Substring Search

Apr. 28, 2015

Acknowledgement: The course slides are adapted from the slides prepared by R. Sedgewick and K. Wayne of Princeton University.
TODAY

- Substring search
- Brute force
- Knuth-Morris-Pratt
- Boyer-Moore
- Rabin-Karp
Goal. Find pattern of length $M$ in a text of length $N$. 

- **Pattern**: NEEDLE
- **Text**: INAHAYSTACK NEEDLE INA

Match:
Substring search applications

**Goal.** Find pattern of length $M$ in a text of length $N$.

\[
\text{pattern} \rightarrow \text{NEEDLE} \\
\text{text} \rightarrow \text{INAHAYSTACK} \text{NEEDLEINA}
\]

Computer forensics. Search memory or disk for signatures, e.g., all URLs or RSA keys that the user has entered.

http://citp.princeton.edu/memory
Substring search applications

Goal. Find pattern of length $M$ in a text of length $N$.

Identify patterns indicative of spam.

- PROFITS
- LOSE WEIGHT
- There is no catch.
- This is a one-time mailing.
- This message is sent in compliance with spam regulations.
Substring search applications

Electronic surveillance.

Need to monitor all internet traffic. (security)

No way! (privacy)

Well, we’re mainly interested in “ATTACK AT DAWN”

OK. Build a machine that just looks for that.

“ATTACK AT DAWN” substring search machine found
Substring search applications

Screen scraping. Extract relevant data from web page.

Ex. Find string delimited by `<b>` and `</b>` after first occurrence of pattern `Last Trade:`.

http://finance.yahoo.com/q?s=goog

```html
<tr>
  <td class= "yfnc_tablehead1" width= "48%">
    Last Trade:
  </td>
  <td class= "yfnc_tabledata1">
    <big><b>452.92</b></big>
  </td>
</tr>
<tr>
  <td class= "yfnc_tablehead1" width= "48%">
    Trade Time:
  </td>
  <td class= "yfnc_tabledata1">
    ...
  </td>
</tr>
```
Screen scraping: Java implementation

Java library. The `indexOf()` method in Java's string library returns the index of the first occurrence of a given string, starting at a given offset.

```java
public class StockQuote {
    public static void main(String[] args) {
        String name = "http://finance.yahoo.com/q?s=";
        In in = new In(name + args[0]);
        String text = in.readAll();
        int start    = text.indexOf("Last Trade:", 0);
        int from     = text.indexOf("<b>",  start);
        int to       = text.indexOf("</b>", from);
        String price = text.substring(from + 3, to);
        StdOut.println(price);
    }
}
```

% java StockQuote goog
582.93

% java StockQuote msft
24.84
SubString Search

- Brute force
- Knuth-Morris-Pratt
- Boyer-Moore
- Rabin-Karp
## Brute-force substring search

Check for pattern starting at each text position.

<table>
<thead>
<tr>
<th>i</th>
<th>j</th>
<th>i+j</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>A</td>
<td>B</td>
<td>A</td>
<td>C</td>
<td>A</td>
<td>D</td>
<td>A</td>
<td>B</td>
<td>R</td>
<td>A</td>
<td>C</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>2</td>
<td>A</td>
<td>B</td>
<td>R</td>
<td>A</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>A</td>
<td>B</td>
<td>R</td>
<td>A</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>3</td>
<td>A</td>
<td>B</td>
<td>R</td>
<td>A</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>3</td>
<td>A</td>
<td>B</td>
<td>R</td>
<td>A</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>5</td>
<td>A</td>
<td>B</td>
<td>R</td>
<td>A</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>5</td>
<td>A</td>
<td>B</td>
<td>R</td>
<td>A</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>4</td>
<td>10</td>
<td>A</td>
<td>B</td>
<td>R</td>
<td>A</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- Entries in red are mismatches.
- Entries in gray are for reference only.
- Entries in black match the text.
- Return $i$ when $j$ is $M$.
- Found match.
Brute-force substring search: Java implementation

Check for pattern starting at each text position.

```
public static int search(String pat, String txt)
{
    int M = pat.length();
    int N = txt.length();
    for (int i = 0; i <= N - M; i++)
    {
        int j;
        for (j = 0; j < M; j++)
            if (txt.charAt(i+j) != pat.charAt(j))
                break;
        if (j == M) return i;
    }
    return N;
}
```
Brute-force substring search: worst case

Brute-force algorithm can be slow if text and pattern are repetitive.

<table>
<thead>
<tr>
<th>i</th>
<th>j</th>
<th>i+j</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>7</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>8</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>10</td>
</tr>
</tbody>
</table>

**txt** → A A A A A A A A A B

**pat** → A A A A A B

Worst case. \( \sim MN \) char compares.
In many applications, we want to avoid backup in text stream.

- Treat input as stream of data.
- Abstract model: standard input.

**Brute-force algorithm needs backup for every mismatch.**

**Approach 1. Maintain buffer of last $M$ characters.**

**Approach 2. Stay tuned.**
Same sequence of char compares as previous implementation.

- \( i \) points to end of sequence of already-matched chars in text.
- \( j \) stores number of already-matched chars (end of sequence in pattern).

```java
public static int search(String pat, String txt) {
    int i, N = txt.length();
    int j, M = pat.length();
    for (i = 0, j = 0; i < N && j < M; i++) {
        if (txt.charAt(i) == pat.charAt(j)) j++;
        else { i -= j; j = 0; }
    }
    if (j == M) return i - M;
    else            return N;
}
```
Algorithmic challenges in substring search

Brute-force is not always good enough.

Theoretical challenge. Linear-time guarantee. ❯ fundamental algorithmic problem

Practical challenge. Avoid backup in text stream. ❯ often no room or time to save text

Now is the time for all people to come to the aid of their party. Now is the time for all good people to come to the aid of their party. Now is the time for many good people to come to the aid of their party. Now is the time for a lot of good people to come to the aid of their party. Now is the time for all of the good people to come to the aid of their party. Now is the time for all good people to come to the aid of their party. Now is the time for each good person to come to the aid of their party. Now is the time for all good people to come to the aid of their party. Now is the time for all good Republicans to come to the aid of their party. Now is the time for all good Democrats to come to the aid of their party. Now is the time for all good Republicans to come to the aid of their party. Now is the time for many or all good people to come to the aid of their party. Now is the time for all good people to come to the aid of their party. Now is the time for all good Democrats to come to the aid of their party.
Substring Search

- Brute force
- Knuth-Morris-Pratt
- Boyer-Moore
- Rabin-Karp
Knuth-Morris-Pratt substring search

**Intuition.** Suppose we are searching in text for pattern `BBBBBBBBB`.  
• Suppose we match 5 chars in pattern, with mismatch on 6th char.  
• We know previous 6 chars in text are `BBBBB`.  
• Don't need to back up text pointer!

**Knuth-Morris-Pratt algorithm.** Clever method to always avoid backup. (!)
Deterministic finite state automaton (DFA)

DFA is abstract string-searching machine.
- Finite number of states (including start and halt).
- Exactly one transition for each char in alphabet.
- Accept if sequence of transitions leads to halt state.

**internal representation**

<table>
<thead>
<tr>
<th>pat.charAt(j)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>B</td>
<td>A</td>
<td>B</td>
<td>A</td>
<td>C</td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>B</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>C</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>6</td>
</tr>
</tbody>
</table>

If in state j reading char c:
- if j is 6 halt and accept
- else move to state dfa[c][j]

**graphical representation**
DFA simulation

\begin{tabular}{cccccc}
0 & 1 & 2 & 3 & 4 & 5 \\
\hline
A & B & A & B & A & C \\
A & 1 & 1 & 3 & 1 & 5 & 1 \\
B & 0 & 2 & 0 & 4 & 0 & 4 \\
C & 0 & 0 & 0 & 0 & 0 & 6 \\
\end{tabular}

\begin{center}
\begin{tikzpicture}

\node (0) at (0,0) {0};
\node (1) at (2,0) {1};
\node (2) at (4,0) {2};
\node (3) at (6,0) {3};
\node (4) at (8,0) {4};
\node (5) at (10,0) {5};
\node (6) at (12,0) {6};

\path[->]
(0) edge [loop above] node {A} (0)
(0) edge [bend right] node {B, C} (1)
(1) edge [loop above] node {A} (1)
(1) edge [bend right] node {B} (2)
(2) edge [loop above] node {A} (2)
(2) edge [bend right] node {B, C} (3)
(3) edge [loop above] node {B} (3)
(3) edge [bend right] node {A} (4)
(4) edge [loop above] node {A} (4)
(4) edge [bend right] node {B} (5)
(5) edge [loop above] node {C} (5)
(5) edge [bend right] node {B} (6)
(6) edge [loop above] node {C} (6);
\end{tikzpicture}
\end{center}

\texttt{pat.charAt(j)}
DFA simulation

A A B A C A A B A B A C A A

<table>
<thead>
<tr>
<th>pat.charAt(j)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>B</td>
<td>A</td>
<td>B</td>
<td>A</td>
<td>C</td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>B</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>C</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>6</td>
</tr>
</tbody>
</table>

0 1 2 3 4 5

B, C

A

B

A

B

A

B

A

B

A

B
DFA simulation

\[
\begin{array}{cccccc}
A & A & B & A & C & A \\
\end{array}
\]

\[
\text{pat.charAt(j)}
\begin{array}{cccccc}
0 & 1 & 2 & 3 & 4 & 5 \\
A & B & A & B & A & C \\
A & 1 & 1 & 3 & 1 & 5 & 1 \\
B & 0 & 2 & 0 & 4 & 0 & 4 \\
C & 0 & 0 & 0 & 0 & 0 & 6 \\
\end{array}
\]
DFA simulation

```plaintext
 pat.charAt(j)      | 0  | 1  | 2  | 3  | 4  | 5  
-------------------|----|----|----|----|----|---- 
 A                 | B  | A  | B  | A  | A  | C  
 A                 | 1  | 1  | 3  | 1  | 5  | 1  
 B                 | 0  | 2  | 0  | 4  | 0  | 4  
 C                 | 0  | 0  | 0  | 0  | 0  | 6  
```
DFA simulation

A A B A C A A B A B A C A A

<table>
<thead>
<tr>
<th>pat.charAt(j)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>B</td>
<td>A</td>
<td>B</td>
<td>A</td>
<td>C</td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>B</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>C</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>6</td>
</tr>
</tbody>
</table>

```
pat.charAt(j) = dfa[][j]
```
DFA simulation

\[
\begin{array}{cccccccc}
\end{array}
\]

\[
\begin{array}{cccccccc}
0 & 1 & 2 & 3 & 4 & 5 \\
pat.charAt(j) & A & B & A & B & A & C \\
A & 1 & 1 & 3 & 1 & 5 & 1 \\
B & 0 & 2 & 0 & 4 & 0 & 4 \\
C & 0 & 0 & 0 & 0 & 0 & 6 \\
\end{array}
\]
DFA simulation

```
A A B A C A A B A B A C A A
```

```
pat.charAt(j)  0  1  2  3  4  5
A B A B A C
A 1 1 3 1 5 1
B 0 2 0 4 0 4
C 0 0 0 0 0 6
```
DFA simulation

\[ \text{pat.charAt}(j) \]

\[
\begin{array}{cccccc}
0 & 1 & 2 & 3 & 4 & 5 \\
A & B & A & B & A & C \\
A & 1 & 1 & 3 & 1 & 5 & 1 \\
B & 0 & 2 & 0 & 4 & 0 & 4 \\
C & 0 & 0 & 0 & 0 & 0 & 6 \\
\end{array}
\]
DFA simulation

\[
\begin{array}{cccccc}
A & A & B & A & C & A \\
A & B & A & B & A & C \\
A & B & A & B & A & C \\
\end{array}
\]

\[
\begin{array}{cccccc}
0 & 1 & 2 & 3 & 4 & 5 \\
pat.charAt(j) & A & B & A & B & A & C \\
A & 1 & 1 & 3 & 1 & 5 & 1 \\
dfa[][][j] & B & 0 & 2 & 0 & 4 & 0 & 4 \\
C & 0 & 0 & 0 & 0 & 0 & 0 & 6 \\
\end{array}
\]
DFA simulation

A A B A C A A B A B A C A A

<table>
<thead>
<tr>
<th>pat.charAt(j)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>B</td>
<td>A</td>
<td>B</td>
<td>A</td>
<td>C</td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>dfa[][j]</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>C</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>6</td>
</tr>
</tbody>
</table>

0 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6
DFA simulation

```
A A B A C A A B A B A C A A
```

<table>
<thead>
<tr>
<th>pat.charAt(j)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>A</td>
<td>B</td>
<td>A</td>
<td>B</td>
<td>A</td>
<td>C</td>
</tr>
<tr>
<td>B</td>
<td>A</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>C</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>6</td>
</tr>
</tbody>
</table>

```
A
B, C
A
A
B, C
A
B, C
C
B, C
A
A
B, C
A
B
C
```

---

29
DFA simulation

\[\begin{array}{cccccc}
A & A & B & A & C & A \\
A & B & A & B & A & C \\
A & C & A & A & & \\
\end{array}\]

\[
\begin{array}{cccccc}
0 & 1 & 2 & 3 & 4 & 5 \\
A & B & A & B & A & C \\
A & 1 & 1 & 3 & 1 & 5 \\
B & 0 & 2 & 0 & 4 & 0 \\
C & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

\[
pat.charAt(j) \\
dfa[][][j]
\]

Graph representation of a DFA simulation.
DFA simulation

A A B A C A A B A B A C A A

pat.charAt(j) | 0 1 2 3 4 5
---|---|---|---|---|---|---
A | B | A | B | A | C |
A | 1 | 1 | 3 | 1 | 5 | 1
B | 0 | 2 | 0 | 4 | 0 | 4
C | 0 | 0 | 0 | 0 | 0 | 6

A

A

B, C

A

B, C

A

B, C

C

B, C

B

C

A

0

1

2

3

4

5

6
DFA simulation

A A B A C A A B A B A C A A

pat.charAt(j) | 0 1 2 3 4 5
A B A B A C
A 1 1 3 1 5 1
B 0 2 0 4 0 4
C 0 0 0 0 0 6

dfa[][][j]
Q. What is interpretation of DFA state after reading in $\text{txt}[i]$?
A. State = number of characters in pattern that have been matched.

Ex. DFA is in state 3 after reading in $\text{txt}[0..6]$.

<table>
<thead>
<tr>
<th>i</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>txt</td>
<td>B</td>
<td>C</td>
<td>B</td>
<td>A</td>
<td>A</td>
<td>B</td>
<td>A</td>
<td>C</td>
<td>A</td>
</tr>
<tr>
<td>pat</td>
<td>A</td>
<td>B</td>
<td>A</td>
<td>B</td>
<td>A</td>
<td>C</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- length of longest prefix of pat[] that is a suffix of txt[0..i]
- suffix of text[0..6]
- prefix of pat[]

Diagram of DFA states:

- State 0, Action: A, Inputs: B, C
- State 1, Action: A, Inputs: A, B
- State 2, Action: A, Inputs: A
- State 3, Action: B, Inputs: B, C
- State 4, Action: A, Inputs: A
- State 5, Action: C, Inputs: B, C
- State 6
Knuth-Morris-Pratt substring search: Java implementation

Key differences from brute-force implementation.

- Need to precompute $\mathsf{dfa}[][]$ from pattern.
- Text pointer $i$ never decrements.

```java
public int search(String txt)
{
    int i, j, N = txt.length();
    for (i = 0, j = 0; i < N && j < M; i++)
        j = dfa[txt.charAt(i)][j];
    if (j == M) return i - M;
    else        return N;
}
```

Running time.

- Simulate DFA on text: at most $N$ character accesses.
- Build DFA: how to do efficiently? [warning: tricky algorithm ahead]
Knuth-Morris-Pratt substring search: Java implementation

Key differences from brute-force implementation.

• Need to precompute $\text{dfa}[][]$ from pattern.
• Text pointer $i$ never decrements.
• Could use input stream.

```java
public int search(In in) {
    int i, j;
    for (i = 0, j = 0; !in.isEmpty() && j < M; i++)
        j = dfa[in.readChar()][j];
    if (j == M) return i - M;
    else        return NOT_FOUND;
}
```

Constructing the DFA for KMP substring search for $\text{A B A B A C}$

<p>| | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>A</td>
<td>B</td>
<td>A</td>
<td>B</td>
<td>A</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>6</td>
</tr>
</tbody>
</table>

$\text{dfa}[][$j$]$
Knuth-Morris-Pratt construction

Include one state for each character in pattern (plus accept state).

```
<table>
<thead>
<tr>
<th>pat.charAt(j)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>A</td>
<td>B</td>
<td>A</td>
<td>B</td>
<td>A</td>
<td>C</td>
</tr>
</tbody>
</table>
```

```
| dfa[][][j]     |   |
|               | A |
|               | B |
|               | C |
```

Constructing the DFA for KMP substring search for A B A B A C

0 1 2 3 4 5 6
Knuth-Morris-Pratt construction

**Match transition.** If in state $j$ and next char $c = \text{pat.charAt}(j)$, go to $j + 1$.

- First $j$ characters of pattern have already been matched.
- Next char matches.
- Now first $j+1$ characters of pattern have been matched.

<table>
<thead>
<tr>
<th>$\text{pat.charAt}(j)$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>B</td>
<td>A</td>
<td>B</td>
<td>A</td>
<td>C</td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>2</td>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C</td>
<td></td>
<td></td>
<td>6</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Constructing the DFA for KMP substring search for A B A B A C.
Mismatch transition: back up if $c \neq \text{pat.charAt}(j)$.

Constructing the DFA for KMP substring search for A B A B A C
Knuth-Morris-Pratt construction

Mismatch transition: back up if `c != pat.charAt(j)`.

Constructing the DFA for KMP substring search for A B A B A C

```
<table>
<thead>
<tr>
<th>pat.charAt(j)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>B</td>
<td>A</td>
<td>B</td>
<td>A</td>
<td>C</td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>dfa[][][j]</td>
<td>B</td>
<td>0</td>
<td>2</td>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
```

Diagram:

```
0 -> A -> 1 -> B -> 2 -> A -> 3 -> B -> 4 -> A -> 5 -> C -> 6
```

B, C, A, j
Mismatch transition: back up if `c != pat.charAt(j)`.

Knuth-Morris-Pratt construction

Constructing the DFA for KMP substring search for `A B A B A C`
Knuth-Morris-Pratt construction

Mismatch transition: back up if \( c \neq \text{pat.charAt}(j) \).

\[
\begin{array}{c|cccccc}
\text{pat.charAt}(j) & 0 & 1 & 2 & 3 & 4 & 5 \\
\hline
A & B & A & B & A & C \\
A & 1 & 1 & 3 & 1 & 5 \\
B & 0 & 2 & 0 & 4 \\
C & 0 & 0 & 0 & 0 & 6 \\
\end{array}
\]

Constructing the DFA for KMP substring search for \( A B A B A C \)
Knuth-Morris-Pratt construction

Mismatch transition: back up if \( c \neq \text{pat.charAt}(j) \).

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{pat.charAt}(j) )</td>
<td>A</td>
<td>B</td>
<td>A</td>
<td>B</td>
<td>A</td>
<td>C</td>
</tr>
<tr>
<td>A</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>( \text{dfa[][]}[j] )</td>
<td>B</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>C</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>6</td>
</tr>
</tbody>
</table>

Constructing the DFA for KMP substring search for A B A B A C
Mismatch transition: back up if $c \neq \text{pat.charAt}(j)$.

Constructing the DFA for KMP substring search for $A B A B A C$
Knuth-Morris-Pratt construction

Constructing the DFA for KMP substring search for A B A B A C

<table>
<thead>
<tr>
<th>pat.charAt(j)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>A</td>
</tr>
<tr>
<td>B</td>
<td>A</td>
<td>B</td>
<td>A</td>
<td>B</td>
<td>A</td>
<td>C</td>
</tr>
<tr>
<td>A</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>B</td>
<td></td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>C</td>
<td></td>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>dfa[][][j]</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>6</td>
</tr>
</tbody>
</table>
How to build DFA from pattern?

Include one state for each character in pattern (plus accept state).
How to build DFA from pattern?

**Match transition.** If in state \( j \) and next char \( c == \text{pat.charAt}(j) \), go to \( j+1 \).

- first \( j \) characters of pattern have already been matched
- next char matches
- now first \( j+1 \) characters of pattern have been matched

<table>
<thead>
<tr>
<th>pat.charAt(j)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>B</td>
<td>A</td>
<td>B</td>
<td>A</td>
<td>C</td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>2</td>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C</td>
<td></td>
<td>6</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[
\begin{array}{ccccccc}
0 & A & 1 & B & 2 & A & 3 & B & 4 & A & 5 & C & 6 \\
\end{array}
\]
**Mismatch transition.** If in state \( j \) and next char \( c \neq \text{pat.charAt}(j) \), then the last \( j-1 \) characters of input are \( \text{pat}[1..j-1] \), followed by \( c \).

To compute \( \text{dfa}[c][j] \): Simulate \( \text{pat}[1..j-1] \) on DFA and take transition \( c \).

**Running time.** Seems to require \( j \) steps.

**Ex.**
\[
\text{dfa['A'][5] = 1; dfa['B'][5] = 4}
\]

simulate BABA; take transition 'A' = dfa['A'][3]

simulate BABA; take transition 'B' = dfa['B'][3]
Mismatch transition. If in state $j$ and next char $c \neq \text{pat.charAt}(j)$, then the last $j-1$ characters of input are $\text{pat}[1..j-1]$, followed by $c$.

To compute $\text{dfa}[c][j]$: Simulate $\text{pat}[1..j-1]$ on DFA and take transition $c$.

Running time. Takes only constant time if we maintain state $X$.

Ex. $\text{dfa[}'A'[5] = 1;$ from state $X$, take transition 'A' = $\text{dfa[}'A'[X]$ $\text{dfa[}'B'[5] = 4;}$ from state $X$, take transition 'B' = $\text{dfa[}'B'[X]$ $\text{X'} = 0$ from state $X$, take transition 'C' = $\text{dfa[}'C'[X]$
Knuth-Morris-Pratt construction (in linear time)

Include one state for each character in pattern (plus accept state).

Constructing the DFA for KMP substring search for A B A B A C
**Knuth-Morris-Pratt construction (in linear time)**

**Match transition.** For each state $j$, $\text{dfa[pat.charAt(j)][j]} = j+1$.

- **first $j$ characters of pattern have already been matched**
- **now first $j+1$ characters of pattern have been matched**

<table>
<thead>
<tr>
<th>$\text{pat.charAt(j)}$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>B</td>
<td>A</td>
<td>B</td>
<td>A</td>
<td>C</td>
<td></td>
</tr>
<tr>
<td>$\text{dfa[][][j]}$</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>B</td>
<td></td>
<td></td>
<td>2</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>6</td>
</tr>
<tr>
<td>C</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Constructing the DFA for KMP substring search for $A B A B A C$
Mismatch transition. For state 0 and char \( c \neq \text{pat.charAt}(j) \), set \( \text{dfa}[c][0] = 0 \).

Knuth-Morris-Pratt construction (in linear time)

Constructing the DFA for KMP substring search for \( \text{A B A B A C} \)
Mismatch transition. For each state $j$ and char $c \neq \text{pat.charAt}(j)$, set $\text{dfa}[c][j] = \text{dfa}[c][X]$; then update $X = \text{dfa}[\text{pat.charAt}(j)][X]$.

Knuth-Morris-Pratt construction (in linear time)

Constructing the DFA for KMP substring search for A B A B A C
Mismatch transition. For each state \( j \) and char \( c \neq \text{pat.charAt}(j) \), set \( \text{dfa}[c][j] = \text{dfa}[c][X] \); then update \( X = \text{dfa[pat.charAt(j)]}[X] \).
Mismatch transition. For each state $j$ and character $c \neq \text{pat.charAt}(j)$, set $\text{dfa}[c][j] = \text{dfa}[c][X]$; then update $X = \text{dfa}[\text{pat.charAt}(j)][X]$.

Knuth-Morris-Pratt construction (in linear time)

Constructing the DFA for KMP substring search for A B A B A C
Mismatch transition. For each state $j$ and char $c \neq \text{pat.charAt}(j)$, set $\text{dfa}[c][j] = \text{dfa}[c][X]$; then update $X = \text{dfa[pat.charAt(j)]}[X]$.

Knuth-Morris-Pratt construction (in linear time)

Constructing the DFA for KMP substring search for $A B A B A C$
Mismatch transition. For each state $j$ and char $c \neq \text{pat.charAt}(j)$, set $\text{dfa}[c][j] = \text{dfa}[c][X]$; then update $X = \text{dfa[pat.charAt(j)]}[X]$.

Knuth-Morris-Pratt construction (in linear time)

Constructing the DFA for KMP substring search for A B A B A C

$X = \text{simulation of B A B A}$
Mismatch transition. For each state $j$ and char $c \neq \text{pat.charAt}(j)$, set $\text{dfa}[c][j] = \text{dfa}[c][X]$; then update $X = \text{dfa}[\text{pat.charAt}(j)][X]$.

Knuth-Morris-Pratt construction (in linear time)

Constructing the DFA for KMP substring search for $\text{A B A B A C}$

$X = \text{simulation of B A B A C}$

<table>
<thead>
<tr>
<th>$\text{pat.charAt}(j)$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>B</td>
<td>A</td>
<td>B</td>
<td>A</td>
<td>C</td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>B</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>C</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>6</td>
</tr>
</tbody>
</table>

Diagram of DFA states and transitions:

- State 0: Initial state.
- State 1: Transition on 'A' to state 2.
- State 2: Transition on 'B' to state 1.
- State 3: Transition on 'A' to state 4.
- State 4: Transition on 'B' to state 5.
- State 5: Transition on 'C' to state 6.
- State 6: Final state.
Knuth-Morris-Pratt construction (in linear time)

Constructing the DFA for KMP substring search for A B A B A C

<table>
<thead>
<tr>
<th>pat.charAt(j)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>B</td>
<td>A</td>
<td>B</td>
<td>A</td>
<td>C</td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>B</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>C</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>6</td>
</tr>
</tbody>
</table>
Constructing the DFA for KMP substring search: Java implementation

For each state $j$:

- Copy $\text{dfa}[][X]$ to $\text{dfa}[][j]$ for mismatch case.
- Set $\text{dfa}[\text{pat.charAt}(j)][j]$ to $j+1$ for match case.
- Update $X$.

```java
public KMP(String pat) {
    this.pat = pat;
    M = pat.length();
    dfa = new int[R][M];
    dfa[pat.charAt(0)][0] = 1;
    for (int X = 0, j = 1; j < M; j++) {
        for (int c = 0; c < R; c++)
            dfa[c][j] = dfa[c][X];
        dfa[pat.charAt(j)][j] = j+1;
        X = dfa[pat.charAt(j)][X];
    }
}
```

Running time. $M$ character accesses (but space proportional to $R \times M$).
KMP substring search analysis

**Proposition.** KMP substring search accesses no more than $M + N$ chars to search for a pattern of length $M$ in a text of length $N$.

**Pf.** Each pattern char accessed once when constructing the DFA; each text char accessed once (in the worst case) when simulating the DFA.

**Proposition.** KMP constructs $\text{dfa}[][]$ in time and space proportional to $R M$.

**Larger alphabets.** Improved version of KMP constructs $\text{nfa}[]$ in time and space proportional to $M$.  

![KMP NFA for ABABAC]
Knuth-Morris-Pratt: brief history

- Independently discovered by two theoreticians and a hacker.
  - Knuth: inspired by esoteric theorem, discovered linear-time algorithm
  - Pratt: made running time independent of alphabet size
  - Morris: built a text editor for the CDC 6400 computer

- Theory meets practice.

---

**FAST PATTERN MATCHING IN STRINGS**

DONALD E. KNUTH†, JAMES H. MORRIS, JR.‡ AND VAUGHAN R. PRATT¶

Abstract. An algorithm is presented which finds all occurrences of one given string within another, in running time proportional to the sum of the lengths of the strings. The constant of proportionality is low enough to make this algorithm of practical use, and the procedure can also be extended to deal with some more general pattern-matching problems. A theoretical application of the algorithm shows that the set of concatenations of even palindromes, i.e., the language \( \{aa^n\}^* \), can be recognized in linear time. Other algorithms which run even faster on the average are also considered.
Substring Search

- Brute force
- Knuth-Morris-Pratt
- Boyer-Moore
- Rabin-Karp
### Intuition.

- Scan characters in pattern from right to left.
- Can skip as many as \( M \) text chars when finding one not in the pattern.

![Mismatched character heuristic for right-to-left (Boyer-Moore) substring search](image)

```plaintext
<table>
<thead>
<tr>
<th>i</th>
<th>j</th>
<th>text</th>
<th>pattern</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>5</td>
<td>NEEDLE</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>NEEDLE</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>4</td>
<td></td>
<td>NEEDLE</td>
</tr>
<tr>
<td>15</td>
<td>0</td>
<td></td>
<td>NEEDLE</td>
</tr>
</tbody>
</table>

return i = 15
**Boyer-Moore: mismatched character heuristic**

Q. How much to skip?

**Case 1.** Mismatch character not in pattern.

Before

```
| txt | . . . . . . . . T L E . . . . . . |
| pat |   N E E D L E |
```

Mismatch character 'T' not in pattern: increment i one character beyond 'T'

After

```
| txt | . . . . . . . . T L E . . . . . . |
| pat |   N E E D L E |
```
Boyer-Moore: mismatched character heuristic

Q. How much to skip?

Case 2a. Mismatch character in pattern.

```
before

      i
  txt . . . . . . . N L E . . . . .
  pat               N E E D L E

after

      i
  txt . . . . . . . N L E . . . . .
  pat       N E E D L E
```

mismatch character 'N' in pattern: align text 'N' with rightmost pattern 'N'
Q. How much to skip?

Case 2b. Mismatch character in pattern (but heuristic no help).

before

\[
\begin{array}{ccccccc}
\text{txt} & \ldots & \ldots & \ldots & E & \text{L} & E & \ldots & \ldots & \ldots \\
\text{pat} & N & E & E & D & \text{L} & E
\end{array}
\]

aligned with rightmost E?

\[
\begin{array}{ccccccc}
\text{txt} & \ldots & \ldots & \ldots & E & \text{L} & E & \ldots & \ldots & \ldots \\
\text{pat} & N & E & E & D & \text{L} & E
\end{array}
\]

mismatch character 'E' in pattern: align text 'E' with rightmost pattern 'E'?
**Q.** How much to skip?

**Case 2b.** Mismatch character in pattern (but heuristic no help).

Before

\[
\begin{array}{c}
\text{txt} \quad \ldots \ldots \ldots\quad \color{red}{E}\quad \color{red}{L}\quad \color{red}{E}\quad \ldots \ldots \ldots \\
\text{pat} \quad \color{red}{N}\quad \color{red}{E}\quad \color{red}{E}\quad \color{red}{D}\quad \color{red}{L}\quad \color{red}{E}
\end{array}
\]

After

\[
\begin{array}{c}
\text{txt} \quad \ldots \ldots \ldots\quad \color{red}{E}\quad \color{red}{L}\quad \color{red}{E}\quad \ldots \ldots \ldots \\
\text{pat} \quad \color{red}{N}\quad \color{red}{E}\quad \color{red}{E}\quad \color{red}{D}\quad \color{red}{L}\quad \color{red}{E}
\end{array}
\]

mismatch character 'E' in pattern: increment i by 1
Boyer-Moore: mismatched character heuristic

Q. How much to skip?

A. Precompute index of rightmost occurrence of character $c$ in pattern (-1 if character not in pattern).

```
right = new int[R];
for (int c = 0; c < R; c++)
    right[c] = -1;
for (int j = 0; j < M; j++)
    right[pat.charAt(j)] = j;
```
public int search(String txt) {
    int N = txt.length();
    int M = pat.length();
    int skip;
    for (int i = 0; i <= N-M; i += skip) {
        skip = 0;
        for (int j = M-1; j >= 0; j--) {
            if (pat.charAt(j) != txt.charAt(i+j)) {
                skip = Math.max(1, j - right[txt.charAt(i+j)]);
                break;
            }
        }
        if (skip == 0) return i;
    }
    return N;
}
Boyer-Moore: analysis

Property. Substring search with the Boyer-Moore mismatched character heuristic takes about $\sim \frac{N}{M}$ character compares to search for a pattern of length $M$ in a text of length $N$.

Worst-case. Can be as bad as $\sim MN$.

Boyer-Moore variant. Can improve worst case to $\sim 3N$ by adding a KMP-like rule to guard against repetitive patterns.
Substring Search

- Brute force
- Knuth-Morris-Pratt
- Boyer-Moore
- Rabin-Karp
Rabin-Karp fingerprint search

Basic idea = modular hashing.
• Compute a hash of pattern characters 0 to $M - 1$.
• For each $i$, compute a hash of text characters $i$ to $M + i - 1$.
• If pattern hash = text substring hash, check for a match.

<table>
<thead>
<tr>
<th>pat.charAt(i)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>i</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>6</td>
<td>5</td>
<td>3</td>
<td>5</td>
</tr>
</tbody>
</table>

% 997 = 613

<table>
<thead>
<tr>
<th>txt.charAt(i)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>i</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
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<td>15</td>
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<td>3</td>
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<tr>
<td></td>
<td>% 997 = 508</td>
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<tr>
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<td>9</td>
<td>3</td>
</tr>
<tr>
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<td>1</td>
<td>4</td>
<td>1</td>
<td>5</td>
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<td>7</td>
<td>9</td>
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<td>6</td>
</tr>
<tr>
<td></td>
<td>% 997 = 201</td>
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<td>1</td>
<td>5</td>
<td>9</td>
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<td>5</td>
<td>9</td>
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<td>9</td>
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<td>9</td>
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<td>6</td>
<td>5</td>
<td>3</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>6</td>
<td>return $i = 6$</td>
<td>2</td>
<td>6</td>
<td>5</td>
<td>3</td>
<td>5</td>
<td>8</td>
<td>9</td>
<td>7</td>
<td>9</td>
<td>3</td>
<td>6</td>
<td>5</td>
<td>3</td>
<td>5</td>
<td>5</td>
</tr>
</tbody>
</table>
% 997 = 613

match
Efficiently computing the hash function

**Modular hash function.** Using the notation $t_i$ for `txt.charAt(i)`, we wish to compute

$$x_i = t_i R^{M-1} + t_{i+1} R^{M-2} + \ldots + t_{i+M-1} R^0 \pmod{Q}$$

**Intuition.** $M$-digit, base-$R$ integer, modulo $Q$.

**Horner's method.** Linear-time method to evaluate degree-$M$ polynomial.

---

```
// Compute hash for M-digit key
private long hash(String key, int M) {
    long h = 0;
    for (int j = 0; j < M; j++)
        h = (R * h + key.charAt(j)) % Q;
    return h;
}
```
**Challenge.** How to efficiently compute \( x_{i+1} \) given that we know \( x_i \).

\[
\begin{align*}
  x_i &= t_i R^{M-1} + t_{i+1} R^{M-2} + \ldots + t_{i+M-1} R^0 \\
  x_{i+1} &= t_{i+1} R^{M-1} + t_{i+2} R^{M-2} + \ldots + t_{i+M} R^0
\end{align*}
\]

**Key property.** Can update hash function in constant time!

\[
\begin{align*}
  x_{i+1} &= (x_i - t_i R^{M-1}) R + t_{i+M} \\
  \text{(can precompute } R^{M-2})
\end{align*}
\]

---

**Text:**

- Current value
- Subtract leading digit
- Multiply by radix
- Add new trailing digit

**Table:**

<table>
<thead>
<tr>
<th>i</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>current value</strong></td>
<td>1</td>
<td>4</td>
<td>1</td>
<td>5</td>
<td>9</td>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td><strong>new value</strong></td>
<td>4</td>
<td>1</td>
<td>5</td>
<td>9</td>
<td>2</td>
<td>6</td>
<td>5</td>
</tr>
</tbody>
</table>

| \begin{align*}
  4 & 1 & 5 & 9 & 2 & \text{current value} \\
  4 & 0 & 0 & 0 & 0 & \\
  1 & 5 & 9 & 2 & \text{subtract leading digit} \\
  \ast & 1 & 0 & \text{multiply by radix} \\
  1 & 5 & 9 & 2 & 0 & \\
  + & 6 & \text{add new trailing digit} \\
  1 & 5 & 9 & 2 & 6 & \text{new value}
\end{align*}
Rabin-Karp substring search example

<table>
<thead>
<tr>
<th>i</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1</td>
<td>4</td>
<td>1</td>
<td>5</td>
<td>9</td>
<td>2</td>
<td>6</td>
<td>5</td>
<td>3</td>
<td>5</td>
<td>8</td>
<td>9</td>
<td>7</td>
<td>9</td>
<td>3</td>
<td></td>
</tr>
</tbody>
</table>

0 3 % 997 = 3
1 3 1 % 997 = (3*10 + 1) % 997 = 31
2 3 1 4 % 997 = (31*10 + 4) % 997 = 314
3 3 1 4 1 % 997 = (314*10 + 1) % 997 = 150
4 3 1 4 1 5 % 997 = (150*10 + 5) % 997 = 508
5 1 4 1 5 9 % 997 = (508 + 3*(997 - 30))*10 + 9) % 997 = 201
6 4 1 5 9 2 % 997 = ((201 + 1*(997 - 30))*10 + 2) % 997 = 715
7 1 5 9 2 6 % 997 = ((715 + 4*(997 - 30))*10 + 6) % 997 = 971
8 5 9 2 6 5 % 997 = ((971 + 1*(997 - 30))*10 + 5) % 997 = 442
9 9 2 6 5 3 % 997 = ((442 + 5*(997 - 30))*10 + 3) % 997 = 929
10 return i-M+1 = 6

2 6 5 3 5 % 997 = ((929 + 9*(997 - 30))*10 + 5) % 997 = 613

match
Rabin-Karp: Java implementation

```java
public class RabinKarp {
    private long patHash; // pattern hash value
    private int M; // pattern length
    private long Q; // modulus
    private int R; // radix
    private long RM; // R^(M-1) % Q

    public RabinKarp(String pat) {
        M = pat.length();
        R = 256;
        Q = longRandomPrime();

        RM = 1;
        for (int i = 1; i <= M-1; i++)
            RM = (R * RM) % Q;
        patHash = hash(pat, M);
    }

    private long hash(String key, int M) {
        /* as before */
    }

    public int search(String txt) {
        /* see next slide */
    }
}
```

- a large prime (but avoid overflow)
- precompute $R^{M-1} \pmod{Q}$
Rabin-Karp: Java implementation (continued)

Monte Carlo version. Return match if hash match.

```java
public int search(String txt) {
    int N = txt.length();
    int txtHash = hash(txt, M);
    if (patHash == txtHash) return 0;
    for (int i = M; i < N; i++) {
        txtHash = (txtHash + Q - RM*txt.charAt(i-M) % Q) % Q;
        txtHash = (txtHash*R + txt.charAt(i)) % Q;
        if (patHash == txtHash) return i - M + 1;
    }
    return N;
}
```

Las Vegas version. Check for substring match if hash match; continue search if false collision.
Rabin-Karp analysis

Theory. If $Q$ is a sufficiently large random prime (about $MN^2$), then the probability of a false collision is about $1/N$.

Practice. Choose $Q$ to be a large prime (but not so large as to cause overflow). Under reasonable assumptions, probability of a collision is about $1/Q$.

Monte Carlo version.
• Always runs in linear time.
• Extremely likely to return correct answer (but not always!).

Las Vegas version.
• Always returns correct answer.
• Extremely likely to run in linear time (but worst case is $MN$).
Rabin-Karp fingerprint search

Advantages.
- Extends to 2d patterns.
- Extends to finding multiple patterns.

Disadvantages.
- Arithmetic ops slower than char compares.
- Las Vegas version requires backup.
- Poor worst-case guarantee.
Cost of searching for an $M$-character pattern in an $N$-character text.

### Substring search cost summary

<table>
<thead>
<tr>
<th>algorithm</th>
<th>version</th>
<th>operation count</th>
<th>backup in input?</th>
<th>correct?</th>
<th>extra space</th>
</tr>
</thead>
<tbody>
<tr>
<td>brute force</td>
<td>—</td>
<td>$MN$</td>
<td>yes</td>
<td>yes</td>
<td>1</td>
</tr>
<tr>
<td>Knuth-Morris-Pratt</td>
<td>full DFA (Algorithm 5.6)</td>
<td>$2N$</td>
<td>no</td>
<td>yes</td>
<td>MR</td>
</tr>
<tr>
<td></td>
<td>mismatch transitions only</td>
<td>$3N$</td>
<td>no</td>
<td>yes</td>
<td>$M$</td>
</tr>
<tr>
<td>Boyer-Moore</td>
<td>full algorithm</td>
<td>$3N$</td>
<td>yes</td>
<td>yes</td>
<td>$R$</td>
</tr>
<tr>
<td></td>
<td>mismatched char heuristic only (Algorithm 5.7)</td>
<td>$MN$</td>
<td>yes</td>
<td>yes</td>
<td>$R$</td>
</tr>
<tr>
<td>Rabin-Karp†</td>
<td>Monte Carlo (Algorithm 5.8)</td>
<td>$7N$</td>
<td>no</td>
<td>yes†</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Las Vegas</td>
<td>$7N$†</td>
<td>yes</td>
<td>yes</td>
<td>1</td>
</tr>
</tbody>
</table>

† probabilistic guarantee, with uniform hash function