BBM 202 - ALGORITHMS

HACETTEPE UNIVERSITY

DEPT. OF COMPUTER ENGINEERING

ERKUT ERDEM

BALANCED TREES

Mar. 19, 2015

Acknowledgement: The course slides are adapted from the slides prepared by R. Sedgewick and K. Wayne of Princeton University.

BALANCED SEARCH TREES

- 2-3 search trees
- Red-black BSTs
- B-trees
- Geometric applications of BSTs

Text

implementation	worst-case cost (after N inserts)			average case (after N random inserts)			ordered iteration?	key interface
	search	insert	delete	search hit	insert	delete		
sequential search (unordered list)	N	N	N	N/2	N	N/2	no	`equals()`
binary search (ordered array)	lg N	N	N	lg N	N/2	N/2	yes	`compareTo()`
BST	N	N	N	1.39 lg N	1.39 lg N	?	yes	`compareTo()`
goal	log N	log N	log N	log N	log N	log N	yes	`compareTo()`

- **Challenge.** Guarantee performance.

BALANCED SEARCH TREES

- 2-3 search trees
- Red-black BSTs
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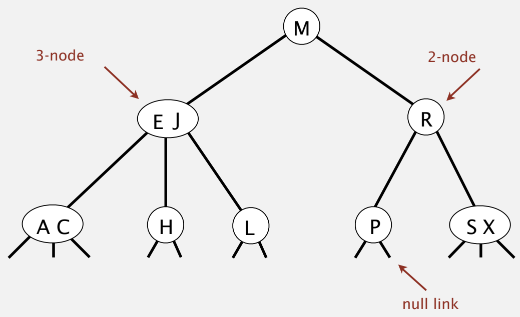
2-3 tree

Allow 1 or 2 keys per node.

- 2-node: one key, two children.
- 3-node: two keys, three children.



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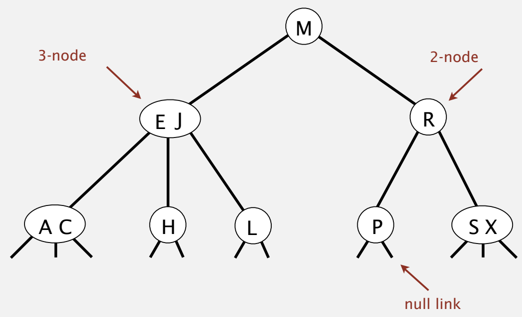

2-3 tree

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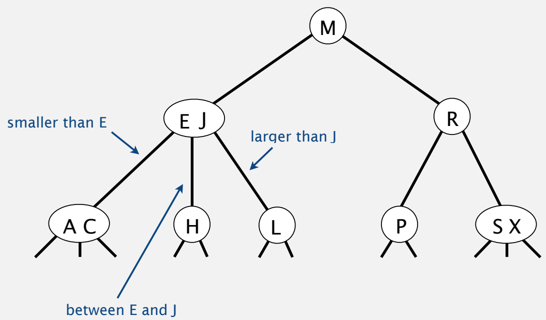
Perfect balance. Every path from root to null link has same length.
Symmetric order. Inorder traversal yields keys in ascending order.

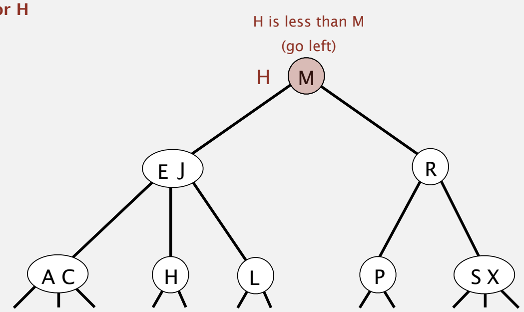


2-3 tree demo

Search.

- Compare search key against keys in node.
- Find interval containing search key.
- Follow associated link (recursively).

search for H



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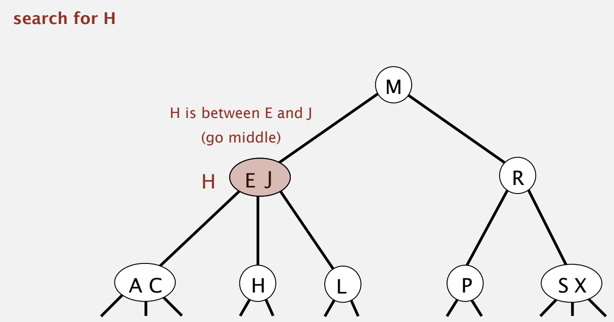

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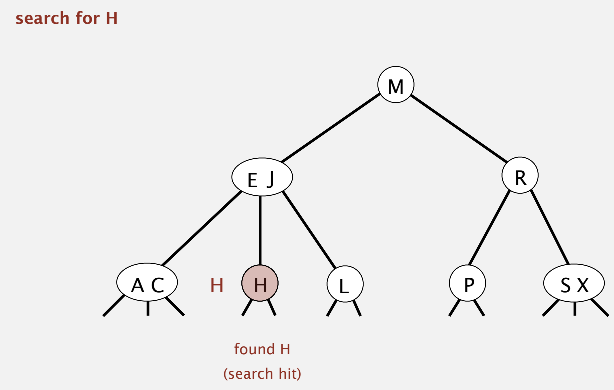

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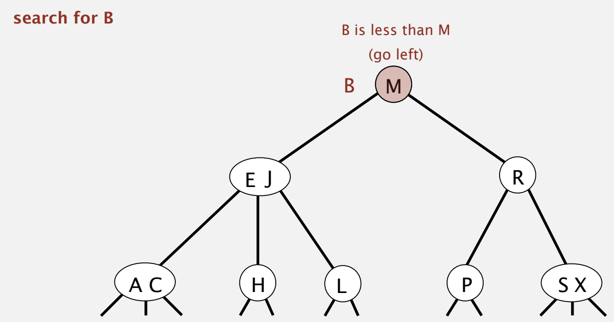

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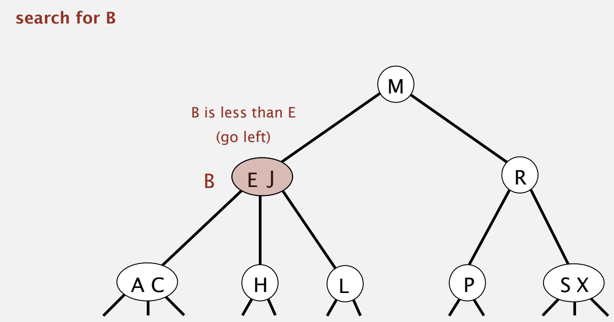

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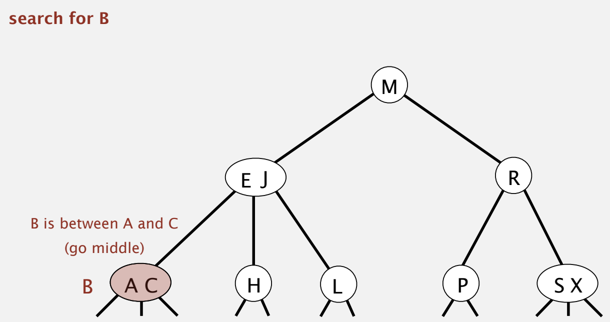

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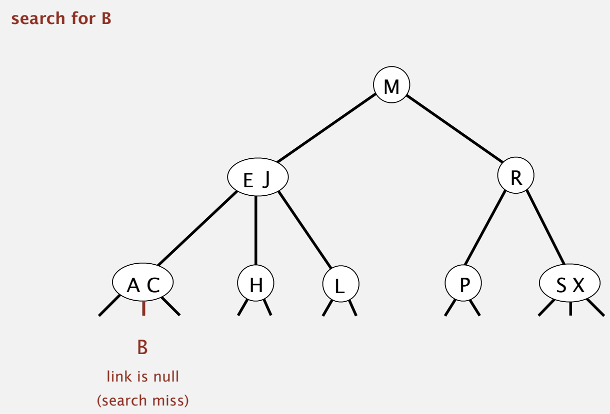
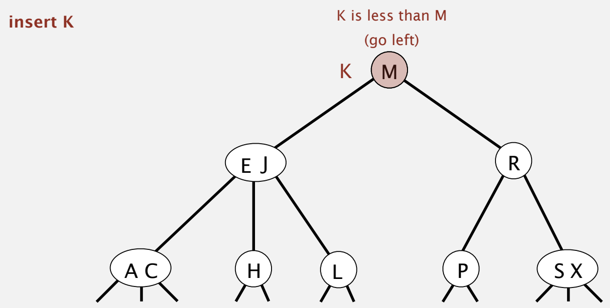


2-3 tree demo

Insert into a 2-node at bottom.

- Search for key, as usual.
- Replace 2-node with 3-node.



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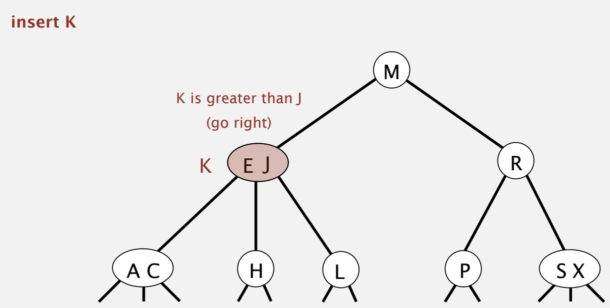

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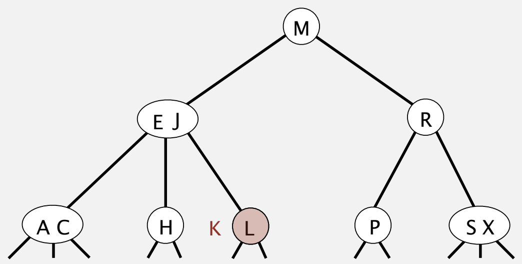
insert K



search ends here

2-3 tree demo

Insert into a 2-node at bottom.

- Search for key, as usual.
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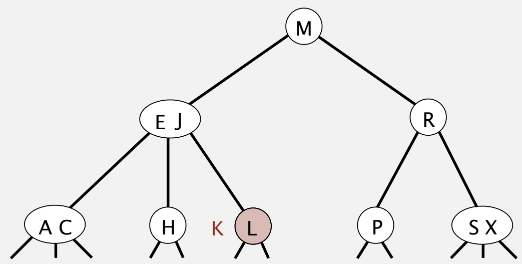
insert K



replace 2-node with
3-node containing K

2-3 tree demo

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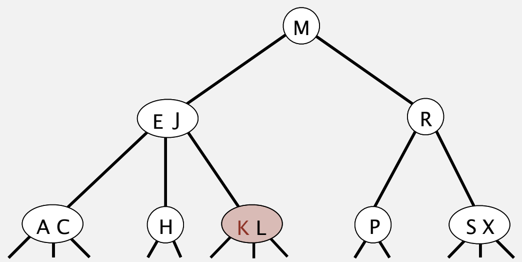
insert K



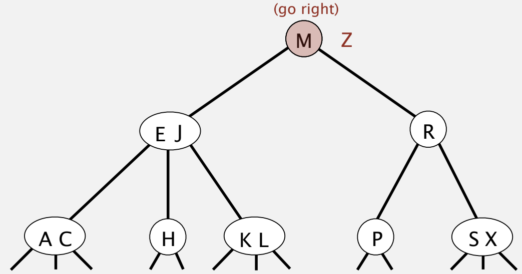
2-3 tree demo

Insert into a 3-node at bottom.

- Add new key to 3-node to create temporary 4-node.
- Move middle key in 4-node into parent.

insert Z

Z is greater than M
(go right)



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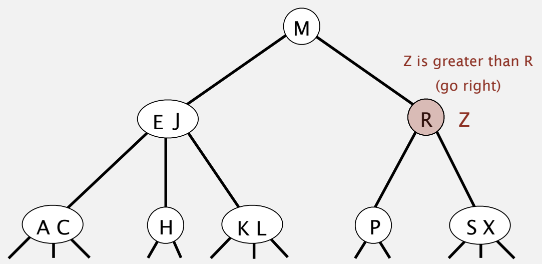

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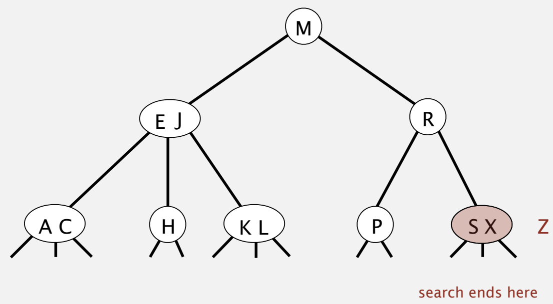

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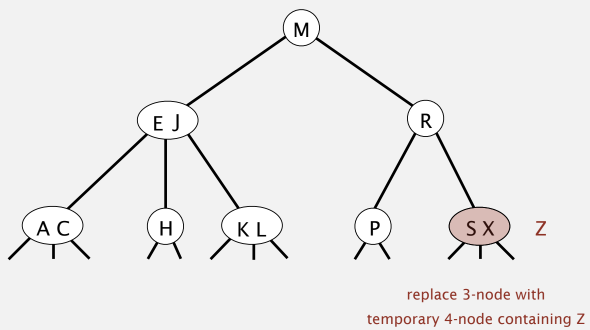

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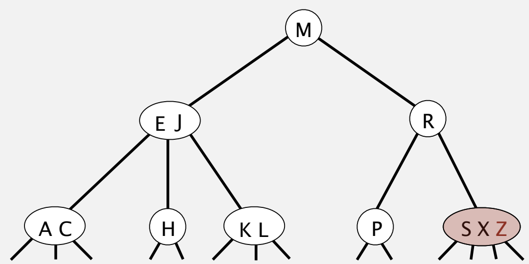

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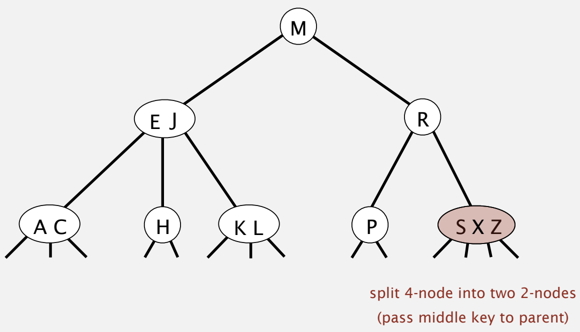

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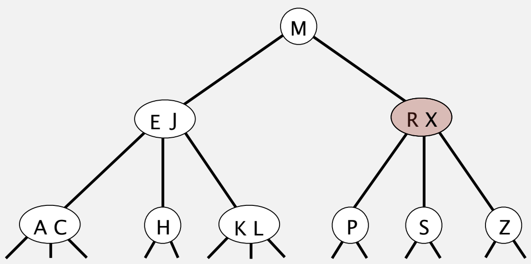

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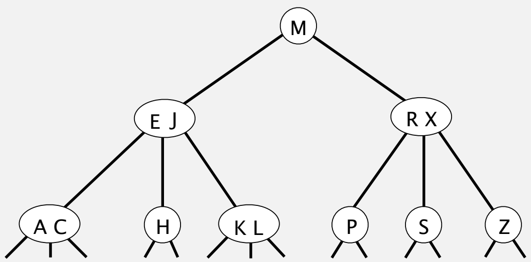

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Insert into a 3-node at bottom.

- Add new key to 3-node to create temporary 4-node.
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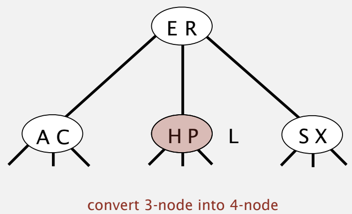
- Repeat up the tree, as necessary.
- If you reach the root and it's a 4-node, split it into three 2-nodes.

insert L



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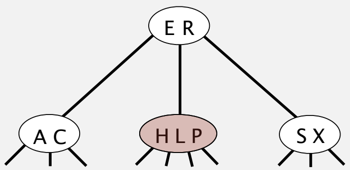

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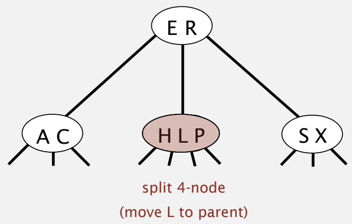

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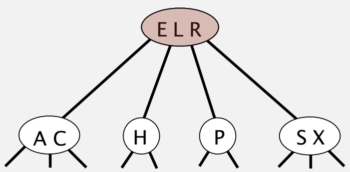

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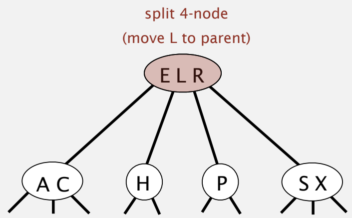

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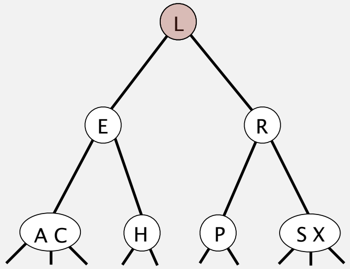
height of tree increases by 1

insert L



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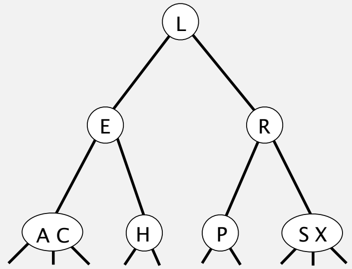
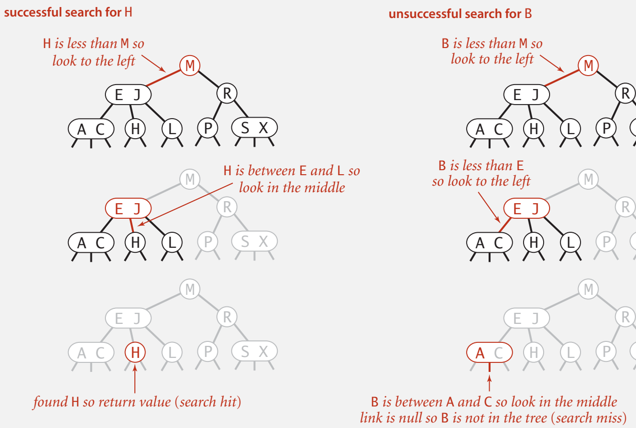


Search in a 2-3 tree

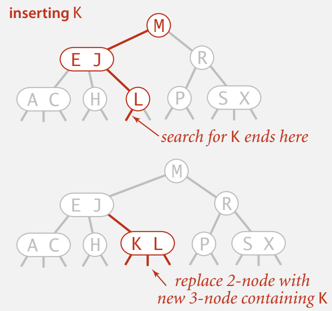
- Compare search key against keys in node.
- Find interval containing search key.
- Follow associated link (recursively).



Insertion in a 2-3 tree

Case 1. Insert into a 2-node at bottom.

- Search for key, as usual.
- Replace 2-node with 3-node.



Insertion in a 2-3 tree

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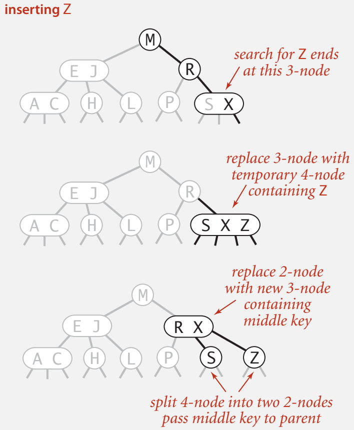

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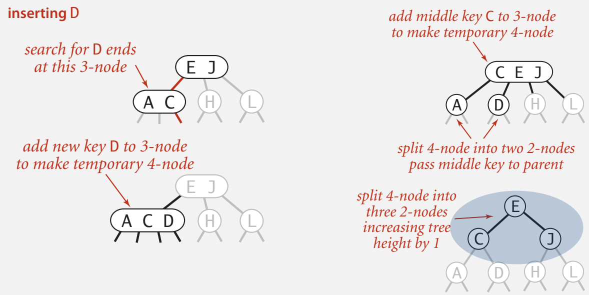
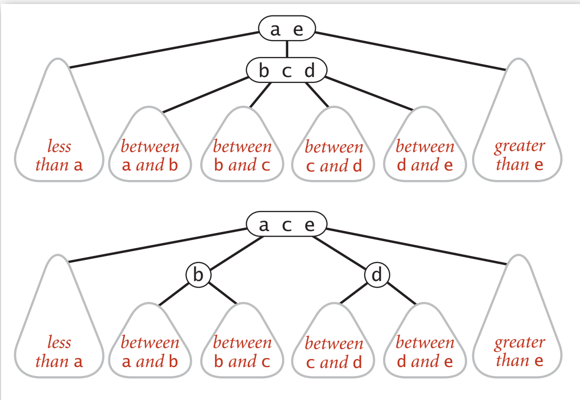
increases height by 1



Local transformations in a 2-3 tree

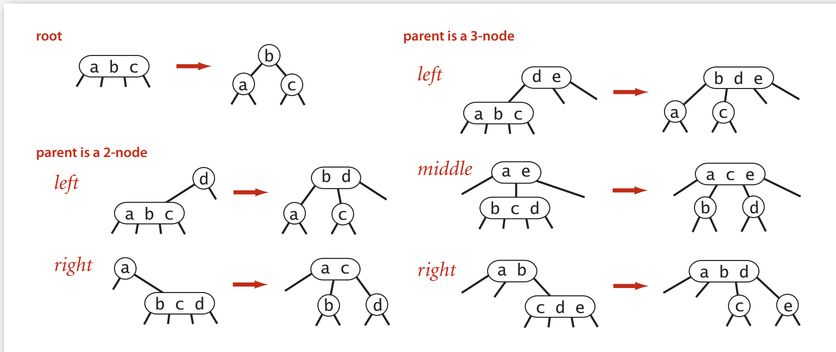
Splitting a 4-node is a local transformation: constant number of operations.



Global properties in a 2-3 tree

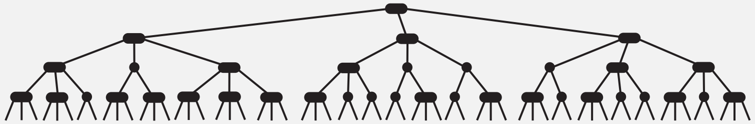
Invariants. Maintains symmetric order and perfect balance.

Pf. Each transformation maintains symmetric order and perfect balance.



2-3 tree: performance

Perfect balance. Every path from root to null link has same length.

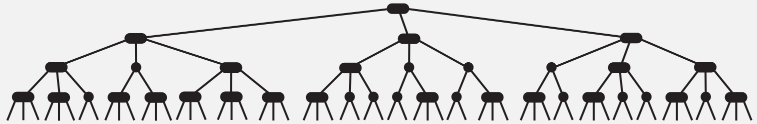


Tree height.

- Worst case:
- Best case:

2-3 tree: performance

Perfect balance. Every path from root to null link has same length.



Tree height.

- Worst case: $\lg N$. [all 2-nodes]
- Best case: $\log_3 N \approx .631 \lg N$. [all 3-nodes]
- Between 12 and 20 for a million nodes.
- Between 18 and 30 for a billion nodes.

Guaranteed logarithmic performance for search and insert.

ST implementations: summary

implementation	worst-case cost (after N inserts)			average case (after N random inserts)			ordered iteration?	key interface
	search	insert	delete	search hit	insert	delete		
sequential search (unordered list)	N	N	N	N/2	N	N/2	no	`equals()`
binary search (ordered array)	lg N	N	N	lg N	N/2	N/2	yes	`compareTo()`
BST	N	N	N	1.39 lg N	1.39 lg N	?	yes	`compareTo()`
2-3 tree	c lg N	c lg N	c lg N	c lg N	c lg N	c lg N	yes	`compareTo()`

constants depend upon implementation

2-3 tree: implementation?

Direct implementation is complicated, because:

- Maintaining multiple node types is cumbersome.
- Need multiple compares to move down tree.
- Need to move back up the tree to split 4-nodes.
- Large number of cases for splitting.

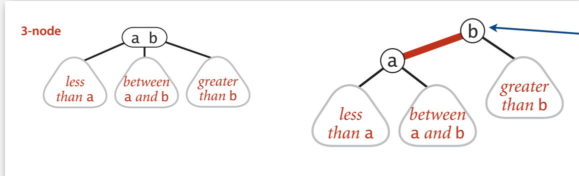
Bottom line. Could do it, but there's a better way.

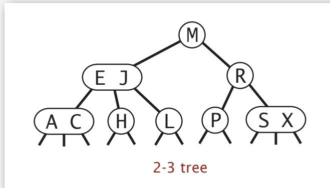
Balanced Search Trees

- 2-3 search trees
- **Red-black BSTs**
- B-trees
- Geometric applications of BSTs

Left-leaning red-black BSTs (Guibas-Sedgewick 1979 and Sedgewick 2007)

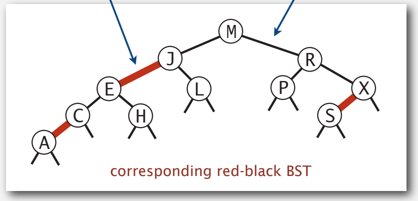
1. Represent 2–3 tree as a BST.
2. Use "internal" left-leaning links as "glue" for 3–nodes.





red links "glue" nodes within a 3-node

black links connect 2-nodes and 3-nodes

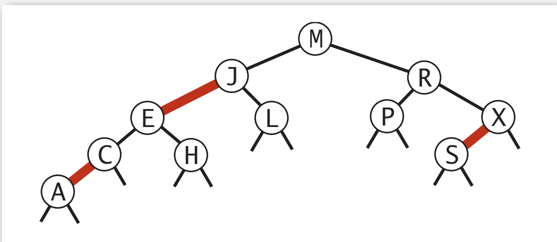


An equivalent definition

A BST such that:

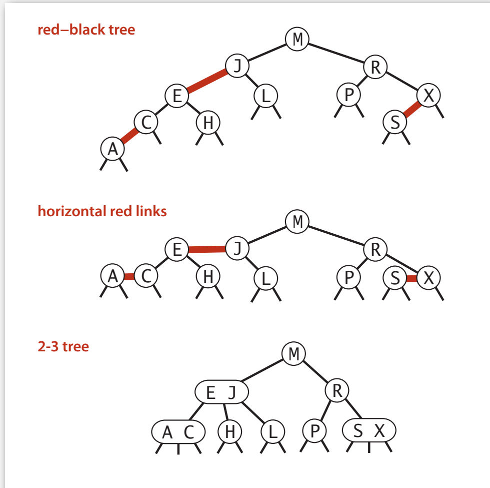
- No node has two red links connected to it.
- Every path from root to null link has the same number of black links.
- Red links lean left.

"perfect black balance"



Left-leaning red-black BSTs: 1-1 correspondence with 2-3 trees

Key property. 1–1 correspondence between 2–3 and LLRB.

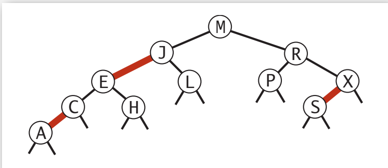


Search implementation for red-black BSTs

Observation. Search is the same as for elementary BST (ignore color).

but runs faster because of better balance

```
public Val get(Key key)
{
   Node x = root;
   while (x != null)
   {
      int cmp = key.compareTo(x.key);
      if      (cmp  < 0) x = x.left;
      else if (cmp  > 0) x = x.right;
      else if (cmp == 0) return x.val;
   }
   return null;
}
```



Remark. Most other ops (e.g., ceiling, selection, iteration) are also identical.

Red-black BST representation

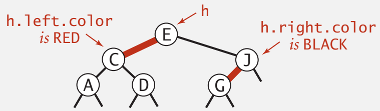
Each node is pointed to by precisely one link (from its parent) ⇒ can encode color of links in nodes.

```
private static final boolean RED   = true;
private static final boolean BLACK = false;

private class Node
{
   Key key;
   Value val;
   Node left, right;
   boolean color;    // color of parent link
}

private boolean isRed(Node x)
{
   if (x == null) return false;
   return x.color == RED;
}
```

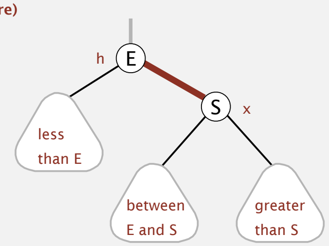
null links are black



Elementary red-black BST operations

Left rotation. Orient a (temporarily) right-leaning red link to lean left.

rotate E left
(before)



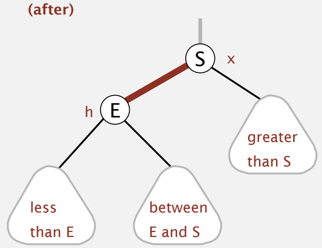

```
private Node rotateLeft(Node h)
{
   assert isRed(h.right);
   Node x = h.right;
   h.right = x.left;
   x.left = h;
   x.color = h.color;
   h.color = RED;
   return x;
}
```

Invariants. Maintains symmetric order and perfect black balance.

Elementary red-black BST operations

Left rotation. Orient a (temporarily) right-leaning red link to lean left.

rotate E left
(after)

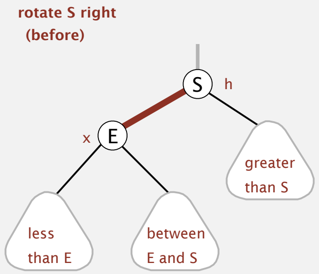



```
private Node rotateLeft(Node h)
{
   assert isRed(h.right);
   Node x = h.right;
   h.right = x.left;
   x.left = h;
   x.color = h.color;
   h.color = RED;
   return x;
}
```

Invariants. Maintains symmetric order and perfect black balance.

Elementary red-black BST operations

Right rotation. Orient a left-leaning red link to (temporarily) lean right.

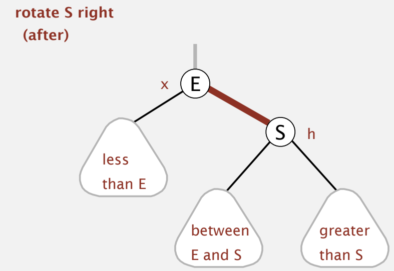



```
private Node rotateRight(Node h)
{
   assert isRed(h.left);
   Node x = h.left;
   h.left = x.right;
   x.right = h;
   x.color = h.color;
   h.color = RED;
   return x;
}
```

Invariants. Maintains symmetric order and perfect black balance.

Elementary red-black BST operations

Right rotation. Orient a left-leaning red link to (temporarily) lean right.

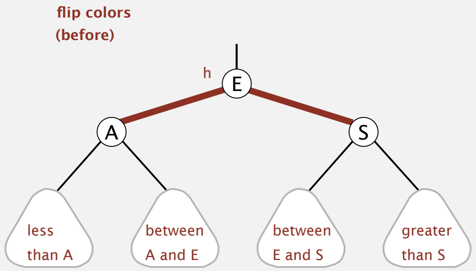



```
private Node rotateRight(Node h)
{
   assert isRed(h.left);
   Node x = h.left;
   h.left = x.right;
   x.right = h;
   x.color = h.color;
   h.color = RED;
   return x;
}
```

Invariants. Maintains symmetric order and perfect black balance.

Elementary red-black BST operations

Color flip. Recolor to split a (temporary) 4-node.

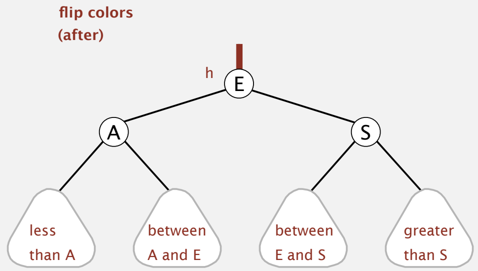



```
private void flipColors(Node h)
{
   assert !isRed(h);
   assert isRed(h.left);
   asset isRed(h.right);
   h.color = RED;
   h.left.color = BLACK;
   h.right.color = BLACK;
}
```

Invariants. Maintains symmetric order and perfect black balance.

Elementary red-black BST operations

Color flip. Recolor to split a (temporary) 4-node.

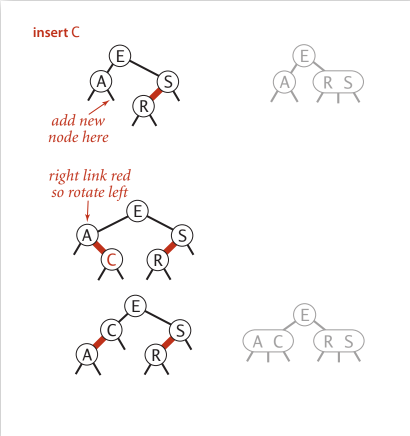



```
private void flipColors(Node h)
{
   assert !isRed(h);
   assert isRed(h.left);
   asset isRed(h.right);
   h.color = RED;
   h.left.color = BLACK;
   h.right.color = BLACK;
}
```

Invariants. Maintains symmetric order and perfect black balance.

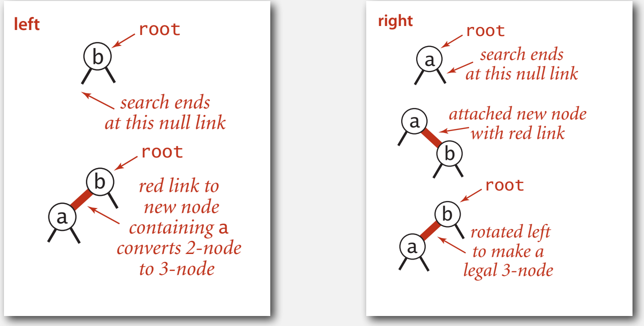
Insertion in a LLRB tree: overview

Basic strategy. Maintain 1-1 correspondence with 2-3 trees by applying elementary red-black BST operations.



Insertion in a LLRB tree

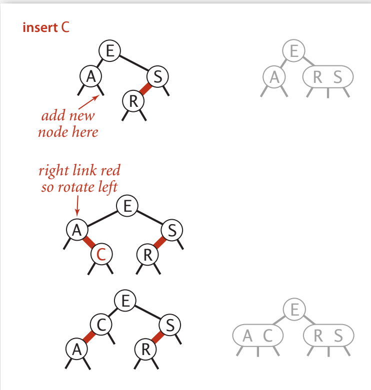
Warmup 1. Insert into a tree with exactly 1 node.



Insertion in a LLRB tree

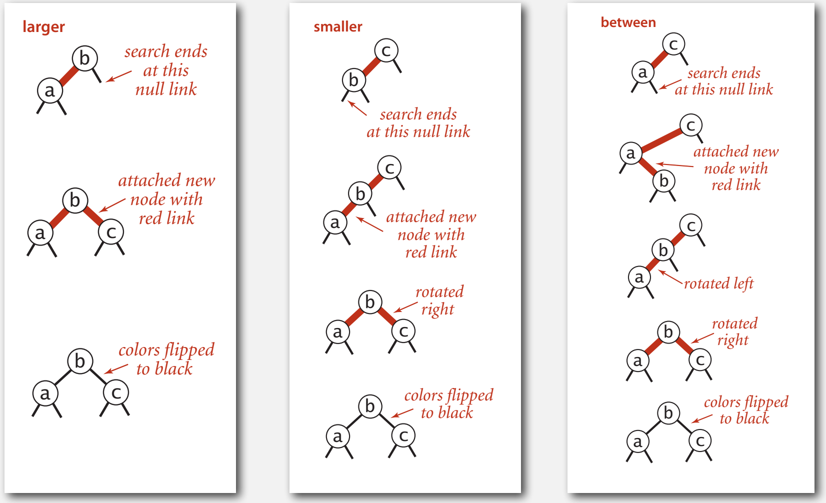
Case 1. Insert into a 2-node at the bottom.

- Do standard BST insert; color new link red.
- If new red link is a right link, rotate left.



Insertion in a LLRB tree

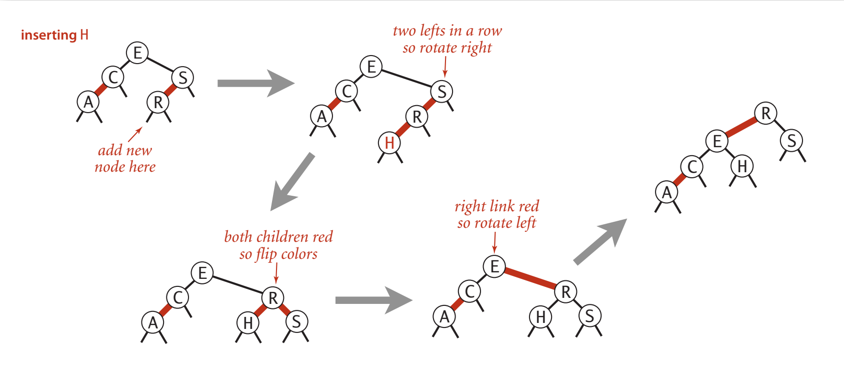
Warmup 2. Insert into a tree with exactly 2 nodes.



Insertion in a LLRB tree

Case 2. Insert into a 3-node at the bottom.

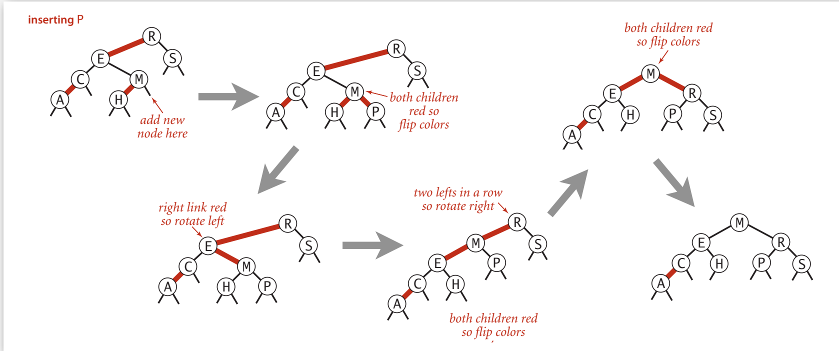
- Do standard BST insert; color new link red.
- Rotate to balance the 4-node (if needed).
- Flip colors to pass red link up one level.
- Rotate to make lean left (if needed).



Insertion in a LLRB tree: passing red links up the tree

Case 2. Insert into a 3-node at the bottom.

- Do standard BST insert; color new link red.
- Rotate to balance the 4-node (if needed).
- Flip colors to pass red link up one level.
- Rotate to make lean left (if needed).
- Repeat case 1 or case 2 up the tree (if needed).

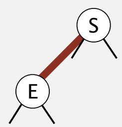


Red-black BST insertion

insert S

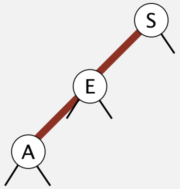
Red-black BST insertion

insert E



Red-black BST insertion

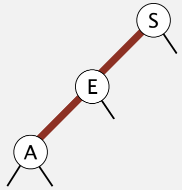
insert A



Red-black BST insertion

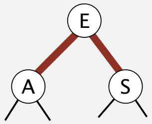
insert A

two left reds in a row
(rotate S right)



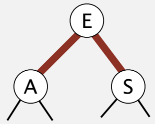
Red-black BST insertion

both children red
(flip colors)



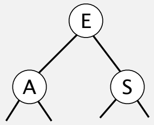
Red-black BST insertion

both children red
(flip colors)



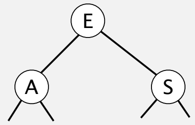
Red-black BST insertion

red-black BST



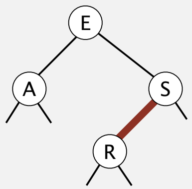
Red-black BST insertion

red-black BST



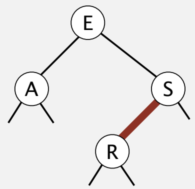
Red-black BST insertion

insert R



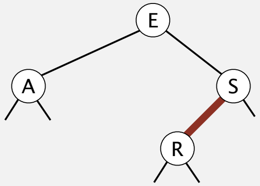
Red-black BST insertion

red-black BST



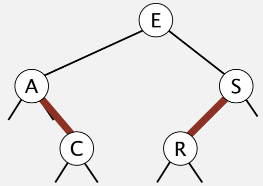
Red-black BST insertion

red–black BST



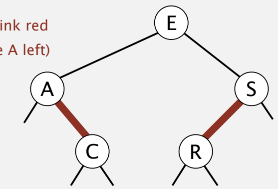
Red-black BST insertion

insert C



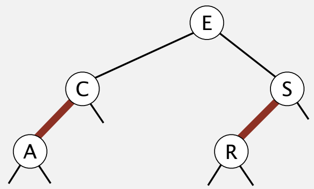
Red-black BST insertion

right link red
(rotate A left)



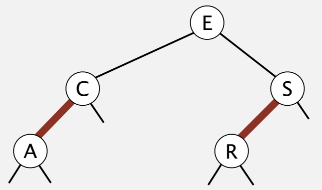
Red-black BST insertion

red–black BST



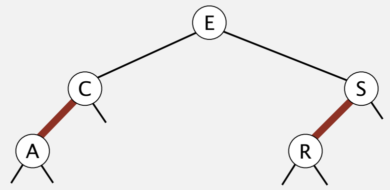
Red-black BST insertion

red–black BST



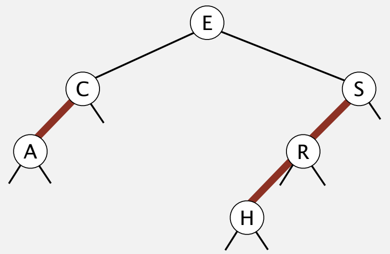
Red-black BST insertion

red–black BST

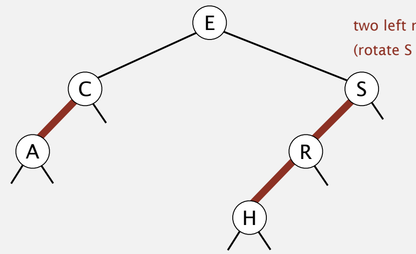


Red-black BST insertion

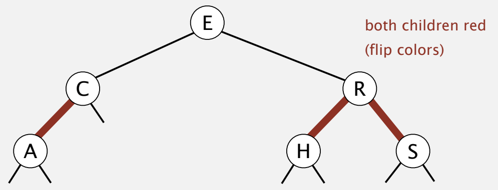
insert H



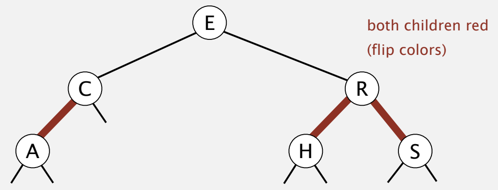
Red-black BST insertion



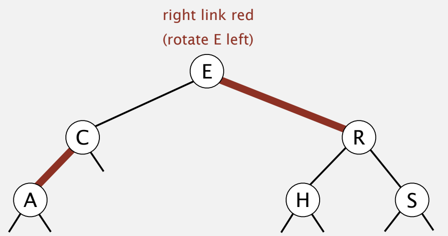
Red-black BST insertion



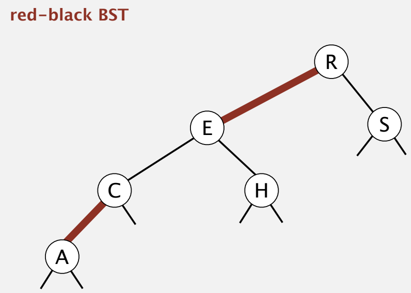
Red-black BST insertion



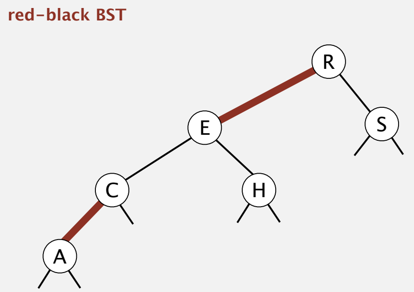
Red-black BST insertion



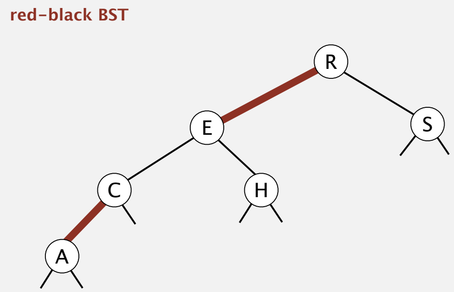
Red-black BST insertion



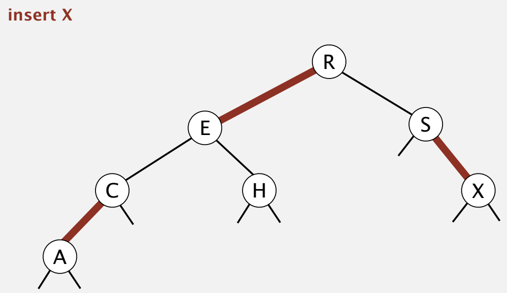
Red-black BST insertion



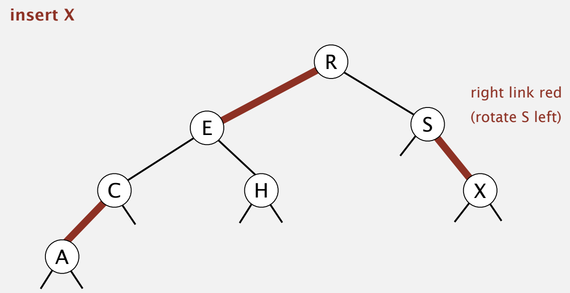
Red-black BST insertion



Red-black BST insertion

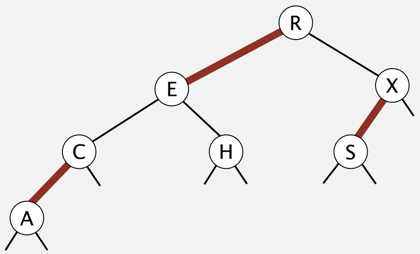


Red-black BST insertion



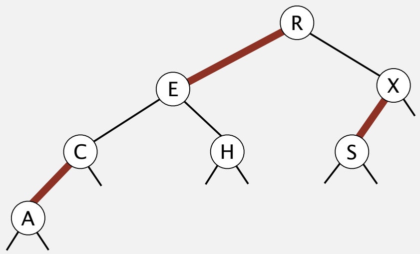
Red-black BST insertion

red-black BST



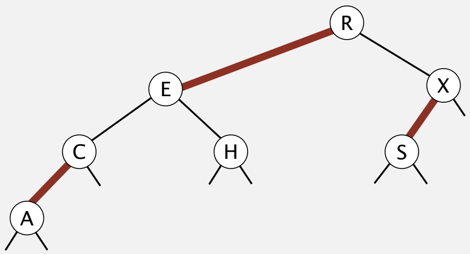
Red-black BST insertion

red-black BST



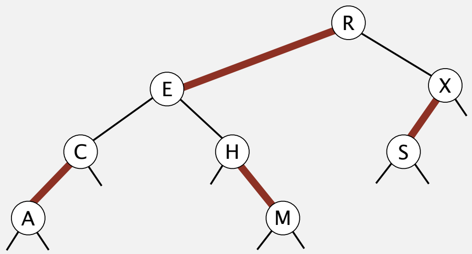
Red-black BST insertion

red-black BST



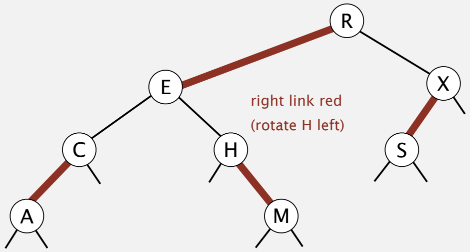
Red-black BST insertion

insert M



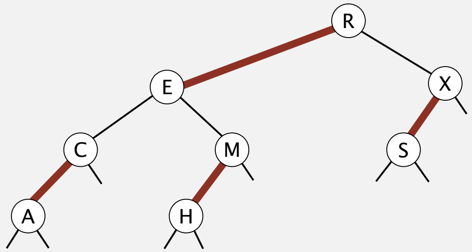
Red-black BST insertion

insert M



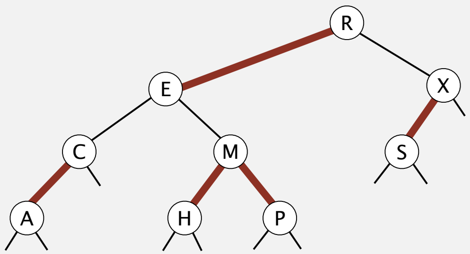
Red-black BST insertion

red–black BST



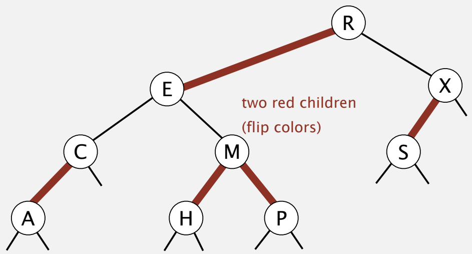
Red-black BST insertion

insert P



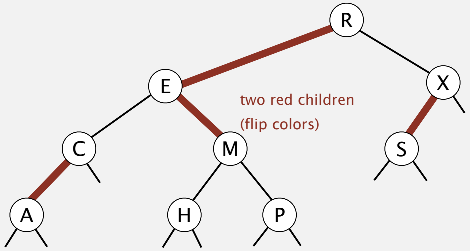
Red-black BST insertion

insert P

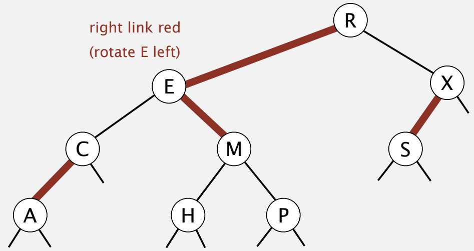


Red-black BST insertion

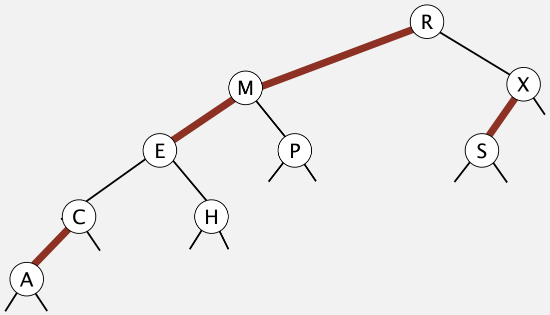
insert P



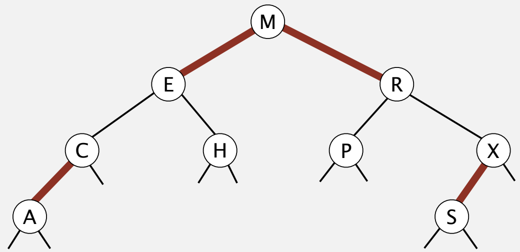
Red-black BST insertion



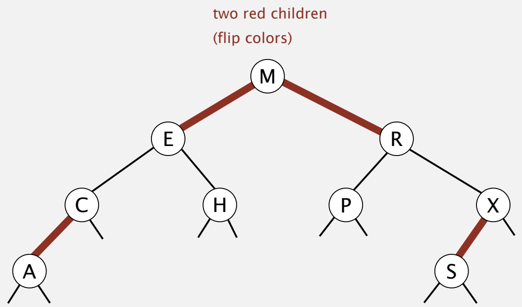
Red-black BST insertion



Red-black BST insertion

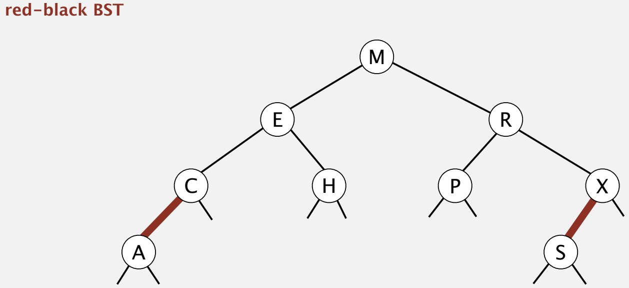


Red-black BST insertion



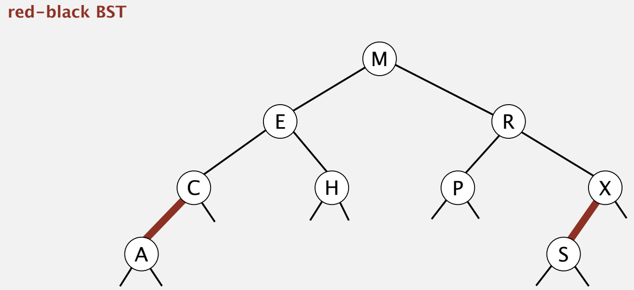
101

Red-black BST insertion



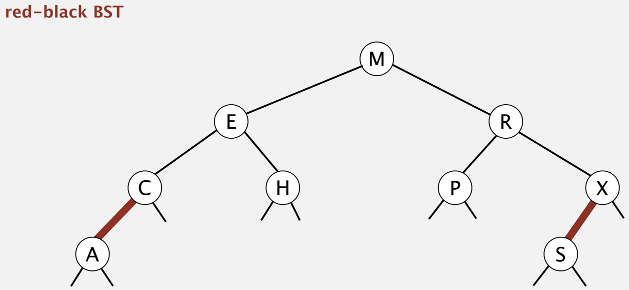
102

Red-black BST insertion



103

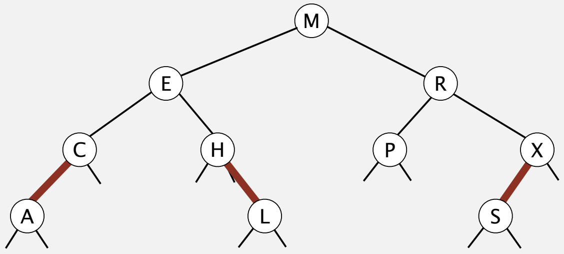
Red-black BST insertion



104

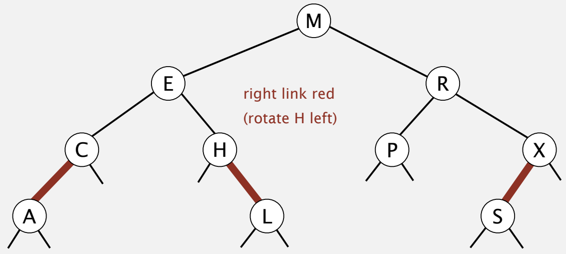
Red-black BST insertion

insert L



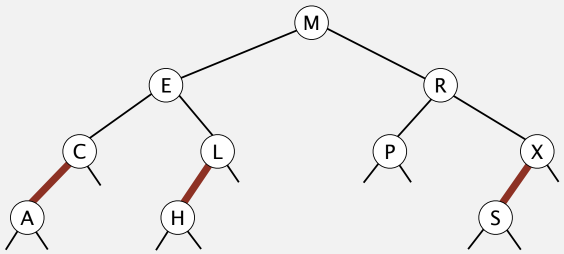
Red-black BST insertion

insert L



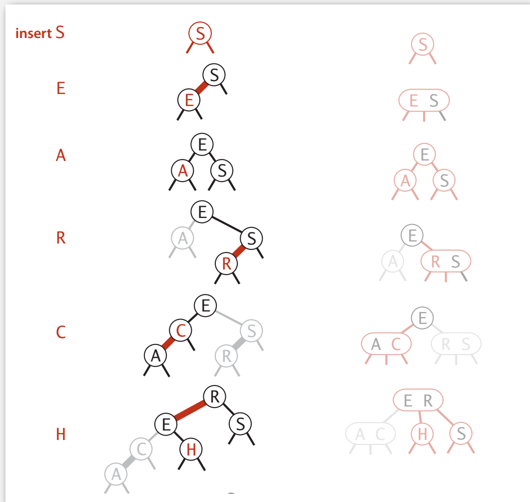
Red-black BST insertion

red-black BST



LLRB tree insertion trace

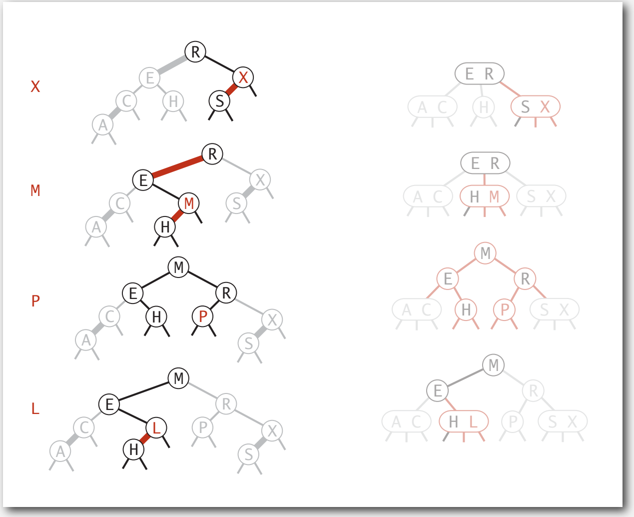
Standard indexing client.



red-black BST corresponding 2-3 tree

LLRB tree insertion trace

Standard indexing client (continued).

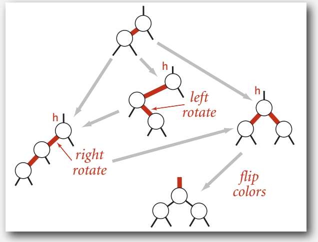


red-black BST corresponding 2-3 tree

Insertion in a LLRB tree: Java implementation

Same code for both cases.

- Right child red, left child black: rotate left.
- Left child, left-left grandchild red: rotate right.
- Both children red: flip colors.




```
private Node put(Node h, Key key, Value val)
{
   if (h == null) return new Node(key, val, RED);
   int cmp = key.compareTo(h.key);
   if      (cmp  < 0) h.left  = put(h.left,  key, val);
   else if (cmp  > 0) h.right = put(h.right, key, val);
   else if (cmp == 0) h.val = val;

   if (isRed(h.right)  && !isRed(h.left))      h = rotateLeft(h);
   if (isRed(h.left)   && isRed(h.left.left))  h = rotateRight(h);
   if (isRed(h.left)   && isRed(h.right))      flipColors(h);

   return h;
}
```

insert at bottom (and color red)

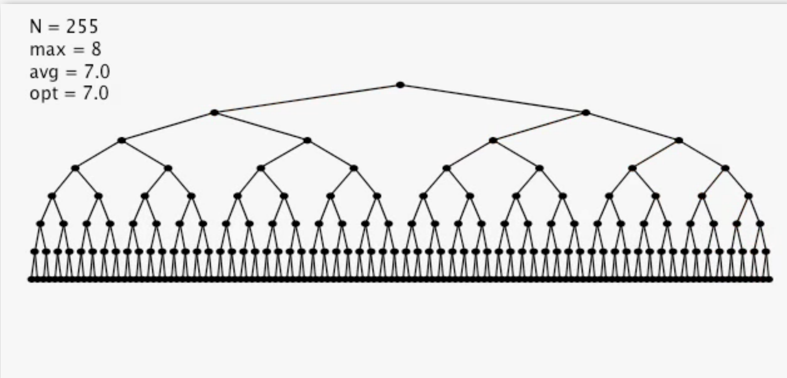
lean left

balance 4-node

split 4-node

only a few extra lines of code provides near-perfect balance

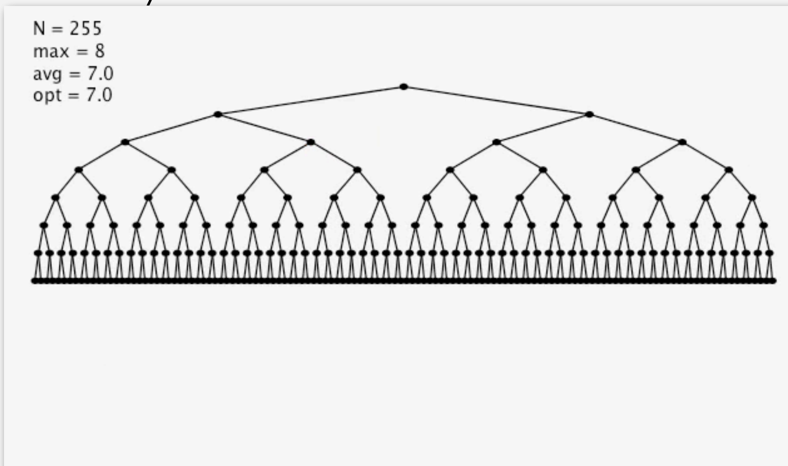
Insertion in a LLRB tree: visualization



255 insertions in ascending order

Insertion in a LLRB tree: visualization

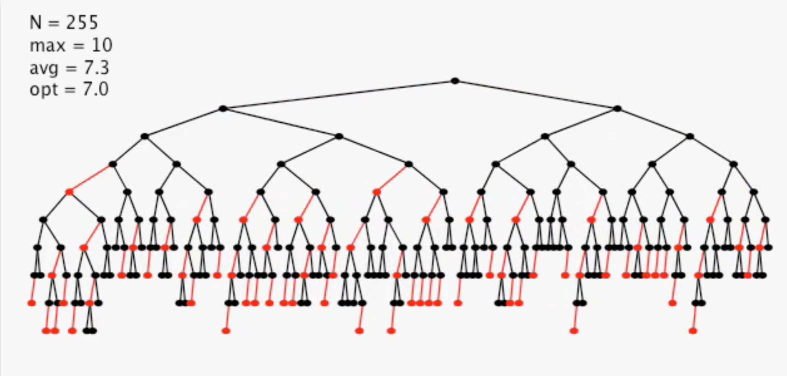
Remark. Only a few extra lines of code to standard BST insert.



255 insertions in descending order

Insertion in a LLRB tree: visualization

Remark. Only a few extra lines of code to standard BST insert.



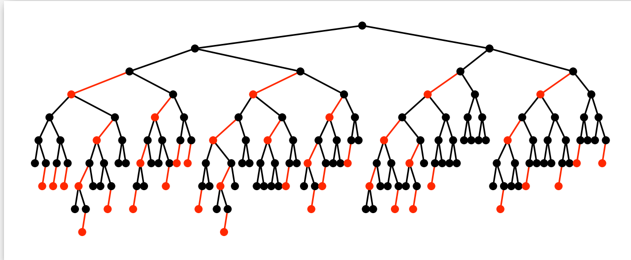
255 random insertions

Balance in LLRB trees

Proposition. Height of tree is $\leq 2 \lg N$ in the worst case.

Pf.

- Every path from root to null link has same number of black links.
- Never two red links in-a-row.



Property. Height of tree is $\sim 1.00 \lg N$ in typical applications.

ST implementations: summary

implementation	worst-case cost (after N inserts)			average case (after N random inserts)			ordered iteration?	key interface
	search	insert	delete	search hit	insert	delete		
sequential search (unordered list)	N	N	N	N/2	N	N/2	no	`equals()`
binary search (ordered array)	lg N	N	N	lg N	N/2	N/2	yes	`compareTo()`
BST	N	N	N	1.39 lg N	1.39 lg N	?	yes	`compareTo()`
2-3 tree	c lg N	c lg N	c lg N	c lg N	c lg N	c lg N	yes	`compareTo()`
red-black BST	2 lg N	2 lg N	2 lg N	1.00 lg N *	1.00 lg N *	1.00 lg N *	yes	`compareTo()`

* exact value of coefficient unknown but extremely close to 1

War story: why red-black?

Xerox PARC innovations. [1970s]

- Alto.
- GUI.
- Ethernet.
- Smalltalk.
- InterPress.
- Laser printing.
- Bitmapped display.
- WYSIWYG text editor.
- ...

XEROX. PARC

Xerox Alto

A DICHROMATIC FRAMEWORK FOR BALANCED TREES

Leo J. Guibas
Xerox Palo Alto Research Center,
Palo Alto, California, and
Carnegie-Mellon University

and

Robert Sedgewick*
Program in Computer Science
Brown University
Providence, R. I.

ABSTRACT

In this paper we present a uniform framework for the implementation and study of balanced tree algorithms. We show how to imbed in this

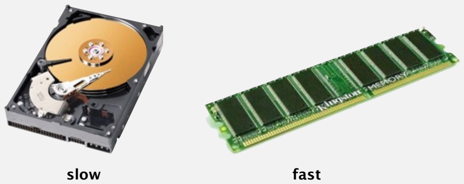
the way down towards a leaf. As we will see, this has a number of significant advantages over the older methods. We shall examine a number of variations on a common theme and exhibit full implementations which are notable for their brevity. One implementation is examined carefully, and some properties about its

Balanced Search Trees

- 2-3 search trees
- Red-black BSTs
- **B-trees**
- Geometric applications of BSTs

File system model

Page. Contiguous block of data (e.g., a file or 4,096-byte chunk).
Probe. First access to a page (e.g., from disk to memory).



Property. Time required for a probe is much larger than time to access data within a page.

Cost model. Number of probes.

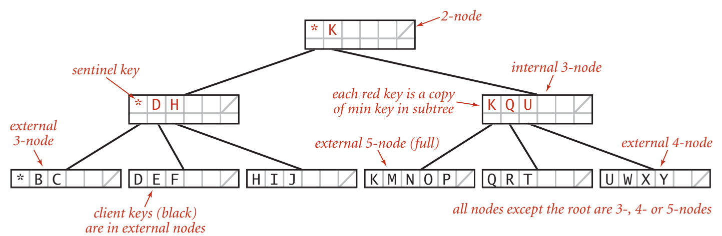
Goal. Access data using minimum number of probes.

B-trees (Bayer-McCreight, 1972)

B-tree. Generalize 2-3 trees by allowing up to $M - 1$ key-link pairs per node.

- At least 2 key-link pairs at root.
- At least $M / 2$ key-link pairs in other nodes.
- External nodes contain client keys.
- Internal nodes contain copies of keys to guide search.

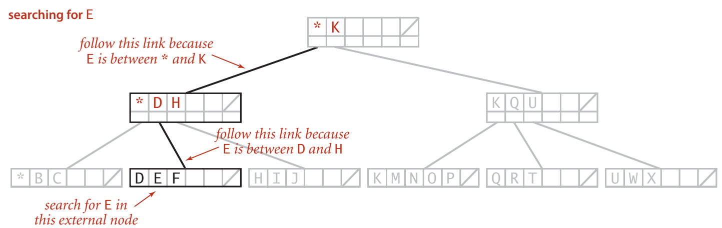
choose M as large as possible so that M links fit in a page, e.g., M = 1024



Anatomy of a B-tree set (M = 6)

Searching in a B-tree

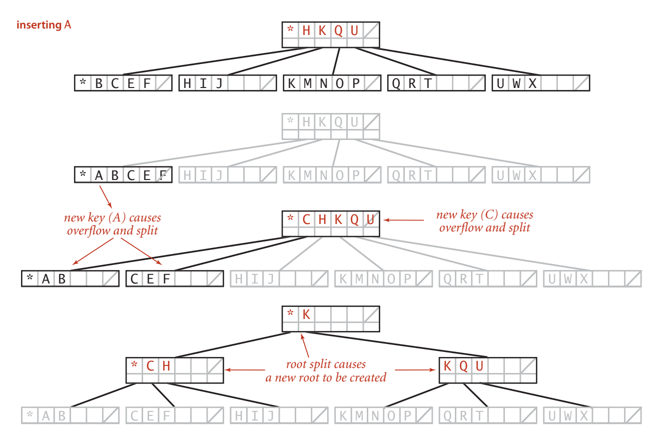
- Start at root.
- Find interval for search key and take corresponding link.
- Search terminates in external node.



Searching in a B-tree set (M = 6)

Insertion in a B-tree

- Search for new key.
- Insert at bottom.
- Split nodes with M key-link pairs on the way up the tree.



Inserting a new key into a B-tree set

Balance in B-tree

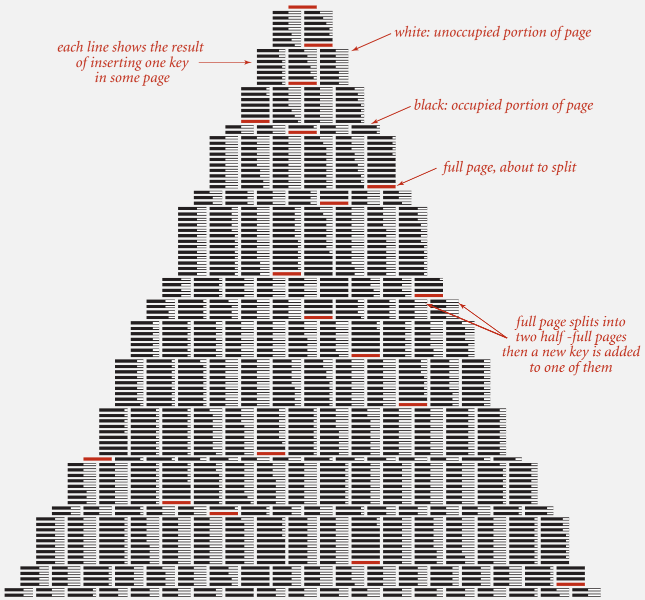
Proposition. A search or an insertion in a B-tree of order M with N keys requires between $\log_{M-1} N$ and $\log_{M/2} N$ probes.

Pf. All internal nodes (besides root) have between $M / 2$ and $M - 1$ links.

In practice. Number of probes is at most 4. ← M = 1024; N = 62 billion; $\log_{M/2} N \le 4$

Optimization. Always keep root page in memory.

Building a large B tree



Balanced trees in the wild

Red-black trees are widely used as system symbol tables.

- Java: `java.util.TreeMap`, `java.util.TreeSet`.
- C++ STL: map, multimap, multiset.
- Linux kernel: completely fair scheduler, `linux/rbtree.h`.

B-tree variants. B+ tree, B*tree, B# tree, …

B-trees (and variants) are widely used for file systems and databases.

- Windows: HPFS.
- Mac: HFS, HFS+.
- Linux: ReiserFS, XFS, Ext3FS, JFS.
- Databases: ORACLE, DB2, INGRES, SQL, PostgreSQL.

Balanced Search Trees

- 2-3 search trees
- Red-black BSTs
- B-trees
- Geometric applications of BSTs

Geometric applications of BSTs

- kd trees

2-d orthogonal range search

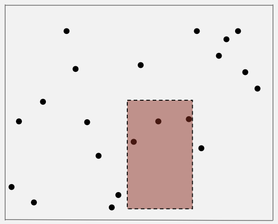
Extension of ordered symbol-table to 2d keys.

- Insert a 2d key.
- Delete a 2d key.
- Search for a 2d key.
- Range search: find all keys that lie in a 2d range.
- Range count: number of keys that lie in a 2d range.

Geometric interpretation.

- Keys are point in the plane.
- Find/count points in a given h–v rectangle.

rectangle is axis-aligned

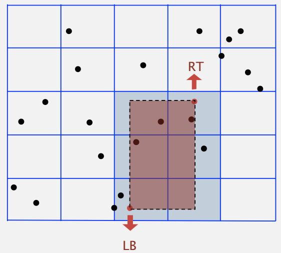


Applications. Networking, circuit design, databases,...

2d orthogonal range search: grid implementation

Grid implementation.

- Divide space into M-by-M grid of squares.
- Create list of points contained in each square.
- Use 2d array to directly index relevant square.
- Insert: add (x, y) to list for corresponding square.
- Range search: examine only those squares that intersect 2d range query.



2d orthogonal range search: grid implementation costs

Space-time tradeoff.

- Space: $M^2 + N$.
- Time: $1 + N / M^2$ per square examined, on average.

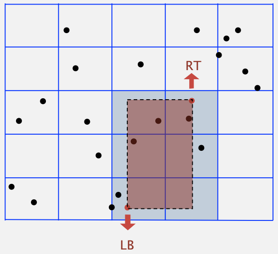
Choose grid square size to tune performance.

- Too small: wastes space.
- Too large: too many points per square.
- Rule of thumb: $\sqrt{N}$-by-$\sqrt{N}$ grid.

Running time. [if points are evenly distributed]

- Initialize data structure: N.
- Insert point: 1.
- Range search: 1 per point in range.

choose M ~ √N

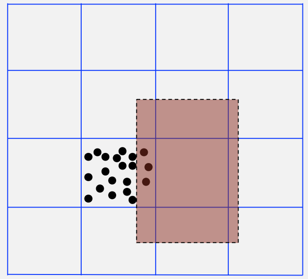


Clustering

Grid implementation. Fast and simple solution for evenly-distributed points.

Problem. Clustering a well-known phenomenon in geometric data.

- Lists are too long, even though average length is short.
- Need data structure that gracefully adapts to data.



Clustering

Grid implementation. Fast and simple solution for evenly-distributed points.

Problem. Clustering a well-known phenomenon in geometric data.

Ex. USA map data.

13,000 points, 1000 grid squares

Space-partitioning trees

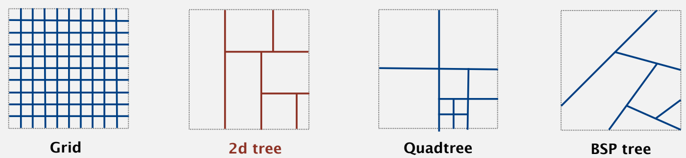
Use a tree to represent a recursive subdivision of 2d space.

Grid. Divide space uniformly into squares.

2d tree. Recursively divide space into two halfplanes.

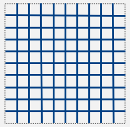
Quadtree. Recursively divide space into four quadrants.

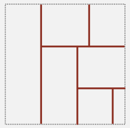
BSP tree. Recursively divide space into two regions.

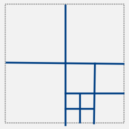


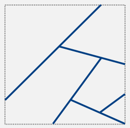
Space-partitioning trees: applications

Applications.
- Ray tracing.
- 2d range search.
- Flight simulators.
- N-body simulation.
- Collision detection.
- Astronomical databases.
- Nearest neighbor search.
- Adaptive mesh generation.
- Accelerate rendering in Doom.
- Hidden surface removal and shadow casting.


Grid


2d tree

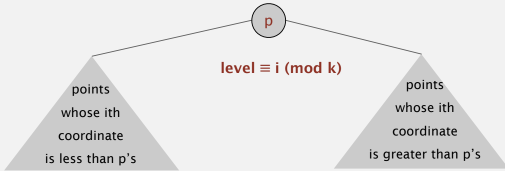

Quadtree

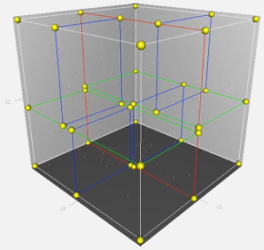

BSP tree

Kd tree

Kd tree. Recursively partition k-dimensional space into 2 halfspaces.

Implementation. BST, but cycle through dimensions ala 2d trees.



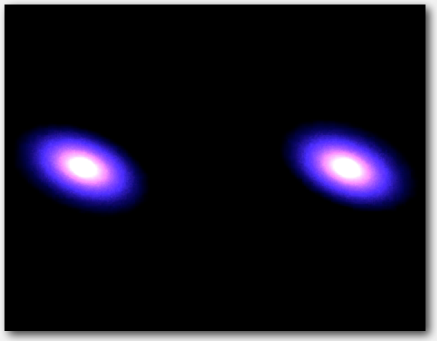


Efficient, simple data structure for processing k-dimensional data.
- Widely used.
- Adapts well to high-dimensional and clustered data.
- Discovered by an undergrad in an algorithms class!


Jon Bentley

N-body simulation

Goal. Simulate the motion of N particles, mutually affected by gravity.


http://www.youtube.com/watch?v=ua7YIN4eL_w

Brute force. For each pair of particles, compute force. $F = \frac{G\, m_1\, m_2}{r^2}$

Appel algorithm for N-body simulation

Key idea. Suppose particle is far, far away from cluster of particles.
- Treat cluster of particles as a single aggregate particle.
- Compute force between particle and center of mass of aggregate particle.

Appel algorithm for N-body simulation

- Build 3d-tree with N particles as nodes.
- Store center-of-mass of subtree in each node.
- To compute total force acting on a particle, traverse tree, but stop as soon as distance from particle to subdivision is sufficiently large.

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008

AN EFFICIENT PROGRAM FOR MANY-BODY SIMULATION*

ANDREW W. APPEL†

Abstract. The simulation of N particles interacting in a gravitational force field is useful in astrophysics, but such simulations become costly for large N. Representing the universe as a tree structure with the particles at the leaves and internal nodes labeled with the centers of mass of their descendants allows several simultaneous attacks on the computation time required by the problem. These approaches range from algorithmic changes (replacing an $O(N^2)$ algorithm with an algorithm whose time-complexity is believed to be $O(N \log N)$) to data structure modifications, code-tuning, and hardware modifications. The changes reduced the running time of a large problem ($N = 10{,}000$) by a factor of four hundred. This paper describes both the particular program and the methodology underlying such speedups.

Impact. Running time per step is $N \log N$ instead of N^2 $\Rightarrow$ enables new research.