BBM 202 - ALGORITHMS

HACETTEPE UNIVERSITY

DEPT. OF COMPUTER ENGINEERING

ERKUT ERDEM

QUICKSORT

Mar. 4, 2014

Acknowledgement: The course slides are adapted from the slides prepared by R. Sedgewick and K. Wayne of Princeton University.

Quicksort

Basic plan.

- Shuffle the array.
- Partition so that, for some `j`
  - entry `a[j]` is in place
  - no larger entry to the left of `j`
  - no smaller entry to the right of `j`
- Sort each piece recursively.

Sir Charles Antony Richard Hoare
1980 Turing Award

input	Q	U	I	C	K	S	O	R	T	E	X	A	M	P	L	E
shuffle	K	R	A	T	E	L	E	P	U	I	M	Q	C	X	O	S
partition	E	C	A	I	E	K	L	P	U	T	M	Q	R	X	O	S
sort left	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
sort right	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X
result	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X

partitioning item

not greater *not less*

Quicksort partitioning

Repeat until `i` and `j` pointers cross.

- Scan `i` from left to right so long as `a[i] < a[lo]`.
- Scan `j` from right to left so long as `a[j] > a[lo]`.
- Exchange `a[i]` with `a[j]`.

K	R	A	T	E	L	E	P	U	I	M	Q	C	X	O	S
lo	i														j

stop `i` scan because `a[i] >= a[lo]`

Quicksort partitioning

Repeat until `i` and `j` pointers cross.

- Scan `i` from left to right so long as `a[i] < a[lo]`.
- Scan `j` from right to left so long as `a[j] > a[lo]`.
- Exchange `a[i]` with `a[j]`.

K	R	A	T	E	L	E	P	U	I	M	Q	C	X	O	S
lo	i														j

Quicksort partitioning

Repeat until `i` and `j` pointers cross.

- Scan `i` from left to right so long as `a[i] < a[lo]`.
- Scan `j` from right to left so long as `a[j] > a[lo]`.
- Exchange `a[i]` with `a[j]`.

K	R	A	T	E	L	E	P	U	I	M	Q	C	X	O	S
lo	i												j		

Quicksort partitioning

Repeat until `i` and `j` pointers cross.

- Scan `i` from left to right so long as `a[i] < a[lo]`.
- Scan `j` from right to left so long as `a[j] > a[lo]`.
- Exchange `a[i]` with `a[j]`.

K	R	A	T	E	L	E	P	U	I	M	Q	C	X	O	S
lo	i											j			

stop `j` scan and exchange `a[i]` with `a[j]`

Quicksort partitioning

Repeat until `i` and `j` pointers cross.

- Scan `i` from left to right so long as `a[i] < a[lo]`.
- Scan `j` from right to left so long as `a[j] > a[lo]`.
- Exchange `a[i]` with `a[j]`.

K	C	A	T	E	L	E	P	U	I	M	Q	R	X	O	S
lo	i											j			

Quicksort partitioning

Repeat until `i` and `j` pointers cross.

- Scan `i` from left to right so long as `a[i] < a[lo]`.
- Scan `j` from right to left so long as `a[j] > a[lo]`.
- Exchange `a[i]` with `a[j]`.

K	C	A	T	E	L	E	P	U	I	M	Q	R	X	O	S
lo		i										j			

Quicksort partitioning

Repeat until `i` and `j` pointers cross.

- Scan `i` from left to right so long as `a[i] < a[lo]`.
- Scan `j` from right to left so long as `a[j] > a[lo]`.
- Exchange `a[i]` with `a[j]`.

K	C	A	T	E	L	E	P	U	I	M	Q	R	X	O	S
↑ lo			↑ i									↑ j			

stop `i` scan because `a[i] >= a[lo]`

Quicksort partitioning

Repeat until `i` and `j` pointers cross.

- Scan `i` from left to right so long as `a[i] < a[lo]`.
- Scan `j` from right to left so long as `a[j] > a[lo]`.
- Exchange `a[i]` with `a[j]`.

K	C	A	T	E	L	E	P	U	I	M	Q	R	X	O	S
↑ lo			↑ i								↑ j				

Quicksort partitioning

Repeat until `i` and `j` pointers cross.

- Scan `i` from left to right so long as `a[i] < a[lo]`.
- Scan `j` from right to left so long as `a[j] > a[lo]`.
- Exchange `a[i]` with `a[j]`.

K	C	A	T	E	L	E	P	U	I	M	Q	R	X	O	S
↑ lo			↑ i							↑ j					

Quicksort partitioning

Repeat until `i` and `j` pointers cross.

- Scan `i` from left to right so long as `a[i] < a[lo]`.
- Scan `j` from right to left so long as `a[j] > a[lo]`.
- Exchange `a[i]` with `a[j]`.

K	C	A	T	E	L	E	P	U	I	M	Q	R	X	O	S
↑ lo			↑ i						↑ j						

stop `j` scan and exchange `a[i]` with `a[j]`

Quicksort partitioning

Repeat until `i` and `j` pointers cross.

- Scan `i` from left to right so long as `a[i] < a[lo]`.
- Scan `j` from right to left so long as `a[j] > a[lo]`.
- Exchange `a[i]` with `a[j]`.

K	C	A	I	E	L	E	P	U	T	M	Q	R	X	O	S
↑ lo			↑ i						↑ j						

Quicksort partitioning

Repeat until `i` and `j` pointers cross.

- Scan `i` from left to right so long as `a[i] < a[lo]`.
- Scan `j` from right to left so long as `a[j] > a[lo]`.
- Exchange `a[i]` with `a[j]`.

K	C	A	I	E	L	E	P	U	T	M	Q	R	X	O	S
↑ lo				↑ i					↑ j						

Quicksort partitioning

Repeat until `i` and `j` pointers cross.

- Scan `i` from left to right so long as `a[i] < a[lo]`.
- Scan `j` from right to left so long as `a[j] > a[lo]`.
- Exchange `a[i]` with `a[j]`.

K	C	A	I	E	L	E	P	U	T	M	Q	R	X	O	S
↑ lo					↑ i				↑ j						

stop `i` scan because `a[i] >= a[lo]`

Quicksort partitioning

Repeat until `i` and `j` pointers cross.

- Scan `i` from left to right so long as `a[i] < a[lo]`.
- Scan `j` from right to left so long as `a[j] > a[lo]`.
- Exchange `a[i]` with `a[j]`.

K	C	A	I	E	L	E	P	U	T	M	Q	R	X	O	S
↑ lo					↑ i			↑ j							

Quicksort partitioning

Repeat until `i` and `j` pointers cross.

- Scan `i` from left to right so long as `a[i] < a[lo]`.
- Scan `j` from right to left so long as `a[j] > a[lo]`.
- Exchange `a[i]` with `a[j]`.

K	C	A	I	E	L	E	P	U	T	M	Q	R	X	O	S
lo					i		j								

Quicksort partitioning

Repeat until `i` and `j` pointers cross.

- Scan `i` from left to right so long as `a[i] < a[lo]`.
- Scan `j` from right to left so long as `a[j] > a[lo]`.
- Exchange `a[i]` with `a[j]`.

K	C	A	I	E	L	E	P	U	T	M	Q	R	X	O	S
lo					i	j									

stop `j` scan and exchange `a[i]` with `a[j]`

Quicksort partitioning

Repeat until `i` and `j` pointers cross.

- Scan `i` from left to right so long as `a[i] < a[lo]`.
- Scan `j` from right to left so long as `a[j] > a[lo]`.
- Exchange `a[i]` with `a[j]`.

K	C	A	I	E	E	L	P	U	T	M	Q	R	X	O	S
lo					i	j									

Quicksort partitioning

Repeat until `i` and `j` pointers cross.

- Scan `i` from left to right so long as `a[i] < a[lo]`.
- Scan `j` from right to left so long as `a[j] > a[lo]`.
- Exchange `a[i]` with `a[j]`.

K	C	A	I	E	E	L	P	U	T	M	Q	R	X	O	S
lo						i j									

stop `i` scan because `a[i] >= a[lo]`

Quicksort partitioning

Repeat until `i` and `j` pointers cross.

- Scan `i` from left to right so long as `a[i] < a[lo]`.
- Scan `j` from right to left so long as `a[j] > a[lo]`.
- Exchange `a[i]` with `a[j]`.

K	C	A	I	E	E	L	P	U	T	M	Q	R	X	O	S
↑ lo					↑ j	↑ i									

stop `j` scan because `a[j] <= a[lo]`

Quicksort partitioning

Repeat until `i` and `j` pointers cross.

- Scan `i` from left to right so long as `a[i] < a[lo]`.
- Scan `j` from right to left so long as `a[j] > a[lo]`.
- Exchange `a[i]` with `a[j]`.

When pointers cross.

- Exchange `a[lo]` with `a[j]`.

K	C	A	I	E	E	L	P	U	T	M	Q	R	X	O	S
↑ lo					↑ j	↑ i									

pointers cross: exchange `a[lo]` with `a[j]`

Quicksort partitioning

Repeat until `i` and `j` pointers cross.

- Scan `i` from left to right so long as `a[i] < a[lo]`.
- Scan `j` from right to left so long as `a[j] > a[lo]`.
- Exchange `a[i]` with `a[j]`.

When pointers cross.

- Exchange `a[lo]` with `a[j]`.

E	C	A	I	E	K	L	P	U	T	M	Q	R	X	O	S
↑ lo					↑ j										↑ hi

partitioned!

Quicksort partitioning

Basic plan.

- Scan `i` from left for an item that belongs on the right.
- Scan `j` from right for an item that belongs on the left.
- Exchange `a[i]` and `a[j]`.
- Repeat until pointers cross.

	i	j	v	a[i] 0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
initial values	0	16		K	R	A	T	E	L	E	P	U	I	M	Q	C	X	O	S
scan left, scan right	1	12		K	R	A	T	E	L	E	P	U	I	M	Q	C	X	O	S
exchange	1	12		K	C	A	T	E	L	E	P	U	I	M	Q	R	X	O	S
scan left, scan right	3	9		K	C	A	T	E	L	E	P	U	I	M	Q	R	X	O	S
exchange	3	9		K	C	A	I	E	L	E	P	U	T	M	Q	R	X	O	S
scan left, scan right	5	6		K	C	A	I	E	L	E	P	U	T	M	Q	R	X	O	S
exchange	5	6		K	C	A	I	E	E	L	P	U	T	M	Q	R	X	O	S
scan left, scan right	6	5		K	C	A	I	E	E	L	P	U	T	M	Q	R	X	O	S
final exchange	6	5		E	C	A	I	E	K	L	P	U	T	M	Q	R	X	O	S
result	6	5		E	C	A	I	E	K	L	P	U	T	M	Q	R	X	O	S

Partitioning trace (array contents before and after each exchange)

Quicksort: Java code for partitioning

```
private static int partition(Comparable[] a, int lo, int hi)
{
   int i = lo, j = hi+1;
   while (true)
   {
      while (less(a[++i], a[lo]))                 find item on left to swap
         if (i == hi) break;

      while (less(a[lo], a[--j]))                 find item on right to swap
         if (j == lo) break;

      if (i >= j) break;                          check if pointers cross
      exch(a, i, j);                              swap
   }

   exch(a, lo, j);                                swap with partitioning item
   return j;                                      return index of item now known to be in place
}
```

Quicksort: Java implementation

```
public class Quick
{
   private static int partition(Comparable[] a, int lo, int hi)
   {  /* see previous slide */  }

   public static void sort(Comparable[] a)
   {
      StdRandom.shuffle(a);
      sort(a, 0, a.length - 1);
   }

   private static void sort(Comparable[] a, int lo, int hi)
   {
      if (hi <= lo) return;
      int j = partition(a, lo, hi);
      sort(a, lo, j-1);
      sort(a, j+1, hi);
   }
}
```

shuffle needed for performance guarantee (stay tuned)

Quicksort trace

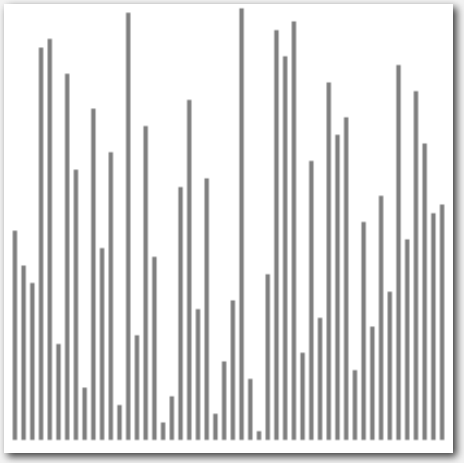
	lo	j	hi	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
initial values				Q	U	I	C	K	S	O	R	T	E	X	A	M	P	L	E
random shuffle				K	R	A	T	E	L	E	P	U	I	M	Q	C	X	O	S
	0	5	15	E	C	A	I	E	K	L	P	U	T	M	Q	R	X	O	S
	0	3	4	E	C	A	E	I	K	L	P	U	T	M	Q	R	X	O	S
	0	2	2	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
	0	0	1	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
	1		1	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
	4		4	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
	6	6	15	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
	7	9	15	A	C	E	E	I	K	L	M	O	P	T	Q	R	X	U	S
	7	7	8	A	C	E	E	I	K	L	M	O	P	T	Q	R	X	U	S
	8		8	A	C	E	E	I	K	L	M	O	P	T	Q	R	X	U	S
	10	13	15	A	C	E	E	I	K	L	M	O	P	S	Q	R	T	U	X
	10	12	12	A	C	E	E	I	K	L	M	O	P	R	Q	S	T	U	X
	10	11	11	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X
	10		10	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X
	14	14	15	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X
	15		15	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X
result				A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X

no partition for subarrays of size 1

Quicksort trace (array contents after each partition)

Quicksort animation

50 random items



http://www.sorting-algorithms.com/quick-sort

- algorithm position
- in order
- current subarray
- not in order

Quicksort: implementation details

Partitioning in-place. Using an extra array makes partitioning easier (and stable), but is not worth the cost.

Terminating the loop. Testing whether the pointers cross is a bit trickier than it might seem.

Staying in bounds. The `(j == lo)` test is redundant (why?), but the `(i == hi)` test is not.

Preserving randomness. Shuffling is needed for performance guarantee.

Equal keys. When duplicates are present, it is (counter-intuitively) better to stop on keys equal to the partitioning item's key.

Quicksort: empirical analysis

Running time estimates:

- Home PC executes 10^8 compares/second.
- Supercomputer executes 10^{12} compares/second.

	insertion sort (N			mergesort (N log N)			quicksort (N log N)		
computer	thousand	million	billion	thousand	million	billion	thousand	million	billion
home	instant	2.8 hours	317 years	instant	1 second	18 min	instant	0.6 sec	12 min
super	instant	1 second	1 week	instant	instant	instant	instant	instant	instant

Lesson 1. Good algorithms are better than supercomputers.
Lesson 2. Great algorithms are better than good ones.

Quicksort: best-case analysis

Best case. Number of compares is $\sim N \lg N$.

			a[ ]														
lo	j	hi	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
initial values			H	A	C	B	F	E	G	D	L	I	K	J	N	M	O
random shuffle			H	A	C	B	F	E	G	D	L	I	K	J	N	M	O
0	7	14	D	A	C	B	F	E	G	H	L	I	K	J	N	M	O
0	3	6	B	A	C	D	F	E	G	H	L	I	K	J	N	M	O
0	1	2	A	B	C	D	F	E	G	H	L	I	K	J	N	M	O
0		0	A	B	C	D	F	E	G	H	L	I	K	J	N	M	O
2		2	A	B	C	D	F	E	G	H	L	I	K	J	N	M	O
4	5	6	A	B	C	D	E	F	G	H	L	I	K	J	N	M	O
4		4	A	B	C	D	E	F	G	H	L	I	K	J	N	M	O
6		6	A	B	C	D	E	F	G	H	L	I	K	J	N	M	O
8	11	14	A	B	C	D	E	F	G	H	J	I	K	L	N	M	O
8	9	10	A	B	C	D	E	F	G	H	I	J	K	L	N	M	O
8		8	A	B	C	D	E	F	G	H	I	J	K	L	N	M	O
10		10	A	B	C	D	E	F	G	H	I	J	K	L	N	M	O
12	13	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
12		12	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
14		14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
			A	B	C	D	E	F	G	H	I	J	K	L	M	N	O

Quicksort: worst-case analysis

Worst case. Number of compares is $\sim \frac{1}{2} N^2$.

			a[ ]														
lo	j	hi	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
initial values			A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
random shuffle			A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
0	0	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
1	1	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
2	2	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
3	3	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
4	4	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
5	5	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
6	6	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
7	7	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
8	8	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
9	9	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
10	10	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
11	11	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
12	12	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
13	13	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
14		14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
			A	B	C	D	E	F	G	H	I	J	K	L	M	N	O

Quicksort: average-case analysis

Proposition. The average number of compares C_N to quicksort an array of N distinct keys is $\sim 2N \ln N$ (and the number of exchanges is $\sim \frac{1}{3} N \ln N$).

Pf 1. C_N satisfies the recurrence $C_0 = C_1 = 0$ and for $N \geq 2$:

$$C_N = \underbrace{(N+1)}_{\text{partitioning}} + \left(\frac{C_0 + C_{N-1}}{N}\right) + \left(\frac{\overset{\text{left}}{C_1} + \overset{\text{right}}{C_{N-2}}}{N}\right) + \ldots + \left(\frac{C_{N-1} + C_0}{N}\right)$$

partitioning probability

- Multiply both sides by N and collect terms:

$$NC_N = N(N+1) + 2(C_0 + C_1 + \ldots + C_{N-1})$$

- Subtract this from the same equation for $N - 1$:

$$NC_N - (N-1)C_{N-1} = 2N + 2C_{N-1}$$

- Rearrange terms and divide by $N(N+1)$:

$$\frac{C_N}{N+1} = \frac{C_{N-1}}{N} + \frac{2}{N+1}$$

Quicksort: average-case analysis

- Repeatedly apply above equation:

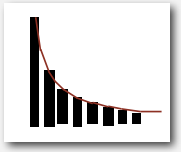
$$\begin{aligned}
\frac{C_N}{N+1} &= \frac{C_{N-1}}{N} + \frac{2}{N+1} \\
&= \frac{C_{N-2}}{N-1} + \frac{2}{N} + \frac{2}{N+1} \\
&= \frac{C_{N-3}}{N-2} + \frac{2}{N-1} + \frac{2}{N} + \frac{2}{N+1} \\
&= \frac{2}{3} + \frac{2}{4} + \frac{2}{5} + \ldots + \frac{2}{N+1}
\end{aligned}$$

previous equation

substitute previous equation

- Approximate sum by an integral:

$$\begin{aligned}
C_N &= 2(N+1)\left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots \frac{1}{N+1}\right) \\
&\sim 2(N+1)\int_3^{N+1} \frac{1}{x}\,dx
\end{aligned}$$



- Finally, the desired result:

$$C_N \sim 2(N+1)\ln N \approx 1.39 N \lg N$$

Quicksort: summary of performance characteristics

Worst case. Number of compares is quadratic.

- $N + (N-1) + (N-2) + \ldots + 1 \sim \frac{1}{2} N^2$.
- More likely that your computer is struck by lightning bolt.

Average case. Number of compares is $\sim 1.39\, N \lg N$.

- 39% more compares than mergesort.
- But faster than mergesort in practice because of less data movement.

Random shuffle.

- Probabilistic guarantee against worst case.
- Basis for math model that can be validated with experiments.

Caveat emptor. Many textbook implementations go quadratic if array

- Is sorted or reverse sorted.
- Has many duplicates (even if randomized!)

Quicksort: practical improvements

Insertion sort small subarrays.

- Even quicksort has too much overhead for tiny subarrays.
- Cutoff to insertion sort for ≈ 10 items.
- Note: could delay insertion sort until one pass at end.

```
private static void sort(Comparable[] a, int lo, int hi)
{
   if (hi <= lo + CUTOFF - 1)
   {
      Insertion.sort(a, lo, hi);
      return;
   }
   int j = partition(a, lo, hi);
   sort(a, lo, j-1);
   sort(a, j+1, hi);
}
```

Quicksort: practical improvements

Median of sample.

- Best choice of pivot item = median.
- Estimate true median by taking median of sample.
- Median-of-3 (random) items.

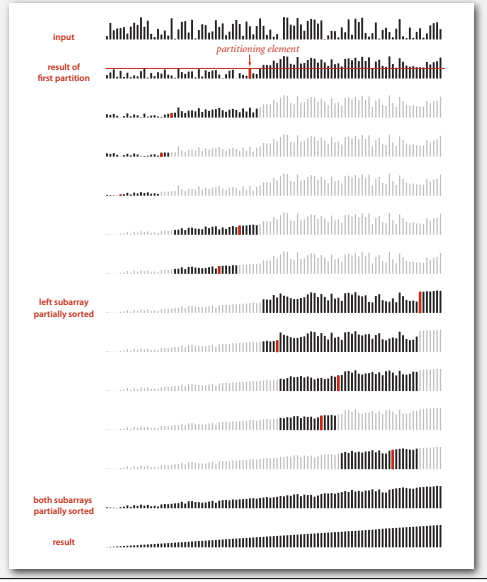
~ 12/7 N ln N compares (slightly fewer)
~ 12/35 N ln N exchanges (slightly more)

```
private static void sort(Comparable[] a, int lo, int hi)
{
   if (hi <= lo) return;

   int m = medianOf3(a, lo, lo + (hi - lo)/2, hi);
   swap(a, lo, m);

   int j = partition(a, lo, hi);
   sort(a, lo, j-1);
   sort(a, j+1, hi);
}
```


Quicksort with median-of-3 and cutoff to insertion sort: visualization



Duplicate keys

Often, purpose of sort is to bring items with equal keys together.

- Sort population by age.
- Find collinear points.
- Remove duplicates from mailing list.
- Sort job applicants by college attended.

Typical characteristics of such applications.

- Huge array.
- Small number of key values.

```
Chicago 09:25:52
Chicago 09:03:13
Chicago 09:21:05
Chicago 09:19:46
Chicago 09:19:32
Chicago 09:00:00
Chicago 09:35:21
Chicago 09:00:59
Houston 09:01:10
Houston 09:00:13
Phoenix 09:37:44
Phoenix 09:00:03
Phoenix 09:14:25
Seattle 09:10:25
Seattle 09:36:14
Seattle 09:22:43
Seattle 09:10:11
Seattle 09:22:54
```

key

Duplicate keys

Mergesort with duplicate keys.

Always between ½ $N \lg N$ and $N \lg N$ compares.

Quicksort with duplicate keys.

- Algorithm goes quadratic unless partitioning stops on equal keys!
- 1990s C user found this defect in `qsort()`.

several textbook and system implementation also have this defect

S T O P O N E Q U A L K E Y S

swap

if we don't stop on equal keys

if we stop on equal keys

Duplicate keys: the problem

Mistake. Put all items equal to the partitioning item on one side.
Consequence. $\sim \frac{1}{2} N^2$ compares when all keys equal.

B A A B A B B B C C C A A A A A A A A A A A

Recommended. Stop scans on items equal to the partitioning item.
Consequence. $\sim N \lg N$ compares when all keys equal.

B A A B A B C C B C B A A A A A A A A A A A

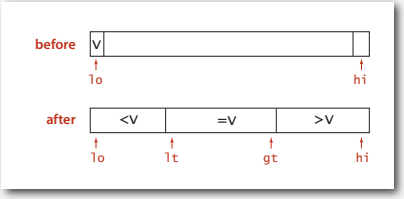
Desirable. Put all items equal to the partitioning item in place.

A A A B B B B B C C C A A A A A A A A A A A

3-way partitioning

Goal. Partition array into 3 parts so that:

- Entries between `lt` and `gt` equal to partition item `v`.
- No larger entries to left of `lt`.
- No smaller entries to right of `gt`.



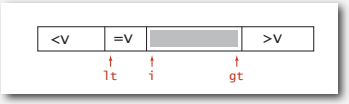
Dutch national flag problem. [Edsger Dijkstra]

- Conventional wisdom until mid 1990s: not worth doing.
- New approach discovered when fixing mistake in C library `qsort()`.
- Now incorporated into `qsort()` and Java system sort.

Dijkstra 3-way partitioning

- Let `v` be partitioning item `a[lo]`.
- Scan `i` from left to right.
  - `(a[i] < v)`: exchange `a[lt]` with `a[i]` and increment both `lt` and `i`
  - `(a[i] > v)`: exchange `a[gt]` with `a[i]` and decrement `gt`
  - `(a[i] == v)`: increment `i`

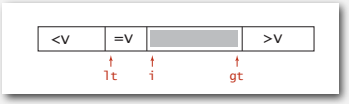
invariant



Dijkstra 3-way partitioning

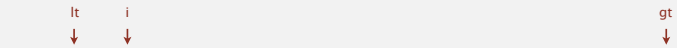
- Let `v` be partitioning item `a[lo]`.
- Scan `i` from left to right.
  - `(a[i] < v)`: exchange `a[lt]` with `a[i]` and increment both `lt` and `i`
  - `(a[i] > v)`: exchange `a[gt]` with `a[i]` and decrement `gt`
  - `(a[i] == v)`: increment `i`

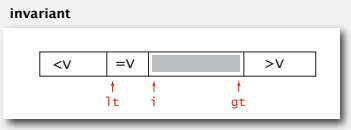
invariant



Dijkstra 3-way partitioning

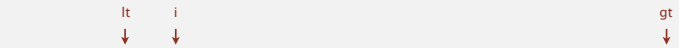
- Let `v` be partitioning item `a[lo]`.
- Scan `i` from left to right.
  - `(a[i] < v)`: exchange `a[lt]` with `a[i]` and increment both `lt` and `i`
  - `(a[i] > v)`: exchange `a[gt]` with `a[i]` and decrement `gt`
  - `(a[i] == v)`: increment `i`

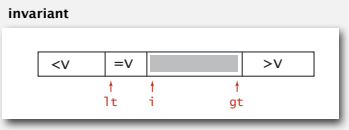




Dijkstra 3-way partitioning

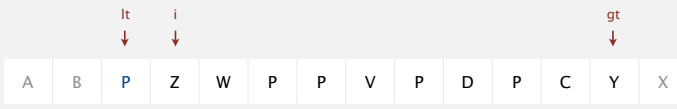
- Let `v` be partitioning item `a[lo]`.
- Scan `i` from left to right.
  - `(a[i] < v)`: exchange `a[lt]` with `a[i]` and increment both `lt` and `i`
  - `(a[i] > v)`: exchange `a[gt]` with `a[i]` and decrement `gt`
  - `(a[i] == v)`: increment `i`

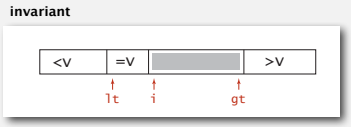




Dijkstra 3-way partitioning

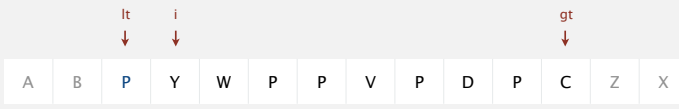
- Let `v` be partitioning item `a[lo]`.
- Scan `i` from left to right.
  - `(a[i] < v)`: exchange `a[lt]` with `a[i]` and increment both `lt` and `i`
  - `(a[i] > v)`: exchange `a[gt]` with `a[i]` and decrement `gt`
  - `(a[i] == v)`: increment `i`

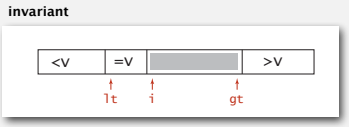




Dijkstra 3-way partitioning

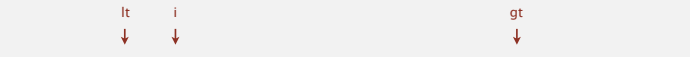
- Let `v` be partitioning item `a[lo]`.
- Scan `i` from left to right.
  - `(a[i] < v)`: exchange `a[lt]` with `a[i]` and increment both `lt` and `i`
  - `(a[i] > v)`: exchange `a[gt]` with `a[i]` and decrement `gt`
  - `(a[i] == v)`: increment `i`

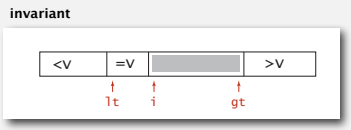




Dijkstra 3-way partitioning

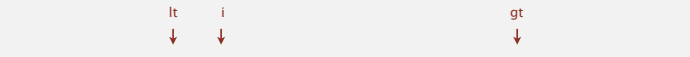
- Let `v` be partitioning item `a[lo]`.
- Scan `i` from left to right.
  - `(a[i] < v)`: exchange `a[lt]` with `a[i]` and increment both `lt` and `i`
  - `(a[i] > v)`: exchange `a[gt]` with `a[i]` and decrement `gt`
  - `(a[i] == v)`: increment `i`

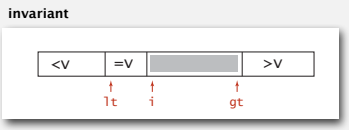




Dijkstra 3-way partitioning

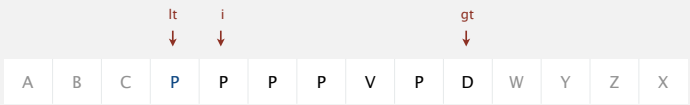
- Let `v` be partitioning item `a[lo]`.
- Scan `i` from left to right.
  - `(a[i] < v)`: exchange `a[lt]` with `a[i]` and increment both `lt` and `i`
  - `(a[i] > v)`: exchange `a[gt]` with `a[i]` and decrement `gt`
  - `(a[i] == v)`: increment `i`

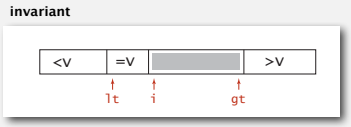




Dijkstra 3-way partitioning

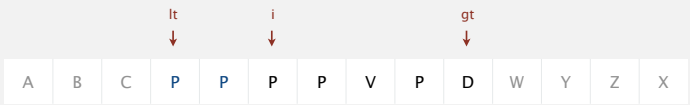
- Let `v` be partitioning item `a[lo]`.
- Scan `i` from left to right.
  - `(a[i] < v)`: exchange `a[lt]` with `a[i]` and increment both `lt` and `i`
  - `(a[i] > v)`: exchange `a[gt]` with `a[i]` and decrement `gt`
  - `(a[i] == v)`: increment `i`

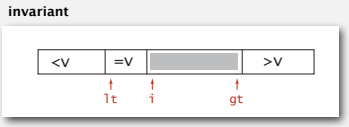




Dijkstra 3-way partitioning

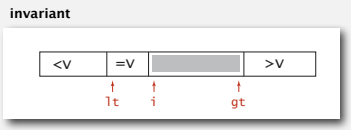
- Let `v` be partitioning item `a[lo]`.
- Scan `i` from left to right.
  - `(a[i] < v)`: exchange `a[lt]` with `a[i]` and increment both `lt` and `i`
  - `(a[i] > v)`: exchange `a[gt]` with `a[i]` and decrement `gt`
  - `(a[i] == v)`: increment `i`





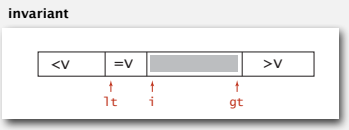
Dijkstra 3-way partitioning

- Let `v` be partitioning item `a[lo]`.
- Scan `i` from left to right.
  - `(a[i] < v)`: exchange `a[lt]` with `a[i]` and increment both `lt` and `i`
  - `(a[i] > v)`: exchange `a[gt]` with `a[i]` and decrement `gt`
  - `(a[i] == v)`: increment `i`



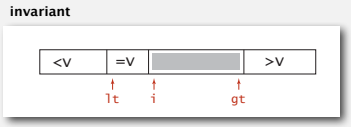
Dijkstra 3-way partitioning

- Let `v` be partitioning item `a[lo]`.
- Scan `i` from left to right.
  - `(a[i] < v)`: exchange `a[lt]` with `a[i]` and increment both `lt` and `i`
  - `(a[i] > v)`: exchange `a[gt]` with `a[i]` and decrement `gt`
  - `(a[i] == v)`: increment `i`



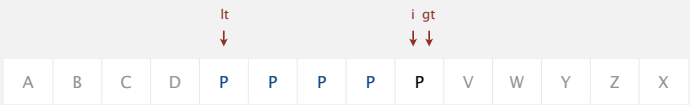
Dijkstra 3-way partitioning

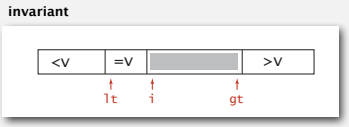
- Let `v` be partitioning item `a[lo]`.
- Scan `i` from left to right.
  - `(a[i] < v)`: exchange `a[lt]` with `a[i]` and increment both `lt` and `i`
  - `(a[i] > v)`: exchange `a[gt]` with `a[i]` and decrement `gt`
  - `(a[i] == v)`: increment `i`



Dijkstra 3-way partitioning

- Let `v` be partitioning item `a[lo]`.
- Scan `i` from left to right.
  - `(a[i] < v)`: exchange `a[lt]` with `a[i]` and increment both `lt` and `i`
  - `(a[i] > v)`: exchange `a[gt]` with `a[i]` and decrement `gt`
  - `(a[i] == v)`: increment `i`

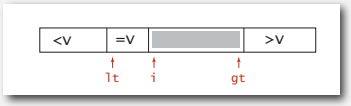




Dijkstra 3-way partitioning

- Let `v` be partitioning item `a[lo]`.
- Scan `i` from left to right.
  - `(a[i] < v)`: exchange `a[lt]` with `a[i]` and increment both `lt` and `i`
  - `(a[i] > v)`: exchange `a[gt]` with `a[i]` and decrement `gt`
  - `(a[i] == v)`: increment `i`

invariant



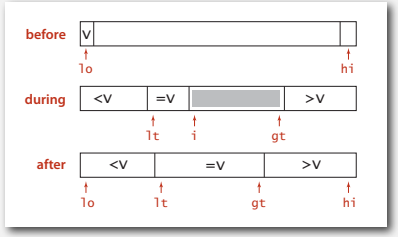
Dijkstra 3-way partitioning algorithm

3-way partitioning.

- Let `v` be partitioning item `a[lo]`.
- Scan `i` from left to right.
  - `a[i]` less than `v`: exchange `a[lt]` with `a[i]` and increment both `lt` and `i`
  - `a[i]` greater than `v`: exchange `a[gt]` with `a[i]` and decrement `gt`
  - `a[i]` equal to `v`: increment `i`

Most of the right properties.

- In-place.
- Not much code.
- Linear time if keys are all equal.



Dijkstra's 3-way partitioning: trace

a[]

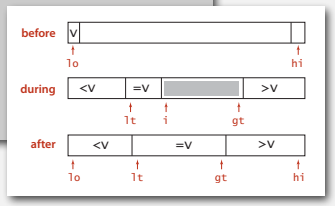
lt	i	gt	v	0	1	2	3	4	5	6	7	8	9	10	11
0	0	11		R	B	W	W	R	W	B	R	R	W	B	R
0	1	11		R	B	W	W	R	W	B	R	R	W	B	R
1	2	11		B	R	W	W	R	W	B	R	R	W	B	R
1	2	10		B	R	R	W	R	W	B	R	R	W	B	W
1	3	10		B	R	R	W	R	W	B	R	R	W	B	W
1	3	9		B	R	R	B	R	W	B	R	R	W	W	W
2	4	9		B	B	R	R	R	W	B	R	R	W	W	W
2	5	9		B	B	R	R	R	W	B	R	R	W	W	W
2	5	8		B	B	R	R	R	W	B	R	R	W	W	W
2	5	7		B	B	R	R	R	R	B	R	W	W	W	W
2	6	7		B	B	R	R	R	R	B	R	W	W	W	W
3	7	7		B	B	B	R	R	R	R	R	W	W	W	W
3	8	7		B	B	B	R	R	R	R	R	W	W	W	W
3	8	7		B	B	B	R	R	R	R	R	W	W	W	W

3-way partitioning trace (array contents after each loop iteration)

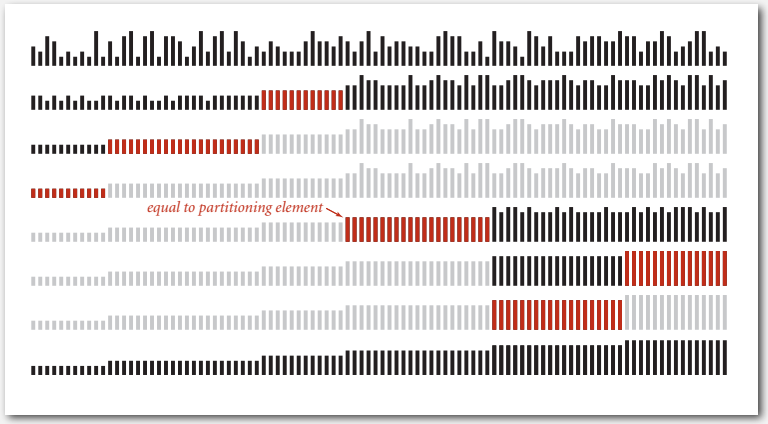
3-way quicksort: Java implementation

```
private static void sort(Comparable[] a, int lo, int hi)
{
   if (hi <= lo) return;
   int lt = lo, gt = hi;
   Comparable v = a[lo];
   int i = lo;
   while (i <= gt)
   {
      int cmp = a[i].compareTo(v);
      if      (cmp < 0) exch(a, lt++, i++);
      else if (cmp > 0) exch(a, i, gt--);
      else              i++;
   }

   sort(a, lo, lt - 1);
   sort(a, gt + 1, hi);
}
```



3-way quicksort: visual trace



Sorting summary

	inplace?	stable?	worst	average	best	remarks
selection	✔		N	N	N	*N* exchanges
insertion	✔	✔	N	N	N	use for small *N* or partially ordered
shell	✔		?	?	N	tight code, subquadratic
merge		✔	N lg N	N lg N	N lg N	*N* log *N* guarantee, stable
quick	✔		N	2 N ln N	N lg N	*N* log *N* probabilistic guarantee fastest in practice
3-way quick	✔		N	2 N ln N	N	improves quicksort in presence of duplicate keys
???	✔	✔	N lg N	N lg N	N lg N	holy sorting grail