BBM 202 - ALGORITHMS

HACETTEPE UNIVERSITY

DEPT. OF COMPUTER ENGINEERING

ERKUT ERDEM

DIRECTED GRAPHS

Mar. 31, 2015

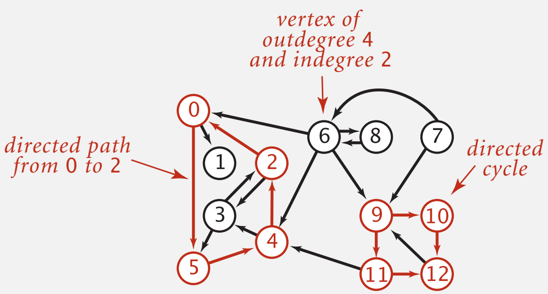
Acknowledgement: The course slides are adapted from the slides prepared by R. Sedgewick and K. Wayne of Princeton University.

TODAY

- Directed Graphs
- Digraph API
- Digraph search
- Topological sort
- Strong components

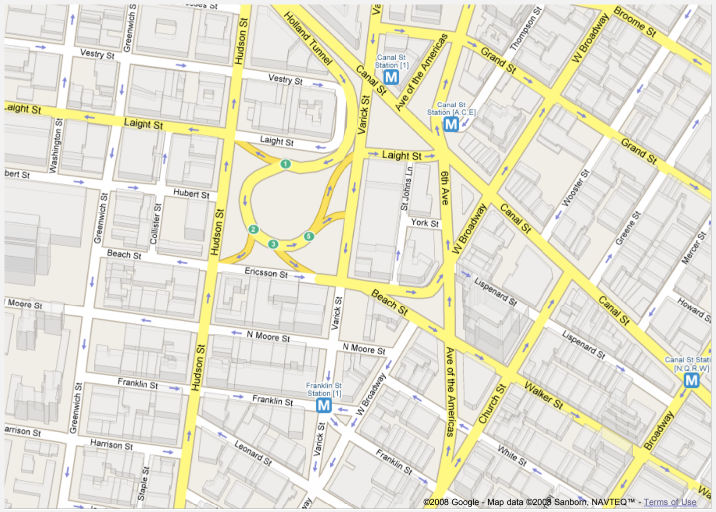
Directed graphs

Digraph. Set of vertices connected pairwise by directed edges.



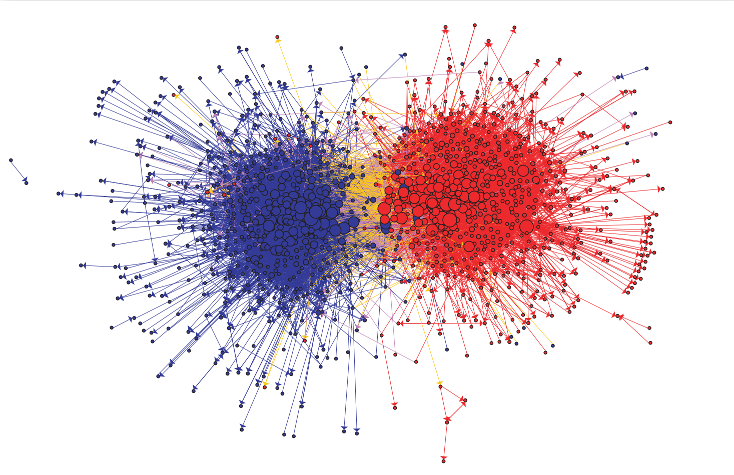
Road network

Vertex = intersection; edge = one-way street.



Political blogosphere graph

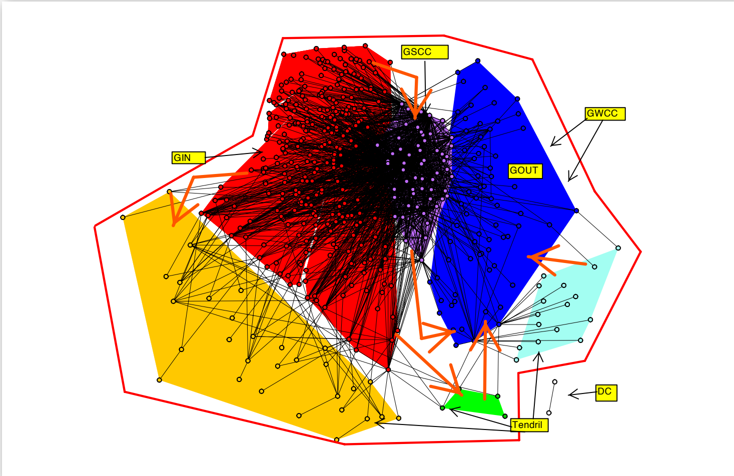
Vertex = political blog; edge = link.



The Political Blogosphere and the 2004 U.S. Election: Divided They Blog, Adamic and Glance, 2005

Overnight interbank loan graph

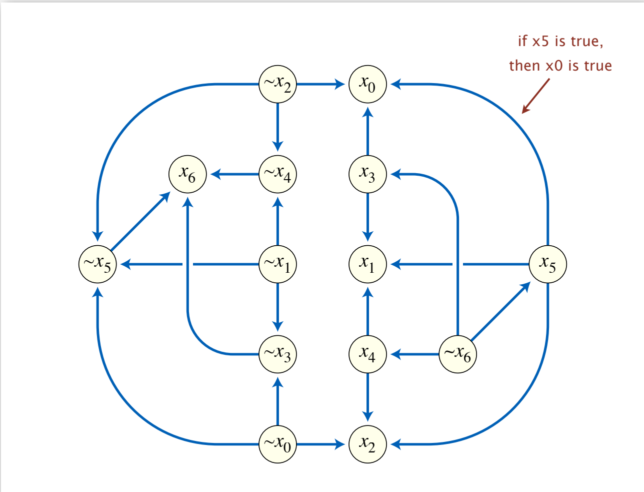
Vertex = bank; edge = overnight loan.



The Topology of the Federal Funds Market, Bech and Atalay, 2008

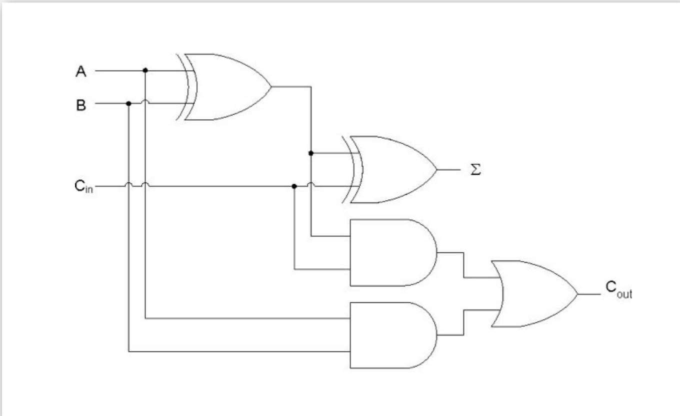
Implication graph

Vertex = variable; edge = logical implication.



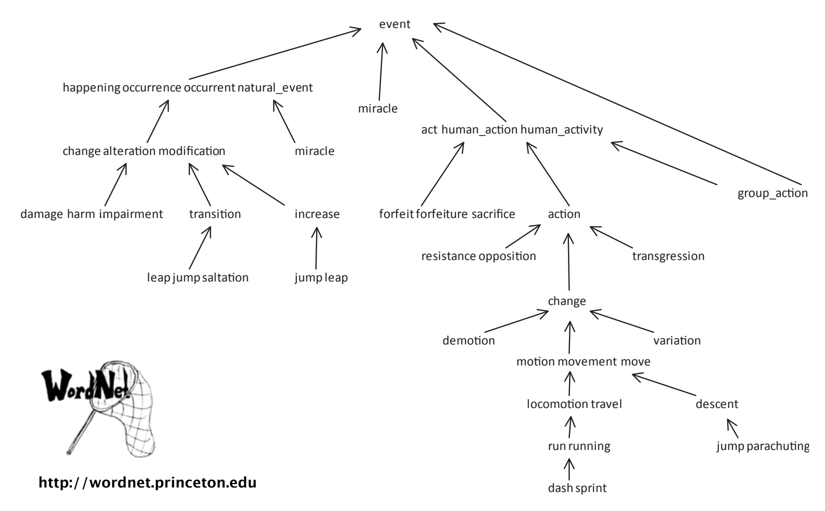
Combinational circuit

Vertex = logical gate; edge = wire.



WordNet graph

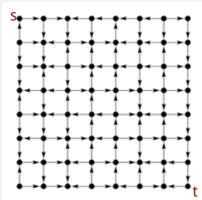
Vertex = synset; edge = hypernym relationship.



Digraph applications

digraph	vertex	directed edge
transportation	street intersection	one-way street
web	web page	hyperlink
food web	species	predator-prey relationship
WordNet	synset	hypernym
scheduling	task	precedence constraint
financial	bank	transaction
cell phone	person	placed call
infectious disease	person	infection
game	board position	legal move
citation	journal article	citation
object graph	object	pointer
inheritance hierarchy	class	inherits from
control flow	code block	jump

Some digraph problems



Path. Is there a directed path from s to t ?

Shortest path. What is the shortest directed path from s to t ?

Topological sort. Can you draw the digraph so that all edges point upwards?

Strong connectivity. Is there a directed path between all pairs of vertices?

Transitive closure. For which vertices v and w is there a path from v to w ?

PageRank. What is the importance of a web page?

Directed Graphs

- Digraph API
- Digraph search
- Topological sort
- Strong components

Digraph API

public class Digraph		
	`Digraph(int V)`	*create an empty digraph with V vertices*
	`Digraph(In in)`	*create a digraph from input stream*
`void`	`addEdge(int v, int w)`	*add a directed edge v→w*
`Iterable<Integer>`	`adj(int v)`	*vertices pointing from v*
`int`	`V()`	*number of vertices*
`int`	`E()`	*number of edges*
`Digraph`	`reverse()`	*reverse of this digraph*
`String`	`toString()`	*string representation*

```
In in = new In(args[0]);
Digraph G = new Digraph(in);

for (int v = 0; v < G.V(); v++)
   for (int w : G.adj(v))
      StdOut.println(v + "->" + w);
```

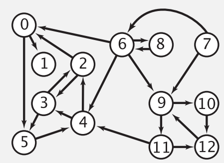
read digraph from input stream

print out each edge (once)

Digraph API

tinyDG.txt

```
V → 13
    22 ← E
     4  2
     2  3
     3  2
     6  0
     0  1
     2  0
    11 12
    12  9
     9 10
     9 11
     7  9
    10 12
    11  4
     4  3
     3  5
     6  8
     8  6
     ⋮
```




```
% java Digraph tinyDG.txt
0->5
0->1
2->0
2->3
3->5
3->2
4->3
4->2
5->4
⋮
11->4
11->12
12-9
```

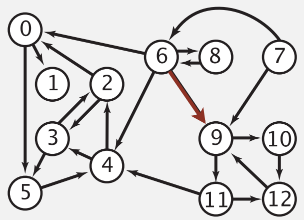
```
In in = new In(args[0]);
Digraph G = new Digraph(in);

for (int v = 0; v < G.V(); v++)
   for (int w : G.adj(v))
      StdOut.println(v + "->" + w);
```

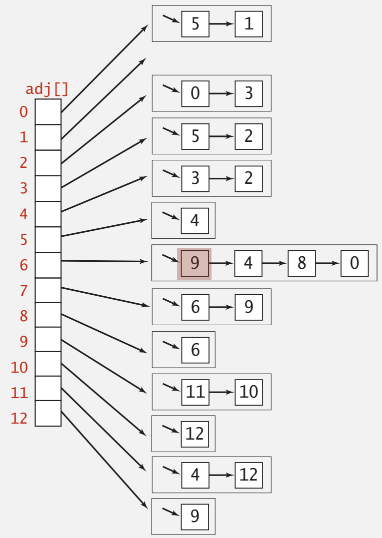
read digraph from input stream

print out each edge (once)

Adjacency-lists digraph representation

Maintain vertex-indexed array of lists.





Adjacency-lists graph representation: Java implementation

```
public class Graph
{
   private final int V;
   private final Bag<Integer>[] adj;

   public Graph(int V)
   {
      this.V = V;
      adj = (Bag<Integer>[]) new Bag[V];
      for (int v = 0; v < V; v++)
         adj[v] = new Bag<Integer>();
   }

   public void addEdge(int v, int w)
   {
      adj[v].add(w);
      adj[w].add(v);
   }

   public Iterable<Integer> adj(int v)
   {  return adj[v];  }
}
```

adjacency lists

create empty graph with V vertices

add edge v–w

iterator for vertices adjacent to v

Adjacency-lists digraph representation: Java implementation

```
public class Digraph
{
   private final int V;
   private final Bag<Integer>[] adj;                    adjacency lists

   public Digraph(int V)                                create empty digraph
   {                                                    with V vertices
      this.V = V;
      adj = (Bag<Integer>[]) new Bag[V];
      for (int v = 0; v < V; v++)
         adj[v] = new Bag<Integer>();
   }

   public void addEdge(int v, int w)                    add edge v→w
   {
      adj[v].add(w);

   }

   public Iterable<Integer> adj(int v)                  iterator for vertices
   {  return adj[v];  }                                 pointing from v
}
```


Digraph representations

In practice. Use adjacency-lists representation.

- Algorithms based on iterating over vertices pointing from v.
- Real-world digraphs tend to be sparse.

huge number of vertices,
small average vertex degree

representation	space	insert edge from v to w	edge from v to w?	iterate over vertices pointing from v?
list of edges	E	1	E	E
adjacency matrix	V^2	1 †	1	V
adjacency lists	E + V	1	outdegree(v)	outdegree(v)

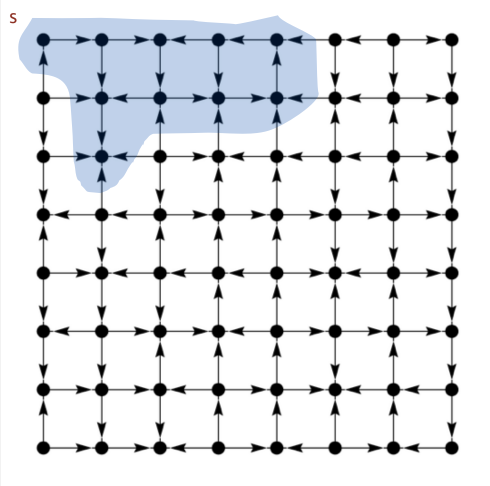
† disallows parallel edges

Directed Graphs

- Digraph API
- **Digraph search**
- Topological sort
- Strong components

Reachability

Problem. Find all vertices reachable from s along a directed path.



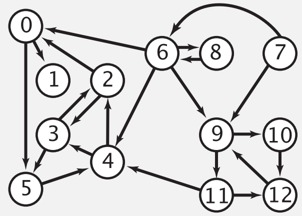
Depth-first search in digraphs

Same method as for undirected graphs.

- Every undirected graph is a digraph (with edges in both directions).
- DFS is a digraph algorithm.

DFS (to visit a vertex v)

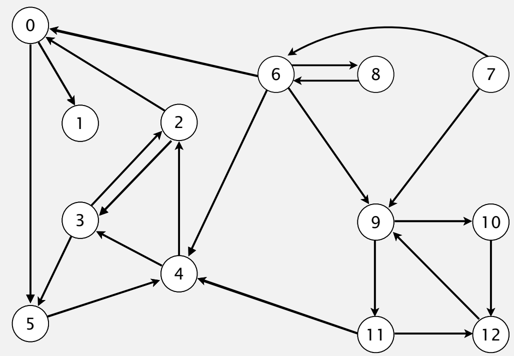
Mark v as visited.
Recursively visit all unmarked
vertices w pointing from v.



Depth-first search

To visit a vertex v :

- Mark vertex v as visited.
- Recursively visit all unmarked vertices pointing from v.



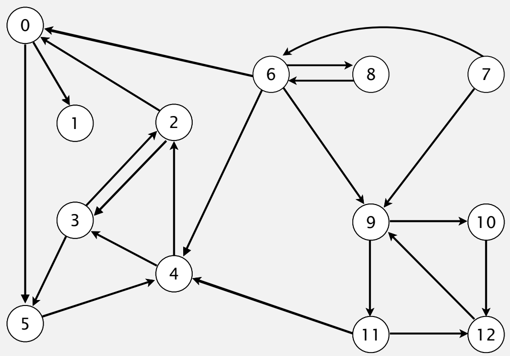
a directed graph

4→2
2→3
3→2
6→0
0→1
2→0
11→12
12→9
9→10
9→11
8→9
10→12
11→4
4→3
3→5
6→8
8→6
5→4
0→5
6→4
6→9
7→6

Depth-first search

To visit a vertex v :

- Mark vertex v as visited.
- Recursively visit all unmarked vertices pointing from v.



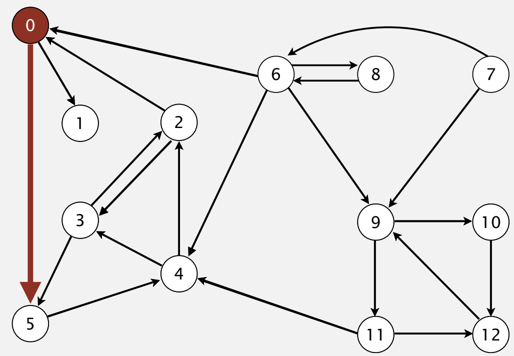
a directed graph

v	marked[]	edgeTo[]
0	F	–
1	F	–
2	F	–
3	F	–
4	F	–
5	F	–
6	F	–
7	F	–
8	F	–
9	F	–
10	F	–
11	F	–
12	F	–

Depth-first search

To visit a vertex v :

- Mark vertex v as visited.
- Recursively visit all unmarked vertices pointing from v.



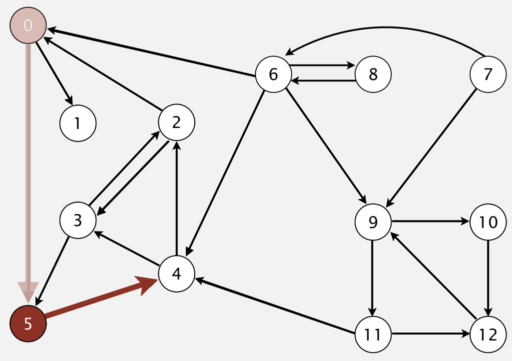
visit 0

v	marked[]	edgeTo[]
0	T	–
1	F	–
2	F	–
3	F	–
4	F	–
5	F	–
6	F	–
7	F	–
8	F	–
9	F	–
10	F	–
11	F	–
12	F	–

Depth-first search

To visit a vertex v :

- Mark vertex v as visited.
- Recursively visit all unmarked vertices pointing from v.



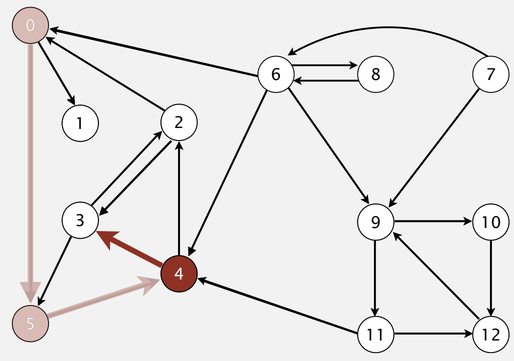
v	marked[]	edgeTo[]
0	T	–
1	F	–
2	F	–
3	F	–
4	F	–
5	T	0
6	F	–
7	F	–
8	F	–
9	F	–
10	F	–
11	F	–
12	F	–

visit 5

Depth-first search

To visit a vertex v :

- Mark vertex v as visited.
- Recursively visit all unmarked vertices pointing from v.



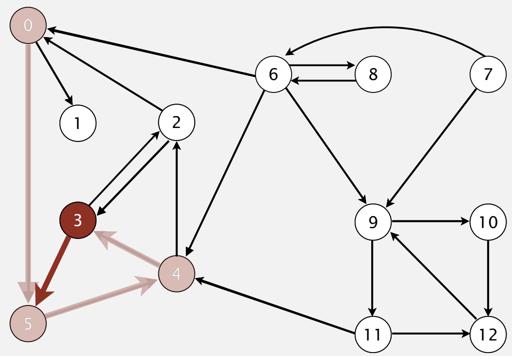
v	marked[]	edgeTo[]
0	T	–
1	F	–
2	F	–
3	F	–
4	T	5
5	T	0
6	F	–
7	F	–
8	F	–
9	F	–
10	F	–
11	F	–
12	F	–

visit 4

Depth-first search

To visit a vertex v :

- Mark vertex v as visited.
- Recursively visit all unmarked vertices pointing from v.



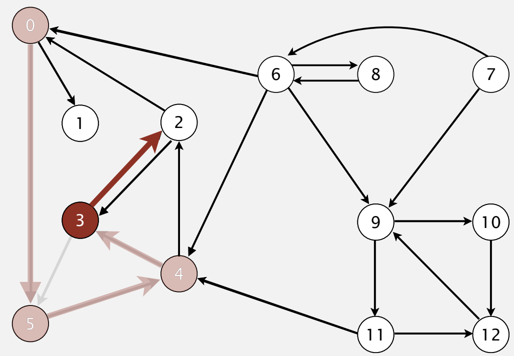
v	marked[]	edgeTo[]
0	T	–
1	F	–
2	F	–
3	T	4
4	T	5
5	T	0
6	F	–
7	F	–
8	F	–
9	F	–
10	F	–
11	F	–
12	F	–

visit 3

Depth-first search

To visit a vertex v :

- Mark vertex v as visited.
- Recursively visit all unmarked vertices pointing from v.



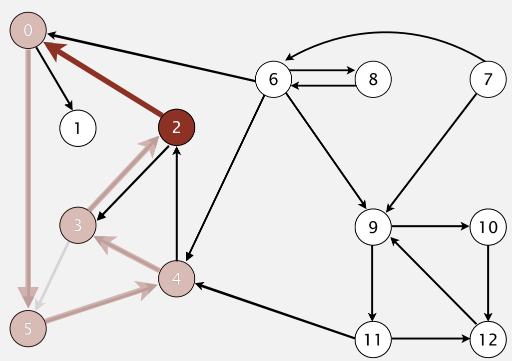
v	marked[]	edgeTo[]
0	T	–
1	F	–
2	F	–
3	T	4
4	T	5
5	T	0
6	F	–
7	F	–
8	F	–
9	F	–
10	F	–
11	F	–
12	F	–

visit 3

Depth-first search

To visit a vertex v :

- Mark vertex v as visited.
- Recursively visit all unmarked vertices pointing from v.



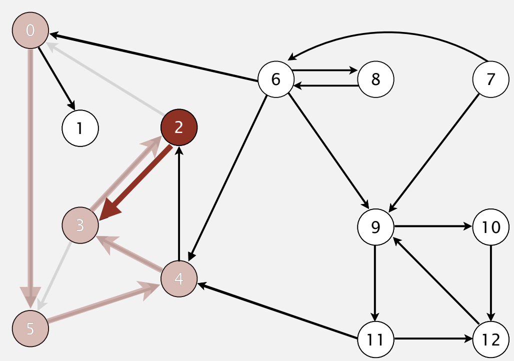
v	marked[]	edgeTo[]
0	T	–
1	F	–
2	T	3
3	T	4
4	T	5
5	T	0
6	F	–
7	F	–
8	F	–
9	F	–
10	F	–
11	F	–
12	F	–

visit 2

Depth-first search

To visit a vertex v :

- Mark vertex v as visited.
- Recursively visit all unmarked vertices pointing from v.



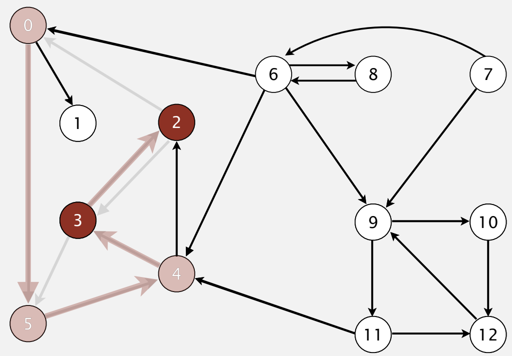
v	marked[]	edgeTo[]
0	T	–
1	F	–
2	T	3
3	T	4
4	T	5
5	T	0
6	F	–
7	F	–
8	F	–
9	F	–
10	F	–
11	F	–
12	F	–

visit 2

Depth-first search

To visit a vertex v :

- Mark vertex v as visited.
- Recursively visit all unmarked vertices pointing from v.



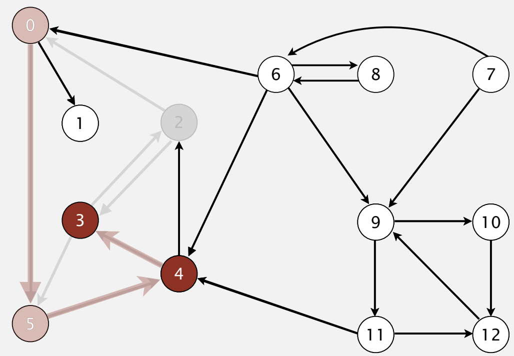
v	marked[]	edgeTo[]
0	T	–
1	F	–
2	T	3
3	T	4
4	T	5
5	T	0
6	F	–
7	F	–
8	F	–
9	F	–
10	F	–
11	F	–
12	F	–

done 2

Depth-first search

To visit a vertex v :

- Mark vertex v as visited.
- Recursively visit all unmarked vertices pointing from v.

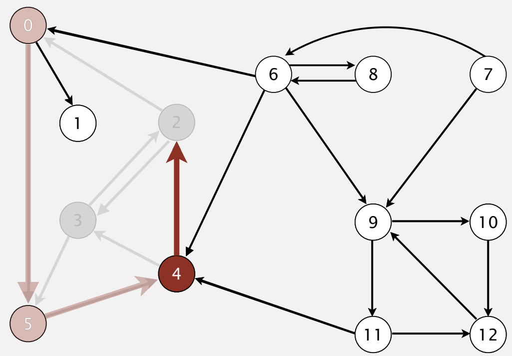


v	marked[]	edgeTo[]
0	T	–
1	F	–
2	T	3
3	T	4
4	T	5
5	T	0
6	F	–
7	F	–
8	F	–
9	F	–
10	F	–
11	F	–
12	F	–

done 3

Depth-first search

To visit a vertex v :
- Mark vertex v as visited.
- Recursively visit all unmarked vertices pointing from v.

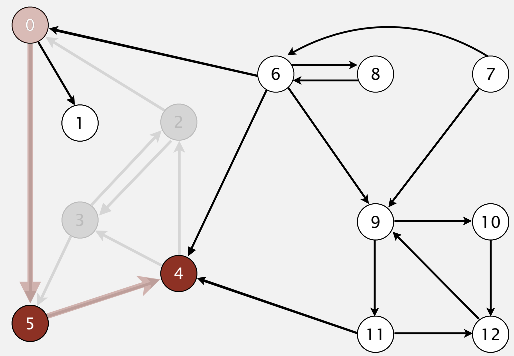


v	marked[]	edgeTo[]
0	T	–
1	F	–
2	T	3
3	T	4
4	T	5
5	T	0
6	F	–
7	F	–
8	F	–
9	F	–
10	F	–
11	F	–
12	F	–

visit 4

Depth-first search

To visit a vertex v :
- Mark vertex v as visited.
- Recursively visit all unmarked vertices pointing from v.

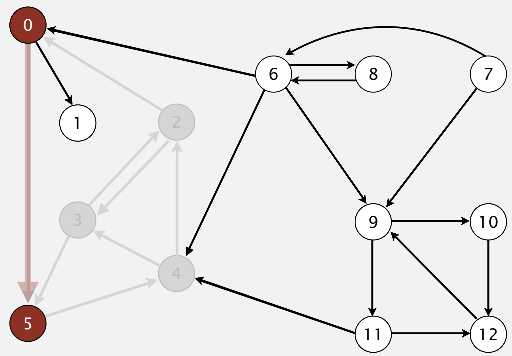


v	marked[]	edgeTo[]
0	T	–
1	F	–
2	T	3
3	T	4
4	T	5
5	T	0
6	F	–
7	F	–
8	F	–
9	F	–
10	F	–
11	F	–
12	F	–

done 4

Depth-first search

To visit a vertex v :
- Mark vertex v as visited.
- Recursively visit all unmarked vertices pointing from v.

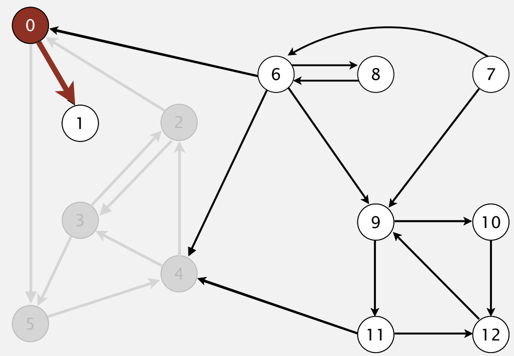


v	marked[]	edgeTo[]
0	T	–
1	F	–
2	T	3
3	T	4
4	T	5
5	T	0
6	F	–
7	F	–
8	F	–
9	F	–
10	F	–
11	F	–
12	F	–

done 5

Depth-first search

To visit a vertex v :
- Mark vertex v as visited.
- Recursively visit all unmarked vertices pointing from v.

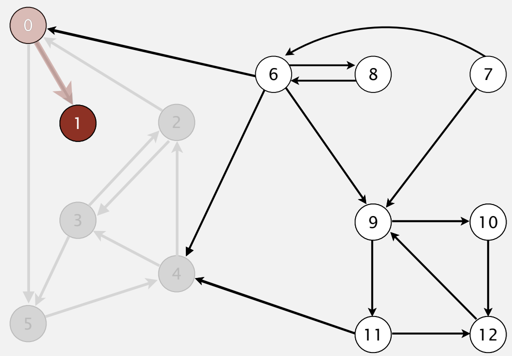


v	marked[]	edgeTo[]
0	T	–
1	F	–
2	T	3
3	T	4
4	T	5
5	T	0
6	F	–
7	F	–
8	F	–
9	F	–
10	F	–
11	F	–
12	F	–

visit 0

Depth-first search

To visit a vertex v :
- Mark vertex v as visited.
- Recursively visit all unmarked vertices pointing from v.

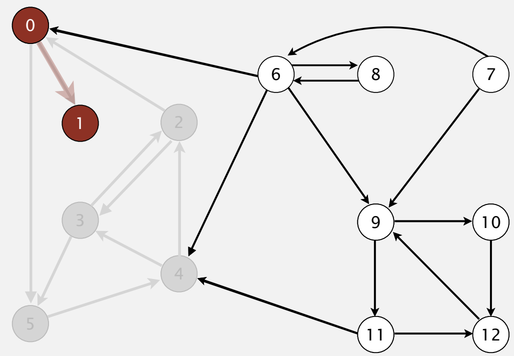


v	marked[]	edgeTo[]
0	T	–
1	T	0
2	T	3
3	T	4
4	T	5
5	T	0
6	F	–
7	F	–
8	F	–
9	F	–
10	F	–
11	F	–
12	F	–

visit 1

Depth-first search

To visit a vertex v :
- Mark vertex v as visited.
- Recursively visit all unmarked vertices pointing from v.

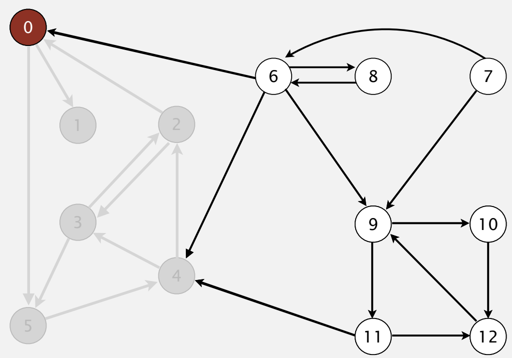


v	marked[]	edgeTo[]
0	T	–
1	T	0
2	T	3
3	T	4
4	T	5
5	T	0
6	F	–
7	F	–
8	F	–
9	F	–
10	F	–
11	F	–
12	F	–

done 1

Depth-first search

To visit a vertex v :
- Mark vertex v as visited.
- Recursively visit all unmarked vertices pointing from v.

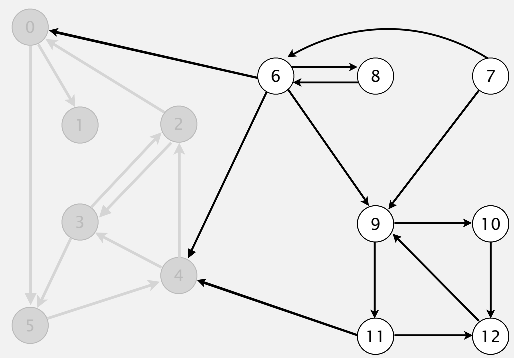


v	marked[]	edgeTo[]
0	T	–
1	T	0
2	T	3
3	T	4
4	T	5
5	T	0
6	F	–
7	F	–
8	F	–
9	F	–
10	F	–
11	F	–
12	F	–

done 0

Depth-first search

To visit a vertex v :
- Mark vertex v as visited.
- Recursively visit all unmarked vertices pointing from v.

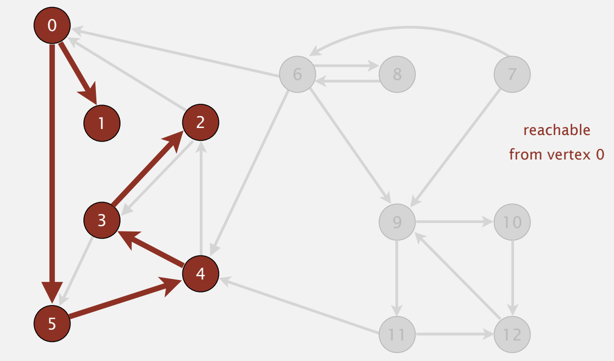


v	marked[]	edgeTo[]
0	T	–
1	T	0
2	T	3
3	T	4
4	T	5
5	T	0
6	F	–
7	F	–
8	F	–
9	F	–
10	F	–
11	F	–
12	F	–

done

Depth-first search

To visit a vertex v :
- Mark vertex v as visited.
- Recursively visit all unmarked vertices pointing from v.



reachable from 0

reachable from vertex 0

v	marked[]	edgeTo[]
0	T	-
1	T	0
2	T	3
3	T	4
4	T	5
5	T	0
6	F	-
7	F	-
8	F	-
9	F	-
10	F	-
11	F	-
12	F	-

Depth-first search (in undirected graphs)

Recall code for undirected graphs.

```
public class DepthFirstSearch
{
   private boolean[] marked;

   public DepthFirstSearch(Graph G, int s)
   {
      marked = new boolean[G.V()];
      dfs(G, s);
   }

   private void dfs(Graph G, int v)
   {
      marked[v] = true;
      for (int w : G.adj(v))
         if (!marked[w]) dfs(G, w);
   }

   public boolean visited(int v)
   {  return marked[v];  }
}
```

true if path to s

constructor marks vertices connected to s

recursive DFS does the work

client can ask whether any vertex is connected to s

Depth-first search (in directed graphs)

Code for directed graphs identical to undirected one.
[substitute `Digraph` for `Graph`]

```
public class DirectedDFS
{
   private boolean[] marked;

   public DirectedDFS(Digraph G, int s)
   {
      marked = new boolean[G.V()];
      dfs(G, s);
   }

   private void dfs(Digraph G, int v)
   {
      marked[v] = true;
      for (int w : G.adj(v))
         if (!marked[w]) dfs(G, w);
   }

   public boolean visited(int v)
   {  return marked[v];  }
}
```

true if path from s

constructor marks vertices reachable from s

recursive DFS does the work

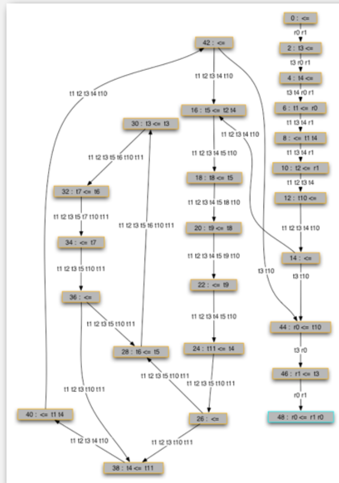
client can ask whether any vertex is reachable from s

Reachability application: program control-flow analysis

Every program is a digraph.
- Vertex = basic block of instructions (straight-line program).
- Edge = jump.

Dead-code elimination.
Find (and remove) unreachable code.

Infinite-loop detection.
Determine whether exit is unreachable.



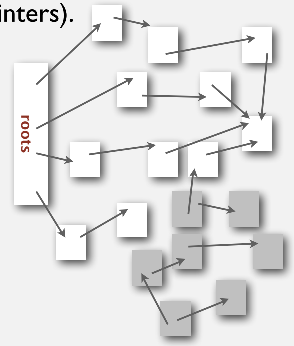
Reachability application: mark-sweep garbage collector

Every data structure is a digraph.

- Vertex = object.
- Edge = reference.

Roots. Objects known to be directly accessible by program (e.g., stack).

Reachable objects. Objects indirectly accessible by program (starting at a root and following a chain of pointers).

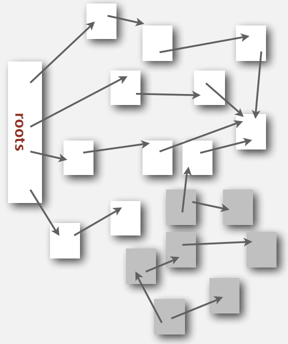


Reachability application: mark-sweep garbage collector

Mark-sweep algorithm. [McCarthy, 1960]

- Mark: mark all reachable objects.
- Sweep: if object is unmarked, it is garbage (so add to free list).

Memory cost. Uses 1 extra mark bit per object (plus DFS stack).



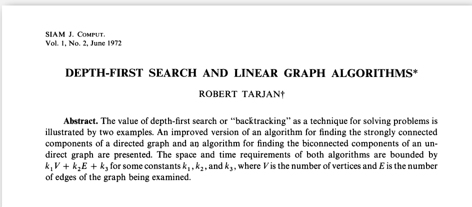
Depth-first search in digraphs summary

DFS enables direct solution of simple digraph problems.

- ✓ Reachability.
- Path finding.
- Topological sort.
- Directed cycle detection.

Basis for solving difficult digraph problems.

- 2-satisfiability.
- Directed Euler path.
- Strongly-connected components.



SIAM J. Comput.
Vol. 1, No. 2, June 1972

DEPTH-FIRST SEARCH AND LINEAR GRAPH ALGORITHMS*

ROBERT TARJAN†

Abstract. The value of depth-first search or "backtracking" as a technique for solving problems is illustrated by two examples. An improved version of an algorithm for finding the strongly connected components of a directed graph and an algorithm for finding the biconnected components of an undirect graph are presented. The space and time requirements of both algorithms are bounded by $k_1V + k_2E + k_3$ for some constants k_1, k_2, and k_3, where V is the number of vertices and E is the number of edges of the graph being examined.

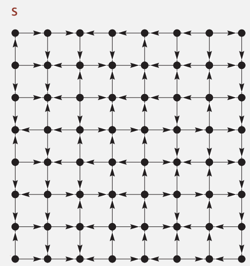
Breadth-first search in digraphs

Same method as for undirected graphs.

- Every undirected graph is a digraph (with edges in both directions).
- BFS is a digraph algorithm.

BFS (from source vertex s)

Put s onto a FIFO queue, and mark s as visited.
Repeat until the queue is empty:
- **remove the least recently added vertex v**
- **for each unmarked vertex pointing from v: add to queue and mark as visited.**



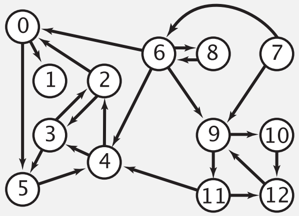
Proposition. BFS computes shortest paths (fewest number of edges) from s to all other vertices n a digraph in time proportional to $E+V$.

Multiple-source shortest paths

Multiple-source shortest paths. Given a digraph and a set of source vertices, find shortest path from any vertex in the set to each other vertex.

Ex. $S = \{ 1, 7, 10 \}$.

- Shortest path to 4 is $7 \to 6 \to 4$.
- Shortest path to 5 is $7 \to 6 \to 0 \to 5$.
- Shortest path to 12 is $10 \to 12$.



Q. How to implement multi-source constructor?

A. Use BFS, but initialize by enqueuing all source vertices.

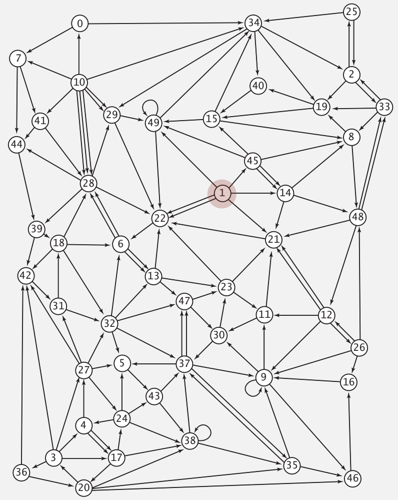
Breadth-first search in digraphs application: web crawler

Goal. Crawl web, starting from some root web page, say `www.princeton.edu`.

Solution. BFS with implicit graph.

BFS.

- Choose root web page as source s.
- Maintain a `Queue` of websites to explore.
- Maintain a `SET` of discovered websites.
- Dequeue the next website and enqueue websites to which it links (provided you haven't done so before).



Q. Why not use DFS?

Bare-bones web crawler: Java implementation

```
Queue<String> queue = new Queue<String>();
SET<String> discovered = new SET<String>();

String root = "http://www.princeton.edu";
queue.enqueue(root);
discovered.add(root);

while (!queue.isEmpty())
{
   String v = queue.dequeue();
   StdOut.println(v);
   In in = new In(v);
   String input = in.readAll();

   String regexp = "http://(\\w+\\.)*(\\w+)";
   Pattern pattern = Pattern.compile(regexp);
   Matcher matcher = pattern.matcher(input);
   while (matcher.find())
   {
      String w = matcher.group();
      if (!discovered.contains(w))
      {
         discovered.add(w);
         queue.enqueue(w);
      }
   }
}
```

queue of websites to crawl

set of discovered websites

start crawling from root website

read in raw html from next website in queue

use regular expression to find all URLs in website of form `http://xxx.yyy.zzz` [crude pattern misses relative URLs]

if undiscovered, mark it as discovered and put on queue

DIRECTED GRAPHS

- Digraph API
- Digraph search
- **Topological sort**
- Strong components

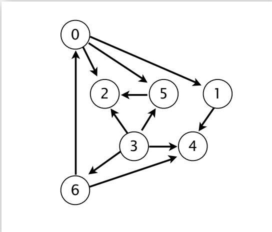
Precedence scheduling

Goal. Given a set of tasks to be completed with precedence constraints, in which order should we schedule the tasks?

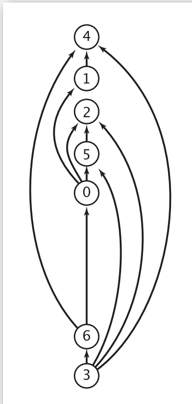
Digraph model. vertex = task; edge = precedence constraint.

0. Algorithms
1. Complexity Theory
2. Artificial Intelligence
3. Intro to CS
4. Cryptography
5. Scientific Computing
6. Advanced Programming

tasks



precedence constraint graph



feasible schedule

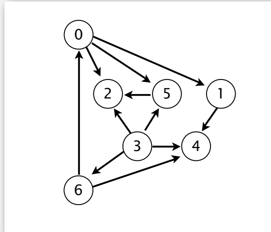
Topological sort

DAG. Directed acyclic graph.

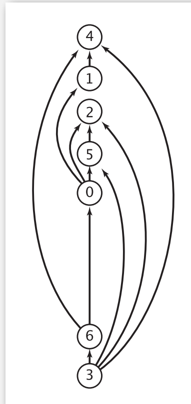
Topological sort. Redraw DAG so all edges point upwards.

0→5	0→2
0→1	3→6
3→5	3→4
5→4	6→4
6→0	3→2
1→4	

directed edges



DAG

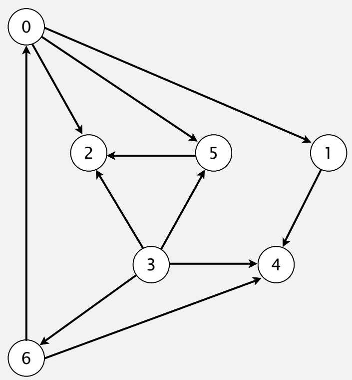


topological order

Solution. DFS. What else?

Topological sort algorithm

- Run depth-first search.
- Return vertices in reverse postorder.

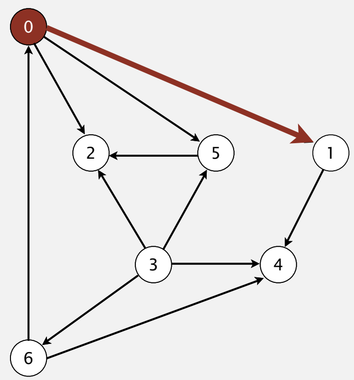


a directed acyclic graph

0→5
0→2
0→1
3→6
3→5
3→4
5→4
6→4
6→0
3→2
1→4

Topological sort algorithm

- Run depth-first search.
- Return vertices in reverse postorder.

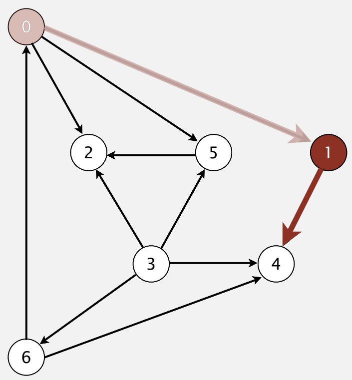


postorder

visit 0

Topological sort algorithm

- Run depth-first search.
- Return vertices in reverse postorder.

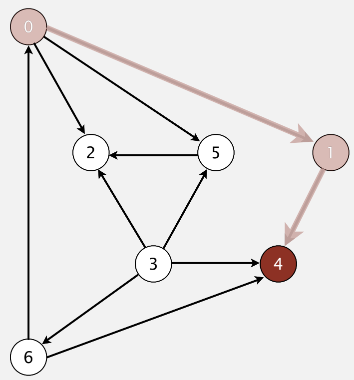


postorder

visit 1

Topological sort algorithm

- Run depth-first search.
- Return vertices in reverse postorder.

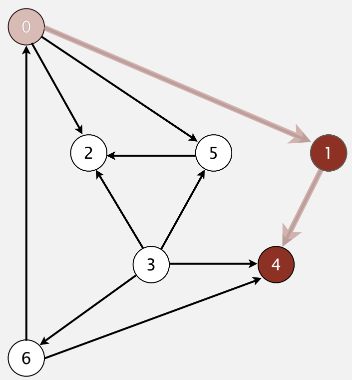


postorder

visit 4

Topological sort algorithm

- Run depth-first search.
- Return vertices in reverse postorder.



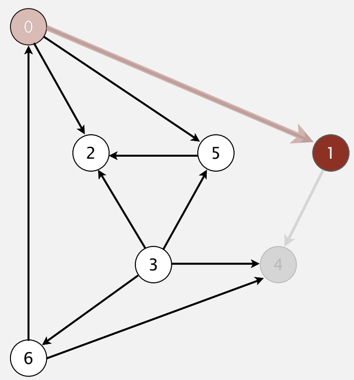
postorder

4

4 done

Topological sort algorithm

- Run depth-first search.
- Return vertices in reverse postorder.



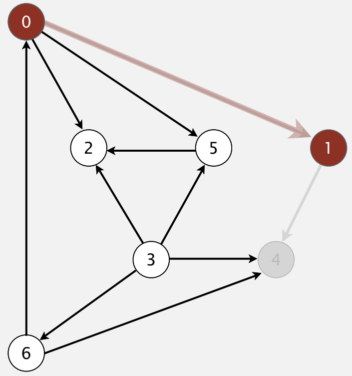
postorder

4

visit 1

Topological sort algorithm

- Run depth-first search.
- Return vertices in reverse postorder.



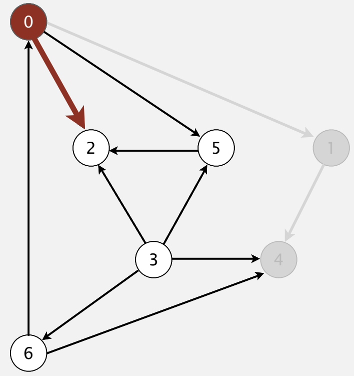
postorder

4 1

1 done

Topological sort algorithm

- Run depth-first search.
- Return vertices in reverse postorder.



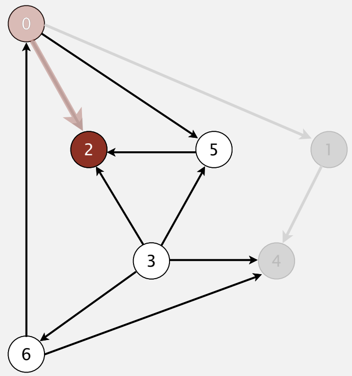
postorder

4 1

visit 0

Topological sort algorithm

- Run depth-first search.
- Return vertices in reverse postorder.



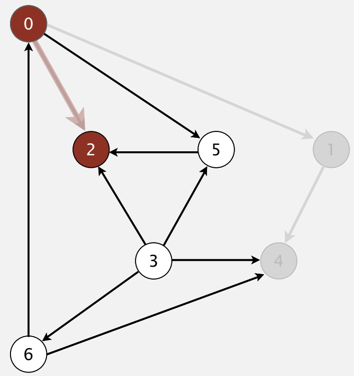
postorder

4 1

visit 2

Topological sort algorithm

- Run depth-first search.
- Return vertices in reverse postorder.



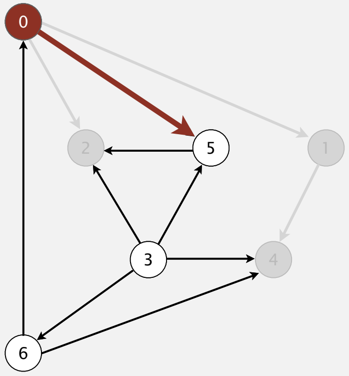
postorder

4 1 2

2 done

Topological sort algorithm

- Run depth-first search.
- Return vertices in reverse postorder.



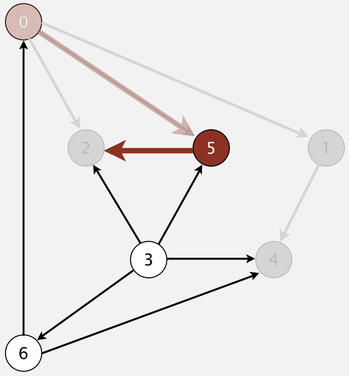
postorder

4 1 2

visit 0

Topological sort algorithm

- Run depth-first search.
- Return vertices in reverse postorder.



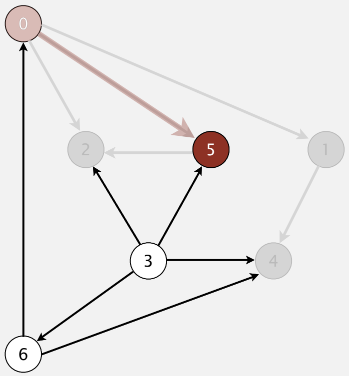
postorder

4 1 2

visit 5

Topological sort algorithm

- Run depth-first search.
- Return vertices in reverse postorder.



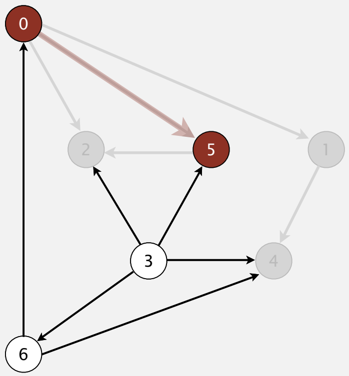
postorder

4 1 2

visit 5

Topological sort algorithm

- Run depth-first search.
- Return vertices in reverse postorder.



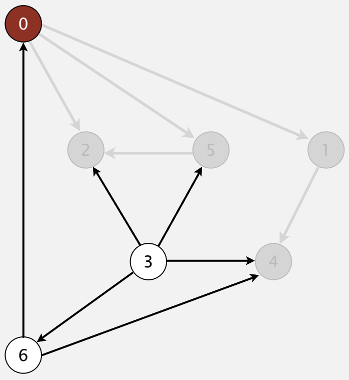
postorder

4 1 2 5

5 done

Topological sort algorithm

- Run depth-first search.
- Return vertices in reverse postorder.



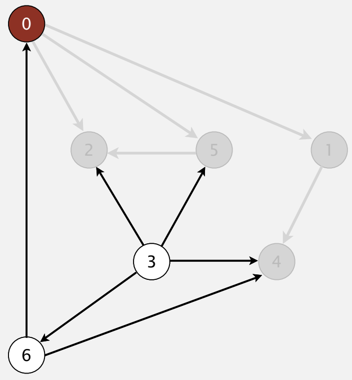
postorder

4 1 2 5

visit 0

Topological sort algorithm

- Run depth-first search.
- Return vertices in reverse postorder.



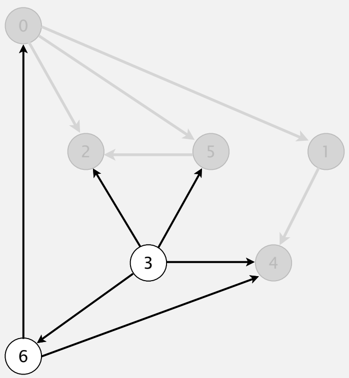
postorder

4 1 2 5 0

0 done

Topological sort algorithm

- Run depth-first search.
- Return vertices in reverse postorder.



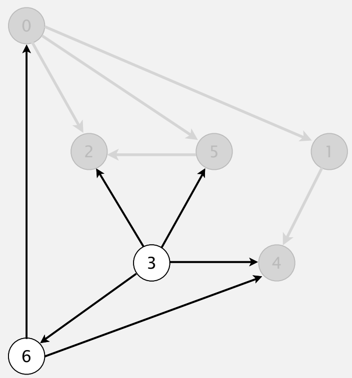
postorder

4 1 2 5 0

check 1

Topological sort algorithm

- Run depth-first search.
- Return vertices in reverse postorder.



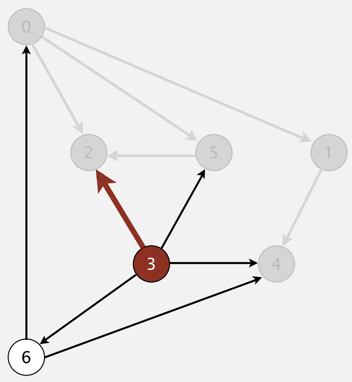
postorder

4 1 2 5 0

check 2

Topological sort algorithm

- Run depth-first search.
- Return vertices in reverse postorder.



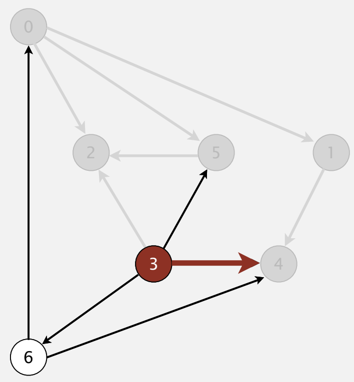
postorder

4 1 2 5 0

visit 3

Topological sort algorithm

- Run depth-first search.
- Return vertices in reverse postorder.



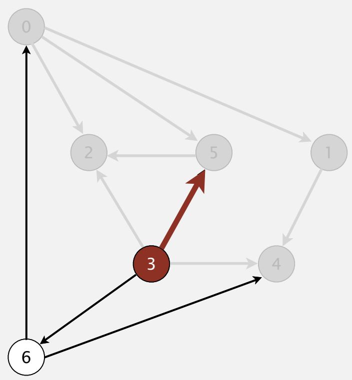
postorder

4 1 2 5 0

visit 3

Topological sort algorithm

- Run depth-first search.
- Return vertices in reverse postorder.



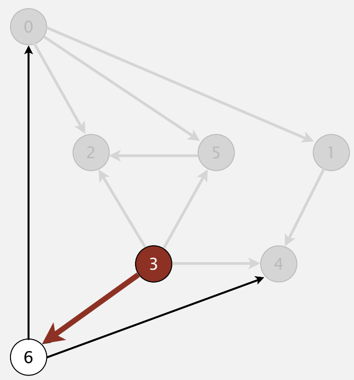
postorder

4 1 2 5 0

visit 3

Topological sort algorithm

- Run depth-first search.
- Return vertices in reverse postorder.



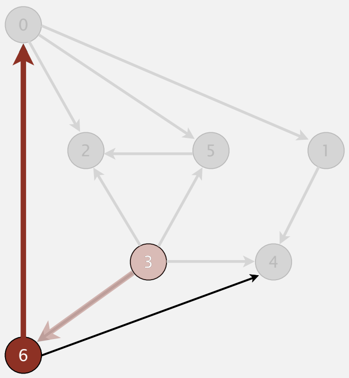
postorder

4 1 2 5 0

visit 3

Topological sort algorithm

- Run depth-first search.
- Return vertices in reverse postorder.



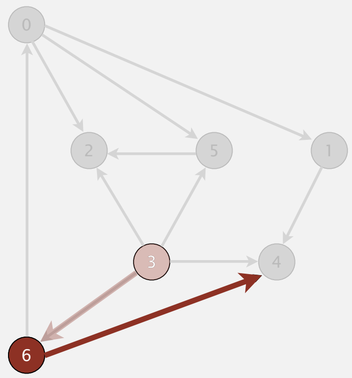
postorder

4 1 2 5 0

visit 6

Topological sort algorithm

- Run depth-first search.
- Return vertices in reverse postorder.



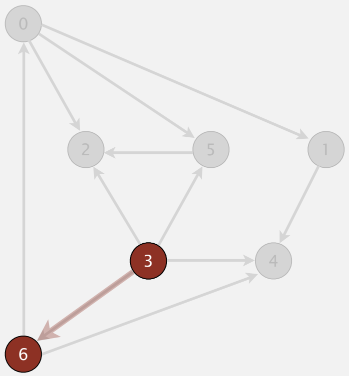
postorder

4 1 2 5 0

visit 6

Topological sort algorithm

- Run depth-first search.
- Return vertices in reverse postorder.



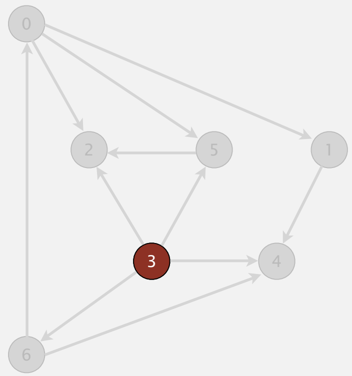
postorder

4 1 2 5 0 6

6 done

Topological sort algorithm

- Run depth-first search.
- Return vertices in reverse postorder.



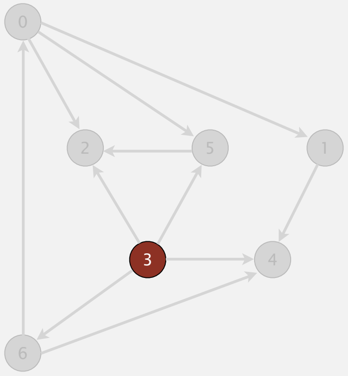
postorder

4 1 2 5 0 6

visit 3

Topological sort algorithm

- Run depth-first search.
- Return vertices in reverse postorder.



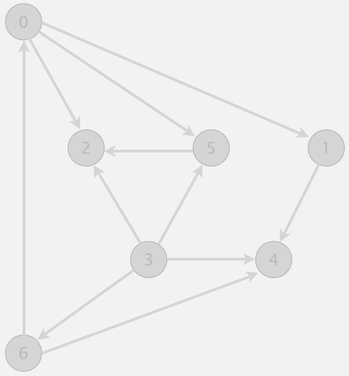
postorder

4 1 2 5 0 6 3

3 done

Topological sort algorithm

- Run depth-first search.
- Return vertices in reverse postorder.



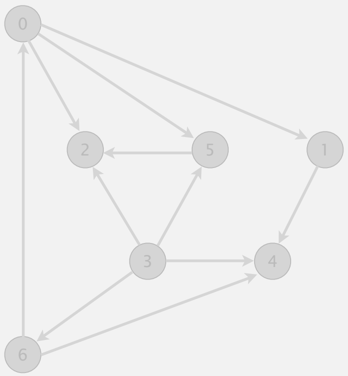
postorder

4 1 2 5 0 6 3

check 4

Topological sort algorithm

- Run depth-first search.
- Return vertices in reverse postorder.



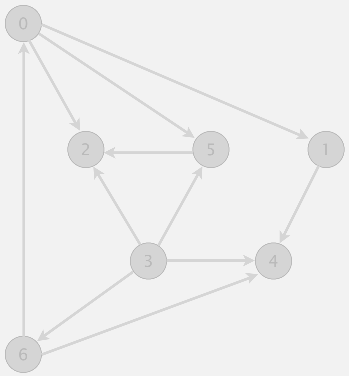
postorder

4 1 2 5 0 6 3

check 5

Topological sort algorithm

- Run depth-first search.
- Return vertices in reverse postorder.



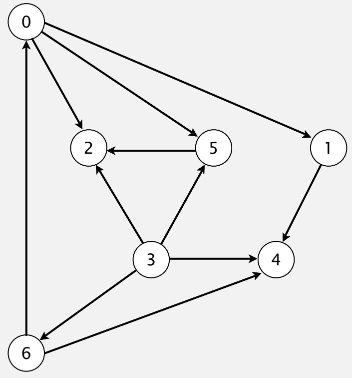
postorder

4 1 2 5 0 6 3

check 6

Topological sort algorithm

- Run depth-first search.
- Return vertices in reverse postorder.



postorder

4 1 2 5 0 6 3

topological order

3 6 0 5 2 1 4

done

Depth-first search order

```
public class DepthFirstOrder
{
   private boolean[] marked;
   private Stack<Integer> reversePost;

   public DepthFirstOrder(Digraph G)
   {
      reversePost = new Stack<Integer>();
      marked = new boolean[G.V()];
      for (int v = 0; v < G.V(); v++)
         if (!marked[v]) dfs(G, v);
   }

   private void dfs(Digraph G, int v)
   {
      marked[v] = true;
      for (int w : G.adj(v))
         if (!marked[w]) dfs(G, w);
      reversePost.push(v);
   }

   public Iterable<Integer> reversePost()
   {  return reversePost;  }
}
```

returns all vertices in "reverse DFS postorder"

Topological sort in a DAG: correctness proof

Proposition. Reverse DFS postorder of a DAG is a topological order.

Pf. Consider any edge $v \to w$. When `dfs(v)` is called:

- Case 1: `dfs(w)` has already been called and returned.
  Thus, w was done before v.
- Case 2: `dfs(w)` has not yet been called.
  `dfs(w)` will get called directly or indirectly by `dfs(v)` and will finish before `dfs(v)`.
  Thus, w will be done before v.
- Case 3: `dfs(w)` has already been called, but has not yet returned.
  Can't happen in a DAG: function call stack contains path from w to v, so $v \to w$ would complete a cycle.

```
dfs(0)
  dfs(1)
    dfs(4)
    4 done
  1 done
  dfs(2)
  2 done
  dfs(5)
    check 2
  5 done
0 done
check 1
check 2
dfs(3)
  check 2
  check 4
  check 5
  dfs(6)
  6 done
3 done
check 4
check 5
check 6
done
```

Ex:

case 1

case 2

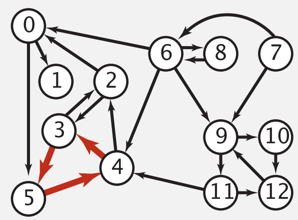
all vertices pointing from 3 are done before 3 is done, so they appear after 3 in topological order

Directed cycle detection

Proposition. A digraph has a topological order iff no directed cycle.

Pf.

- If directed cycle, topological order impossible.
- If no directed cycle, DFS-based algorithm finds a topological order.



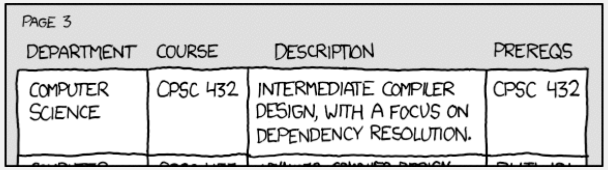
a digraph with a directed cycle

Goal. Given a digraph, find a directed cycle.

Solution. DFS. What else? See textbook.

Directed cycle detection application: precedence scheduling

Scheduling. Given a set of tasks to be completed with precedence constraints, in what order should we schedule the tasks?



http://xkcd.com/754

Remark. A directed cycle implies scheduling problem is infeasible.

Directed cycle detection application: cyclic inheritance

The Java compiler does cycle detection.

```
public class A extends B
{
   ...
}
```

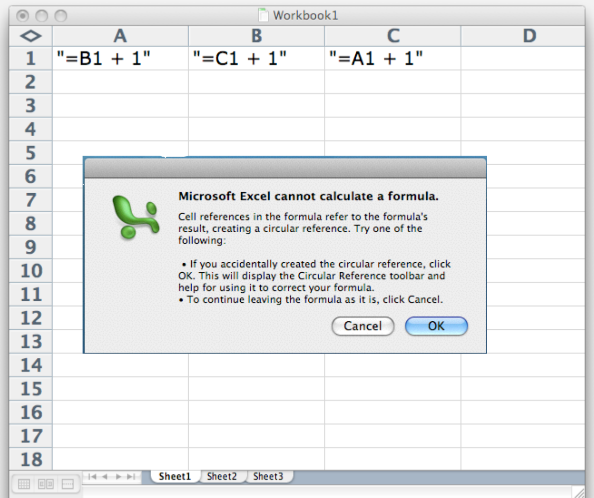
```
public class B extends C
{
   ...
}
```

```
public class C extends A
{
   ...
}
```

```
% javac A.java
A.java:1: cyclic inheritance
involving A
public class A extends B { }
       ^
1 error
```


Directed cycle detection application: spreadsheet recalculation

Microsoft Excel does cycle detection (and has a circular reference toolbar!)



Directed cycle detection applications

- Causalities.
- Email loops.
- Compilation units.
- Class inheritance.
- Course prerequisites.
- Deadlocking detection.
- Precedence scheduling.
- Temporal dependencies.
- Pipeline of computing jobs.
- Check for symbolic link loop.
- Evaluate formula in spreadsheet.

Directed Graphs

- Digraph API
- Digraph search
- Topological sort
- **Strong components**

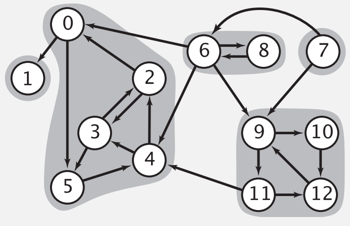
Strongly-connected components

Def. Vertices v and w are strongly connected if there is a directed path from v to w and a directed path from w to v.

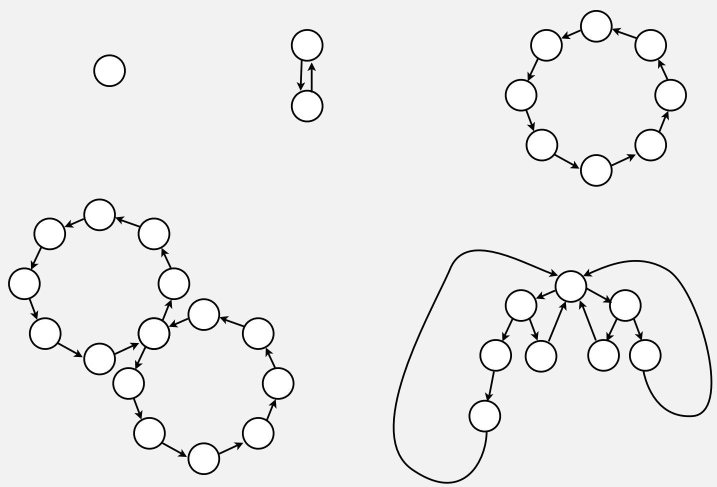
Key property. Strong connectivity is an equivalence relation:

- v is strongly connected to v.
- If v is strongly connected to w, then w is strongly connected to v.
- If v is strongly connected to w and w to x, then v is strongly connected to x.

Def. A strong component is a maximal subset of strongly-connected vertices.

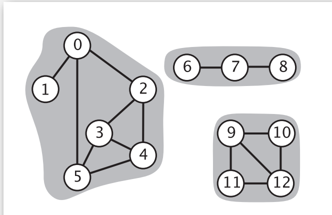


Examples of strongly-connected digraphs



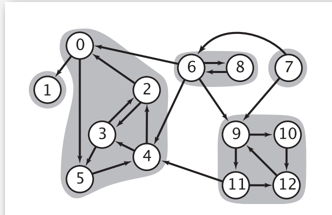
Connected components vs. strongly-connected components

- v and w are connected if there is a path between v and w



3 connected components

- v and w are strongly connected if there is a directed path from v to w and a directed path from w to v



5 strongly-connected components

connected component id (easy to compute with DFS)

	0	1	2	3	4	5	6	7	8	9	10	11	12
cc[]	0	0	0	0	0	0	1	1	1	2	2	2	2

strongly-connected component id (how to compute?)

	0	1	2	3	4	5	6	7	8	9	10	11	12
scc[]	1	0	1	1	1	1	3	4	3	2	2	2	2

```
public int connected(int v, int w)
{  return cc[v] == cc[w];  }
```

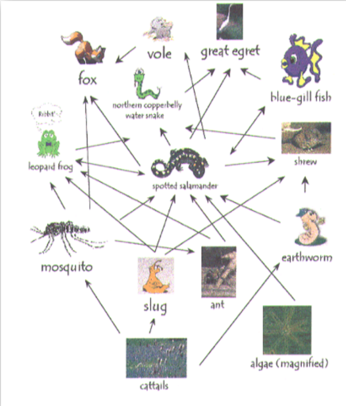
constant-time client connectivity query

```
public int stronglyConnected(int v, int w)
{  return scc[v] == scc[w];  }
```

constant-time client strong-connectivity query

Strong component application: ecological food webs

Food web graph. Vertex = species; edge = from producer to consumer.



http://www.twingroves.district96.k12.il.us/Wetlands/Salamander/SalGraphics/salfoodweb.gif

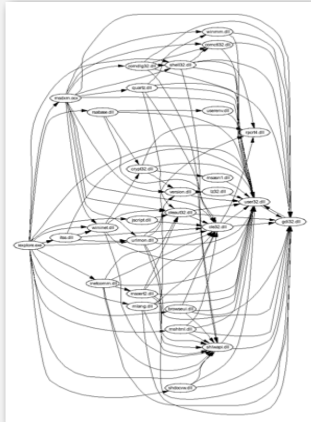
Strong component. Subset of species with common energy flow.

Strong component application: software modules

Software module dependency graph.

- Vertex = software module.
- Edge: from module to dependency.

Firefox



Internet Explorer

Strong component. Subset of mutually interacting modules.
Approach 1. Package strong components together.
Approach 2. Use to improve design!

Strong components algorithms: brief history

1960s: Core OR problem.

- Widely studied; some practical algorithms.
- Complexity not understood.

1972: linear-time DFS algorithm (Tarjan).

- Classic algorithm.
- Level of difficulty: Algs4++.
- Demonstrated broad applicability and importance of DFS.

1980s: easy two-pass linear-time algorithm (Kosaraju-Sharir).

- Forgot notes for lecture; developed algorithm in order to teach it!
- Later found in Russian scientific literature (1972).

1990s: more easy linear-time algorithms.

- Gabow: fixed old OR algorithm.
- Cheriyan-Mehlhorn: needed one-pass algorithm for LEDA.

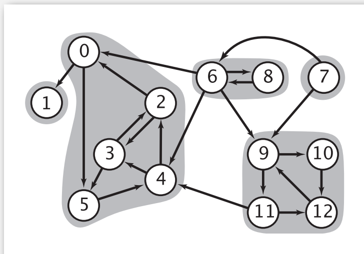
Kosaraju's algorithm: intuition

Reverse graph. Strong components in G are same as in G^R.

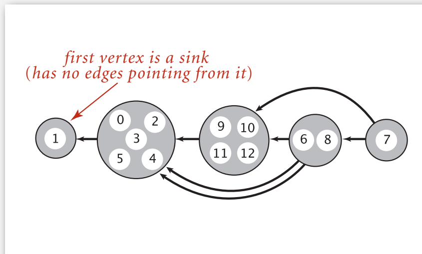
Kernel DAG. Contract each strong component into a single vertex.

Idea. how to compute?

- Compute topological order (reverse postorder) in kernel DAG.
- Run DFS, considering vertices in reverse topological order.



digraph G and its strong components



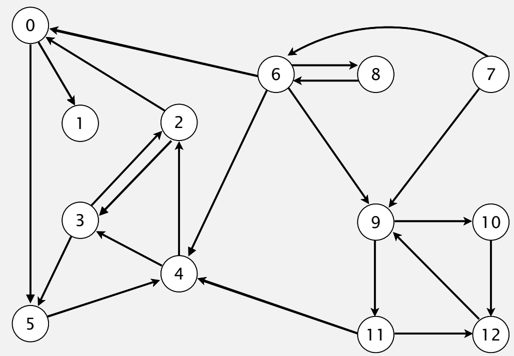
kernel DAG of G (in reverse topological order)

Kosaraju's algorithm

- DFS in reverse graph
- DFS in original graph

Kosaraju-Sharir

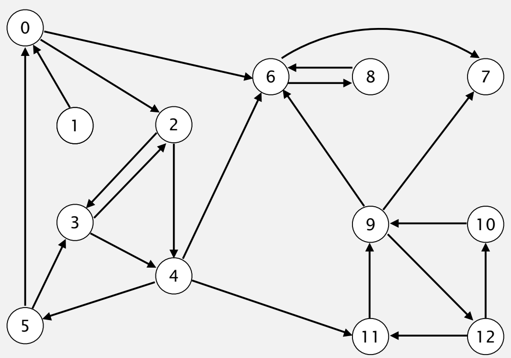
Phase 1. Compute reverse postorder in G^R.



digraph G

Kosaraju-Sharir

Phase 1. Compute reverse postorder in G^R.

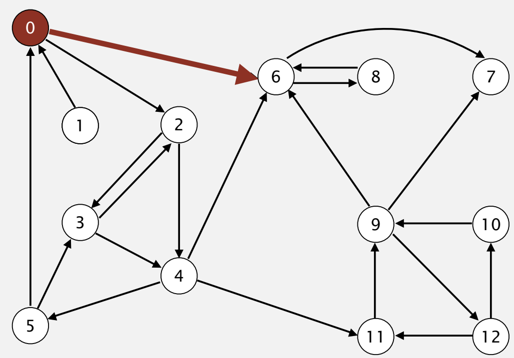


v	marked[v]
0	–
1	–
2	–
3	–
4	–
5	–
6	–
7	–
8	–
9	–
10	–
11	–
12	–

reverse digraph G^R

Kosaraju-Sharir

Phase 1. Compute reverse postorder in G^R.

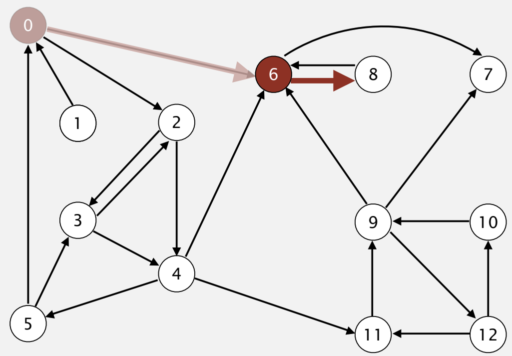


v	marked[v]
0	T
1	F
2	F
3	F
4	F
5	F
6	F
7	F
8	F
9	F
10	F
11	F
12	F

visit 0

Kosaraju-Sharir

Phase 1. Compute reverse postorder in G^R.

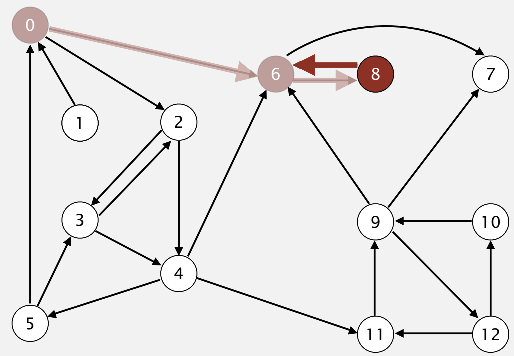


v	marked[v]
0	T
1	F
2	F
3	F
4	F
5	F
6	T
7	F
8	F
9	F
10	F
11	F
12	F

visit 6

Kosaraju-Sharir

Phase 1. Compute reverse postorder in G^R.



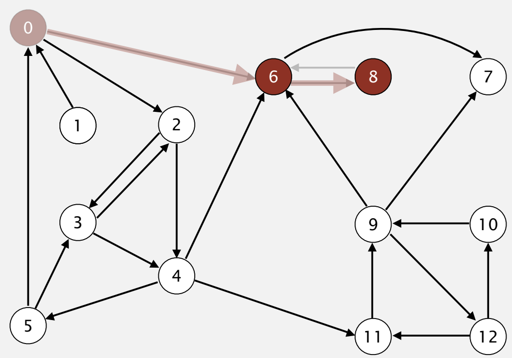
v	marked[v]
0	T
1	F
2	F
3	F
4	F
5	F
6	T
7	F
8	T
9	F
10	F
11	F
12	F

visit 8

Kosaraju-Sharir

Phase 1. Compute reverse postorder in G^R.

8



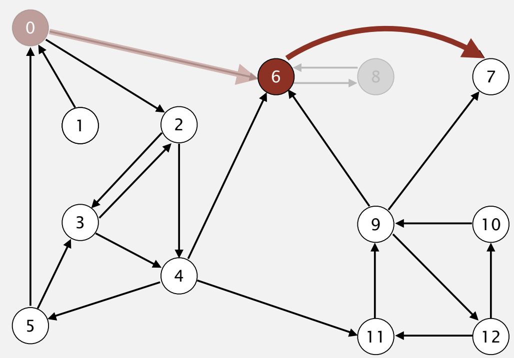
v	marked[v]
0	T
1	F
2	F
3	F
4	F
5	F
6	T
7	F
8	T
9	F
10	F
11	F
12	F

8 done

Kosaraju-Sharir

Phase 1. Compute reverse postorder in G^R.

8



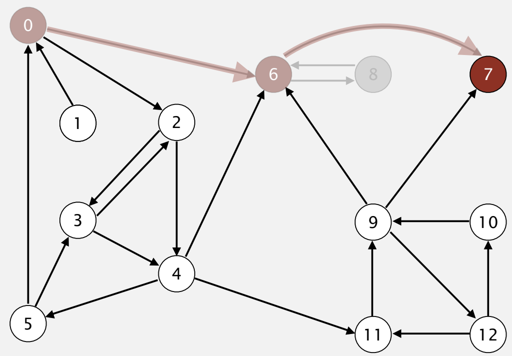
v	marked[v]
0	T
1	F
2	F
3	F
4	F
5	F
6	T
7	F
8	T
9	F
10	F
11	F
12	F

visit 6

Kosaraju-Sharir

Phase 1. Compute reverse postorder in G^R.

8



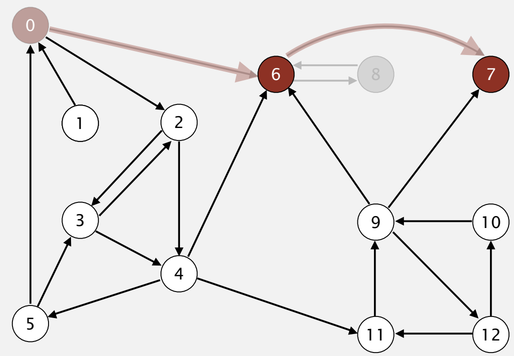
v	marked[v]
0	T
1	F
2	F
3	F
4	F
5	F
6	T
7	T
8	T
9	F
10	F
11	F
12	F

visit 7

Kosaraju-Sharir

Phase 1. Compute reverse postorder in G^R.

7 8



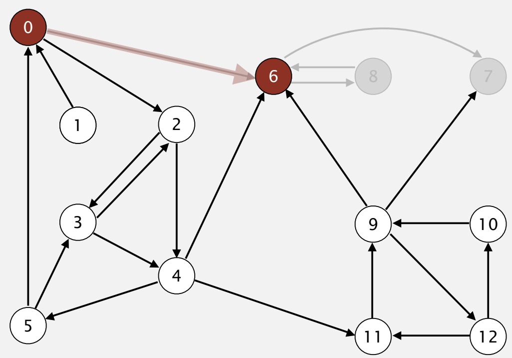
v	marked[v]
0	T
1	F
2	F
3	F
4	F
5	F
6	T
7	T
8	T
9	F
10	F
11	F
12	F

7 done

Kosaraju-Sharir

Phase 1. Compute reverse postorder in G^R.

6 7 8



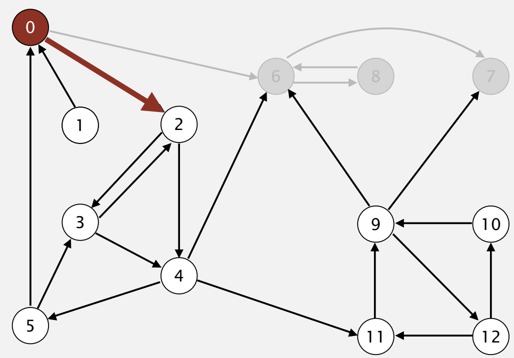
v	marked[v]
0	T
1	F
2	F
3	F
4	F
5	F
6	T
7	T
8	T
9	F
10	F
11	F
12	F

6 done

Kosaraju-Sharir

Phase 1. Compute reverse postorder in G^R.

6 7 8



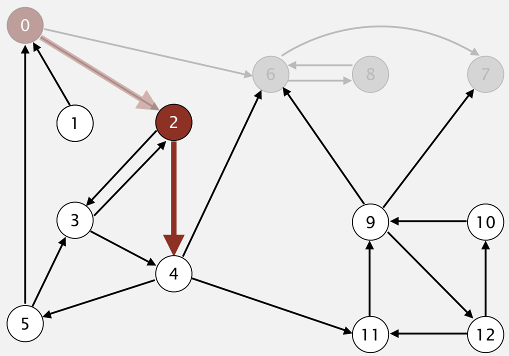
v	marked[v]
0	T
1	F
2	F
3	F
4	F
5	F
6	T
7	T
8	T
9	F
10	F
11	F
12	F

visit 0

Kosaraju-Sharir

Phase 1. Compute reverse postorder in G^R.

6 7 8



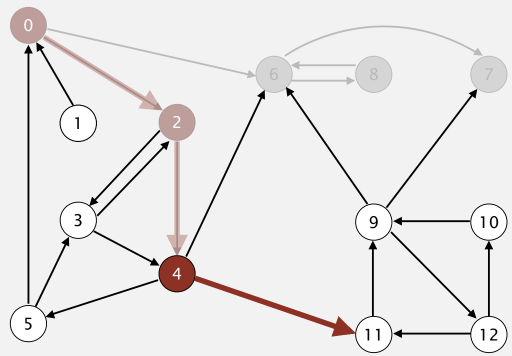
v	marked[v]
0	T
1	F
2	T
3	F
4	F
5	F
6	T
7	T
8	T
9	F
10	F
11	F
12	F

visit 2

Kosaraju-Sharir

Phase 1. Compute reverse postorder in G^R.

6 7 8



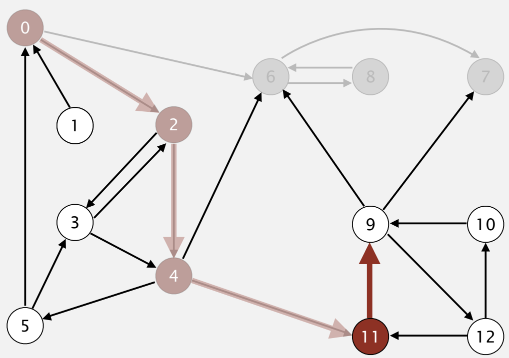
v	marked[v]
0	T
1	F
2	T
3	F
4	T
5	F
6	T
7	T
8	T
9	F
10	F
11	F
12	F

visit 4

Kosaraju-Sharir

Phase 1. Compute reverse postorder in G^R.

6 7 8



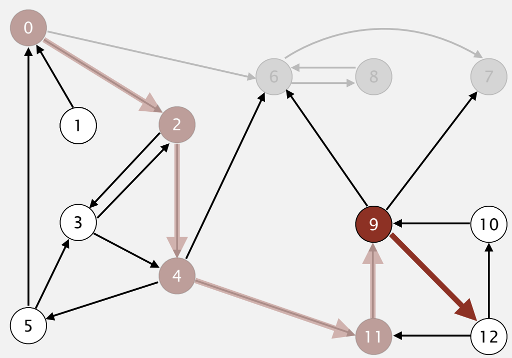
v	marked[v]
0	T
1	F
2	T
3	F
4	T
5	F
6	T
7	T
8	T
9	F
10	F
11	T
12	F

visit 11

Kosaraju-Sharir

Phase 1. Compute reverse postorder in G^R.

6 7 8



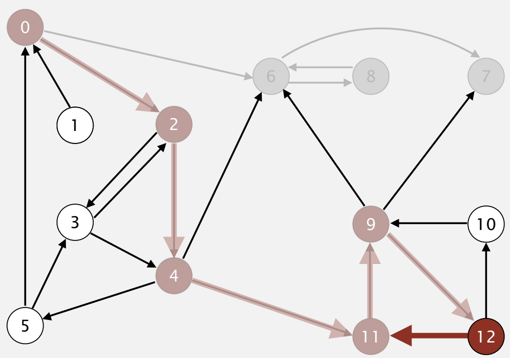
v	marked[v]
0	T
1	F
2	T
3	F
4	T
5	F
6	T
7	T
8	T
9	T
10	F
11	T
12	F

visit 9

Kosaraju-Sharir

Phase 1. Compute reverse postorder in G^R.

6 7 8



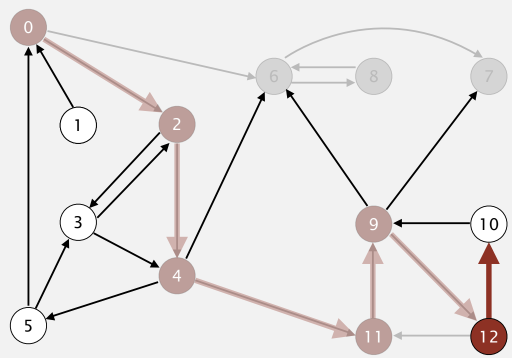
v	marked[v]
0	T
1	F
2	T
3	F
4	T
5	F
6	T
7	T
8	T
9	T
10	F
11	T
12	T

visit 12

Kosaraju-Sharir

Phase 1. Compute reverse postorder in G^R.

6 7 8



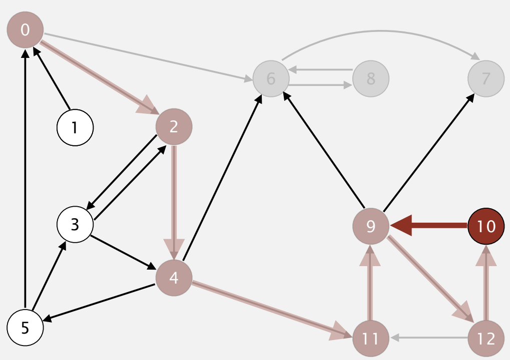
v	marked[v]
0	T
1	F
2	T
3	F
4	T
5	F
6	T
7	T
8	T
9	T
10	F
11	T
12	T

visit 12

Kosaraju-Sharir

Phase 1. Compute reverse postorder in G^R.

6 7 8



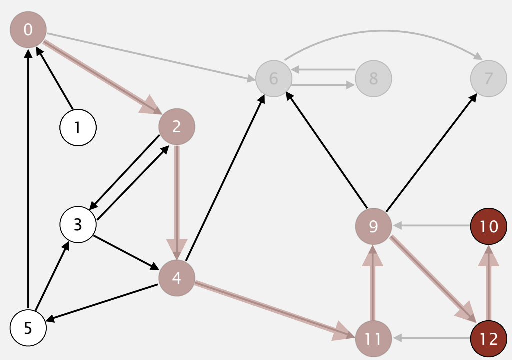
v	marked[v]
0	T
1	F
2	T
3	F
4	T
5	F
6	T
7	T
8	T
9	T
10	T
11	T
12	T

visit 10

Kosaraju-Sharir

Phase 1. Compute reverse postorder in G^R.

10 6 7 8



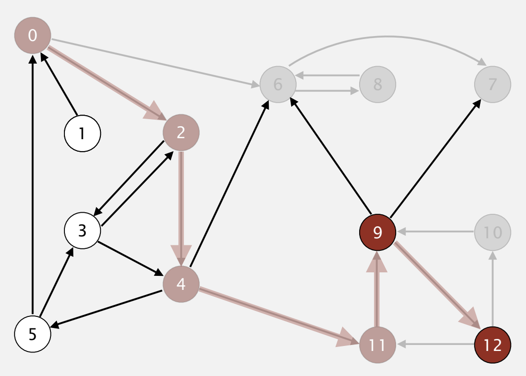
v	marked[v]
0	T
1	F
2	T
3	F
4	T
5	F
6	T
7	T
8	T
9	T
10	T
11	T
12	T

10 done

Kosaraju-Sharir

Phase 1. Compute reverse postorder in G^R.

12 10 6 7 8



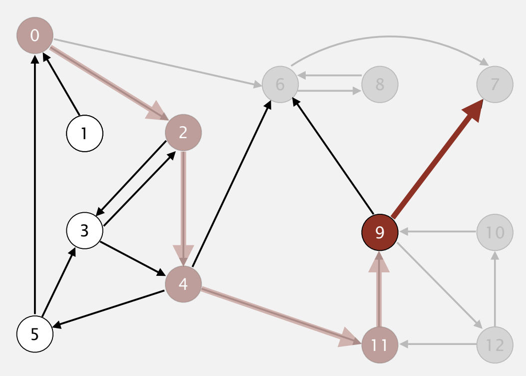
12 done

v	marked[v]
0	T
1	F
2	T
3	F
4	T
5	F
6	T
7	T
8	T
9	T
10	T
11	T
12	T

Kosaraju-Sharir

Phase 1. Compute reverse postorder in G^R.

12 10 6 7 8



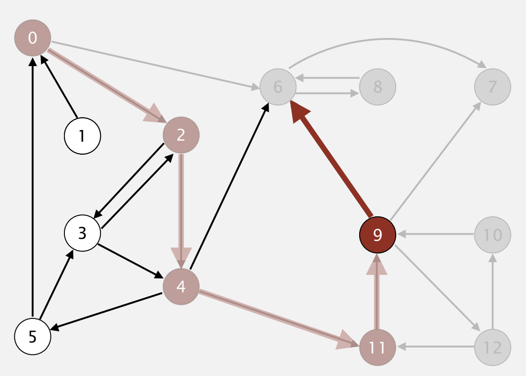
visit 9

v	marked[v]
0	T
1	F
2	T
3	F
4	T
5	F
6	T
7	T
8	T
9	T
10	T
11	T
12	T

Kosaraju-Sharir

Phase 1. Compute reverse postorder in G^R.

12 10 6 7 8



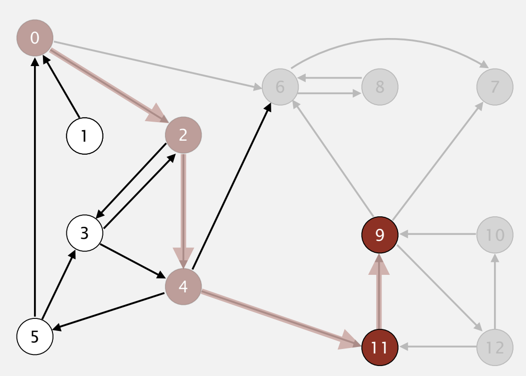
visit 9

v	marked[v]
0	T
1	F
2	T
3	F
4	T
5	F
6	T
7	T
8	T
9	T
10	T
11	T
12	T

Kosaraju-Sharir

Phase 1. Compute reverse postorder in G^R.

9 12 10 6 7 8



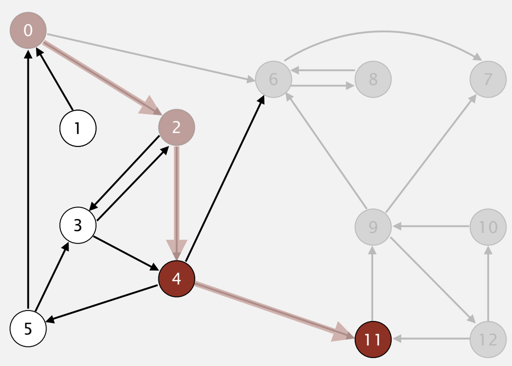
9 done

v	marked[v]
0	T
1	F
2	T
3	F
4	T
5	F
6	T
7	T
8	T
9	T
10	T
11	T
12	T

Kosaraju-Sharir

Phase 1. Compute reverse postorder in G^R.

11 9 12 10 6 7 8



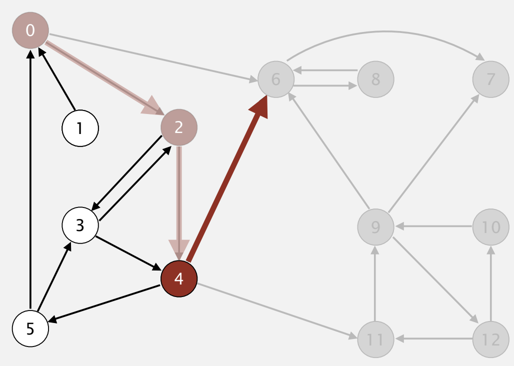
v	marked[v]
0	T
1	F
2	T
3	F
4	T
5	F
6	T
7	T
8	T
9	T
10	T
11	T
12	T

11 done

Kosaraju-Sharir

Phase 1. Compute reverse postorder in G^R.

11 9 12 10 6 7 8



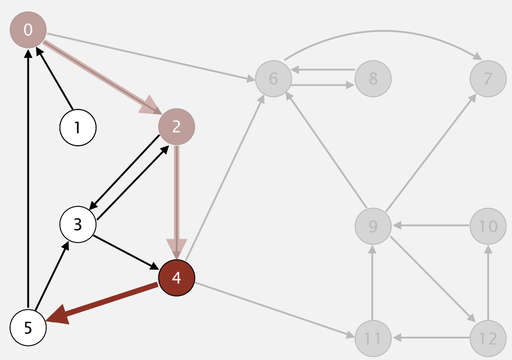
v	marked[v]
0	T
1	F
2	T
3	F
4	T
5	F
6	T
7	T
8	T
9	T
10	T
11	T
12	T

visit 4

Kosaraju-Sharir

Phase 1. Compute reverse postorder in G^R.

11 9 12 10 6 7 8



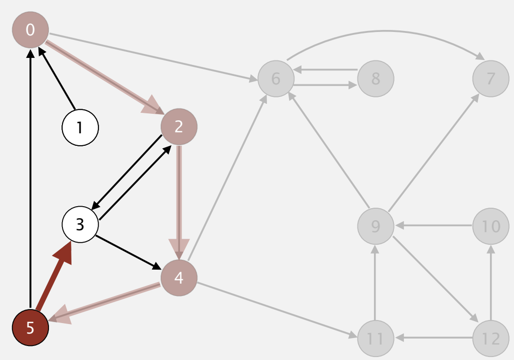
v	marked[v]
0	T
1	F
2	T
3	F
4	T
5	F
6	T
7	T
8	T
9	T
10	T
11	T
12	T

visit 4

Kosaraju-Sharir

Phase 1. Compute reverse postorder in G^R.

11 9 12 10 6 7 8



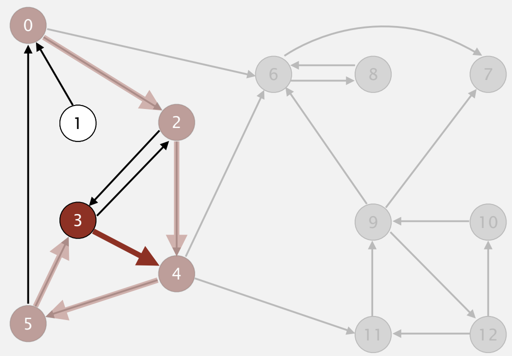
v	marked[v]
0	T
1	F
2	T
3	F
4	T
5	T
6	T
7	T
8	T
9	T
10	T
11	T
12	T

visit 5

Kosaraju-Sharir

Phase 1. Compute reverse postorder in G^R.

11 9 12 10 6 7 8



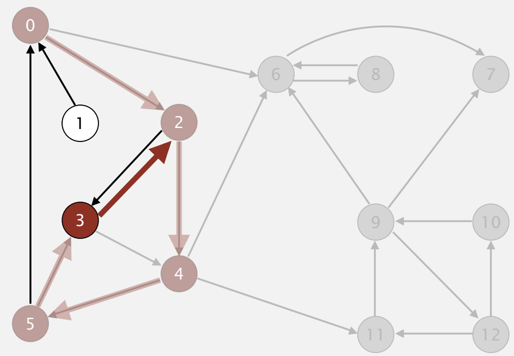
v	marked[v]
0	T
1	F
2	T
3	T
4	T
5	T
6	T
7	T
8	T
9	T
10	T
11	T
12	T

visit 3

Kosaraju-Sharir

Phase 1. Compute reverse postorder in G^R.

11 9 12 10 6 7 8



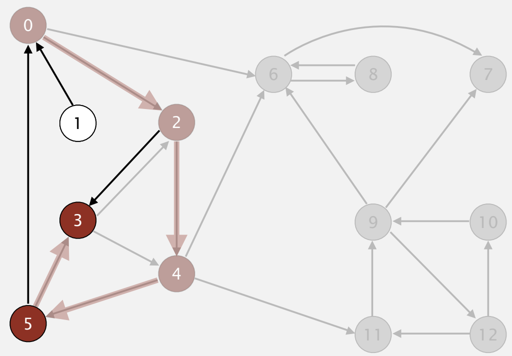
v	marked[v]
0	T
1	F
2	T
3	T
4	T
5	T
6	T
7	T
8	T
9	T
10	T
11	T
12	T

visit 3

Kosaraju-Sharir

Phase 1. Compute reverse postorder in G^R.

3 11 9 12 10 6 7 8



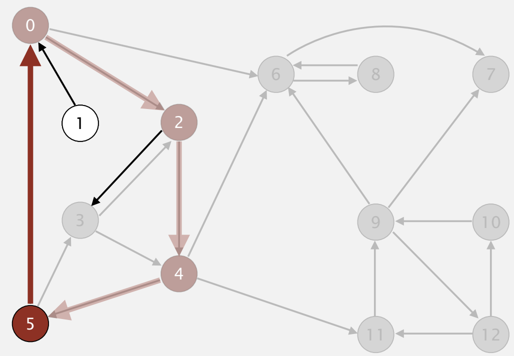
v	marked[v]
0	T
1	F
2	T
3	T
4	T
5	T
6	T
7	T
8	T
9	T
10	T
11	T
12	T

3 done

Kosaraju-Sharir

Phase 1. Compute reverse postorder in G^R.

3 11 9 12 10 6 7 8



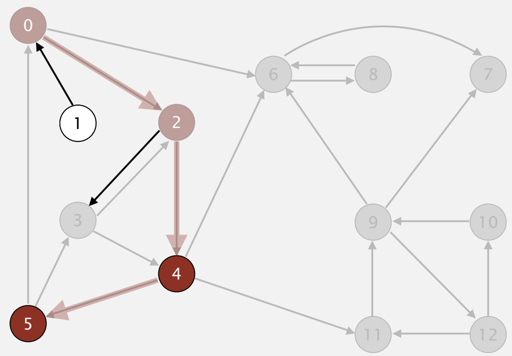
v	marked[v]
0	T
1	F
2	T
3	T
4	T
5	T
6	T
7	T
8	T
9	T
10	T
11	T
12	T

visit 5

Kosaraju-Sharir

Phase 1. Compute reverse postorder in G^R.

5 3 11 9 12 10 6 7 8



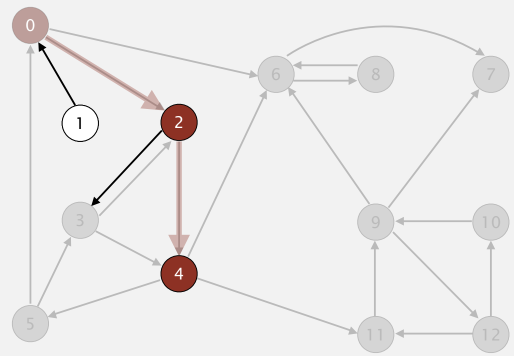
v	marked[v]
0	T
1	F
2	T
3	T
4	T
5	T
6	T
7	T
8	T
9	T
10	T
11	T
12	T

5 done

Kosaraju-Sharir

Phase 1. Compute reverse postorder in G^R.

4 5 3 11 9 12 10 6 7 8



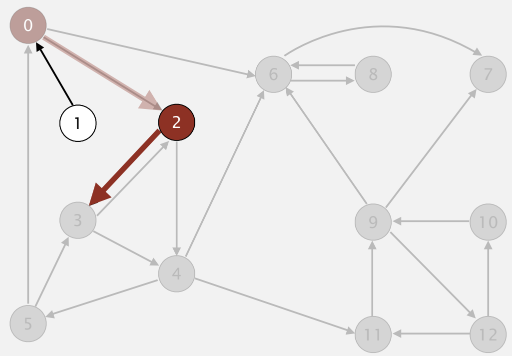
v	marked[v]
0	T
1	F
2	T
3	T
4	T
5	T
6	T
7	T
8	T
9	T
10	T
11	T
12	T

4 done

Kosaraju-Sharir

Phase 1. Compute reverse postorder in G^R.

4 5 3 11 9 12 10 6 7 8



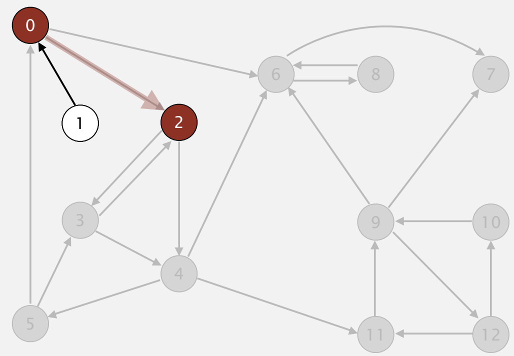
v	marked[v]
0	T
1	F
2	T
3	T
4	T
5	T
6	T
7	T
8	T
9	T
10	T
11	T
12	T

visit 2

Kosaraju-Sharir

Phase 1. Compute reverse postorder in G^R.

2 4 5 3 11 9 12 10 6 7 8



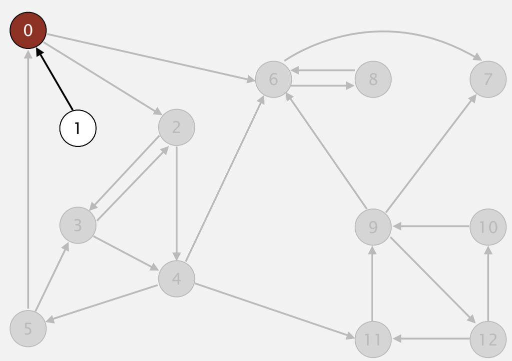
v	marked[v]
0	T
1	F
2	T
3	T
4	T
5	T
6	T
7	T
8	T
9	T
10	T
11	T
12	T

2 done

Kosaraju-Sharir

Phase 1. Compute reverse postorder in G^R.

0 2 4 5 3 11 9 12 10 6 7 8



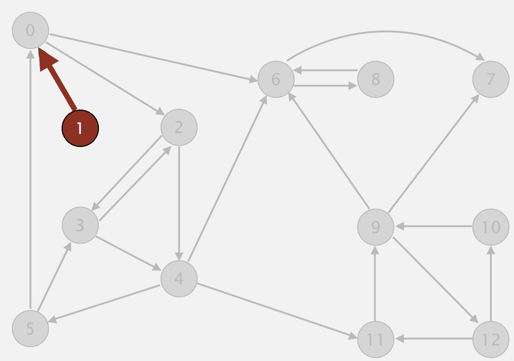
v	marked[v]
0	T
1	F
2	T
3	T
4	T
5	T
6	T
7	T
8	T
9	T
10	T
11	T
12	T

0 done

Kosaraju-Sharir

Phase 1. Compute reverse postorder in G^R.

0 2 4 5 3 11 9 12 10 6 7 8



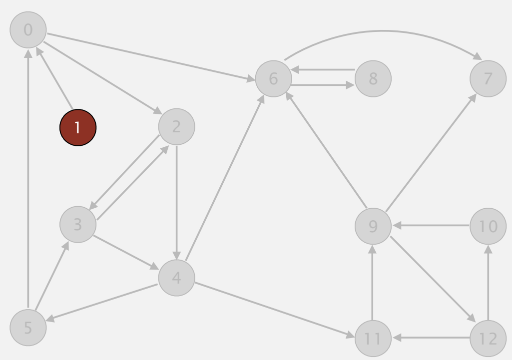
v	marked[v]
0	T
1	T
2	T
3	T
4	T
5	T
6	T
7	T
8	T
9	T
10	T
11	T
12	T

visit 1

Kosaraju-Sharir

Phase 1. Compute reverse postorder in G^R.

1 0 2 4 5 3 11 9 12 10 6 7 8



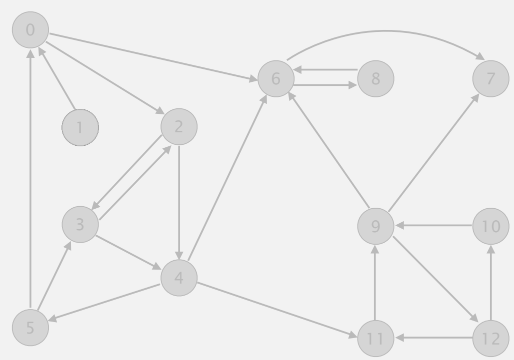
v	marked[v]
0	T
1	T
2	T
3	T
4	T
5	T
6	T
7	T
8	T
9	T
10	T
11	T
12	T

1 done

Kosaraju-Sharir

Phase 1. Compute reverse postorder in G^R.

1 0 2 4 5 3 11 9 12 10 6 7 8



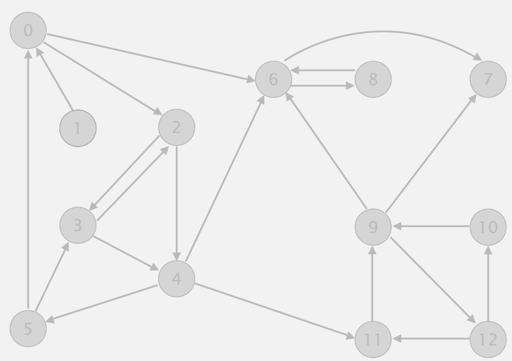
v	marked[v]
0	T
1	T
2	T
3	T
4	T
5	T
6	T
7	T
8	T
9	T
10	T
11	T
12	T

check 2 3 4 5 6 7 8 9 10 11 12

Kosaraju-Sharir

Phase 1. Compute reverse postorder in G^R.

1 0 2 4 5 3 11 9 12 10 6 7 8



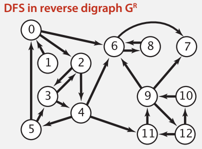
reverse digraph G^R

Kosaraju's algorithm

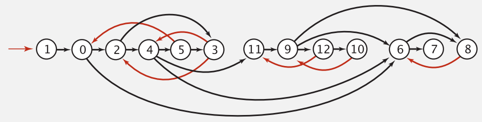
Simple (but mysterious) algorithm for computing strong components.

- Run DFS on G^R to compute reverse postorder.
- Run DFS on G, considering vertices in order given by first DFS.

DFS in reverse digraph G^R



check unmarked vertices in the order
0 1 2 3 4 5 6 7 8 9 10 11 12



reverse postorder for use in second `dfs()`
1 0 2 4 5 3 11 9 12 10 6 7 8

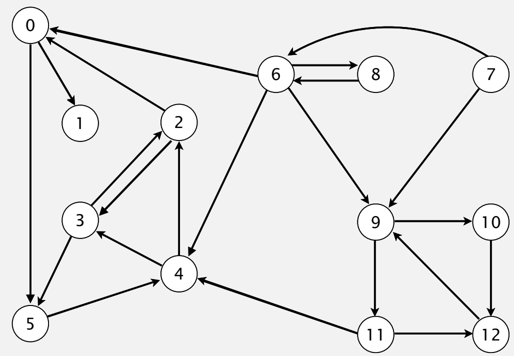
```
dfs(0)
  dfs(6)
    dfs(8)
      check 6
    8 done
    dfs(7)
    7 done
  6 done
  dfs(2)
    dfs(4)
      dfs(11)
        dfs(9)
          dfs(12)
            check 11
            dfs(10)
              check 9
            10 done
          12 done
          check 7
          check 6
...
```


Kosaraju's algorithm

- DFS in reverse graph
- **DFS in original graph**

Kosaraju-Sharir

Phase 2. Run DFS in G, visiting unmarked vertices in reverse postorder of G^R. **1 0 2 4 5 3 11 9 12 10 6 7 8**

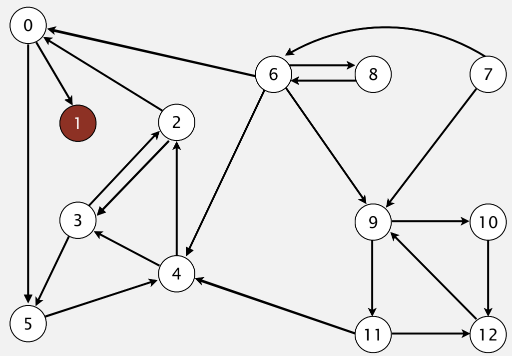


original digraph G

v	scc[v]
0	-
1	-
2	-
3	-
4	-
5	-
6	-
7	-
8	-
9	-
10	-
11	-
12	-

Kosaraju-Sharir

Phase 2. Run DFS in G, visiting unmarked vertices in reverse postorder of G^R. 1 0 2 4 5 3 11 9 12 10 6 7 8

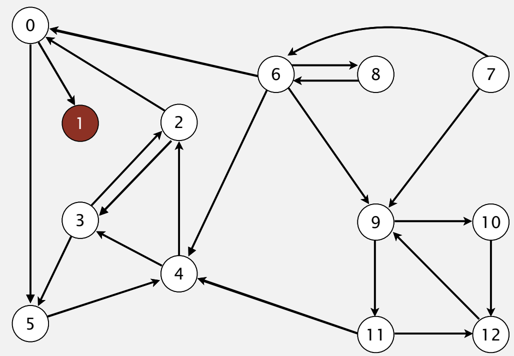


v	scc[v]
0	–
1	0
2	–
3	–
4	–
5	–
6	–
7	–
8	–
9	–
10	–
11	–
12	–

visit 1

Kosaraju-Sharir

Phase 2. Run DFS in G, visiting unmarked vertices in reverse postorder of G^R. 1 0 2 4 5 3 11 9 12 10 6 7 8

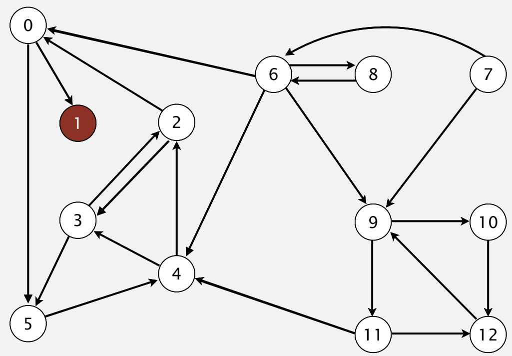


v	scc[v]
0	–
1	0
2	–
3	–
4	–
5	–
6	–
7	–
8	–
9	–
10	–
11	–
12	–

1 done

Kosaraju-Sharir

Phase 2. Run DFS in G, visiting unmarked vertices in reverse postorder of G^R. 1 0 2 4 5 3 11 9 12 10 6 7 8

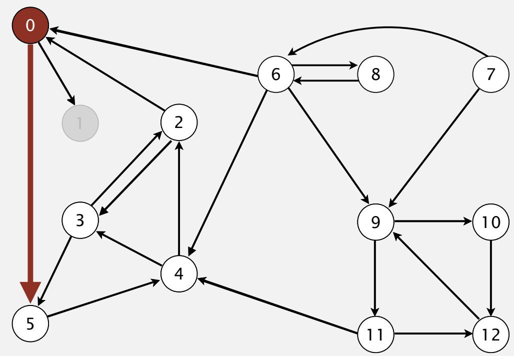


v	scc[v]
0	–
1	0
2	–
3	–
4	–
5	–
6	–
7	–
8	–
9	–
10	–
11	–
12	–

strong component: 1

Kosaraju-Sharir

Phase 2. Run DFS in G, visiting unmarked vertices in reverse postorder of G^R. 1 0 2 4 5 3 11 9 12 10 6 7 8

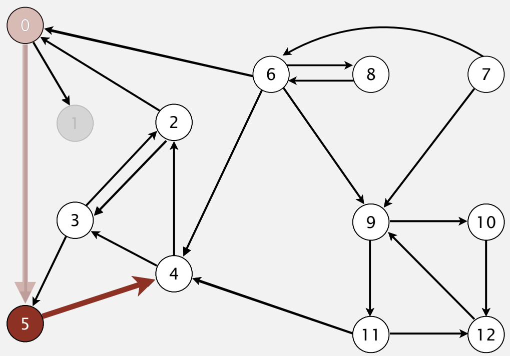


v	scc[v]
0	1
1	0
2	–
3	–
4	–
5	–
6	–
7	–
8	–
9	–
10	–
11	–
12	–

visit 0

Kosaraju-Sharir

Phase 2. Run DFS in G, visiting unmarked vertices in reverse postorder of G^R. 1 0 2 4 5 3 11 9 12 10 6 7 8

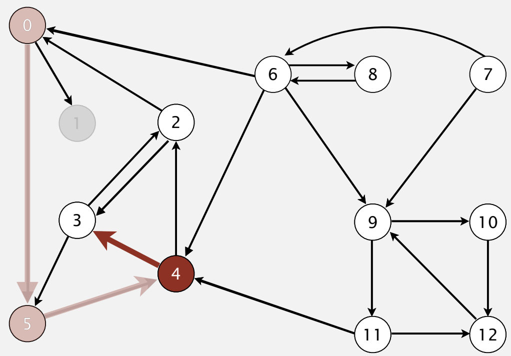


v	scc[v]
0	1
1	0
2	–
3	–
4	–
5	1
6	–
7	–
8	–
9	–
10	–
11	–
12	–

visit 5

Kosaraju-Sharir

Phase 2. Run DFS in G, visiting unmarked vertices in reverse postorder of G^R. 1 0 2 4 5 3 11 9 12 10 6 7 8

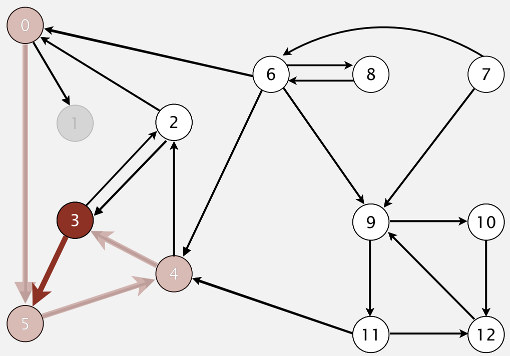


v	scc[v]
0	1
1	0
2	–
3	–
4	1
5	1
6	–
7	–
8	–
9	–
10	–
11	–
12	–

visit 4

Kosaraju-Sharir

Phase 2. Run DFS in G, visiting unmarked vertices in reverse postorder of G^R. 1 0 2 4 5 3 11 9 12 10 6 7 8

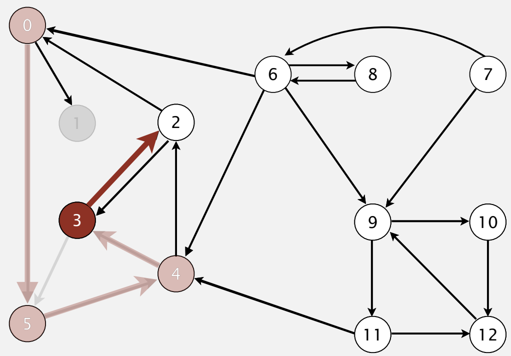


v	scc[v]
0	1
1	0
2	–
3	1
4	1
5	1
6	–
7	–
8	–
9	–
10	–
11	–
12	–

visit 3

Kosaraju-Sharir

Phase 2. Run DFS in G, visiting unmarked vertices in reverse postorder of G^R. 1 0 2 4 5 3 11 9 12 10 6 7 8

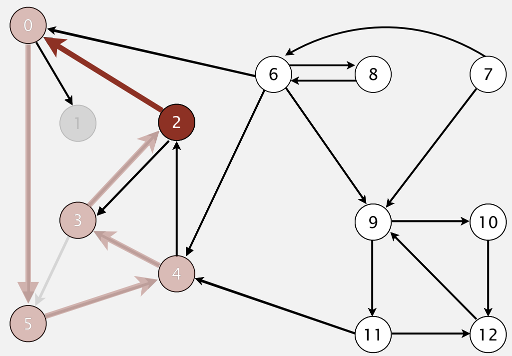


v	scc[v]
0	1
1	0
2	–
3	1
4	1
5	1
6	–
7	–
8	–
9	–
10	–
11	–
12	–

visit 3

Kosaraju-Sharir

Phase 2. Run DFS in G, visiting unmarked vertices in reverse postorder of G^R. 1 0 2 4 5 3 11 9 12 10 6 7 8

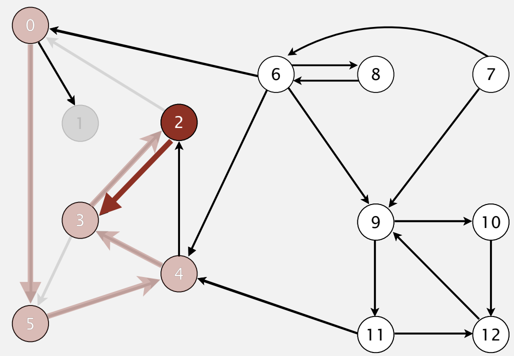


v	scc[v]
0	1
1	0
2	1
3	1
4	1
5	1
6	–
7	–
8	–
9	–
10	–
11	–
12	–

visit 2

Kosaraju-Sharir

Phase 2. Run DFS in G, visiting unmarked vertices in reverse postorder of G^R. 1 0 2 4 5 3 11 9 12 10 6 7 8

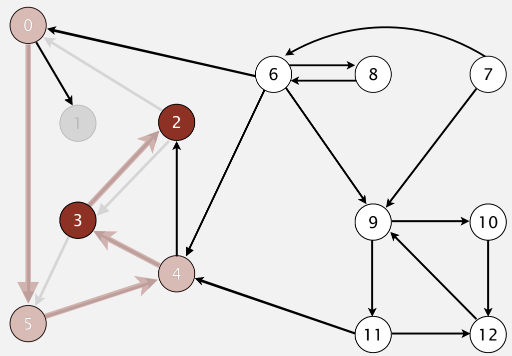


v	scc[v]
0	1
1	0
2	1
3	1
4	1
5	1
6	–
7	–
8	–
9	–
10	–
11	–
12	–

visit 2

Kosaraju-Sharir

Phase 2. Run DFS in G, visiting unmarked vertices in reverse postorder of G^R. 1 0 2 4 5 3 11 9 12 10 6 7 8

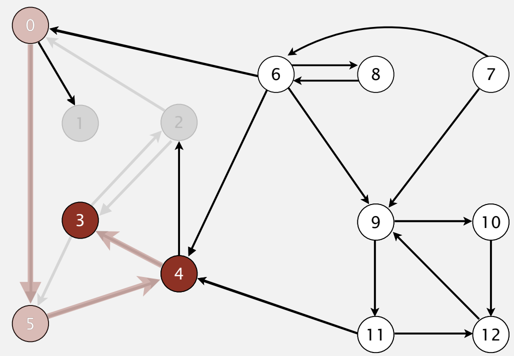


v	scc[v]
0	1
1	0
2	1
3	1
4	1
5	1
6	–
7	–
8	–
9	–
10	–
11	–
12	–

2 done

Kosaraju-Sharir

Phase 2. Run DFS in G, visiting unmarked vertices in reverse postorder of G^R. 1 0 2 4 5 3 11 9 12 10 6 7 8

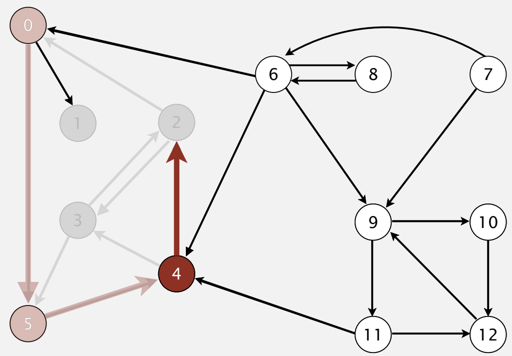


v	scc[v]
0	1
1	0
2	1
3	1
4	1
5	1
6	–
7	–
8	–
9	–
10	–
11	–
12	–

3 done

Kosaraju-Sharir

Phase 2. Run DFS in G, visiting unmarked vertices in reverse postorder of G^R. 1 0 2 4 5 3 11 9 12 10 6 7 8

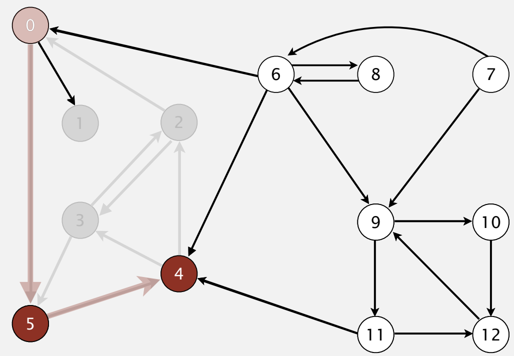


v	scc[v]
0	1
1	0
2	1
3	1
4	1
5	1
6	-
7	-
8	-
9	-
10	-
11	-
12	-

visit 4

Kosaraju-Sharir

Phase 2. Run DFS in G, visiting unmarked vertices in reverse postorder of G^R. 1 0 2 4 5 3 11 9 12 10 6 7 8

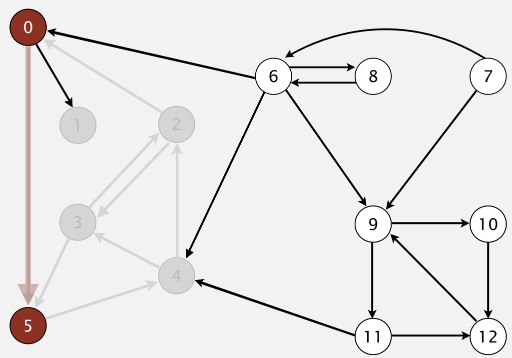


v	scc[v]
0	1
1	0
2	1
3	1
4	1
5	1
6	-
7	-
8	-
9	-
10	-
11	-
12	-

4 done

Kosaraju-Sharir

Phase 2. Run DFS in G, visiting unmarked vertices in reverse postorder of G^R. 1 0 2 4 5 3 11 9 12 10 6 7 8

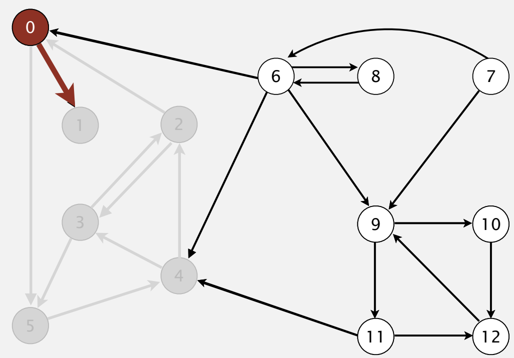


v	scc[v]
0	1
1	0
2	1
3	1
4	1
5	1
6	-
7	-
8	-
9	-
10	-
11	-
12	-

5 done

Kosaraju-Sharir

Phase 2. Run DFS in G, visiting unmarked vertices in reverse postorder of G^R. 1 0 2 4 5 3 11 9 12 10 6 7 8

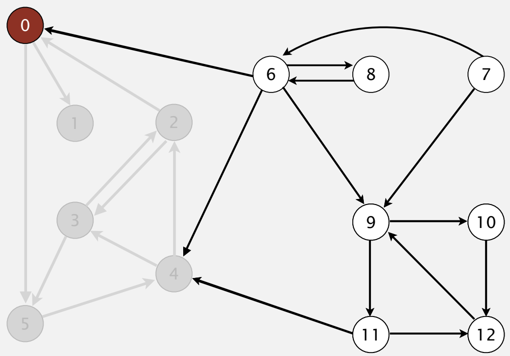


v	scc[v]
0	1
1	0
2	1
3	1
4	1
5	1
6	-
7	-
8	-
9	-
10	-
11	-
12	-

visit 0

Kosaraju-Sharir

Phase 2. Run DFS in G, visiting unmarked vertices in reverse postorder of G^R. 1 0 2 4 5 3 11 9 12 10 6 7 8

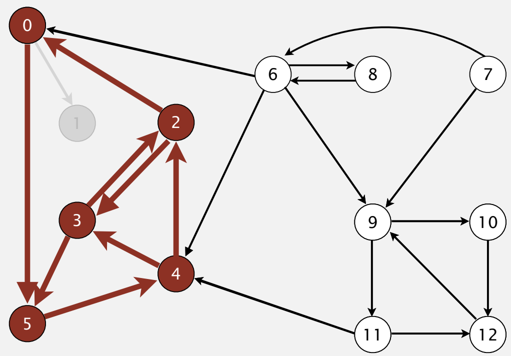


0 done

v	scc[v]
0	1
1	0
2	1
3	1
4	1
5	1
6	–
7	–
8	–
9	–
10	–
11	–
12	–

Kosaraju-Sharir

Phase 2. Run DFS in G, visiting unmarked vertices in reverse postorder of G^R. 1 0 2 4 5 3 11 9 12 10 6 7 8

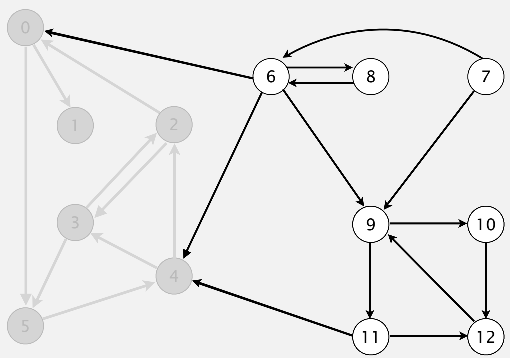


strong component: 0 2 3 4 5

v	scc[v]
0	1
1	0
2	1
3	1
4	1
5	1
6	–
7	–
8	–
9	–
10	–
11	–
12	–

Kosaraju-Sharir

Phase 2. Run DFS in G, visiting unmarked vertices in reverse postorder of G^R. 1 0 2 4 5 3 11 9 12 10 6 7 8

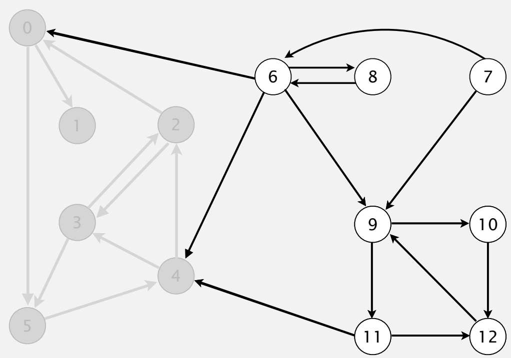


check 2

v	scc[v]
0	1
1	0
2	1
3	1
4	1
5	1
6	–
7	–
8	–
9	–
10	–
11	–
12	–

Kosaraju-Sharir

Phase 2. Run DFS in G, visiting unmarked vertices in reverse postorder of G^R. 1 0 2 4 5 3 11 9 12 10 6 7 8

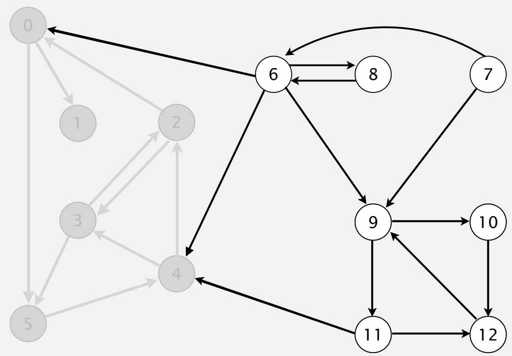


check 4

v	scc[v]
0	1
1	0
2	1
3	1
4	1
5	1
6	–
7	–
8	–
9	–
10	–
11	–
12	–

Kosaraju-Sharir

Phase 2. Run DFS in G, visiting unmarked vertices in reverse postorder of G^R. 1 0 2 4 5 3 11 9 12 10 6 7 8

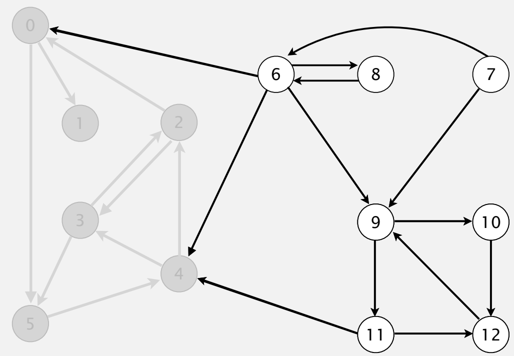


v	scc[v]
0	1
1	0
2	1
3	1
4	1
5	1
6	–
7	–
8	–
9	–
10	–
11	–
12	–

check 5

Kosaraju-Sharir

Phase 2. Run DFS in G, visiting unmarked vertices in reverse postorder of G^R. 1 0 2 4 5 3 11 9 12 10 6 7 8

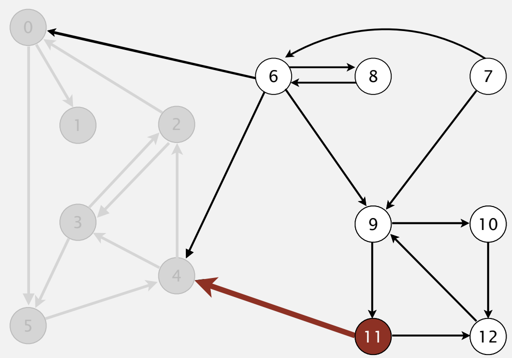


v	scc[v]
0	1
1	0
2	1
3	1
4	1
5	1
6	–
7	–
8	–
9	–
10	–
11	–
12	–

check 3

Kosaraju-Sharir

Phase 2. Run DFS in G, visiting unmarked vertices in reverse postorder of G^R. 1 0 2 4 5 3 11 9 12 10 6 7 8

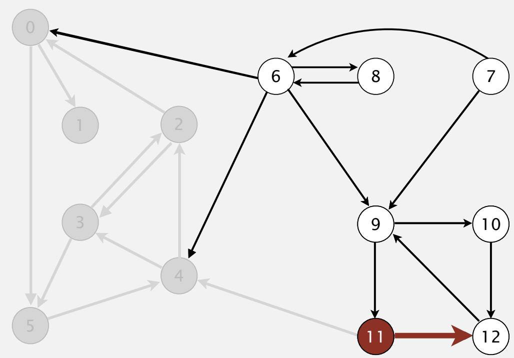


v	scc[v]
0	1
1	0
2	1
3	1
4	1
5	1
6	–
7	–
8	–
9	–
10	–
11	2
12	–

visit 11

Kosaraju-Sharir

Phase 2. Run DFS in G, visiting unmarked vertices in reverse postorder of G^R. 1 0 2 4 5 3 11 9 12 10 6 7 8

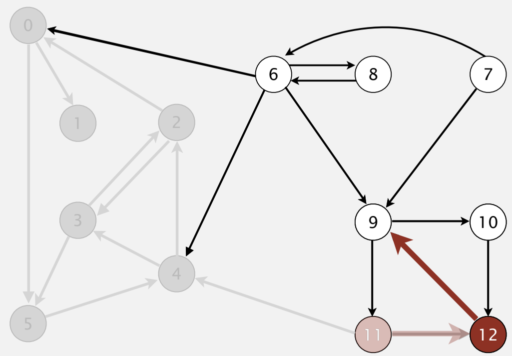


v	scc[v]
0	1
1	0
2	1
3	1
4	1
5	1
6	–
7	–
8	–
9	–
10	–
11	2
12	–

visit 11

Kosaraju-Sharir

Phase 2. Run DFS in G, visiting unmarked vertices in reverse postorder of G^R. 1 0 2 4 5 3 11 9 12 10 6 7 8

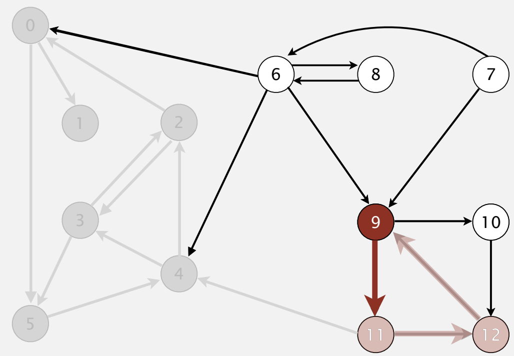


visit 12

v	scc[v]
0	1
1	0
2	1
3	1
4	1
5	1
6	–
7	–
8	–
9	–
10	–
11	2
12	2

Kosaraju-Sharir

Phase 2. Run DFS in G, visiting unmarked vertices in reverse postorder of G^R. 1 0 2 4 5 3 11 9 12 10 6 7 8

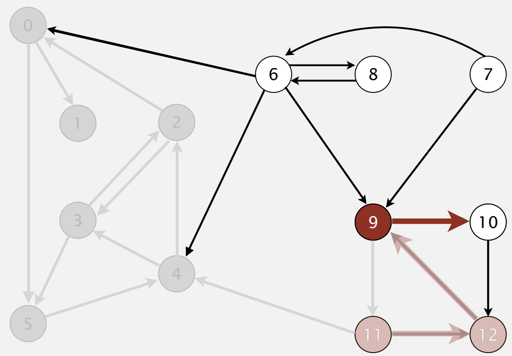


visit 9

v	scc[v]
0	1
1	0
2	1
3	1
4	1
5	1
6	–
7	–
8	–
9	2
10	–
11	2
12	2

Kosaraju-Sharir

Phase 2. Run DFS in G, visiting unmarked vertices in reverse postorder of G^R. 1 0 2 4 5 3 11 9 12 10 6 7 8

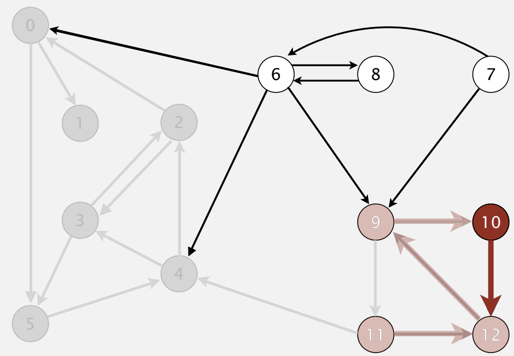


visit 9

v	scc[v]
0	1
1	0
2	1
3	1
4	1
5	1
6	–
7	–
8	–
9	2
10	–
11	2
12	2

Kosaraju-Sharir

Phase 2. Run DFS in G, visiting unmarked vertices in reverse postorder of G^R. 1 0 2 4 5 3 11 9 12 10 6 7 8

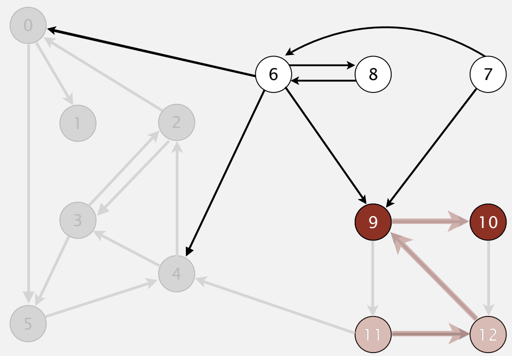


visit 10

v	scc[v]
0	1
1	0
2	1
3	1
4	1
5	1
6	–
7	–
8	–
9	2
10	2
11	2
12	2

Kosaraju-Sharir

Phase 2. Run DFS in G, visiting unmarked vertices in reverse postorder of G^R. 1 0 2 4 5 3 11 9 12 10 6 7 8

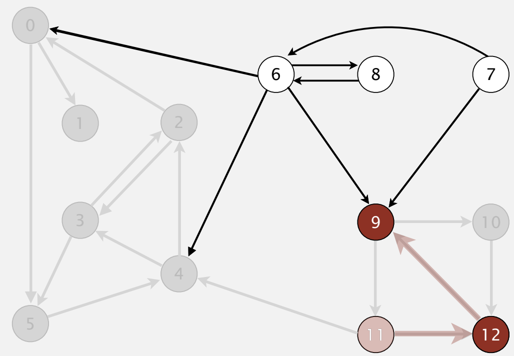


v	scc[v]
0	1
1	0
2	1
3	1
4	1
5	1
6	–
7	–
8	–
9	2
10	2
11	2
12	2

10 done

Kosaraju-Sharir

Phase 2. Run DFS in G, visiting unmarked vertices in reverse postorder of G^R. 1 0 2 4 5 3 11 9 12 10 6 7 8

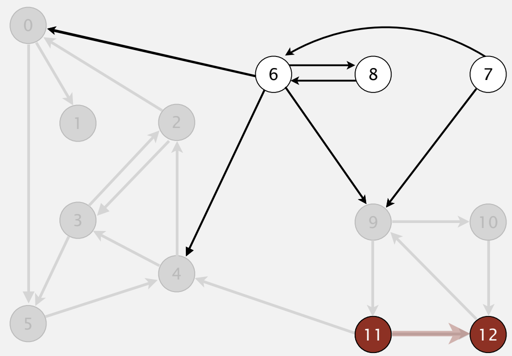


v	scc[v]
0	1
1	0
2	1
3	1
4	1
5	1
6	–
7	–
8	–
9	2
10	2
11	2
12	2

9 done

Kosaraju-Sharir

Phase 2. Run DFS in G, visiting unmarked vertices in reverse postorder of G^R. 1 0 2 4 5 3 11 9 12 10 6 7 8

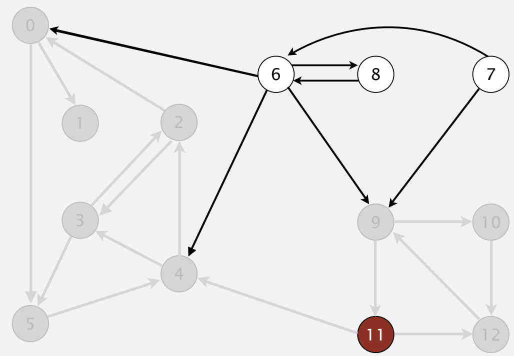


v	scc[v]
0	1
1	0
2	1
3	1
4	1
5	1
6	–
7	–
8	–
9	2
10	2
11	2
12	2

12 done

Kosaraju-Sharir

Phase 2. Run DFS in G, visiting unmarked vertices in reverse postorder of G^R. 1 0 2 4 5 3 11 9 12 10 6 7 8

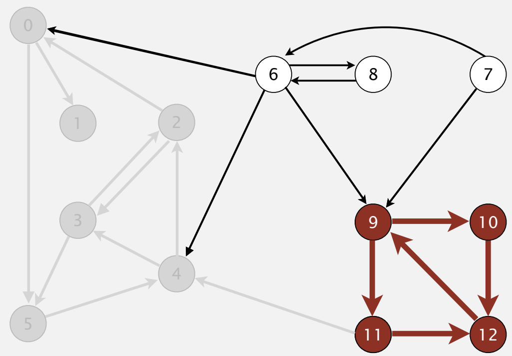


v	scc[v]
0	1
1	0
2	1
3	1
4	1
5	1
6	–
7	–
8	–
9	2
10	2
11	2
12	2

11 done

Kosaraju-Sharir

Phase 2. Run DFS in G, visiting unmarked vertices in reverse postorder of G^R. 1 0 2 4 5 3 11 9 12 10 6 7 8

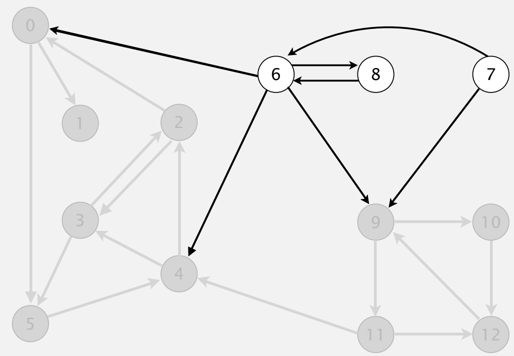


v	scc[v]
0	1
1	0
2	1
3	1
4	1
5	1
6	–
7	–
8	–
9	2
10	2
11	2
12	2

strong component: 9 10 11 12

Kosaraju-Sharir

Phase 2. Run DFS in G, visiting unmarked vertices in reverse postorder of G^R. 1 0 2 4 5 3 11 9 12 10 6 7 8

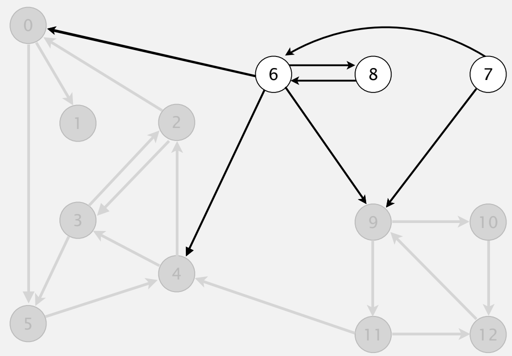


v	scc[v]
0	1
1	0
2	1
3	1
4	1
5	1
6	–
7	–
8	–
9	2
10	2
11	2
12	2

check 9

Kosaraju-Sharir

Phase 2. Run DFS in G, visiting unmarked vertices in reverse postorder of G^R. 1 0 2 4 5 3 11 9 12 10 6 7 8

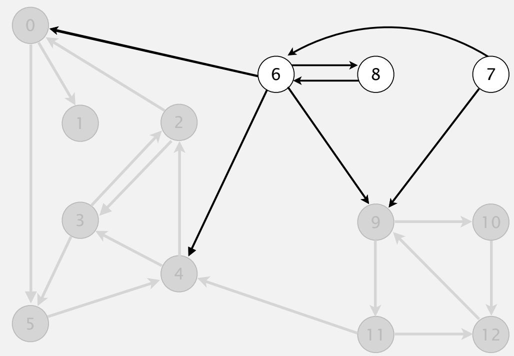


v	scc[v]
0	1
1	0
2	1
3	1
4	1
5	1
6	–
7	–
8	–
9	2
10	2
11	2
12	2

check 12

Kosaraju-Sharir

Phase 2. Run DFS in G, visiting unmarked vertices in reverse postorder of G^R. 1 0 2 4 5 3 11 9 12 10 6 7 8

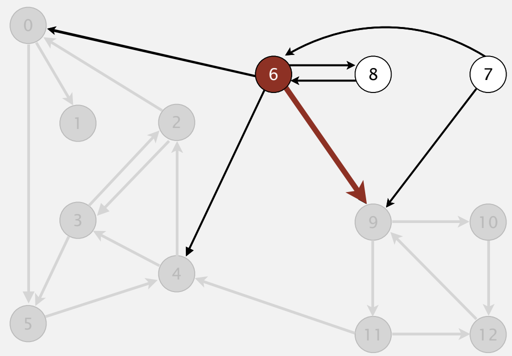


v	scc[v]
0	1
1	0
2	1
3	1
4	1
5	1
6	–
7	–
8	–
9	2
10	2
11	2
12	2

check 10

Kosaraju-Sharir

Phase 2. Run DFS in G, visiting unmarked vertices in reverse postorder of G^R. 1 0 2 4 5 3 11 9 12 10 6 7 8

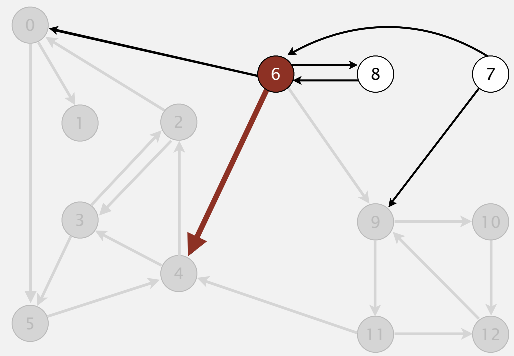


v	scc[v]
0	1
1	0
2	1
3	1
4	1
5	1
6	3
7	–
8	–
9	2
10	2
11	2
12	2

visit 6

Kosaraju-Sharir

Phase 2. Run DFS in G, visiting unmarked vertices in reverse postorder of G^R. 1 0 2 4 5 3 11 9 12 10 6 7 8

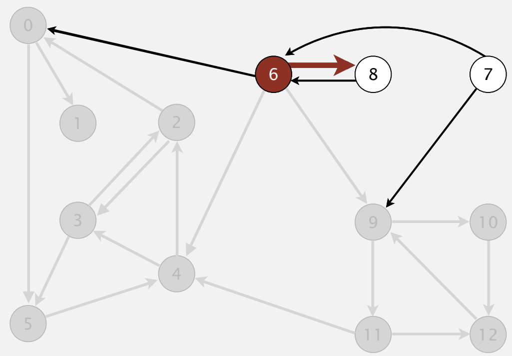


v	scc[v]
0	1
1	0
2	1
3	1
4	1
5	1
6	3
7	–
8	–
9	2
10	2
11	2
12	2

visit 6

Kosaraju-Sharir

Phase 2. Run DFS in G, visiting unmarked vertices in reverse postorder of G^R. 1 0 2 4 5 3 11 9 12 10 6 7 8

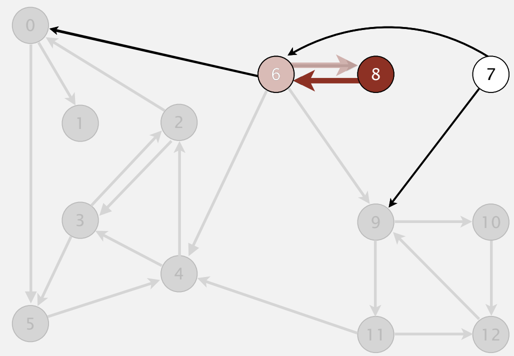


v	scc[v]
0	1
1	0
2	1
3	1
4	1
5	1
6	3
7	–
8	–
9	2
10	2
11	2
12	2

visit 6

Kosaraju-Sharir

Phase 2. Run DFS in G, visiting unmarked vertices in reverse postorder of G^R. 1 0 2 4 5 3 11 9 12 10 6 7 8

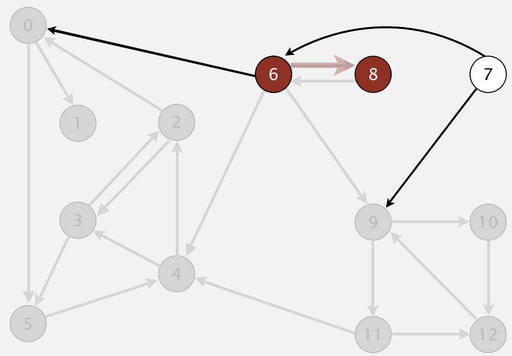


v	scc[v]
0	1
1	0
2	1
3	1
4	1
5	1
6	3
7	–
8	3
9	2
10	2
11	2
12	2

visit 8

Kosaraju-Sharir

Phase 2. Run DFS in G, visiting unmarked vertices in reverse postorder of G^R. 1 0 2 4 5 3 11 9 12 10 6 7 8

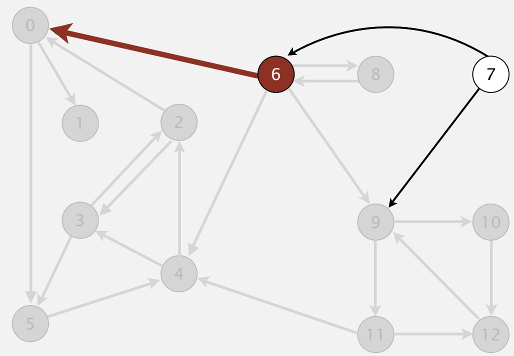


v	scc[v]
0	1
1	0
2	1
3	1
4	1
5	1
6	3
7	–
8	3
9	2
10	2
11	2
12	2

8 done

Kosaraju-Sharir

Phase 2. Run DFS in G, visiting unmarked vertices in reverse postorder of G^R. 1 0 2 4 5 3 11 9 12 10 6 7 8

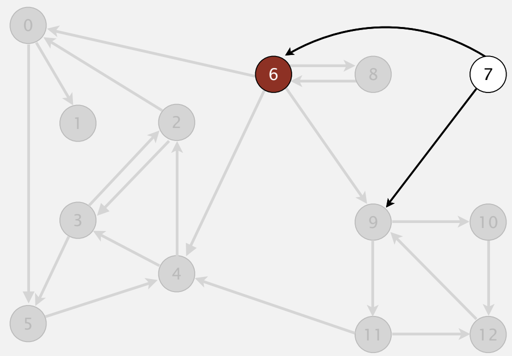


v	scc[v]
0	1
1	0
2	1
3	1
4	1
5	1
6	3
7	–
8	3
9	2
10	2
11	2
12	2

visit 6

Kosaraju-Sharir

Phase 2. Run DFS in G, visiting unmarked vertices in reverse postorder of G^R. 1 0 2 4 5 3 11 9 12 10 6 7 8

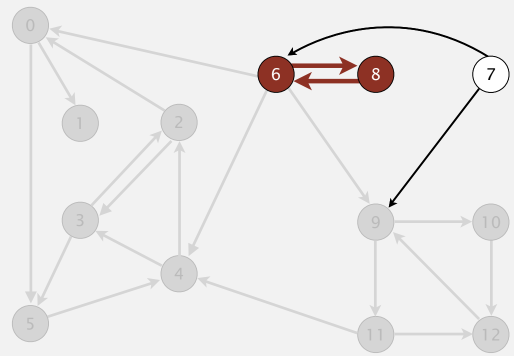


v	scc[v]
0	1
1	0
2	1
3	1
4	1
5	1
6	3
7	–
8	3
9	2
10	2
11	2
12	2

6 done

Kosaraju-Sharir

Phase 2. Run DFS in G, visiting unmarked vertices in reverse postorder of G^R. 1 0 2 4 5 3 11 9 12 10 6 7 8

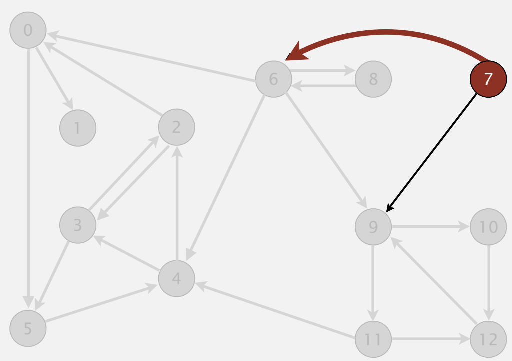


v	scc[v]
0	1
1	0
2	1
3	1
4	1
5	1
6	3
7	–
8	3
9	2
10	2
11	2
12	2

strong component: 6 8

Kosaraju-Sharir

Phase 2. Run DFS in G, visiting unmarked vertices in reverse postorder of G^R. 1 0 2 4 5 3 11 9 12 10 6 7 8

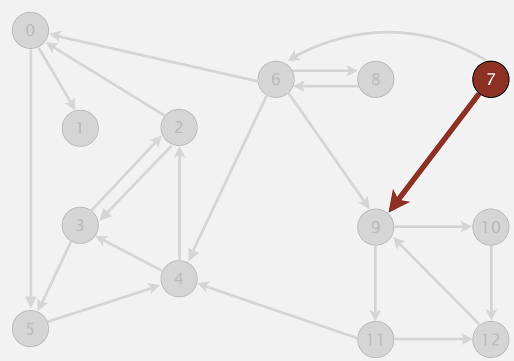


v	scc[v]
0	1
1	0
2	1
3	1
4	1
5	1
6	3
7	4
8	3
9	2
10	2
11	2
12	2

visit 7

Kosaraju-Sharir

Phase 2. Run DFS in G, visiting unmarked vertices in reverse postorder of G^R. 1 0 2 4 5 3 11 9 12 10 6 7 8

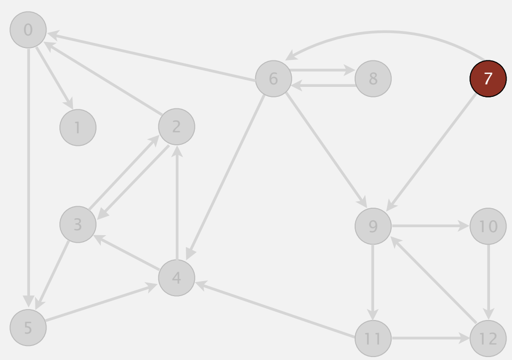


v	scc[v]
0	1
1	0
2	1
3	1
4	1
5	1
6	3
7	4
8	3
9	2
10	2
11	2
12	2

visit 7

Kosaraju-Sharir

Phase 2. Run DFS in G, visiting unmarked vertices in reverse postorder of G^R. 1 0 2 4 5 3 11 9 12 10 6 7 8

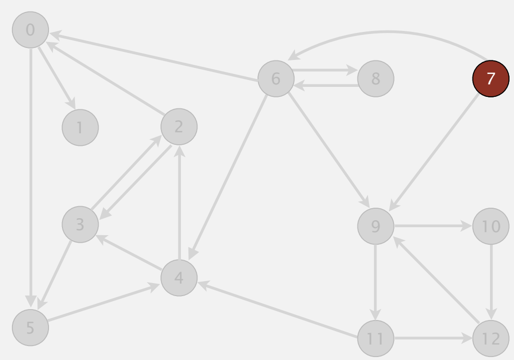


v	scc[v]
0	1
1	0
2	1
3	1
4	1
5	1
6	3
7	4
8	3
9	2
10	2
11	2
12	2

7 done

Kosaraju-Sharir

Phase 2. Run DFS in G, visiting unmarked vertices in reverse postorder of G^R. 1 0 2 4 5 3 11 9 12 10 6 7 8

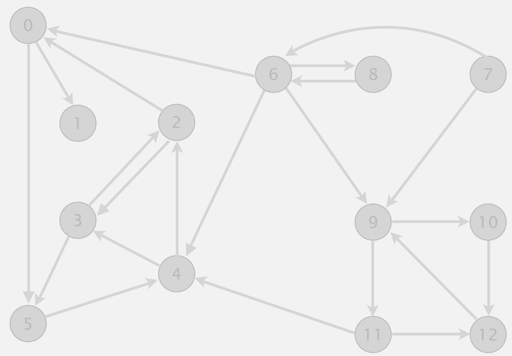


v	scc[v]
0	1
1	0
2	1
3	1
4	1
5	1
6	3
7	4
8	3
9	2
10	2
11	2
12	2

strong component: 7

Kosaraju-Sharir

Phase 2. Run DFS in G, visiting unmarked vertices in reverse postorder of G^R. 1 0 2 4 5 3 11 9 12 10 6 7 8

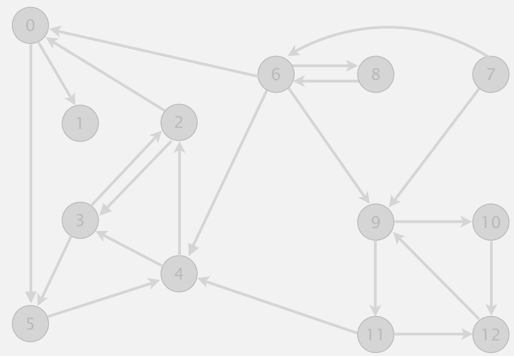


v	scc[v]
0	1
1	0
2	1
3	1
4	1
5	1
6	3
7	4
8	3
9	2
10	2
11	2
12	2

check 8

Kosaraju-Sharir

Phase 2. Run DFS in G, visiting unmarked vertices in reverse postorder of G^R. 1 0 2 4 5 3 11 9 12 10 6 7 8



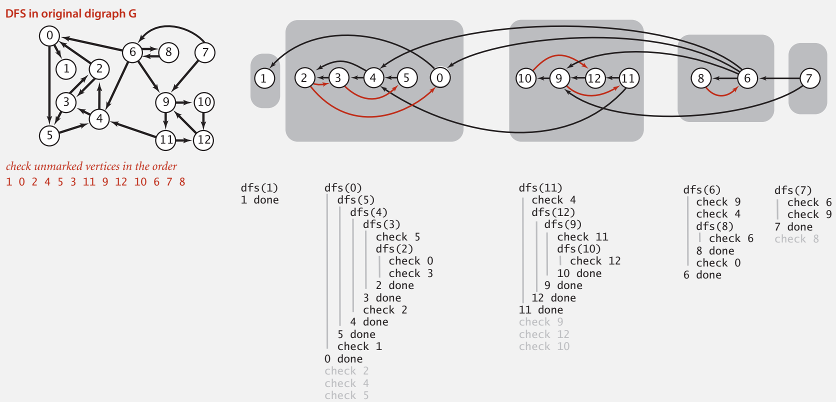
v	scc[v]
0	1
1	0
2	1
3	1
4	1
5	1
6	3
7	4
8	3
9	2
10	2
11	2
12	2

done

Kosaraju's algorithm

Simple (but mysterious) algorithm for computing strong components.

- Run DFS on G^R to compute reverse postorder.
- Run DFS on G, considering vertices in order given by first DFS.



Proposition. Second DFS gives strong components. (!!)

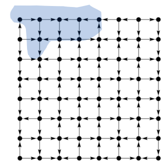
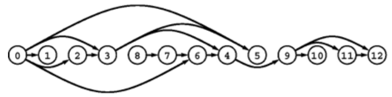
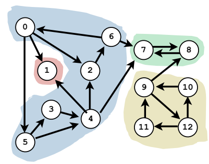
Connected components in an undirected graph (with DFS)

```
public class CC
{
   private boolean marked[];
   private int[] id;
   private int count;

   public CC(Graph G)
   {
      marked = new boolean[G.V()];
      id = new int[G.V()];

      for (int v = 0; v < G.V(); v++)
      {
         if (!marked[v])
         {
            dfs(G, v);
            count++;
         }
      }
   }

   private void dfs(Graph G, int v)
   {
      marked[v] = true;
      id[v] = count;
      for (int w : G.adj(v))
         if (!marked[w])
            dfs(G, w);
   }

   public boolean connected(int v, int w)
   {  return id[v] == id[w];  }
}
```

Strong components in a digraph (with two DFSs)

```
public class KosarajuSCC
{
   private boolean marked[];
   private int[] id;
   private int count;

   public KosarajuSCC(Digraph G)
   {
      marked = new boolean[G.V()];
      id = new int[G.V()];
      DepthFirstOrder dfs = new DepthFirstOrder(G.reverse());
      for (int v : dfs.reversePost())
      {
         if (!marked[v])
         {
            dfs(G, v);
            count++;
         }
      }
   }

   private void dfs(Digraph G, int v)
   {
      marked[v] = true;
      id[v] = count;
      for (int w : G.adj(v))
         if (!marked[w])
            dfs(G, w);
   }

   public boolean stronglyConnected(int v, int w)
   {  return id[v] == id[w];  }
}
```


Digraph-processing summary: algorithms of the day

single-source reachability		DFS
topological sort (DAG)		DFS
strong components		Kosaraju DFS (twice)