Acknowledgement: The course slides are adapted from the slides prepared by R. Sedgewick and K. Wayne of Princeton University.
Balanced Search Trees

- 2-3 search trees
- Red-black BSTs
- B-trees
- Geometric applications of BSTs
<table>
<thead>
<tr>
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- **Challenge.** Guarantee performance.
BALANCED SEARCH TREES

- 2-3 search trees
- Red-black BSTs
- B-trees
- Geometric applications of BSTs
2-3 tree

Allow 1 or 2 keys per node.

- 2-node: one key, two children.
- 3-node: two keys, three children.
Allow 1 or 2 keys per node.

- 2-node: one key, two children.
- 3-node: two keys, three children.

**Perfect balance.** Every path from root to null link has same length.
Allow 1 or 2 keys per node.
- 2-node: one key, two children.
- 3-node: two keys, three children.

**Perfect balance.** Every path from root to null link has same length.

**Symmetric order.** Inorder traversal yields keys in ascending order.
2-3 tree demo

Search.
- Compare search key against keys in node.
- Find interval containing search key.
- Follow associated link (recursively).

search for H

H is less than M
(go left)
Search.
- Compare search key against keys in node.
- Find interval containing search key.
- Follow associated link (recursively).

**search for H**

H is between E and J (go middle)
Search.

- Compare search key against keys in node.
- Find interval containing search key.
- Follow associated link (recursively).

search for H

found H
(search hit)
Search.

- Compare search key against keys in node.
- Find interval containing search key.
- Follow associated link (recursively).

search for B

B is less than M
(go left)
2-3 tree demo

**Search.**
- Compare search key against keys in node.
- Find interval containing search key.
- Follow associated link (recursively).

**search for B**

```
B is less than E
(go left)
```
**Search.**
- Compare search key against keys in node.
- Find interval containing search key.
- Follow associated link (recursively).

*search for B*

- B is between A and C (go middle)
2-3 tree demo

Search.
- Compare search key against keys in node.
- Find interval containing search key.
- Follow associated link (recursively).

search for B

link is null
(search miss)
Insert into a 2-node at bottom.
• Search for key, as usual.
• Replace 2-node with 3-node.
2-3 tree demo

Insert into a 2-node at bottom.
- Search for key, as usual.
- Replace 2-node with 3-node.
2-3 tree demo

Insert into a 2-node at bottom.
- Search for key, as usual.
- Replace 2-node with 3-node.

insert K

search ends here
Insert into a 2-node at bottom.

- Search for key, as usual.
- Replace 2-node with 3-node.
Insert into a 2-node at bottom.

- Search for key, as usual.
- Replace 2-node with 3-node.

**insert K**
Insert into a 3-node at bottom.

- Add new key to 3-node to create temporary 4-node.
- Move middle key in 4-node into parent.

insert Z

Z is greater than M
(go right)
Insert into a 3-node at bottom.

- Add new key to 3-node to create temporary 4-node.
- Move middle key in 4-node into parent.

2-3 tree demo

insert Z
2-3 tree demo

Insert into a 3-node at bottom.
- Add new key to 3-node to create temporary 4-node.
- Move middle key in 4-node into parent.

insert Z

search ends here
2-3 tree demo

Insert into a 3-node at bottom.
- Add new key to 3-node to create temporary 4-node.
- Move middle key in 4-node into parent.

insert Z

replace 3-node with temporary 4-node containing Z
Insert into a 3-node at bottom.

- Add new key to 3-node to create temporary 4-node.
- Move middle key in 4-node into parent.
**2-3 tree demo**

Insert into a 3-node at bottom.
- Add new key to 3-node to create temporary 4-node.
- Move middle key in 4-node into parent.

**insert Z**

```
    M
   /|
  E  J  R
 /|
A  C  H
 /|
K  L  P
 /|
S  X  Z
```

split 4-node into two 2-nodes
(pass middle key to parent)
2-3 tree demo

Insert into a 3-node at bottom.
- Add new key to 3-node to create temporary 4-node.
- Move middle key in 4-node into parent.

[Diagram of a 2-3 tree with insertions and node labels]
2-3 tree demo

Insert into a 3-node at bottom.
• Add new key to 3-node to create temporary 4-node.
• Move middle key in 4-node into parent.

insert Z
2-3 tree demo

Insert into a 3-node at bottom.
- Add new key to 3-node to create temporary 4-node.
- Move middle key in 4-node into parent.
- Repeat up the tree, as necessary.
- If you reach the root and it's a 4-node, split it into three 2-nodes.

insert L

convert 3-node into 4-node
2-3 tree demo

Insert into a 3-node at bottom.
- Add new key to 3-node to create temporary 4-node.
- Move middle key in 4-node into parent.
- Repeat up the tree, as necessary.
- If you reach the root and it's a 4-node, split it into three 2-nodes.

insert L
Insert into a 3-node at bottom.

- Add new key to 3-node to create temporary 4-node.
- Move middle key in 4-node into parent.
- Repeat up the tree, as necessary.
- If you reach the root and it's a 4-node, split it into three 2-nodes.

\[ \text{insert L} \]
Insert into a 3-node at bottom.
- Add new key to 3-node to create temporary 4-node.
- Move middle key in 4-node into parent.
- Repeat up the tree, as necessary.
- If you reach the root and it's a 4-node, split it into three 2-nodes.

insert L
2-3 tree demo

Insert into a 3-node at bottom.
• Add new key to 3-node to create temporary 4-node.
• Move middle key in 4-node into parent.
• Repeat up the tree, as necessary.
• If you reach the root and it's a 4-node, split it into three 2-nodes.

insert L

split 4-node
(move L to parent)
2-3 tree demo

Insert into a 3-node at bottom.

- Add new key to 3-node to create temporary 4-node.
- Move middle key in 4-node into parent.
- Repeat up the tree, as necessary.
- If you reach the root and it's a 4-node, split it into three 2-nodes.

height of tree increases by 1

insert L
2-3 tree demo

Insert into a 3-node at bottom.
- Add new key to 3-node to create temporary 4-node.
- Move middle key in 4-node into parent.
- Repeat up the tree, as necessary.
- If you reach the root and it's a 4-node, split it into three 2-nodes.

insert L
Search in a 2-3 tree

- Compare search key against keys in node.
- Find interval containing search key.
- Follow associated link (recursively).

**Successful search for H**

1. H is less than M so look to the left

2. H is between E and L so look in the middle

3. Found H so return value (search hit)

**Unsuccessful search for B**

1. B is less than M so look to the left

2. B is less than E so look to the left

3. B is between A and C so look in the middle

4. Link is null so B is not in the tree (search miss)
Case 1. Insert into a 2-node at bottom.

- Search for key, as usual.
- Replace 2-node with 3-node.
**Case 2.** Insert into a 3-node at bottom.

- Add new key to 3-node to create temporary 4-node.
- Move middle key in 4-node into parent.
- Repeat up the tree, as necessary.
Case 2. Insert into a 3-node at bottom.

- Add new key to 3-node to create temporary 4-node.
- Move middle key in 4-node into parent.
- Repeat up the tree, as necessary.
- If you reach the root and it's a 4-node, split it into three 2-nodes.
Local transformations in a 2-3 tree

Splitting a 4-node is a local transformation: constant number of operations.
Invariants. Maintains symmetric order and perfect balance.
Pf. Each transformation maintains symmetric order and perfect balance.
Perfect balance. Every path from root to null link has same length.

Tree height.
- Worst case:
- Best case:
2-3 tree: performance

Perfect balance. Every path from root to null link has same length.

Tree height.
- Worst case: \( \lg N \). [all 2-nodes]
- Best case: \( \log_3 N \approx 0.631 \lg N \). [all 3-nodes]
- Between 12 and 20 for a million nodes.
- Between 18 and 30 for a billion nodes.

Guaranteed logarithmic performance for search and insert.
## ST implementations: summary

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Constants depend upon implementation.
2-3 tree: implementation?

Direct implementation is complicated, because:

- Maintaining multiple node types is cumbersome.
- Need multiple compares to move down tree.
- Need to move back up the tree to split 4-nodes.
- Large number of cases for splitting.

**Bottom line.** Could do it, but there's a better way.
Balanced Search Trees

- 2-3 search trees
- Red-black BSTs
- B-trees
- Geometric applications of BSTs
1. Represent 2–3 tree as a BST.
2. Use "internal" left-leaning links as "glue" for 3–nodes.

Left-leaning red-black BSTs (Guibas-Sedgewick 1979 and Sedgewick 2007)

- Larger key is root
- Red links "glue" nodes within a 3-node
- Black links connect 2-nodes and 3-nodes
- 2-3 tree
- Corresponding red-black BST
An equivalent definition

A BST such that:
- No node has two red links connected to it.
- Every path from root to null link has the same number of black links.
- Red links lean left.

"perfect black balance"
Key property. 1–1 correspondence between 2–3 and LLRB.
Search implementation for red-black BSTs

**Observation.** Search is the same as for elementary BST (ignore color).

but runs faster because of better balance

```java
public Val get(Key key) {
    Node x = root;
    while (x != null) {
        int cmp = key.compareTo(x.key);
        if (cmp < 0) x = x.left;
        else if (cmp > 0) x = x.right;
        else if (cmp == 0) return x.val;
    }
    return null;
}
```

**Remark.** Most other ops (e.g., ceiling, selection, iteration) are also identical.
Red-black BST representation

Each node is pointed to by precisely one link (from its parent) \Rightarrow can encode color of links in nodes.

```java
private static final boolean RED   = true;
private static final boolean BLACK = false;

private class Node
{
    Key key;
    Value val;
    Node left, right;
    boolean color;  // color of parent link
}

private boolean isRed(Node x)
{
    if (x == null) return false;
    return x.color == RED;
}
```

null links are black

h.left.color is RED
h.right.color is BLACK
Elementary red-black BST operations

**Left rotation.** Orient a (temporarily) right-leaning red link to lean left.

```
rotate E left (before)

```

```
private Node rotateLeft(Node h)
{
    assert isRed(h.right);
    Node x = h.right;
    h.right = x.left;
    x.left = h;
    x.color = h.color;
    h.color = RED;
    return x;
}
```

**Invariants.** Maintains symmetric order and perfect black balance.
**Left rotation.** Orient a (temporarily) right-leaning red link to lean left.

Elementary red-black BST operations

**Invariants.** Maintains symmetric order and perfect black balance.

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private Node rotateLeft(Node h) {
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    Node x = h.right;
    h.right = x.left;
    x.left = h;
    x.color = h.color;
    h.color = RED;
    return x;
}
```
Elementary red-black BST operations

Right rotation. Orient a left-leaning red link to (temporarily) lean right.

```
private Node rotateRight(Node h) {
    assert isRed(h.left);
    Node x = h.left;
    h.left = x.right;
    x.right = h;
    x.color = h.color;
    h.color = RED;
    return x;
}
```

Invariants. Maintains symmetric order and perfect black balance.
Elementary red-black BST operations

Right rotation. Orient a left-leaning red link to (temporarily) lean right.

\[
\text{rotate } S \text{ right} \\
\text{(after)}
\]

\[
\begin{align*}
\text{E} & \quad \text{x} \\
\text{less than E} & \quad \text{between E and S} \\
\text{greater than S} & \quad \text{h}
\end{align*}
\]

private Node rotateRight(Node h)
{
  assert isRed(h.left);
  Node x = h.left;
  h.left = x.right;
  x.right = h;
  x.color = h.color;
  h.color = RED;
  return x;
}

Invariants. Maintains symmetric order and perfect black balance.
Elementary red-black BST operations

**Color flip.** Recolor to split a (temporary) 4-node.

```
private void flipColors(Node h) {
    assert !isRed(h);
    assert isRed(h.left);
    assert isRed(h.right);
    h.color = RED;
    h.left.color = BLACK;
    h.right.color = BLACK;
}
```

**Invariants.** Maintains symmetric order and perfect black balance.
Elementary red-black BST operations

**Color flip.** Recolor to split a (temporary) 4-node.

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    h.color = RED;
    h.left.color = BLACK;
    h.right.color = BLACK;
}
```

**Invariants.** Maintains symmetric order and perfect black balance.
Basic strategy. Maintain 1-1 correspondence with 2-3 trees by applying elementary red-black BST operations.
Insertion in a LLRB tree

Warmup 1. Insert into a tree with exactly 1 node.
Case 1. Insert into a 2-node at the bottom.

• Do standard BST insert; color new link red.
• If new red link is a right link, rotate left.
Warmup 2. Insert into a tree with exactly 2 nodes.

Insertion in a LLRB tree
**Case 2.** Insert into a 3-node at the bottom.

- Do standard BST insert; color new link red.
- Rotate to balance the 4-node (if needed).
- Flip colors to pass red link up one level.
- Rotate to make lean left (if needed).
Case 2. Insert into a 3-node at the bottom.

- Do standard BST insert; color new link red.
- Rotate to balance the 4-node (if needed).
- Flip colors to pass red link up one level.
- Rotate to make lean left (if needed).
- Repeat case 1 or case 2 up the tree (if needed).
Red-black BST insertion

insert $S$
Red-black BST insertion

insert E
Red-black BST insertion

insert A
Red-black BST insertion

insert A

two left reds in a row
(rotate S right)
Red-black BST insertion

both children red
(flip colors)
Red-black BST insertion

both children red
(flip colors)
Red-black BST insertion

red–black BST
Red-black BST insertion

red-black BST
Red-black BST insertion

insert R

```
E
/   \
A    S
 \
R
```
Red-black BST insertion

red–black BST

![Red-black BST diagram](image)
Red-black BST insertion

red–black BST
Red-black BST insertion

insert C
Red-black BST insertion

right link red
(rotate A left)
Red-black BST insertion

red–black BST

![Red-black BST Diagram]
Red-black BST insertion

red–black BST
Red-black BST insertion

red-black BST
Red-black BST insertion

insert H
Red-black BST insertion

two left reds in a row
(rotate S right)
Red-black BST insertion

Both children red
(flip colors)
Red-black BST insertion

both children red
(flip colors)
Right link red
(rotate E left)
Red-black BST insertion

red-black BST
Red-black BST insertion
Red-black BST insertion

red–black BST

![Red-black BST Diagram](image-url)
Red-black BST insertion

insert X

```
    R
   / |`
  E  S
 / |  \
C  H  X
 / |  \
A  
```
Red-black BST insertion

insert X

right link red
(rotate S left)
Red-black BST insertion

red–black BST
Red-black BST insertion

red–black BST

[Diagram of a red-black BST]
Red-black BST insertion

red–black BST
Red-black BST insertion

insert M
Red-black BST insertion

insert M

right link red
(rotate H left)
Red-black BST insertion

red-black BST
Red-black BST insertion

insert P
Red-black BST insertion

insert P

two red children

(flip colors)
Red-black BST insertion

insert P

two red children (flip colors)
Red-black BST insertion

right link red
(rotate E left)
Red-black BST insertion

two left reds in a row
(rotate R right)
Red-black BST insertion

two red children
(flip colors)
Red-black BST insertion

two red children
(flip colors)
Red-black BST insertion

red–black BST
Red-black BST insertion

red–black BST
Red-black BST insertion

red–black BST

![Red-black BST Diagram](image-url)
Red-black BST insertion

insert L
Red-black BST insertion

insert L

right link red (rotate H left)
Red-black BST insertion

red-black BST
LLRB tree insertion trace

Standard indexing client.

red-black BST  

insert S

E

A

R

C

H

corresponding 2–3 tree
LLRB tree insertion trace

Standard indexing client (continued).

red–black BST
corresponding 2–3 tree
Same code for both cases.

- Right child red, left child black: rotate left.
- Left child, left-left grandchild red: rotate right.
- Both children red: flip colors.

```java
private Node put(Node h, Key key, Value val) {
    if (h == null) return new Node(key, val, RED);
    int cmp = key.compareTo(h.key);
    if      (cmp  < 0) h.left  = put(h.left,  key, val);
    else if (cmp  > 0) h.right = put(h.right, key, val);
    else if (cmp == 0) h.val = val;

    if (isRed(h.right) && !isRed(h.left))     h = rotateLeft(h);
    if (isRed(h.left)  && isRed(h.left.left)) h = rotateRight(h);
    if (isRed(h.left)  && isRed(h.right))     flipColors(h);
    return h;
}
```
Insertion in a LLRB tree: visualization

255 insertions in ascending order

N = 255
max = 8
avg = 7.0
opt = 7.0
Remark. Only a few extra lines of code to standard BST insert.

255 insertions in descending order
Remark. Only a few extra lines of code to standard BST insert.

255 random insertions
**Proposition.** Height of tree is $\leq 2 \lg N$ in the worst case.

**Pf.**
- Every path from root to null link has same number of black links.
- Never two red links in-a-row.

**Property.** Height of tree is $\sim 1.00 \lg N$ in typical applications.
### ST implementations: summary

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* exact value of coefficient unknown but extremely close to 1
War story: why red-black?

Xerox PARC innovations. [1970s]

- Alto.
- GUI.
- Ethernet.
- Smalltalk.
- InterPress.
- Laser printing.
- Bitmapped display.
- WYSIWYG text editor.
- ...

---

A DICHROMATIC FRAMEWORK FOR BALANCED TREES

Leo J. Guibas  Robert Sedgewick*
Xerox Palo Alto Research Center,  Program in Computer Science
Palo Alto, California, and  Brown University
Carnegie-Mellon University  Providence, R. I.

ABSTRACT

In this paper we present a uniform framework for the implementation and study of balanced tree algorithms. We show how to imbed in this the way down towards a leaf. As we will see, this has a number of significant advantages over the older methods. We shall examine a number of variations on a common theme and exhibit full implementations which are notable for their brevity. One implementation is examined carefully, and some properties about its
Balanced Search Trees

- 2-3 search trees
- Red-black BSTs
- B-trees
- Geometric applications of BSTs
File system model

**Page.** Contiguous block of data (e.g., a file or 4,096-byte chunk).

**Probe.** First access to a page (e.g., from disk to memory).

**Property.** Time required for a probe is much larger than time to access data within a page.

**Cost model.** Number of probes.

**Goal.** Access data using minimum number of probes.
B-trees (Bayer-McCreight, 1972)

B-tree. Generalize 2-3 trees by allowing up to $M - 1$ key-link pairs per node.

- At least 2 key-link pairs at root.
- At least $M / 2$ key-link pairs in other nodes.
- External nodes contain client keys.
- Internal nodes contain copies of keys to guide search.

choose $M$ as large as possible so that $M$ links fit in a page, e.g., $M = 1024$

Anatomy of a B-tree set ($M = 6$)
Searching in a B-tree

- Start at root.
- Find interval for search key and take corresponding link.
- Search terminates in external node.
• Search for new key.
• Insert at bottom.
• Split nodes with $M$ key-link pairs on the way up the tree.

Inserting a new key into a B-tree set
Balance in B-tree

**Proposition.** A search or an insertion in a B-tree of order $M$ with $N$ keys requires between $\log_{M-1} N$ and $\log_{M/2} N$ probes.

**Pf.** All internal nodes (besides root) have between $M/2$ and $M - 1$ links.

**In practice.** Number of probes is at most 4. \( M = 1024; N = \text{62 billion} \Rightarrow \log_{M/2} N \leq 4 \)

**Optimization.** Always keep root page in memory.
Building a large B tree

each line shows the result of inserting one key in some page

white: unoccupied portion of page
black: occupied portion of page
full page, about to split
full page splits into two half-full pages then a new key is added to one of them
Balanced trees in the wild

Red-black trees are widely used as system symbol tables.
- **Java**: `java.util.TreeMap`, `java.util.TreeSet`.
- **C++ STL**: `map`, `multimap`, `multiset`.
- **Linux kernel**: completely fair scheduler, `linux/rbtree.h`.

**B-tree variants.** B+ tree, B*tree, B# tree, …

**B-trees (and variants) are widely used for file systems and databases.**
- **Windows**: HPFS.
- **Mac**: HFS, HFS+.
- **Linux**: ReiserFS, XFS, Ext3FS, JFS.
- **Databases**: ORACLE, DB2, INGRES, SQL, PostgreSQL.
Balanced Search Trees

- 2-3 search trees
- Red-black BSTs
- B-trees
- Geometric applications of BSTs
GEOMETRIC APPLICATIONS OF BSTs

- kd trees
2-d orthogonal range search

Extension of ordered symbol-table to 2d keys.
• Insert a 2d key.
• Delete a 2d key.
• Search for a 2d key.
• **Range search**: find all keys that lie in a 2d range.
• **Range count**: number of keys that lie in a 2d range.

Geometric interpretation.
• Keys are point in the **plane**.
• Find/count points in a given **$h-v$ rectangle**.

**Applications.** Networking, circuit design, databases,...
2d orthogonal range search: grid implementation

Grid implementation.

- Divide space into $M$-by-$M$ grid of squares.
- Create list of points contained in each square.
- Use 2d array to directly index relevant square.
- Insert: add $(x, y)$ to list for corresponding square.
- Range search: examine only those squares that intersect 2d range query.
2d orthogonal range search: grid implementation costs

Space-time tradeoff.
• Space: $M^2 + N$.
• Time: $1 + N / M^2$ per square examined, on average.

Choose grid square size to tune performance.
• Too small: wastes space.
• Too large: too many points per square.
• Rule of thumb: $\sqrt{N}$-by-$\sqrt{N}$ grid.

Running time. [if points are evenly distributed]
• Initialize data structure: $N$.
• Insert point: 1.
• Range search: 1 per point in range.
Grid implementation. Fast and simple solution for evenly-distributed points.

Problem. Clustering a well-known phenomenon in geometric data.
- Lists are too long, even though average length is short.
- Need data structure that *gracefully* adapts to data.
Grid implementation. Fast and simple solution for evenly-distributed points.

Problem. Clustering a well-known phenomenon in geometric data. 
Ex. USA map data.

13,000 points, 1000 grid squares

- half the squares are empty
- half the points are in 10% of the squares
Use a tree to represent a recursive subdivision of 2d space.

**Grid.** Divide space uniformly into squares.

**2d tree.** Recursively divide space into two halfplanes.

**Quadtree.** Recursively divide space into four quadrants.

**BSP tree.** Recursively divide space into two regions.
Applications.

- Ray tracing.
- 2d range search.
- Flight simulators.
- N-body simulation.
- Collision detection.
- Astronomical databases.
- Nearest neighbor search.
- Adaptive mesh generation.
- Accelerate rendering in Doom.
- Hidden surface removal and shadow casting.

Space-partitioning trees: applications

- Grid
- 2d tree
- Quadtree
- BSP tree
**Kd tree**

Kd tree. Recursively partition $k$-dimensional space into 2 halfspaces.

**Implementation.** BST, but cycle through dimensions ala 2d trees.

Efficient, simple data structure for processing $k$-dimensional data.

- Widely used.
- Adapts well to high-dimensional and clustered data.
- Discovered by an undergrad in an algorithms class!
Goal. Simulate the motion of $N$ particles, mutually affected by gravity.

Brute force. For each pair of particles, compute force. $ F = \frac{G m_1 m_2}{r^2}$

http://www.youtube.com/watch?v=ua7YIN4eL_w
Appel algorithm for N-body simulation

**Key idea.** Suppose particle is far, far away from cluster of particles.

- Treat cluster of particles as a single aggregate particle.
- Compute force between particle and center of mass of aggregate particle.
Appel algorithm for N-body simulation

- Build 3d-tree with $N$ particles as nodes.
- Store center-of-mass of subtree in each node.
- To compute total force acting on a particle, traverse tree, but stop as soon as distance from particle to subdivision is sufficiently large.

**Impact.** Running time per step is $N \log N$ instead of $N^2 \Rightarrow$ enables new research.