int to() {
        return w;
    }

    public int weight() {
        return weight;
    }
}
```
# Edge-weighted digraph API

```java
public class EdgeWeightedDigraph {

    EdgeWeightedDigraph(int V) // edge-weighted digraph with V vertices
    EdgeWeightedDigraph(In in) // edge-weighted digraph from input stream

    void addEdge(DirectedEdge e) // add weighted directed edge e

    Iterable<DirectedEdge> adj(int v) // edges pointing from v

    int V() // number of vertices

    int E() // number of edges

    Iterable<DirectedEdge> edges() // all edges

    String toString() // string representation
```

## Conventions
Allow self-loops and parallel edges.
Edge-weighted digraph: adjacency-lists representation

tinyEWD.txt

V
8
15
4 5 0.35
5 4 0.35
4 7 0.37
5 7 0.28
7 5 0.28
5 1 0.32
0 4 0.38
0 2 0.26
7 3 0.39
1 3 0.29
2 7 0.34
6 2 0.40
3 6 0.52
6 0 0.58
6 4 0.93

adj

Bag objects

reference to a DirectedEdge object
**Edge-weighted digraph: adjacency-lists implementation in Java**

Same as `EdgeWeightedGraph` except replace `Graph` with `Digraph`.

```java
public class EdgeWeightedDigraph {
    private final int V;
    private final Bag<Edge>[] adj;

    public EdgeWeightedDigraph(int V) {
        this.V = V;
        adj = (Bag<DirectedEdge>[]) new Bag[V];
        for (int v = 0; v < V; v++)
            adj[v] = new Bag<DirectedEdge>();
    }

    public void addEdge(DirectedEdge e) {
        int v = e.from();
        adj[v].add(e);
    }

    public Iterable<DirectedEdge> adj(int v) {
        return adj[v];
    }
}
```

Add edge `e = v→w` only to `v`'s adjacency list.
Single-source shortest paths API

Goal. Find the shortest path from \( s \) to every other vertex.

```java
public class SP

SP(EdgeWeightedDigraph G, int s) \hspace{1cm} \text{shortest paths from } s \text{ in graph } G

double distTo(int v) \hspace{1cm} \text{length of shortest path from } s \text{ to } v

Iterable <DirectedEdge> pathTo(int v) \hspace{1cm} \text{shortest path from } s \text{ to } v

boolean hasPathTo(int v) \hspace{1cm} \text{is there a path from } s \text{ to } v?
```

```java
SP sp = new SP(G, s);
for (int v = 0; v < G.V(); v++)
{
   StdOut.printf("%d to %d (%.2f): ", s, v, sp.distTo(v));
   for (DirectedEdge e : sp.pathTo(v))
      StdOut.print(e + "  ");
   StdOut.println();
}
```
Single-source shortest paths API

**Goal.** Find the shortest path from \( s \) to every other vertex.

```java
public class SP

    SP(EdgeWeightedDigraph G, int s)  // shortest paths from s in graph G

    double distTo(int v)  // length of shortest path from s to v

    Iterable <DirectedEdge> pathTo(int v)  // shortest path from s to v

    boolean hasPathTo(int v)  // is there a path from s to v?
```

% java SP tinyEWD.txt 0
0 to 0 (0.00):
0 to 1 (1.05): 0->4 0.38  4->5 0.35  5->1 0.32
0 to 2 (0.26): 0->2 0.26
0 to 3 (0.99): 0->2 0.26  2->7 0.34  7->3 0.39
0 to 4 (0.38): 0->4 0.38
0 to 5 (0.73): 0->4 0.38  4->5 0.35
0 to 6 (1.51): 0->2 0.26  2->7 0.34  7->3 0.39  3->6 0.52
0 to 7 (0.60): 0->2 0.26  2->7 0.34
Shortest Paths

- Edge-weighted digraph API
- Shortest-paths properties
- Dijkstra's algorithm
- Edge-weighted DAGs
- Negative weights
Data structures for single-source shortest paths

Goal. Find the shortest path from $s$ to every other vertex.

Observation. A shortest-paths tree (SPT) solution exists. Why?

Consequence. Can represent the SPT with two vertex-indexed arrays:

- $\text{distTo}[v]$ is length of shortest path from $s$ to $v$.
- $\text{edgeTo}[v]$ is last edge on shortest path from $s$ to $v$. 

![shortest-paths tree from 0]
Goal. Find the shortest path from $s$ to every other vertex.

Observation. A shortest-paths tree (SPT) solution exists. Why?

Consequence. Can represent the SPT with two vertex-indexed arrays:

- $\text{distTo}[v]$ is length of shortest path from $s$ to $v$.
- $\text{edgeTo}[v]$ is last edge on shortest path from $s$ to $v$.

```java
public double distTo(int v)
{
    return distTo[v];
}

public Iterable<DirectedEdge> pathTo(int v)
{
    Stack<DirectedEdge> path = new Stack<DirectedEdge>();
    for (DirectedEdge e = edgeTo[v]; e != null; e = edgeTo[e.from()])
        path.push(e);
    return path;
}
```
Edge relaxation

Relax edge $e = v \rightarrow w$.

- $\text{distTo}[v]$ is length of shortest known path from $s$ to $v$.
- $\text{distTo}[w]$ is length of shortest known path from $s$ to $w$.
- $\text{edgeTo}[w]$ is last edge on shortest known path from $s$ to $w$.
- If $e = v \rightarrow w$ gives shorter path to $w$ through $v$,
  update $\text{distTo}[w]$ and $\text{edgeTo}[w]$.

$v \rightarrow w$ successfully relaxes
Edge relaxation

Relax edge $e = v \rightarrow w$.

- $\text{distTo}[v]$ is length of shortest known path from $s$ to $v$.
- $\text{distTo}[w]$ is length of shortest known path from $s$ to $w$.
- $\text{edgeTo}[w]$ is last edge on shortest known path from $s$ to $w$.
- If $e = v \rightarrow w$ gives shorter path to $w$ through $v$, update $\text{distTo}[w]$ and $\text{edgeTo}[w]$.

```java
private void relax(DirectedEdge e)
{
    int v = e.from(), w = e.to();
    if (distTo[w] > distTo[v] + e.weight())
    {
        distTo[w] = distTo[v] + e.weight();
        edgeTo[w] = e;
    }
}
```
**Proposition.** Let $G$ be an edge-weighted digraph. Then $\text{distTo}[]$ are the shortest path distances from $s$ iff:

- For each vertex $v$, $\text{distTo}[v]$ is the length of some path from $s$ to $v$.
- For each edge $e = v \rightarrow w$, $\text{distTo}[w] \leq \text{distTo}[v] + e$.weight().

**Pf.** $\Leftarrow$ [necessary]

- Suppose that $\text{distTo}[w] > \text{distTo}[v] + e$.weight() for some edge $e = v \rightarrow w$.
- Then, $e$ gives a path from $s$ to $w$ (through $v$) of length less than $\text{distTo}[w]$.
Shortest-paths optimality conditions

**Proposition.** Let $G$ be an edge-weighted digraph. Then $\text{distTo}[]$ are the shortest path distances from $s$ iff:

- For each vertex $v$, $\text{distTo}[v]$ is the length of some path from $s$ to $v$.
- For each edge $e = v \rightarrow w$, $\text{distTo}[w] \leq \text{distTo}[v] + e$.weight().

**Pf.** $\Rightarrow$ [sufficient]

- Suppose that $s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_k = w$ is a shortest path from $s$ to $w$.
- Then,
  
  $\text{distTo}[v_k] \leq \text{distTo}[v_{k-1}] + e_k$.weight()
  
  $\text{distTo}[v_{k-1}] \leq \text{distTo}[v_{k-2}] + e_{k-1}.weight()$

  $\ldots$

  $\text{distTo}[v_1] \leq \text{distTo}[v_0] + e_1$.weight()

- Add inequalities; simplify; and substitute $\text{distTo}[v_0] = \text{distTo}[s] = 0$:
  
  $\text{distTo}[w] = \text{distTo}[v_k] \leq e_k$.weight() + $e_{k-1}.weight() + \ldots + e_1$.weight()

- Thus, $\text{distTo}[w]$ is the weight of shortest path to $w$. ■
**Generic shortest-paths algorithm**

*Generic algorithm (to compute SPT from s)*

Initialize $\text{distTo}[s] = 0$ and $\text{distTo}[v] = \infty$ for all other vertices.

Repeat until optimality conditions are satisfied:

- Relax any edge.

**Proposition.** Generic algorithm computes SPT (if it exists) from $s$.

**Pf sketch.**

- Throughout algorithm, $\text{distTo}[v]$ is the length of a simple path from $s$ to $v$ (and $\text{edgeTo}[v]$ is last edge on path).
- Each successful relaxation decreases $\text{distTo}[v]$ for some $v$.
- The entry $\text{distTo}[v]$ can decrease at most a finite number of times. ■
Generic shortest-paths algorithm

**Generic algorithm (to compute SPT from s)**

- Initialize `distTo[s] = 0` and `distTo[v] = ∞` for all other vertices.
- Repeat until optimality conditions are satisfied:
  - Relax any edge.

**Efficient implementations.** How to choose which edge to relax?

**Ex 1.** Dijkstra's algorithm (nonnegative weights).

**Ex 2.** Topological sort algorithm (no directed cycles).

**Ex 3.** Bellman-Ford algorithm (no negative cycles).
Shortest Paths

- Edge-weighted digraph API
- Shortest-paths properties
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- Edge-weighted DAGs
- Negative weights
“Do only what only you can do.”

“In their capacity as a tool, computers will be but a ripple on the surface of our culture. In their capacity as intellectual challenge, they are without precedent in the cultural history of mankind.”

“The use of COBOL cripples the mind; its teaching should, therefore, be regarded as a criminal offence.”

“It is practically impossible to teach good programming to students that have had a prior exposure to BASIC: as potential programmers they are mentally mutilated beyond hope of regeneration.”

“APL is a mistake, carried through to perfection. It is the language of the future for the programming techniques of the past: it creates a new generation of coding bums.”
"Object-oriented programming is an exceptionally bad idea which could only have originated in California."

-- Edsger Dijkstra
Dijkstra's algorithm

- Consider vertices in increasing order of distance from $s$ (non-tree vertex with the lowest $\text{distTo}[]$ value).
- Add vertex to tree and relax all edges incident from that vertex.

an edge-weighted digraph
Dijkstra's algorithm

- Consider vertices in increasing order of distance from $s$ (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges incident from that vertex.

choose source vertex 0

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<th>edgeTo[]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0</td>
<td>-</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
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<td>3</td>
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<td>4</td>
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<td>5</td>
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<td>6</td>
<td></td>
<td></td>
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<tr>
<td>7</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
• Consider vertices in increasing order of distance from \( s \) (non-tree vertex with the lowest \( \text{distTo}[] \) value).
• Add vertex to tree and relax all edges incident from that vertex.
Dijkstra's algorithm

- Consider vertices in increasing order of distance from $s$ (non-tree vertex with the lowest distTo[] value).
- Add vertex to tree and relax all edges incident from that vertex.
Dijkstra's algorithm

- Consider vertices in increasing order of distance from $s$ (non-tree vertex with the lowest $\text{distTo[]}$ value).
- Add vertex to tree and relax all edges incident from that vertex.

<table>
<thead>
<tr>
<th>$v$</th>
<th>$\text{distTo[]}$</th>
<th>$\text{edgeTo[]}[]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0</td>
<td>-</td>
</tr>
<tr>
<td>1</td>
<td>5.0</td>
<td>0→1</td>
</tr>
<tr>
<td>2</td>
<td>9.0</td>
<td>0→4</td>
</tr>
<tr>
<td>4</td>
<td>8.0</td>
<td>0→7</td>
</tr>
</tbody>
</table>
Dijkstra's algorithm

- Consider vertices in increasing order of distance from \( s \) (non-tree vertex with the lowest \( \text{distTo}[] \) value).
- Add vertex to tree and relax all edges incident from that vertex.

Choose vertex 1

<table>
<thead>
<tr>
<th>( v )</th>
<th>( \text{distTo}[] )</th>
<th>( \text{edgeTo}[] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0</td>
<td>-</td>
</tr>
<tr>
<td>1</td>
<td>5.0</td>
<td>0→1</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
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<tr>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>9.0</td>
<td>0→4</td>
</tr>
<tr>
<td>5</td>
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<td></td>
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<tr>
<td>6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>8.0</td>
<td>0→7</td>
</tr>
</tbody>
</table>
Dijkstra's algorithm

- Consider vertices in increasing order of distance from $s$ (non-tree vertex with the lowest $\text{distTo}[\cdot]$ value).
- Add vertex to tree and relax all edges incident from that vertex.

\[
\begin{array}{cccc}
v & \text{distTo[\cdot]} & \text{edgeTo[\cdot]} \\
0 & 0.0 & - \\
1 & 5.0 & 0\rightarrow1 \\
2 & & \\
3 & & \\
4 & 9.0 & 0\rightarrow4 \\
5 & & \\
6 & & \\
7 & 8.0 & 0\rightarrow7 \\
\end{array}
\]

relax all edges incident from 1
Dijkstra's algorithm

- Consider vertices in increasing order of distance from $s$ (non-tree vertex with the lowest \texttt{distTo[]} value).
- Add vertex to tree and relax all edges incident from that vertex.

\begin{itemize}
  \item \texttt{distTo[0]} = 0
  \item \texttt{distTo[1]} = 5
  \item \texttt{distTo[2]} = 17
  \item \texttt{distTo[3]} = 20
  \item \texttt{distTo[4]} = 9
  \item \texttt{distTo[5]} = ∞
  \item \texttt{distTo[6]} = ∞
  \item \texttt{distTo[7]} = 8
\end{itemize}

\begin{tabular}{c|c|c}
\texttt{v} & \texttt{distTo[]} & \texttt{edgeTo[]} \\
0 & 0.0 & - \\
1 & 5.0 & 0→1 \\
2 & 17.0 & 1→2 \\
3 & 20.0 & 1→3 \\
4 & 9.0 & 0→4 \\
5 & & \\
6 & & \\
7 & 8.0 & 0→7 \\
\end{tabular}

relax all edges incident from 1
Dijkstra's algorithm

- Consider vertices in increasing order of distance from $s$ (non-tree vertex with the lowest $\text{distTo}[]$ value).
- Add vertex to tree and relax all edges incident from that vertex.

<table>
<thead>
<tr>
<th>vertex</th>
<th>$\text{distTo}[]$</th>
<th>$\text{edgeTo}[]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0</td>
<td>-</td>
</tr>
<tr>
<td>1</td>
<td>5.0</td>
<td>0→1</td>
</tr>
<tr>
<td>2</td>
<td>17.0</td>
<td>1→2</td>
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<tr>
<td>3</td>
<td>20.0</td>
<td>1→3</td>
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<tr>
<td>4</td>
<td>9.0</td>
<td>0→4</td>
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<td></td>
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<tr>
<td>6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>8.0</td>
<td>0→7</td>
</tr>
</tbody>
</table>
Dijkstra's algorithm

- Consider vertices in increasing order of distance from \( s \) (non-tree vertex with the lowest \( \text{distTo}[] \) value).
- Add vertex to tree and relax all edges incident from that vertex.

Choose vertex 7

<table>
<thead>
<tr>
<th>( v )</th>
<th>( \text{distTo}[] )</th>
<th>( \text{edgeTo}[] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0</td>
<td>-</td>
</tr>
<tr>
<td>1</td>
<td>5.0</td>
<td>0\rightarrow1</td>
</tr>
<tr>
<td>2</td>
<td>17.0</td>
<td>1\rightarrow2</td>
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<tr>
<td>3</td>
<td>20.0</td>
<td>1\rightarrow3</td>
</tr>
<tr>
<td>4</td>
<td>9.0</td>
<td>0\rightarrow4</td>
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<tr>
<td>5</td>
<td></td>
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<tr>
<td>6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>8.0</td>
<td>0\rightarrow7</td>
</tr>
</tbody>
</table>
Dijkstra's algorithm

- Consider vertices in increasing order of distance from \( s \) (non-tree vertex with the lowest \( \text{distTo}[] \) value).
- Add vertex to tree and relax all edges incident from that vertex.

```
   1
 /   \
|     |
0 <--- 7 <--- 2
 |     |
|     |
5 <--- 3 <--- 6
|     |
|     |
4 <--- 5 <--- 6
|     |
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\         \         \         \         \\
\         \         \         \         \
```
Dijkstra's algorithm

- Consider vertices in increasing order of distance from $s$ (non-tree vertex with the lowest $\text{distTo}[]$ value).
- Add vertex to tree and relax all edges incident from that vertex.

<table>
<thead>
<tr>
<th>$v$</th>
<th>$\text{distTo}[]$</th>
<th>$\text{edgeTo}[]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0</td>
<td>-</td>
</tr>
<tr>
<td>1</td>
<td>5.0</td>
<td>0→1</td>
</tr>
<tr>
<td>2</td>
<td>15.0</td>
<td>7→2</td>
</tr>
<tr>
<td>3</td>
<td>20.0</td>
<td>1→3</td>
</tr>
<tr>
<td>4</td>
<td>9.0</td>
<td>0→4</td>
</tr>
<tr>
<td>5</td>
<td>14.0</td>
<td>7→5</td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>8.0</td>
<td>0→7</td>
</tr>
</tbody>
</table>
Dijkstra's algorithm

- Consider vertices in increasing order of distance from $s$ (non-tree vertex with the lowest $\text{distTo}[]$ value).
- Add vertex to tree and relax all edges incident from that vertex.
Dijkstra's algorithm

- Consider vertices in increasing order of distance from $s$ (non-tree vertex with the lowest $\text{distTo}[i]$ value).
- Add vertex to tree and relax all edges incident from that vertex.

<table>
<thead>
<tr>
<th>v</th>
<th>distTo[]</th>
<th>edgeTo[]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0</td>
<td>-</td>
</tr>
<tr>
<td>1</td>
<td>5.0</td>
<td>0→1</td>
</tr>
<tr>
<td>2</td>
<td>15.0</td>
<td>7→2</td>
</tr>
<tr>
<td>3</td>
<td>20.0</td>
<td>1→3</td>
</tr>
<tr>
<td>4</td>
<td>9.0</td>
<td>0→4</td>
</tr>
<tr>
<td>5</td>
<td>14.0</td>
<td>7→5</td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>8.0</td>
<td>0→7</td>
</tr>
</tbody>
</table>

select vertex 4
Dijkstra's algorithm

- Consider vertices in increasing order of distance from $s$ (non-tree vertex with the lowest $\text{distTo}[]$ value).
- Add vertex to tree and relax all edges incident from that vertex.

\[
\begin{array}{cccc}
\text{v} & \text{distTo[]} & \text{edgeTo[]} \\
0 & 0.0 & - \\
1 & 5.0 & 0\rightarrow1 \\
2 & 15.0 & 7\rightarrow2 \\
3 & 20.0 & 1\rightarrow3 \\
4 & 9.0 & 0\rightarrow4 \\
5 & 14.0 & 7\rightarrow5 \\
6 & & \\
7 & 8.0 & 0\rightarrow7 \\
\end{array}
\]

relax all edges incident from 4
Dijkstra's algorithm

- Consider vertices in increasing order of distance from $s$ (non-tree vertex with the lowest $\text{distTo}[\cdot]$ value).
- Add vertex to tree and relax all edges incident from that vertex.

relax all edges incident from 4
• Consider vertices in increasing order of distance from $s$ (non-tree vertex with the lowest $\text{distTo}[]$ value).
• Add vertex to tree and relax all edges incident from that vertex.

**Dijkstra's algorithm**

- 0: $\text{distTo}[0] = 0.0$, $\text{edgeTo}[0] = -$
- 1: $\text{distTo}[1] = 5.0$, $\text{edgeTo}[1] = 0 \rightarrow 1$
- 2: $\text{distTo}[2] = 15.0$, $\text{edgeTo}[2] = 7 \rightarrow 2$
- 3: $\text{distTo}[3] = 20.0$, $\text{edgeTo}[3] = 1 \rightarrow 3$
- 4: $\text{distTo}[4] = 9.0$, $\text{edgeTo}[4] = 0 \rightarrow 4$
- 5: $\text{distTo}[5] = 13.0$, $\text{edgeTo}[5] = 4 \rightarrow 5$
- 6: $\text{distTo}[6] = 29.0$, $\text{edgeTo}[6] = 4 \rightarrow 6$
- 7: $\text{distTo}[7] = 8.0$, $\text{edgeTo}[7] = 0 \rightarrow 7$
Dijkstra's algorithm

- Consider vertices in increasing order of distance from $s$ (non-tree vertex with the lowest $distTo[]$ value).
- Add vertex to tree and relax all edges incident from that vertex.

```
<table>
<thead>
<tr>
<th>v</th>
<th>distTo[]</th>
<th>edgeTo[]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0</td>
<td>-</td>
</tr>
<tr>
<td>1</td>
<td>5.0</td>
<td>0→1</td>
</tr>
<tr>
<td>2</td>
<td>15.0</td>
<td>7→2</td>
</tr>
<tr>
<td>3</td>
<td>20.0</td>
<td>1→3</td>
</tr>
<tr>
<td>4</td>
<td>9.0</td>
<td>0→4</td>
</tr>
<tr>
<td>5</td>
<td>13.0</td>
<td>4→5</td>
</tr>
<tr>
<td>6</td>
<td>29.0</td>
<td>4→6</td>
</tr>
<tr>
<td>7</td>
<td>8.0</td>
<td>0→7</td>
</tr>
</tbody>
</table>
```

select vertex 5
Dijkstra's algorithm

- Consider vertices in increasing order of distance from $s$ (non-tree vertex with the lowest $\text{distTo}[]$ value).
- Add vertex to tree and relax all edges incident from that vertex.

![Graph Diagram]

<table>
<thead>
<tr>
<th>v</th>
<th>distTo[]</th>
<th>edgeTo[]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0</td>
<td>-</td>
</tr>
<tr>
<td>1</td>
<td>5.0</td>
<td>0→1</td>
</tr>
<tr>
<td>2</td>
<td>15.0</td>
<td>7→2</td>
</tr>
<tr>
<td>3</td>
<td>20.0</td>
<td>1→3</td>
</tr>
<tr>
<td>4</td>
<td>9.0</td>
<td>0→4</td>
</tr>
<tr>
<td>5</td>
<td>13.0</td>
<td>4→5</td>
</tr>
<tr>
<td>6</td>
<td>29.0</td>
<td>4→6</td>
</tr>
<tr>
<td>7</td>
<td>8.0</td>
<td>0→7</td>
</tr>
</tbody>
</table>

relax all edges incident from 5
Dijkstra's algorithm

- Consider vertices in increasing order of distance from \( s \) (non-tree vertex with the lowest \( \text{distTo}[] \) value).
- Add vertex to tree and relax all edges incident from that vertex.

<table>
<thead>
<tr>
<th>v</th>
<th>distTo[]</th>
<th>edgeTo[]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0</td>
<td>-</td>
</tr>
<tr>
<td>1</td>
<td>5.0</td>
<td>0 \rightarrow 1</td>
</tr>
<tr>
<td>2</td>
<td>14.0</td>
<td>5 \rightarrow 2</td>
</tr>
<tr>
<td>3</td>
<td>20.0</td>
<td>1 \rightarrow 3</td>
</tr>
<tr>
<td>4</td>
<td>9.0</td>
<td>0 \rightarrow 4</td>
</tr>
<tr>
<td>5</td>
<td>13.0</td>
<td>4 \rightarrow 5</td>
</tr>
<tr>
<td>6</td>
<td>26.0</td>
<td>5 \rightarrow 6</td>
</tr>
<tr>
<td>7</td>
<td>8.0</td>
<td>0 \rightarrow 7</td>
</tr>
</tbody>
</table>

relax all edges incident from 5
Dijkstra's algorithm

- Consider vertices in increasing order of distance from $s$ (non-tree vertex with the lowest $\text{distTo}[]$ value).
- Add vertex to tree and relax all edges incident from that vertex.
Dijkstra's algorithm

- Consider vertices in increasing order of distance from $s$ (non-tree vertex with the lowest $\text{distTo}[]$ value).
- Add vertex to tree and relax all edges incident from that vertex.

select vertex 2
Dijkstra's algorithm

- Consider vertices in increasing order of distance from $s$ (non-tree vertex with the lowest \texttt{distTo[]} value).
- Add vertex to tree and relax all edges incident from that vertex.

<table>
<thead>
<tr>
<th>$v$</th>
<th>distTo[]</th>
<th>edgeTo[]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0</td>
<td>-</td>
</tr>
<tr>
<td>1</td>
<td>5.0</td>
<td>0→1</td>
</tr>
<tr>
<td>2</td>
<td>14.0</td>
<td>5→2</td>
</tr>
<tr>
<td>3</td>
<td>20.0</td>
<td>1→3</td>
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<tr>
<td>4</td>
<td>9.0</td>
<td>0→4</td>
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<tr>
<td>5</td>
<td>13.0</td>
<td>4→5</td>
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<tr>
<td>6</td>
<td>26.0</td>
<td>5→6</td>
</tr>
<tr>
<td>7</td>
<td>8.0</td>
<td>0→7</td>
</tr>
</tbody>
</table>

relax all edges incident from 2
Dijkstra's algorithm

- Consider vertices in increasing order of distance from $s$ (non-tree vertex with the lowest distTo[] value).
- Add vertex to tree and relax all edges incident from that vertex.

```
<table>
<thead>
<tr>
<th>v</th>
<th>distTo[]</th>
<th>edgeTo[]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0</td>
<td>-</td>
</tr>
<tr>
<td>1</td>
<td>5.0</td>
<td>0→1</td>
</tr>
<tr>
<td>2</td>
<td>14.0</td>
<td>5→2</td>
</tr>
<tr>
<td>3</td>
<td>17.0</td>
<td>2→3</td>
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<tr>
<td>4</td>
<td>9.0</td>
<td>0→4</td>
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<tr>
<td>5</td>
<td>13.0</td>
<td>4→5</td>
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<tr>
<td>6</td>
<td>25.0</td>
<td>2→6</td>
</tr>
<tr>
<td>7</td>
<td>8.0</td>
<td>0→7</td>
</tr>
</tbody>
</table>
```

relax all edges incident from 2
Dijkstra's algorithm

• Consider vertices in increasing order of distance from \( s \) (non-tree vertex with the lowest \( \text{distTo[]} \) value).
• Add vertex to tree and relax all edges incident from that vertex.

\[
\begin{array}{ccc}
\text{v} & \text{distTo[]} & \text{edgeTo[]} \\
0 & 0.0 & - \\
1 & 5.0 & 0\rightarrow 1 \\
2 & 14.0 & 5\rightarrow 2 \\
3 & 17.0 & 2\rightarrow 3 \\
4 & 9.0 & 0\rightarrow 4 \\
5 & 13.0 & 4\rightarrow 5 \\
6 & 25.0 & 2\rightarrow 6 \\
7 & 8.0 & 0\rightarrow 7 \\
\end{array}
\]
Dijkstra's algorithm

- Consider vertices in increasing order of distance from \( s \) (non-tree vertex with the lowest \( \text{distTo}[] \) value).
- Add vertex to tree and relax all edges incident from that vertex.

<table>
<thead>
<tr>
<th>v</th>
<th>( \text{distTo}[] )</th>
<th>( \text{edgeTo}[] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0</td>
<td>-</td>
</tr>
<tr>
<td>1</td>
<td>5.0</td>
<td>0→1</td>
</tr>
<tr>
<td>2</td>
<td>14.0</td>
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</tr>
<tr>
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</tr>
<tr>
<td>4</td>
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</tr>
<tr>
<td>7</td>
<td>8.0</td>
<td>0→7</td>
</tr>
</tbody>
</table>

select vertex 3
Dijkstra's algorithm

- Consider vertices in increasing order of distance from \( s \) (non-tree vertex with the lowest \( \text{distTo}[] \) value).
- Add vertex to tree and relax all edges incident from that vertex.

```
<table>
<thead>
<tr>
<th>v</th>
<th>distTo[]</th>
<th>edgeTo[]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0</td>
<td>-</td>
</tr>
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<td>5.0</td>
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</tr>
<tr>
<td>7</td>
<td>8.0</td>
<td>0→7</td>
</tr>
</tbody>
</table>
```

relax all edges incident from 3
Dijkstra's algorithm

- Consider vertices in increasing order of distance from $s$ (non-tree vertex with the lowest $\text{distTo}[]$ value).
- Add vertex to tree and relax all edges incident from that vertex.

\[
\begin{array}{ccc}
\text{v} & \text{distTo[]} & \text{edgeTo[]} \\
0 & 0.0 & - \\
1 & 5.0 & 0 \rightarrow 1 \\
2 & 14.0 & 5 \rightarrow 2 \\
3 & 17.0 & 2 \rightarrow 3 \\
4 & 9.0 & 0 \rightarrow 4 \\
5 & 13.0 & 4 \rightarrow 5 \\
6 & 25.0 & 2 \rightarrow 6 \\
7 & 8.0 & 0 \rightarrow 7 \\
\end{array}
\]

relax all edges incident from 3
Dijkstra's algorithm

- Consider vertices in increasing order of distance from \( s \) (non-tree vertex with the lowest \( \text{distTo}[] \) value).
- Add vertex to tree and relax all edges incident from that vertex.
Dijkstra's algorithm

- Consider vertices in increasing order of distance from $s$ (non-tree vertex with the lowest $\text{distTo}[]$ value).
- Add vertex to tree and relax all edges incident from that vertex.

```
select vertex 6
```

```
<table>
<thead>
<tr>
<th>v</th>
<th>distTo[]</th>
<th>edgeTo[]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0</td>
<td>-</td>
</tr>
<tr>
<td>1</td>
<td>5.0</td>
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</tr>
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<td>0→7</td>
</tr>
</tbody>
</table>
```
Dijkstra's algorithm

- Consider vertices in increasing order of distance from $s$ (non-tree vertex with the lowest $\text{distTo}[]$ value).
- Add vertex to tree and relax all edges incident from that vertex.

relax all edges incident from 6
Dijkstra's algorithm

- Consider vertices in increasing order of distance from $s$ (non-tree vertex with the lowest $\text{distTo}[]$ value).
- Add vertex to tree and relax all edges incident from that vertex.

**Graph and Table**

- **Graph**: A directed graph with vertices labeled from 0 to 7, interconnected with edges and weights.
- **Table**: A table listing vertices $v$, their distances $\text{distTo}[]$, and the previous vertex in the path $\text{edgeTo}[]$.

<table>
<thead>
<tr>
<th>$v$</th>
<th>$\text{distTo}[]$</th>
<th>$\text{edgeTo}[]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0</td>
<td>-</td>
</tr>
<tr>
<td>1</td>
<td>5.0</td>
<td>0→1</td>
</tr>
<tr>
<td>2</td>
<td>14.0</td>
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<td>8.0</td>
<td>0→7</td>
</tr>
</tbody>
</table>
Dijkstra's algorithm

- Consider vertices in increasing order of distance from $s$ (non-tree vertex with the lowest \textit{distTo[]} value).
- Add vertex to tree and relax all edges incident from that vertex.

```
<table>
<thead>
<tr>
<th>v</th>
<th>distTo[]</th>
<th>edgeTo[]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0</td>
<td>-</td>
</tr>
<tr>
<td>1</td>
<td>5.0</td>
<td>0\to1</td>
</tr>
<tr>
<td>2</td>
<td>14.0</td>
<td>5\to2</td>
</tr>
<tr>
<td>3</td>
<td>17.0</td>
<td>2\to3</td>
</tr>
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<td>4</td>
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<td>2\to6</td>
</tr>
<tr>
<td>7</td>
<td>8.0</td>
<td>0\to7</td>
</tr>
</tbody>
</table>
```

shortest-paths tree from vertex $s$
Dijkstra’s algorithm visualization
Dijkstra’s algorithm visualization
**Dijkstra's algorithm: correctness proof**

**Proposition.** Dijkstra's algorithm computes a SPT in any edge-weighted digraph with nonnegative weights.

**Pf.**

- Each edge $e = v \rightarrow w$ is relaxed exactly once (when $v$ is relaxed), leaving $\text{distTo}[w] \leq \text{distTo}[v] + e.\text{weight}()$.
- Inequality holds until algorithm terminates because:
  - $\text{distTo}[w]$ cannot increase $\iff$ $\text{distTo}[]$ values are monotone decreasing
  - $\text{distTo}[v]$ will not change $\iff$ edge weights are nonnegative and we choose lowest $\text{distTo}[]$ value at each step
- Thus, upon termination, shortest-paths optimality conditions hold. ☑
public class DijkstraSP
{
    private DirectedEdge[] edgeTo;
    private double[] distTo;
    private IndexMinPQ<Double> pq;

    public DijkstraSP(EdgeWeightedDigraph G, int s)
    {
        edgeTo = new DirectedEdge[G.V()];
        distTo = new double[G.V()];
        pq = new IndexMinPQ<Double>(G.V());

        for (int v = 0; v < G.V(); v++)
            distTo[v] = Double.POSITIVE_INFINITY;
        distTo[s] = 0.0;

        pq.insert(s, 0.0);
        while (!pq.isEmpty())
        {
            int v = pq.delMin();
            for (DirectedEdge e : G.adj(v))
                relax(e);
        }
    }
}

relax vertices in order of distance from s
Dijkstra's algorithm: Java implementation

private void relax(DirectedEdge e)
{
    int v = e.from(), w = e.to();
    if (distTo[w] > distTo[v] + e.weight())
    {
        distTo[w] = distTo[v] + e.weight();
        edgeTo[w] = e;
        if (pq.contains(w)) pq.decreaseKey(w, distTo[w]);
        else                pq.insert     (w, distTo[w]);
    }
}
Dijkstra's algorithm: which priority queue?

Depends on PQ implementation: $V$ insert, $V$ delete-min, $E$ decrease-key.

<table>
<thead>
<tr>
<th>PQ implementation</th>
<th>insert</th>
<th>delete-min</th>
<th>decrease-key</th>
<th>total</th>
</tr>
</thead>
<tbody>
<tr>
<td>array</td>
<td>1</td>
<td>V</td>
<td>1</td>
<td>$V^2$</td>
</tr>
<tr>
<td>binary heap</td>
<td>log $V$</td>
<td>log $V$</td>
<td>log $V$</td>
<td>$E \log V$</td>
</tr>
<tr>
<td>d-way heap (Johnson 1975)</td>
<td>$d \log_d V$</td>
<td>$d \log_d V$</td>
<td>$\log_d V$</td>
<td>$E \log \frac{E}{V} \cdot V$</td>
</tr>
<tr>
<td>Fibonacci heap (Fredman-Tarjan 1984)</td>
<td>1 †</td>
<td>log $V$ †</td>
<td>1 †</td>
<td>$E + V \log V$</td>
</tr>
</tbody>
</table>

† amortized

**Bottom line.**
- Array implementation optimal for dense graphs.
- Binary heap much faster for sparse graphs.
- d-way heap worth the trouble in performance-critical situations.
- Fibonacci heap best in theory, but not worth implementing.
Priority-first search

**Insight.** Four of our graph-search methods are the same algorithm!
- Maintain a set of explored vertices $S$.
- Grow $S$ by exploring edges with exactly one endpoint leaving $S$.

**DFS.** Take edge from vertex which was discovered most recently.
**BFS.** Take edge from vertex which was discovered least recently.
**Prim.** Take edge of minimum weight.
**Dijkstra.** Take edge to vertex that is closest to $S$.

**Challenge.** Express this insight in reusable Java code.
Shortest Paths

- Edge-weighted digraph API
- Shortest-paths properties
- Dijkstra's algorithm
- Edge-weighted DAGs
- Negative weights
Acyclic edge-weighted digraphs

Q. Suppose that an edge-weighted digraph has no directed cycles. Is it easier to find shortest paths than in a general digraph?

A. Yes!
Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.

an edge-weighted DAG

0 → 1  5.0
0 → 4  9.0
0 → 7  8.0
1 → 2  12.0
1 → 3  15.0
1 → 7  4.0
2 → 3  3.0
2 → 6  11.0
3 → 6  9.0
4 → 5  4.0
4 → 6  20.0
4 → 7  5.0
5 → 2  1.0
5 → 6  13.0
7 → 5  6.0
7 → 2  7.0
Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.

```
topological order: 0 1 4 7 5 2 3 6
```
Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.

choose vertex 0
Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.

relax all edges incident from 0
Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.

relax all edges incident from 0
Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.

```
0   0.0        -
1   5.0       0→1
2
3
4   9.0       0→4
5
6
7   8.0       0→7
```
Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.

```
choose vertex 1
```

```
0 1 4 7 5 2 3 6
v distTo[] edgeTo[]
0 0.0 -
1 5.0 0→1
2
3
4 9.0 0→4
5
6
7 8.0 0→7
```
Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.

relax all edges incident from 1
• Consider vertices in topological order.
• Relax all edges incident from that vertex.

Topological sort algorithm

relax all edges incident from 1
Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.
Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.

select vertex 4
(Dijkstra would have selected vertex 7)
Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.
Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.

![Graph with vertex labels and edge weights]

### Table

<table>
<thead>
<tr>
<th>v</th>
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</thead>
<tbody>
<tr>
<td>0</td>
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</tr>
</tbody>
</table>

relax all edges incident from 4
Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.

![Graph with vertices labeled 0 to 7 and edges connecting them.]

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Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.

Choose vertex 7

<table>
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<tr>
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</tr>
</tbody>
</table>
Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.

```
relax all edges incident from 7
```
Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.

```
<table>
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<th>v</th>
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</tr>
</tbody>
</table>
```

relax all edges incident from 7
• Consider vertices in topological order.
• Relax all edges incident from that vertex.
Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.

Select vertex 5
Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.

```
relax all edges incident from 5
```
Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.

relax all edges incident from 5
Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.
Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.

### Example Table

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</thead>
<tbody>
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<td>0→4</td>
</tr>
<tr>
<td>5</td>
<td>13.0</td>
<td>4→5</td>
</tr>
<tr>
<td>6</td>
<td>26.0</td>
<td>5→6</td>
</tr>
<tr>
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</tr>
</tbody>
</table>
Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.

```
relax all edges incident from 2
```

```
<table>
<thead>
<tr>
<th>v</th>
<th>distTo[]</th>
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</thead>
<tbody>
<tr>
<td>0</td>
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</tr>
<tr>
<td>7</td>
<td>8.0</td>
<td>0→7</td>
</tr>
</tbody>
</table>
```
Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.

relax all edges incident from 2

\[
\begin{array}{ccccccc}
\text{v} & \text{distTo[]} & \text{edgeTo[]} \\
0 & 0.0 & - \\
1 & 5.0 & 0 \rightarrow 1 \\
2 & 14.0 & 5 \rightarrow 2 \\
3 & 17.0 & 2 \rightarrow 3 \\
4 & 9.0 & 0 \rightarrow 4 \\
5 & 13.0 & 4 \rightarrow 5 \\
6 & 25.0 & 2 \rightarrow 6 \\
7 & 8.0 & 0 \rightarrow 7 \\
\end{array}
\]
Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.

<table>
<thead>
<tr>
<th>v</th>
<th>distTo[]</th>
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</tr>
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<tbody>
<tr>
<td>0</td>
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<tr>
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<td>2→6</td>
</tr>
<tr>
<td>7</td>
<td>8.0</td>
<td>0→7</td>
</tr>
</tbody>
</table>
Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.

```
0 1 4 7 5 2 3 6

<table>
<thead>
<tr>
<th>v</th>
<th>distTo[]</th>
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</tr>
</tbody>
</table>
```

select vertex 3
Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.

<table>
<thead>
<tr>
<th>v</th>
<th>distTo[]</th>
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</thead>
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<tr>
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<td>8.0</td>
<td>0→7</td>
</tr>
</tbody>
</table>

relax all edges incident from 3
Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.

relax all edges incident from 3
Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.
Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.

select vertex 6
Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.

```
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<td>0→7</td>
</tr>
</tbody>
</table>
```
Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.
Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.

```
shortest-paths tree from vertex s
```

```
0     0.0        -
1     5.0       0→1
2     14.0      5→2
3     17.0      2→3
4     9.0       0→4
5     13.0      4→5
6     25.0      2→6
7     8.0       0→7
```
**Proposition.** Topological sort algorithm computes SPT in any edge-weighted DAG in time proportional to $E + V$.

**Pf.**

- Each edge $e = v \rightarrow w$ is relaxed exactly once (when $v$ is relaxed), leaving $\text{distTo}[w] \leq \text{distTo}[v] + e\.\text{weight()}$.

- Inequality holds until algorithm terminates because:
  - $\text{distTo}[w]$ cannot increase
  - $\text{distTo}[v]$ will not change

- Thus, upon termination, shortest-paths optimality conditions hold. ■
public class AcyclicSP
{
    private DirectedEdge[] edgeTo;
    private double[] distTo;

    public AcyclicSP(EdgeWeightedDigraph G, int s)
    {
        edgeTo = new DirectedEdge[G.V()];
        distTo = new double[G.V()];

        for (int v = 0; v < G.V(); v++)
            distTo[v] = Double.POSITIVE_INFINITY;
        distTo[s] = 0.0;

        Topological topological = new Topological(G);
        for (int v : topological.order())
            for (DirectedEdge e : G.adj(v))
                relax(e);
    }
}
Content-aware resizing

Seam carving. [Avidan and Shamir] Resize an image without distortion for display on cell phones and web browsers.
Content-aware resizing

Seam carving. [Avidan and Shamir] Resize an image without distortion for display on cell phones and web browsers.

In the wild. Photoshop CS 5, Imagemagick, GIMP, ...
To find vertical seam:

- Grid DAG: vertex = pixel; edge = from pixel to 3 downward neighbors.
- Weight of pixel = energy function of 8 neighboring pixels.
- Seam = shortest path from top to bottom.
To find vertical seam:

- Grid DAG: vertex = pixel; edge = from pixel to 3 downward neighbors.
- Weight of pixel = energy function of 8 neighboring pixels.
- Seam = shortest path from top to bottom.
To remove vertical seam:

- Delete pixels on seam (one in each row).
To remove vertical seam:

- Delete pixels on seam (one in each row).
Longest paths in edge-weighted DAGs

Formulate as a shortest paths problem in edge-weighted DAGs.

- Negate all weights.
- Find shortest paths.
- Negate weights in result.

### Key point
Topological sort algorithm works even with negative edge weights.

---

**longest paths input**  |  **shortest paths input**
---|---
5->4 0.35  |  5->4 -0.35
4->7 0.37  |  4->7 -0.37
5->7 0.28  |  5->7 -0.28
5->1 0.32  |  5->1 -0.32
4->0 0.38  |  4->0 -0.38
0->2 0.26  |  0->2 -0.26
3->7 0.39  |  3->7 -0.39
1->3 0.29  |  1->3 -0.29
7->2 0.34  |  7->2 -0.34
6->2 0.40  |  6->2 -0.40
3->6 0.52  |  3->6 -0.52
6->0 0.58  |  6->0 -0.58
6->4 0.93  |  6->4 -0.93
Longest paths in edge-weighted DAGs: application

Parallel job scheduling. Given a set of jobs with durations and precedence constraints, schedule the jobs (by finding a start time for each) so as to achieve the minimum completion time, while respecting the constraints.

<table>
<thead>
<tr>
<th>job</th>
<th>duration</th>
<th>must complete before</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>41.0</td>
<td>1 7 9</td>
</tr>
<tr>
<td>1</td>
<td>51.0</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>50.0</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>36.0</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>38.0</td>
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<td>5</td>
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<td>3 8</td>
</tr>
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<td>7</td>
<td>32.0</td>
<td>3 8</td>
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<tr>
<td>8</td>
<td>32.0</td>
<td>2</td>
</tr>
<tr>
<td>9</td>
<td>29.0</td>
<td>4 6</td>
</tr>
</tbody>
</table>

Parallel job scheduling solution
Critical path method

**CPM.** To solve a parallel job-scheduling problem, create edge-weighted DAG:

- Source and sink vertices.
- Two vertices (begin and end) for each job.
- Three edges for each job.
  - begin to end (weighted by duration)
  - source to begin (0 weight)
  - end to sink (0 weight)
- One edge for each precedence constraint (0 weight).

### Example Table

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</thead>
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<tr>
<td>9</td>
<td>29.0</td>
<td>4 6</td>
</tr>
</tbody>
</table>
Critical path method

**CPM.** Use longest path from the source to schedule each job.
Shortest Paths

- Edge-weighted digraph API
- Shortest-paths properties
- Dijkstra's algorithm
- Edge-weighted DAGs
- Negative weights
Shortest paths with negative weights: failed attempts

Dijkstra. Doesn’t work with negative edge weights.

Dijkstra selects vertex 3 immediately after 0.
But shortest path from 0 to 3 is $0 \rightarrow 1 \rightarrow 2 \rightarrow 3$.

Re-weighting. Add a constant to every edge weight doesn’t work.

Adding 9 to each edge weight changes the shortest path from $0 \rightarrow 1 \rightarrow 2 \rightarrow 3$ to $0 \rightarrow 3$.

Bad news. Need a different algorithm.
**Negative cycles**

**Def.** A **negative cycle** is a directed cycle whose sum of edge weights is negative.

**Proposition.** A SPT exists iff no negative cycles.

Assuming all vertices reachable from s.
Bellman-Ford algorithm

Initialize distTo[s] = 0 and distTo[v] = \infty for all other vertices.
Repeat V times:
  - Relax each edge.

for (int i = 0; i < G.V(); i++)
   for (int v = 0; v < G.V(); v++)
      for (DirectedEdge e : G.adj(v))
         relax(e);
Bellman-Ford algorithm demo

Repeat $V$ times: relax all $E$ edges.

an edge-weighted digraph
Bellman-Ford algorithm demo

Repeat $V$ times: relax all $E$ edges.

```
initialize

<table>
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<tr>
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<tbody>
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</tbody>
</table>
```

Bellman-Ford algorithm demo

Repeat $V$ times: relax all $E$ edges.
Repeat $V$ times: relax all $E$ edges.

Bellman-Ford algorithm demo

pass 0

<table>
<thead>
<tr>
<th>$v$</th>
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</tr>
</thead>
<tbody>
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<td>-</td>
</tr>
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<td>1</td>
<td>5.0</td>
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</table>
Repeat $V$ times: relax all $E$ edges.
Bellman-Ford algorithm demo

Repeat $V$ times: relax all $E$ edges.

pass 0

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2
Bellman-Ford algorithm demo

Repeat $V$ times: relax all $E$ edges.

---

**Pass 0**

$0 \rightarrow 1 \ 0 \rightarrow 4 \ 0 \rightarrow 7 \ 1 \rightarrow 2 \ 1 \rightarrow 3 \ 1 \rightarrow 7 \ 2 \rightarrow 3 \ 2 \rightarrow 6 \ 3 \rightarrow 6 \ 4 \rightarrow 5 \ 4 \rightarrow 6 \ 4 \rightarrow 7 \ 5 \rightarrow 2 \ 5 \rightarrow 6 \ 7 \rightarrow 5 \ 7 \rightarrow 2$
Bellman-Ford algorithm demo

Repeat $V$ times: relax all $E$ edges.

Pass 0

$0 \rightarrow 1 \ 0 \rightarrow 4 \ 0 \rightarrow 7 \ 1 \rightarrow 2 \ 1 \rightarrow 3 \ 1 \rightarrow 7 \ 2 \rightarrow 3 \ 2 \rightarrow 6 \ 3 \rightarrow 6 \ 4 \rightarrow 5 \ 4 \rightarrow 6 \ 4 \rightarrow 7 \ 5 \rightarrow 2 \ 5 \rightarrow 6 \ 7 \rightarrow 5 \ 7 \rightarrow 2$
Bellman-Ford algorithm demo

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pass 0

0→1  0→4  0→7  1→2  1→3  1→7  2→3  2→6  3→6  4→5  4→6  4→7  5→2  5→6  7→5  7→2
Bellman-Ford algorithm demo

Repeat $V$ times: relax all $E$ edges.

---

**pass 0**

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Bellman-Ford algorithm demo

Repeat $V$ times: relax all $E$ edges.
Bellman-Ford algorithm demo

Repeat $V$ times: relax all $E$ edges.

### Graph

- Nodes: 0, 1, 2, 3, 4, 5, 6, 7
- Edges and weights:
  - 0→1: 15
  - 1→2: 15
  - 1→3: 20
  - 1→5: ∞
  - 2→3: 17
  - 2→5: ∞
  - 3→4: 9
  - 3→5: 8
  - 4→7: 20
  - 5→6: ∞
  - 6→7: ∞

### Table

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### Pass 0

0→1  0→4  0→7  1→2  1→3  1→7  2→3  2→6  3→6  4→5  4→6  4→7  5→2  5→6  7→5  7→2
Bellman-Ford algorithm demo

Repeat $V$ times: relax all $E$ edges.

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pass 0

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Bellman-Ford algorithm demo

Repeat $V$ times: relax all $E$ edges.

**Pass 0**

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Bellman-Ford algorithm demo

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Bellman-Ford algorithm demo

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### Diagram

- **Pass 0**
  - 0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2
Bellman-Ford algorithm demo

Repeat $V$ times: relax all $E$ edges.

```
pass 0

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2
```

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Bellman-Ford algorithm demo

Repeat $V$ times: relax all $E$ edges.
Bellman-Ford algorithm demo

Repeat $V$ times: relax all $E$ edges.
Bellman-Ford algorithm demo

Repeat $V$ times: relax all $E$ edges.

pass 0

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2
Bellman-Ford algorithm demo

Repeat $V$ times: relax all $E$ edges.

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pass 0

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Bellman-Ford algorithm demo

Repeat $V$ times: relax all $E$ edges.

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pass 0

0→1  0→4  0→7  1→2  1→3  1→7  2→3  2→6  3→6  4→5  4→6  4→7  5→2  5→6  7→5  7→2
Bellman-Ford algorithm demo

Repeat $V$ times: relax all $E$ edges.
Bellman-Ford algorithm demo

Repeat $V$ times: relax all $E$ edges.

**Pass 0**

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

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Bellman-Ford algorithm demo

Repeat $V$ times: relax all $E$ edges.

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pass 0

$0→1$ $0→4$ $0→7$ $1→2$ $1→3$ $1→7$ $2→3$ $2→6$ $3→6$ $4→5$ $4→6$ $4→7$ $5→2$ $5→6$ $7→5$ $7→2$
Bellman-Ford algorithm demo

Repeat $V$ times: relax all $E$ edges.

pass 0

0 → 1  0 → 4  0 → 7  1 → 2  1 → 3  1 → 7  2 → 3  2 → 6  3 → 6  4 → 5  4 → 6  4 → 7  5 → 2  5 → 6  7 → 5  7 → 2
Bellman-Ford algorithm demo

Repeat \( V \) times: relax all \( E \) edges.

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pass 0

\(0\rightarrow1\) \(0\rightarrow4\) \(0\rightarrow7\) \(1\rightarrow2\) \(1\rightarrow3\) \(1\rightarrow7\) \(2\rightarrow3\) \(2\rightarrow6\) \(3\rightarrow6\) \(4\rightarrow5\) \(4\rightarrow6\) \(4\rightarrow7\) \(5\rightarrow2\) \(5\rightarrow6\) \(7\rightarrow5\) \(7\rightarrow2\)
Bellman-Ford algorithm demo

Repeat \( V \) times: relax all \( E \) edges.

\[
\begin{array}{c|c|c}
\text{v} & \text{distTo[]} & \text{edgeTo[]} \\
0 & 0.0 & - \\
1 & 5.0 & 0\rightarrow1 \\
2 & 14.0 & 5\rightarrow2 \\
3 & 20.0 & 1\rightarrow3 \\
4 & 9.0 & 0\rightarrow4 \\
5 & 13.0 & 4\rightarrow5 \\
6 & 26.0 & 5\rightarrow6 \\
7 & 8.0 & 0\rightarrow7 \\
\end{array}
\]

pass 1

0\rightarrow1 0\rightarrow4 0\rightarrow7 1\rightarrow2 1\rightarrow3 1\rightarrow7 2\rightarrow3 2\rightarrow6 3\rightarrow6 4\rightarrow5 4\rightarrow6 4\rightarrow7 5\rightarrow2 5\rightarrow6 7\rightarrow5 7\rightarrow2
Repeat $V$ times: relax all $E$ edges.

**Bellman-Ford algorithm demo**

Pass 1

0→1  0→4  0→7  1→2  1→3  1→7  2→3  2→6  3→6  4→5  4→6  4→7  5→2  5→6  7→5  7→2
Bellman-Ford algorithm demo

Repeat $V$ times: relax all $E$ edges.

\[
\begin{array}{c|c|c}
\text{v} & \text{distTo[]} & \text{edgeTo[]} \\
0 & 0.0 & - \\
1 & 5.0 & 0 \rightarrow 1 \\
2 & 14.0 & 5 \rightarrow 2 \\
3 & 20.0 & 1 \rightarrow 3 \\
4 & 9.0 & 0 \rightarrow 4 \\
5 & 13.0 & 4 \rightarrow 5 \\
6 & 26.0 & 5 \rightarrow 6 \\
7 & 8.0 & 0 \rightarrow 7 \\
\end{array}
\]
Bellman-Ford algorithm demo

Repeat $V$ times: relax all $E$ edges.

\[
\begin{array}{cccc}
 v & \text{distTo[]} & \text{edgeTo[]} \\
 0 & 0.0 & - \\
 1 & 5.0 & 0\rightarrow1 \\
 2 & 14.0 & 5\rightarrow2 \\
 3 & 20.0 & 1\rightarrow3 \\
 4 & 9.0 & 0\rightarrow4 \\
 5 & 13.0 & 4\rightarrow5 \\
 6 & 26.0 & 5\rightarrow6 \\
 7 & 8.0 & 0\rightarrow7 \\
\end{array}
\]

pass 1

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2
Bellman-Ford algorithm demo

Repeat $V$ times: relax all $E$ edges.

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pass 1

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2
Bellman-Ford algorithm demo

Repeat $V$ times: relax all $E$ edges.

### pass 1

\[
\begin{array}{c|c|c}
\text{v} & \text{distTo[]} & \text{edgeTo[]} \\
\hline
0 & 0.0 & - \\
1 & 5.0 & 0\to1 \\
2 & 14.0 & 5\to2 \\
3 & 20.0 & 1\to3 \\
4 & 9.0 & 0\to4 \\
5 & 13.0 & 4\to5 \\
6 & 26.0 & 5\to6 \\
7 & 8.0 & 0\to7 \\
\end{array}
\]
Bellman-Ford algorithm demo

Repeat $V$ times: relax all $E$ edges.

Pass 1

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

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Bellman-Ford algorithm demo

Repeat \( V \) times: relax all \( E \) edges.

2-3 successfully relaxed in pass 1, but not pass 0

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pass 1

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2
Bellman-Ford algorithm demo

Repeat $V$ times: relax all $E$ edges.

pass 1

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2
Bellman-Ford algorithm demo

Repeat $V$ times: relax all $E$ edges.

2-6 successfully relaxed in pass 0 and pass 1

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pass 1

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

26 successfully relaxed in pass 0 and pass 1
Repeat $V$ times: relax all $E$ edges.
Bellman-Ford algorithm demo

Repeat $V$ times: relax all $E$ edges.

**pass 1**

$0\to1$  $0\to4$  $0\to7$  $1\to2$  $1\to3$  $1\to7$  $2\to3$  $2\to6$  $3\to6$  $4\to5$  $4\to6$  $4\to7$  $5\to2$  $5\to6$  $7\to5$  $7\to2$

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Bellman-Ford algorithm demo

Repeat $V$ times: relax all $E$ edges.

pass 1

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Bellman-Ford algorithm demo

Repeat $V$ times: relax all $E$ edges.

pass 1

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Bellman-Ford algorithm demo

Repeat $V$ times: relax all $E$ edges.

### Bellman-Ford algorithm demo

![Graph with labeled edges and vertices](image)

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### Pass 1

0→1  0→4  0→7  1→2  1→3  1→7  2→3  2→6  3→6  4→5  4→6  4→7  5→2  5→6  7→5  7→2
Bellman-Ford algorithm demo

Repeat $V$ times: relax all $E$ edges.

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pass 1

0→1  0→4  0→7  1→2  1→3  1→7  2→3  2→6  3→6  4→5  4→6  4→7  5→2  5→6  7→5  7→2
Bellman-Ford algorithm demo

Repeat $V$ times: relax all $E$ edges.

pass 1

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2
Bellman-Ford algorithm demo

Repeat \( V \) times: relax all \( E \) edges.

<table>
<thead>
<tr>
<th>( v )</th>
<th>distTo[]</th>
<th>edgeTo[]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0</td>
<td>-</td>
</tr>
<tr>
<td>1</td>
<td>5.0</td>
<td>0→1</td>
</tr>
<tr>
<td>2</td>
<td>14.0</td>
<td>0→4</td>
</tr>
<tr>
<td>3</td>
<td>17.0</td>
<td>2→3</td>
</tr>
<tr>
<td>4</td>
<td>9.0</td>
<td>0→4</td>
</tr>
<tr>
<td>5</td>
<td>13.0</td>
<td>4→5</td>
</tr>
<tr>
<td>6</td>
<td>25.0</td>
<td>2→6</td>
</tr>
<tr>
<td>7</td>
<td>8.0</td>
<td>0→7</td>
</tr>
</tbody>
</table>

pass 1

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2
Bellman-Ford algorithm demo

Repeat $V$ times: relax all $E$ edges.

Pass 2, 3, 4, … (no further changes)
Bellman-Ford algorithm demo

Repeat $V$ times: relax all $E$ edges.

<table>
<thead>
<tr>
<th>v</th>
<th>distTo[]</th>
<th>edgeTo[]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0</td>
<td>-</td>
</tr>
<tr>
<td>1</td>
<td>5.0</td>
<td>0→1</td>
</tr>
<tr>
<td>2</td>
<td>14.0</td>
<td>5→2</td>
</tr>
<tr>
<td>3</td>
<td>17.0</td>
<td>2→3</td>
</tr>
<tr>
<td>4</td>
<td>9.0</td>
<td>0→4</td>
</tr>
<tr>
<td>5</td>
<td>13.0</td>
<td>4→5</td>
</tr>
<tr>
<td>6</td>
<td>25.0</td>
<td>2→6</td>
</tr>
<tr>
<td>7</td>
<td>8.0</td>
<td>0→7</td>
</tr>
</tbody>
</table>

shortest-paths tree from vertex $s$
Bellman-Ford algorithm visualization

passes
4
7
10
13
SPT
**Proposition.** Dynamic programming algorithm computes SPT in any edge-weighted digraph with no negative cycles in time proportional to $E \times V$.

**Pf idea.** After pass $i$, found shortest path containing at most $i$ edges.
Bellman-Ford algorithm: practical improvement

Observation. If \( \text{distTo}[v] \) does not change during pass \( i \), no need to relax any edge pointing from \( v \) in pass \( i + 1 \).

FIFO implementation. Maintain queue of vertices whose \( \text{distTo}[\cdot] \) changed.

be careful to keep at most one copy of each vertex on queue (why?)

Overall effect.
- The running time is still proportional to \( E \times V \) in worst case.
- But much faster than that in practice.
public class BellmanFordSP
{
    private double[] distTo;
    private DirectedEdge[] edgeTo;
    private boolean[] onQ;
    private Queue<Integer> queue;

    public BellmanFordSPT(EdgeWeightedDigraph G, int s)
    {
        distTo = new double[G.V()];
        edgeTo = new DirectedEdge[G.V()];
        onq    = new boolean[G.V()];
        queue  = new Queue<Integer>();

        for (int v = 0; v < V; v++)
            distTo[v] = Double.POSITIVE_INFINITY;
        distTo[s] = 0.0;
        queue.enqueue(s);
        while (!queue.isEmpty())
        {
            int v = queue.dequeue();
            onQ[v] = false;
            for (DirectedEdge e : G.adj(v))
                relax(e);
        }
    }
}

private void relax(DirectedEdge e)
{
    int v = e.from(), w = e.to();
    if (distTo[w] > distTo[v] + e.weight())
    {
        distTo[w] = distTo[v] + e.weight();
        edgeTo[w] = e;
        if (!onQ[w])
        {
            queue.enqueue(w);
            onQ[w] = true;
        }
    }
}
### Single source shortest-paths implementation: cost summary

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Restriction</th>
<th>Typical case</th>
<th>Worst case</th>
<th>Extra space</th>
</tr>
</thead>
<tbody>
<tr>
<td>Topological sort</td>
<td>no directed cycles</td>
<td>$E + V$</td>
<td>$E + V$</td>
<td>$V$</td>
</tr>
<tr>
<td>Dijkstra (binary heap)</td>
<td>no negative weights</td>
<td>$E \log V$</td>
<td>$E \log V$</td>
<td>$V$</td>
</tr>
<tr>
<td>Bellman-Ford</td>
<td>no negative cycles</td>
<td>$E$</td>
<td>$E$</td>
<td>$V$</td>
</tr>
<tr>
<td>Bellman-Ford (queue-based)</td>
<td>no negative cycles</td>
<td>$E + V$</td>
<td>$E$</td>
<td>$V$</td>
</tr>
</tbody>
</table>

**Remark 1.** Directed cycles make the problem harder.

**Remark 2.** Negative weights make the problem harder.

**Remark 3.** Negative cycles makes the problem intractable.
Finding a negative cycle

Negative cycle. Add two method to the API for $SP$.

```
boolean hasNegativeCycle()  // is there a negative cycle?
Iterable <DirectedEdge> negativeCycle()  // negative cycle reachable from s
```

digraph
4→5  0.35
5→4  -0.66
4→7  0.37
5→7  0.28
7→5  0.28
5→1  0.32
0→4  0.38
0→2  0.26
7→3  0.39
1→3  0.29
2→7  0.34
6→2  0.40
3→6  0.52
6→0  0.58
6→4  0.93

digraph
5→4→7→5
negative cycle  (-0.66 + 0.37 + 0.28)
0→4→7→5→4→7→5...→1→3→6
shortest path from 0 to 6

negative cycle  (-0.66 + 0.37 + 0.28)
5→4→7→5

**Finding a negative cycle**

**Observation.** If there is a negative cycle, Bellman-Ford gets stuck in loop, updating $\text{distTo}[]$ and $\text{edgeTo}[]$ entries of vertices in the cycle.

![Graph diagram]

**Proposition.** If any vertex $v$ is updated in phase $V$, there exists a negative cycle (and can trace back $\text{edgeTo}[v]$ entries to find it).

**In practice.** Check for negative cycles more frequently.
**Problem.** Given table of exchange rates, is there an arbitrage opportunity?

<table>
<thead>
<tr>
<th></th>
<th>USD</th>
<th>EUR</th>
<th>GBP</th>
<th>CHF</th>
<th>CAD</th>
</tr>
</thead>
<tbody>
<tr>
<td>USD</td>
<td>1</td>
<td>0.741</td>
<td>0.657</td>
<td>1.061</td>
<td>1.011</td>
</tr>
<tr>
<td>EUR</td>
<td>1.350</td>
<td>1</td>
<td>0.888</td>
<td>1.433</td>
<td>1.366</td>
</tr>
<tr>
<td>GBP</td>
<td>1.521</td>
<td>1.126</td>
<td>1</td>
<td>1.614</td>
<td>1.538</td>
</tr>
<tr>
<td>CHF</td>
<td>0.943</td>
<td>0.698</td>
<td>0.620</td>
<td>1</td>
<td>0.953</td>
</tr>
<tr>
<td>CAD</td>
<td>0.995</td>
<td>0.732</td>
<td>0.650</td>
<td>1.049</td>
<td>1</td>
</tr>
</tbody>
</table>

**Ex.** $1,000 \Rightarrow 741$ Euros $\Rightarrow 1,012.206$ Canadian dollars $\Rightarrow 1,007.14497$.

$1000 \times 0.741 \times 1.366 \times 0.995 = 1007.14497$
Currency exchange graph.

- Vertex = currency.
- Edge = transaction, with weight equal to exchange rate.
- Find a directed cycle whose product of edge weights is > 1.

Challenge. Express as a negative cycle detection problem.
Model as a negative cycle detection problem by taking logs.

- Let weight of edge $v \rightarrow w$ be $-\ln$ (exchange rate from currency $v$ to $w$).
- Multiplication turns to addition; $> 1$ turns to $< 0$.
- Find a directed cycle whose sum of edge weights is $< 0$ (negative cycle).

Remark. Fastest algorithm is extraordinarily valuable!
Shortest-paths summary

Dijkstra’s algorithm.
• Nearly linear-time when weights are nonnegative.
• Generalization encompasses DFS, BFS, and Prim.

Acyclic edge-weighted digraphs.
• Arise in applications.
• Faster than Dijkstra’s algorithm.
• Negative weights are no problem.

Negative weights and negative cycles.
• Arise in applications.
• If no negative cycles, can find shortest paths via Bellman-Ford.
• If negative cycles, can find one via Bellman-Ford.

Shortest-paths is a broadly useful problem-solving model.