Illustration: Frank Rosenblatt's Perceptron

FIG. 1 — Organization of a biological brain. (Red areas indicate active cells, responding to the letter X.)

# BBM406

## Fundamentals of Machine Learning

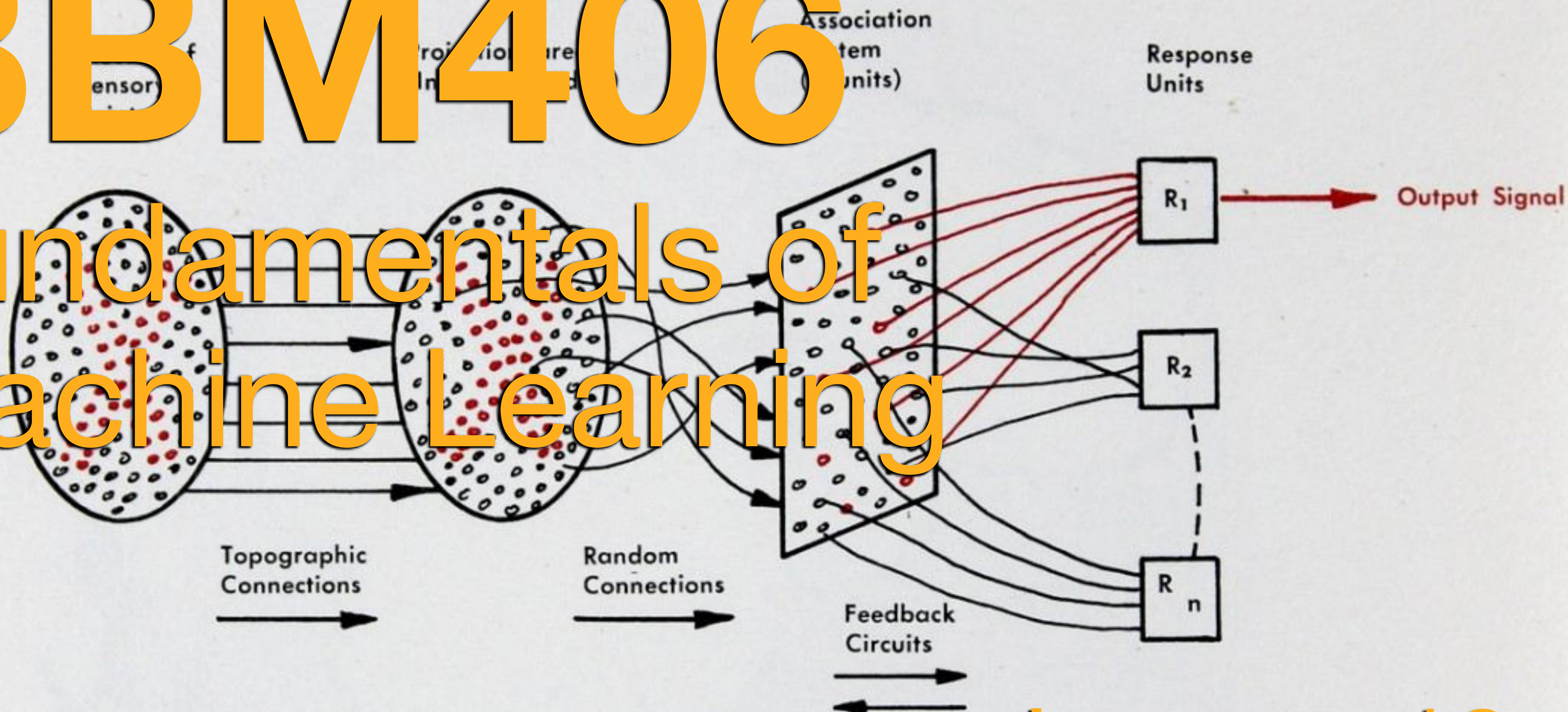



FIG. 2 — Organization of a perceptron.

## Lecture 10:
## Linear Discriminant Functions
## Perceptron


HACETTEPE UNIVERSITY COMPUTER VISION LAB

Erkut Erdem // Hacettepe University // Fall 2021

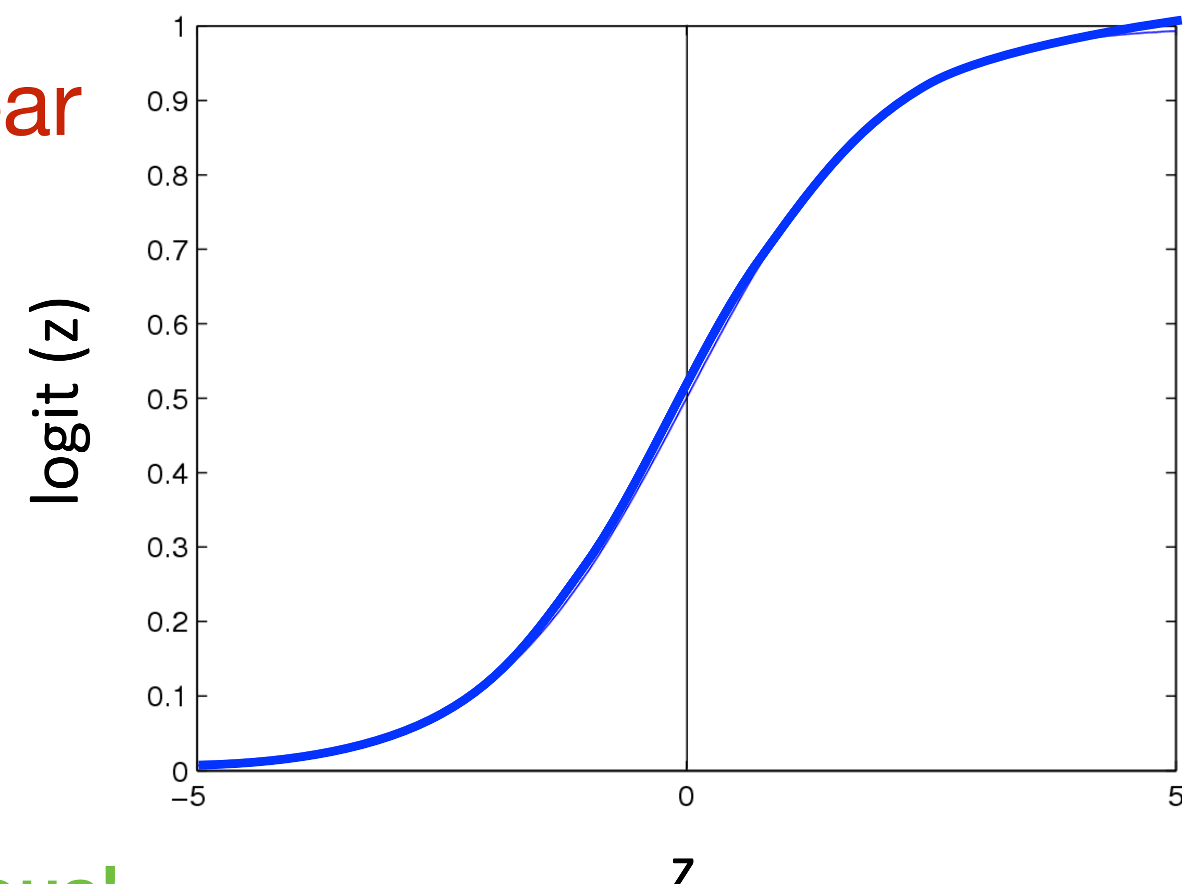
# Last time… **Logistic Regression**

Assumes the following functional form for P(Y|X):

$$P(Y=1|X) = \frac{1}{1+\exp(w_0 + \sum_i w_i X_i)}$$

Logistic function applied to linear function of the data

**Logistic function (or Sigmoid):** $\frac{1}{1+exp(-z)}$




Features can be discrete or continuous!

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# Last time.. **Logistic Regression vs. Gaussian Naïve Bayes**

- LR is a linear classifier
  - decision rule is a hyperplane
- LR optimized by maximizing conditional likelihood
  - no closed-form solution
  - concave ! global optimum with gradient ascent
- Gaussian Naïve Bayes with class-independent variances representationally equivalent to LR
  - Solution differs because of objective (loss) function
- In general, NB and LR make different assumptions
  - NB: Features independent given class! assumption on P(**X**|Y)
  - LR: Functional form of P(Y|**X**), no assumption on P(**X**|Y)
- Convergence rates
  - GNB (usually) needs less data
  - LR (usually) gets to better solutions in the limit

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# Linear Discriminant Functions

# Linear Discriminant Function

- Linear discriminant function for a vector $\mathbf{x}$

$$y(\mathbf{x}) = \mathbf{w}^T\mathbf{x} + w_0$$

where $\mathbf{w}$ is called weight vector, and $w_0$ is a bias.

- The classification function is

$$C(\mathbf{x}) = \text{sign}(\mathbf{w}^T\mathbf{x} + w_0)$$

where step function sign(·) is defined as

$$\text{sign}(a) = \begin{cases} +1, & a \geqslant 0 \\ -1, & a < 0 \end{cases}$$

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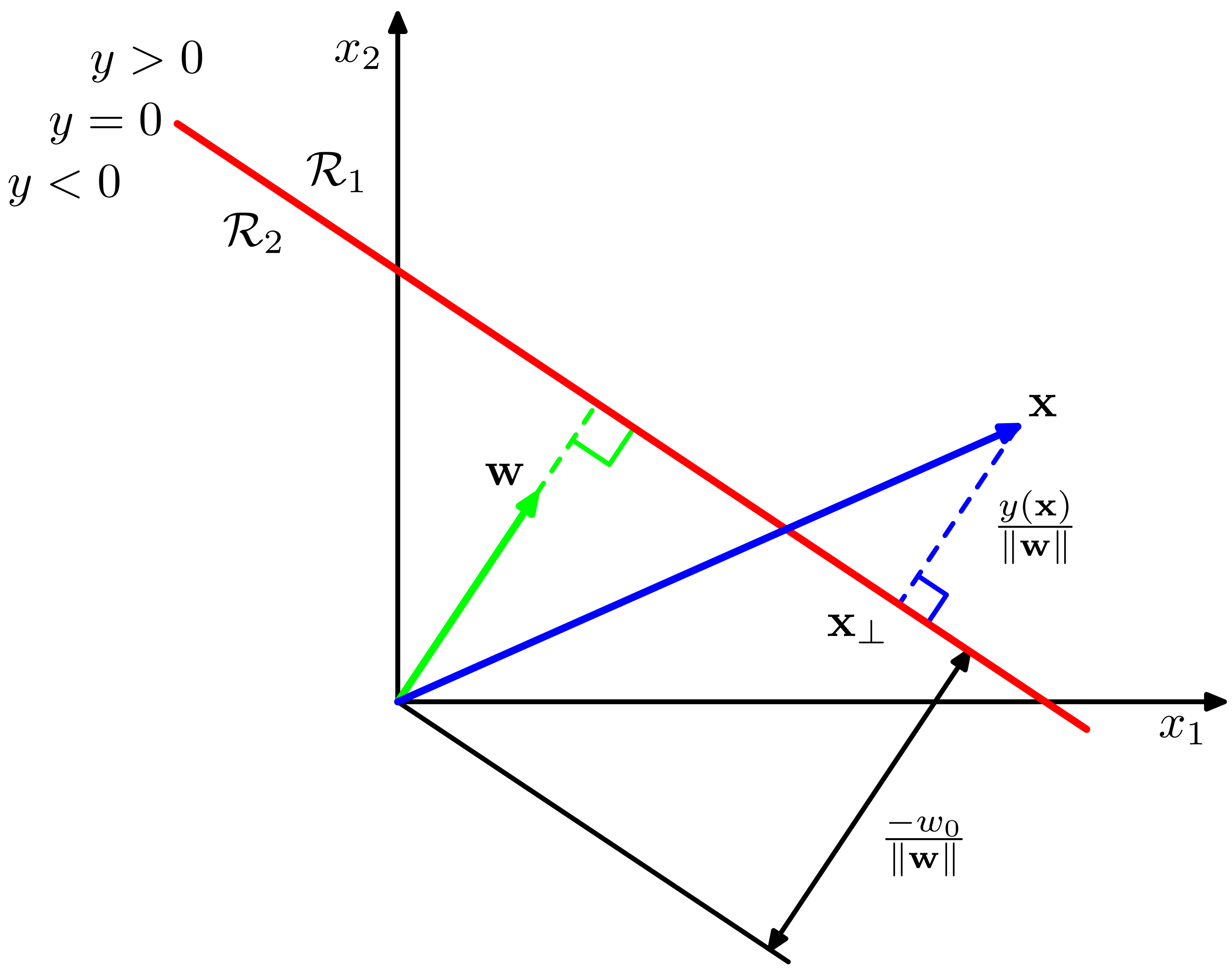
# Properties of Linear Discriminant Functions




- The decision surface, shown in red, is perpendicular to $\mathbf{w}$, and its displacement from the origin is controlled by the bias parameter $w_0$.
- The signed orthogonal distance of a general point $\mathbf{x}$ from the decision surface is given by $y(\mathbf{x})/\|\mathbf{w}\|$
- $y(\mathbf{x})$ gives a signed measure of the perpendicular distance $r$ of the point $\mathbf{x}$ from the decision surface

- $y(\mathbf{x}) = 0$ for $\mathbf{x}$ on the decision surface. The normal distance from the origin to the decision surface is

$$\frac{\mathbf{w}^T\mathbf{x}}{\|\mathbf{w}\|} = -\frac{w_0}{\|\mathbf{w}\|}$$

- So $w_0$ determines the location of the decision surface.

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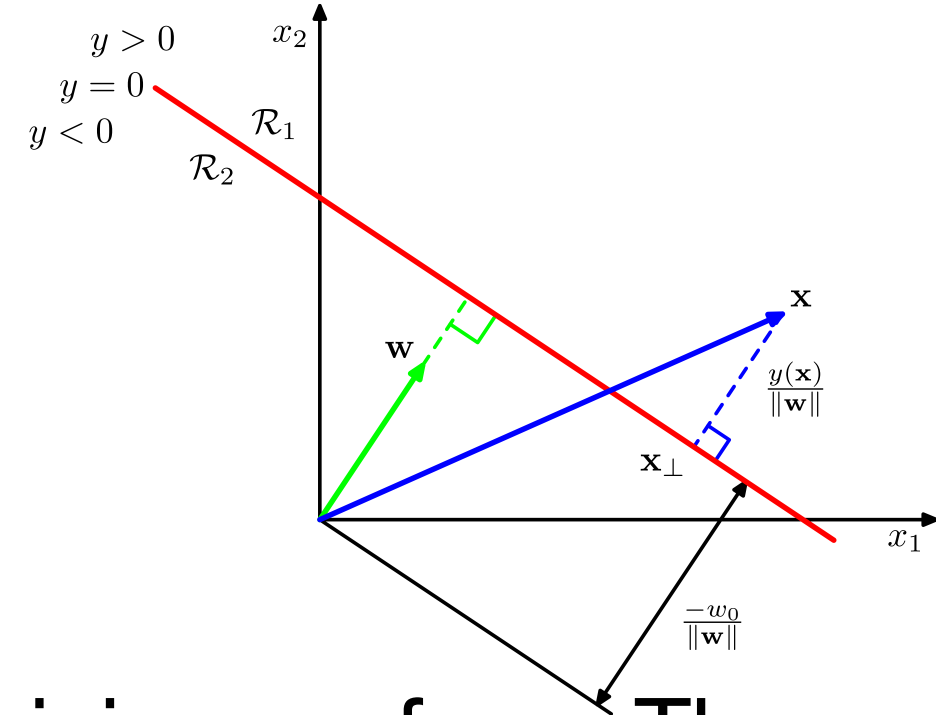
# Properties of Linear Discriminant Functions




- Let

$$\mathbf{x} = \mathbf{x}_\perp + r\frac{\mathbf{w}}{\|\mathbf{w}\|}$$

where $\mathbf{x}_\perp$ is the projection $\mathbf{x}$ on the decision surface. Then

$$\mathbf{w}^T\mathbf{x} = \mathbf{w}^T\mathbf{x}_\perp + r\frac{\mathbf{w}^T\mathbf{w}}{\|\mathbf{w}\|}$$

$$\mathbf{w}^T\mathbf{x} + w_0 = \mathbf{w}^T\mathbf{x}_\perp + w_0 + r\|\mathbf{w}\|$$

$$y(\mathbf{x}) = r\|\mathbf{w}\|$$

$$r = \frac{y(\mathbf{x})}{\|\mathbf{w}\|}$$

- Simpler notion: define $\widetilde{\mathbf{w}} = (w_0, \mathbf{w})$ and $\widetilde{\mathbf{x}} = (1, \mathbf{x})$ so that

$$y(\mathbf{x}) = \widetilde{\mathbf{w}}^T\widetilde{\mathbf{x}}$$

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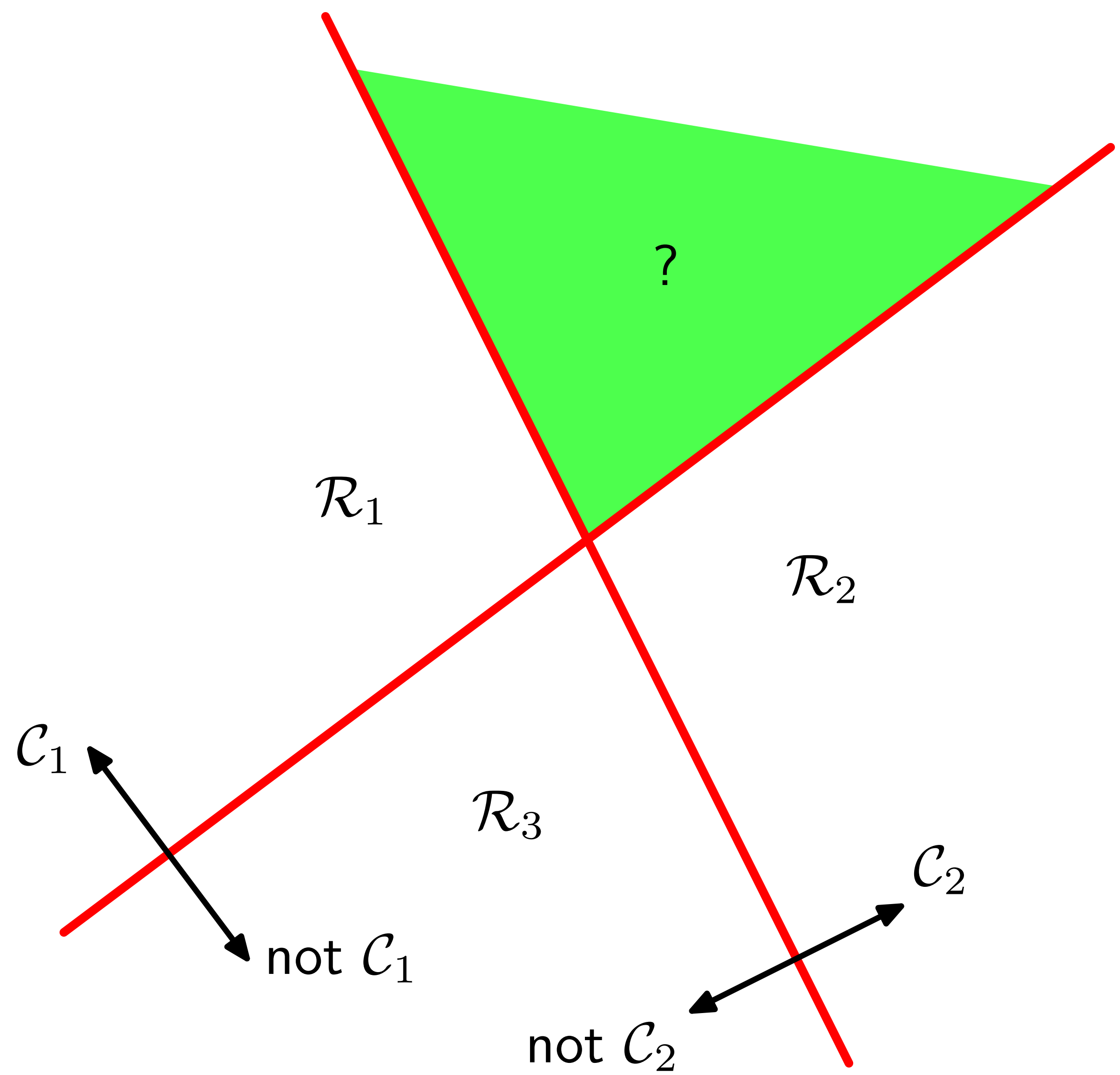
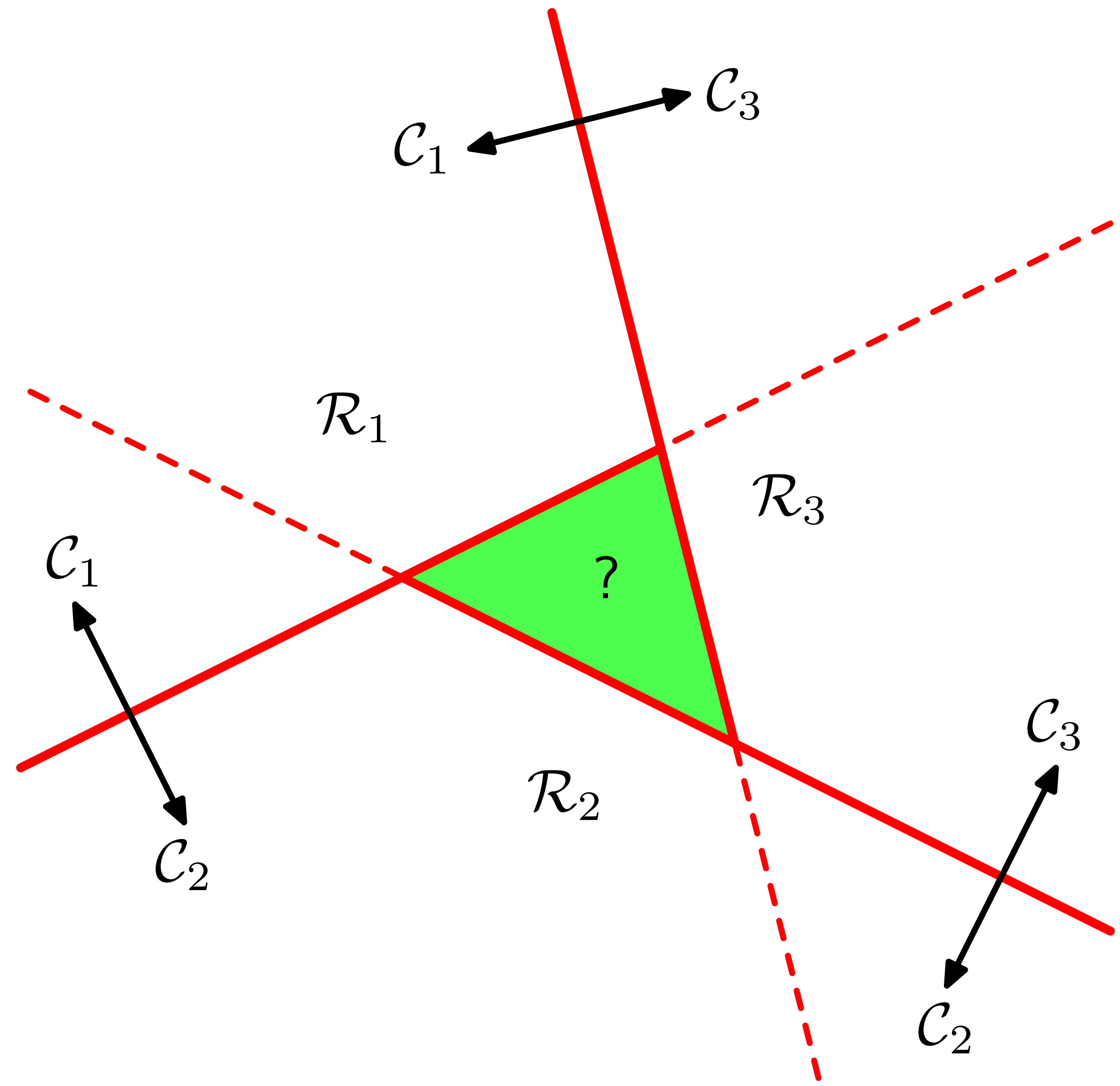
# Multiple Classes: Simple Extension

- **One-versus-the-rest** classifier: classify $C_k$ and samples not in $C_k$. *(K –1 classifiers)*
- **One-versus-one** classifier: classify every pair of classes. K*(K –1)/2 classifiers)*

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# Multiple Classes: K-Class Discriminant

- A single $K$-class discriminant comprising $K$ linear functions

$$y_k(\mathbf{x}) = \mathbf{w}_k^T \mathbf{x} + w_{k0}$$

- Decision function

$$\mathcal{C}(\mathbf{x}) = k, \text{ if } y_k(\mathbf{x}) > y_j(\mathbf{x}) \; \forall \, j \neq k$$

- The decision boundary between class $C_k$ and $C_j$ is given by $y_k(\mathbf{x}) = y_j(\mathbf{x})$

$$(\mathbf{w}_k - \mathbf{w}_j)^T \mathbf{x} + (w_{k0} - w_{j0}) = 0$$

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# Property of the Decision Regions

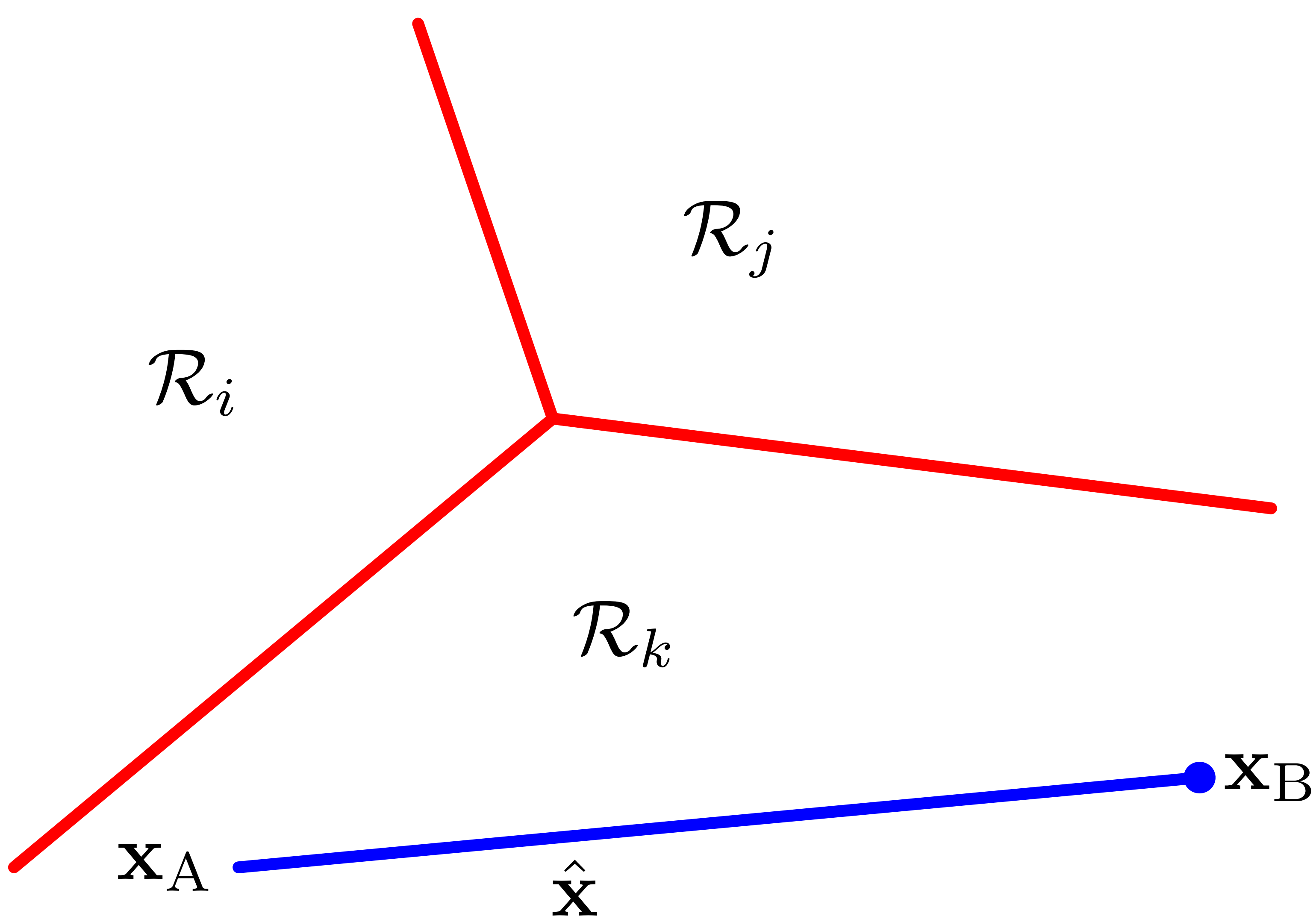
**Theorem**

*The decision regions of the K-class discriminant* $y_k(\mathbf{x}) = \mathbf{w}_k^T\mathbf{x} + w_{k0}$ *are singly connected and convex.*

# Property of the Decision Regions

**Theorem**

*The decision regions of the K-class discriminant $y_k(\mathbf{x}) = \mathbf{w}_k^T\mathbf{x} + w_{k0}$ are singly connected and convex.*




If two points $\mathbf{x}_\mathrm{A}$ and $\mathbf{x}_\mathrm{B}$ both lie inside the same decision region $R_k$, then any point $\mathbf{x}$ that lies on the line connecting these two points must also lie in $R_k$, and hence the decision region must be singly connected and convex.

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# Fisher's Linear Discriminant

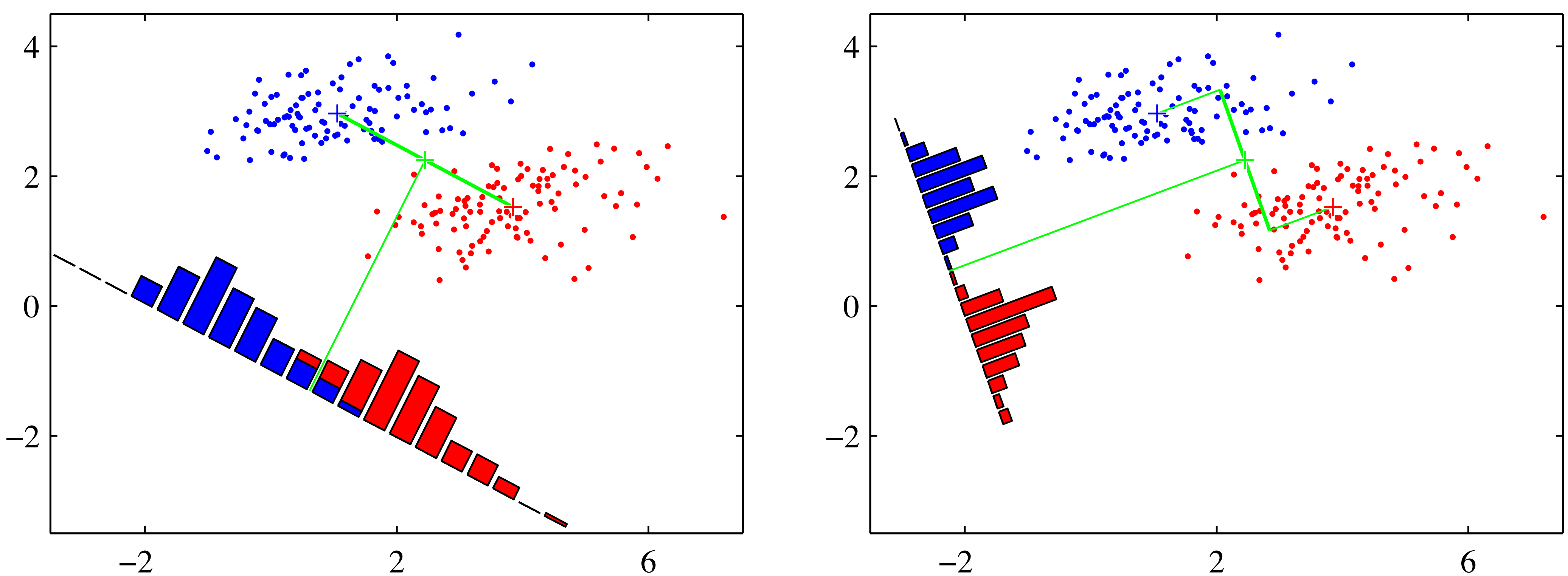
- Pursue the optimal linear projection on which the two classes can be maximally separated

$$y = \mathbf{w}^T \mathbf{x}$$

- The mean vectors of the two classes

$$\mathbf{m}_1 = \frac{1}{N_1} \sum_{n \in \mathcal{C}_1} \mathbf{x}_n, \quad \mathbf{m}_2 = \frac{1}{N_2} \sum_{n \in \mathcal{C}_2} \mathbf{x}_n$$

A way to view a linear classification model is in terms of dimensionality reduction.




Difference of means

Fisher's Linear Discriminant

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# What's a Good Projection?

- After projection, the two classes are separated as much as possible. Measured by the distance between projected center

$$\left(\mathbf{w}^T(\mathbf{m}_1 - \mathbf{m}_2)\right)^2 = \mathbf{w}^T(\mathbf{m}_1 - \mathbf{m}_2)(\mathbf{m}_1 - \mathbf{m}_2)^T\mathbf{w}$$
$$= \mathbf{w}^T\mathbf{S}_B\mathbf{w}$$

  where $\mathbf{S}_B = (\mathbf{m}_1 - \mathbf{m}_2)(\mathbf{m}_1 - \mathbf{m}_2)^T$ is called **between-class** covariance matrix.

- After projection, the variances of the two classes are as small as possible. Measured by the within-class covariance

$$\mathbf{w}^T\mathbf{S}_W\mathbf{w}$$

  where

$$\mathbf{S}_W = \sum_{n \in \mathcal{C}_1}(\mathbf{x}_n - \mathbf{m}_1)(\mathbf{x}_n - \mathbf{m}_1)^T + \sum_{n \in \mathcal{C}_2}(\mathbf{x}_n - \mathbf{m}_2)(\mathbf{x}_n - \mathbf{m}_2)^T$$

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# Fisher's Linear Discriminant

- Fisher criterion: maximize the ratio w.r.t. $\mathbf{w}$

$$J(\mathbf{w}) = \frac{\texttt{Between-class variance}}{\texttt{Within-class variance}} = \frac{\mathbf{w}^T \mathbf{S}_B \mathbf{w}}{\mathbf{w}^T \mathbf{S}_W \mathbf{w}}$$

- Recall the quotient rule: for $f(x) = \frac{g(x)}{h(x)}$

$$f'(x) = \frac{g'(x)h(x) - g(x)h'(x)}{h^2(x)}$$

- Setting $\nabla J(w) = 0$, we obtain

$$(\mathbf{w}^T \mathbf{S}_B \mathbf{w})\mathbf{S}_W \mathbf{w} = (\mathbf{w}^T \mathbf{S}_W \mathbf{w})\mathbf{S}_B \mathbf{w}$$

$$(\mathbf{w}^T \mathbf{S}_B \mathbf{w})\mathbf{S}_W \mathbf{w} = (\mathbf{w}^T \mathbf{S}_W \mathbf{w})(\mathbf{m}_2 - \mathbf{m}_1)\Big((\mathbf{m}_2 - \mathbf{m}_1)^T \mathbf{w}\Big)$$

- Terms $\mathbf{w}^T\mathbf{S}_B\mathbf{w}$, $\mathbf{w}^T\mathbf{S}_W\mathbf{w}$ and $(\mathbf{m}_2 - \mathbf{m}_1)^T\mathbf{w}$ are scalars, and we only care about directions. So the scalars are dropped. Therefore

$$\mathbf{w} \propto \mathbf{S}_W^{-1}(\mathbf{m}_2 - \mathbf{m}_1)$$

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# From Fisher's Linear Discriminant to Classifiers

- Fisher's Linear Discriminant is not a classifier; it only decides on an optimal projection to convert high-dimensional classification problem to 1D.

- A bias (threshold) is needed to form a linear classifier (multiple thresholds lead to nonlinear classifiers). The final classifier has the form

$$y(\mathbf{x}) = \mathrm{sign}(\mathbf{w}^T\mathbf{x} + w_0)$$

  where the nonlinear activation function sign(·) is a step function

$$\mathrm{sign}(a) = \begin{cases} +1, & a \geqslant 0 \\ -1, & a < 0 \end{cases}$$

- How to decide the bias $w_0$?

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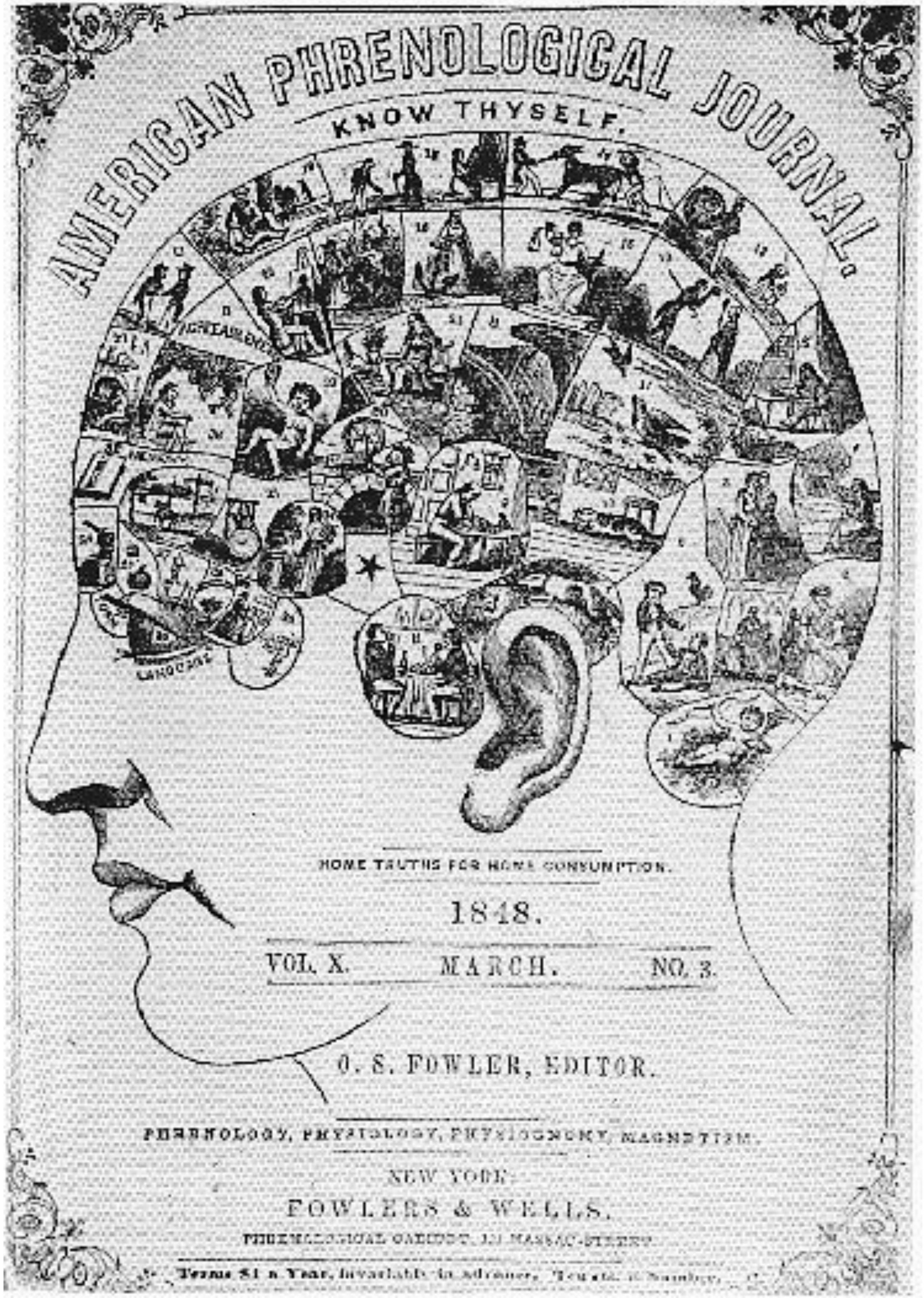
# Perceptron




early theories
of the brain

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# Biology and Learning

- Basic Idea
  - Good behavior should be rewarded, bad behavior punished (or not rewarded). This improves system fitness.
  - Killing a sabertooth tiger should be rewarded ...
  - Correlated events should be combined.
  - Pavlov’s salivating dog.

- Training mechanisms
  - Behavioral modification of individuals (learning)
    Successful behavior is rewarded (e.g. food).
  - Hard-coded behavior in the genes (instinct)
    The wrongly coded animal does not reproduce.

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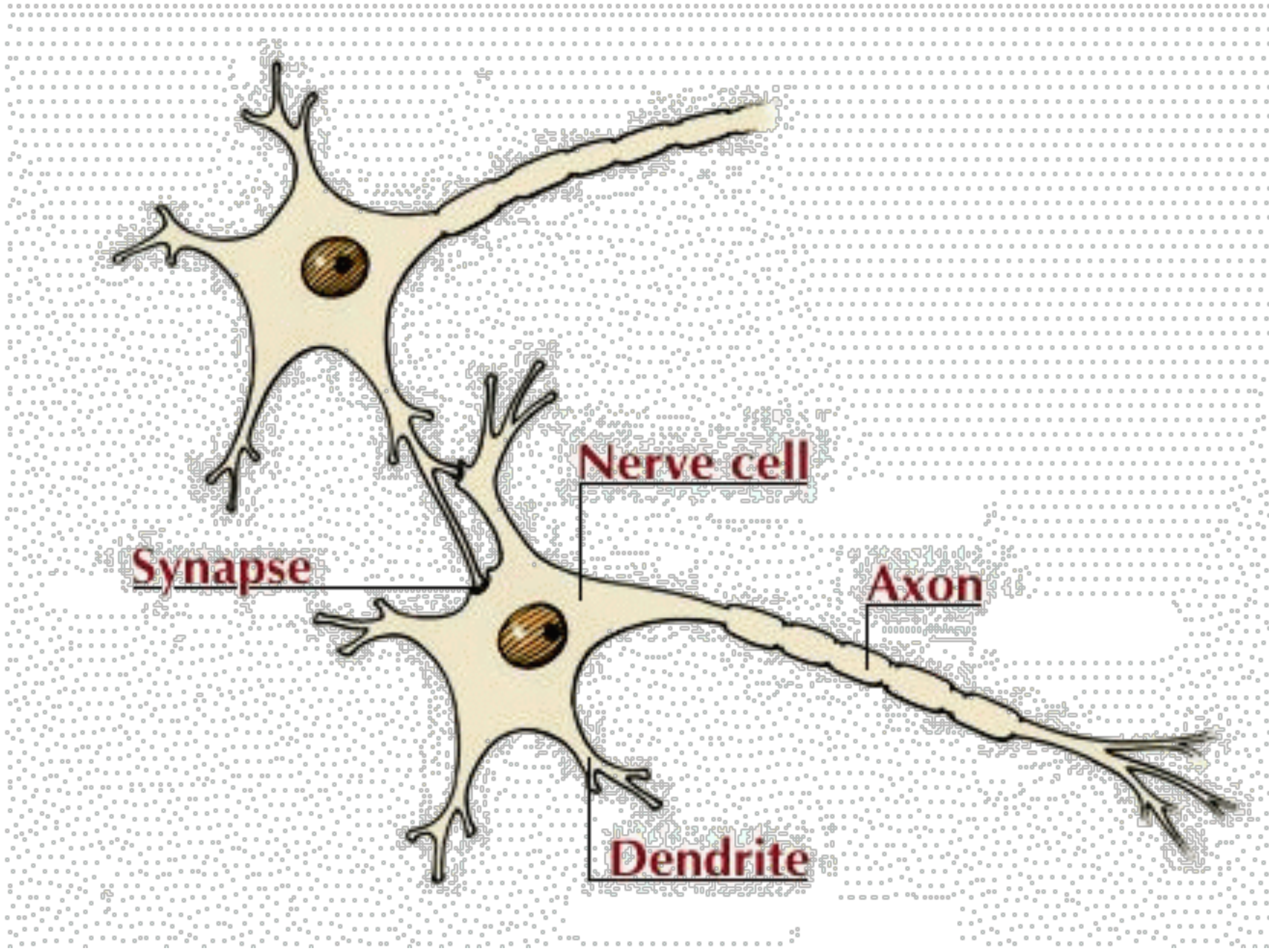
# Neurons




- Soma (CPU)
  Cell body - combines signals

- Dendrite (input bus)
  Combines the inputs from several other nerve cells

- Synapse (interface)
  Interface and parameter store between neurons

- Axon (cable)
  May be up to 1m long and will transport the activation signal to neurons at different locations

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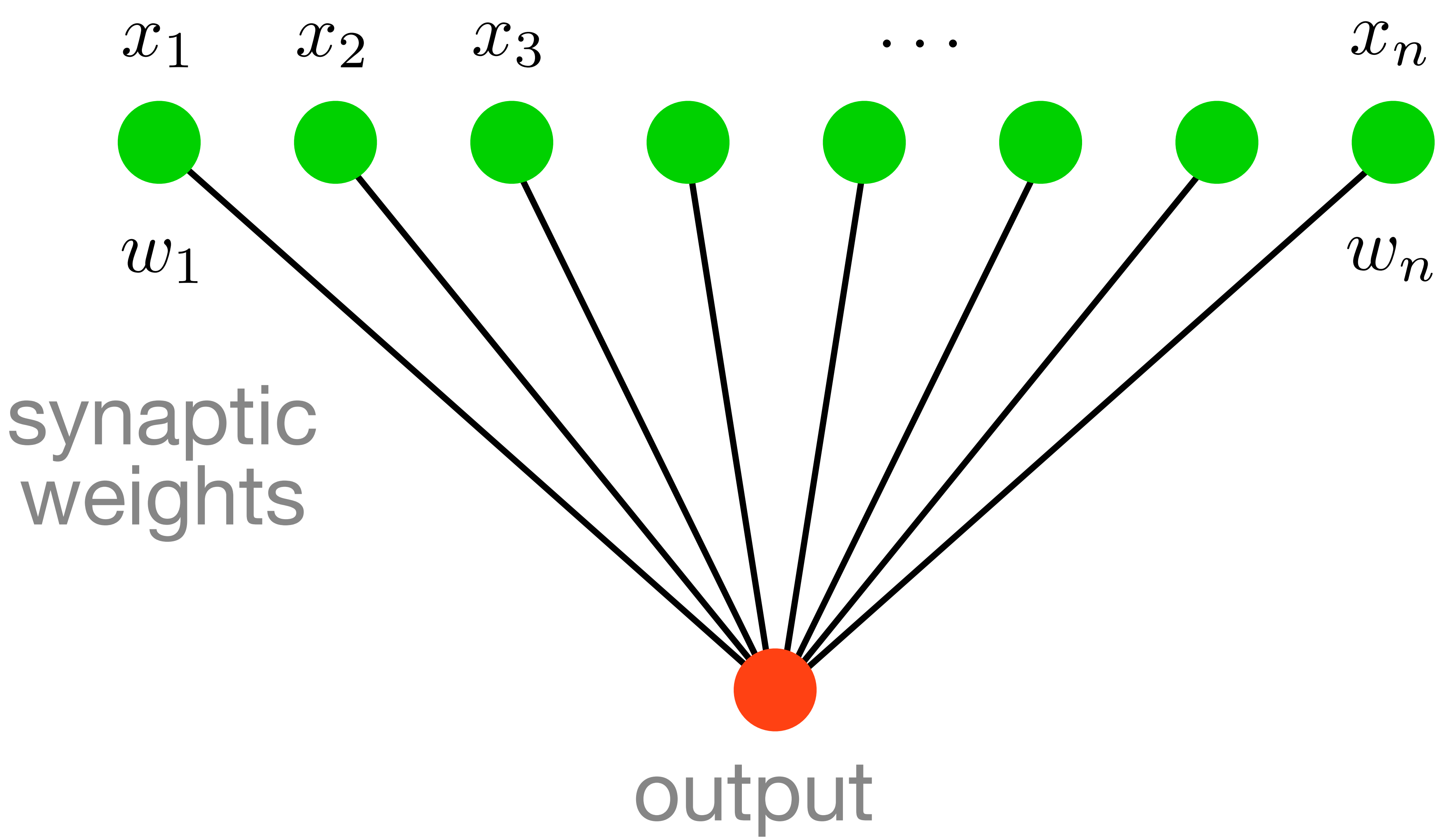
# Neurons




$$f(x) = \sum_i w_i x_i = \langle w, x \rangle$$

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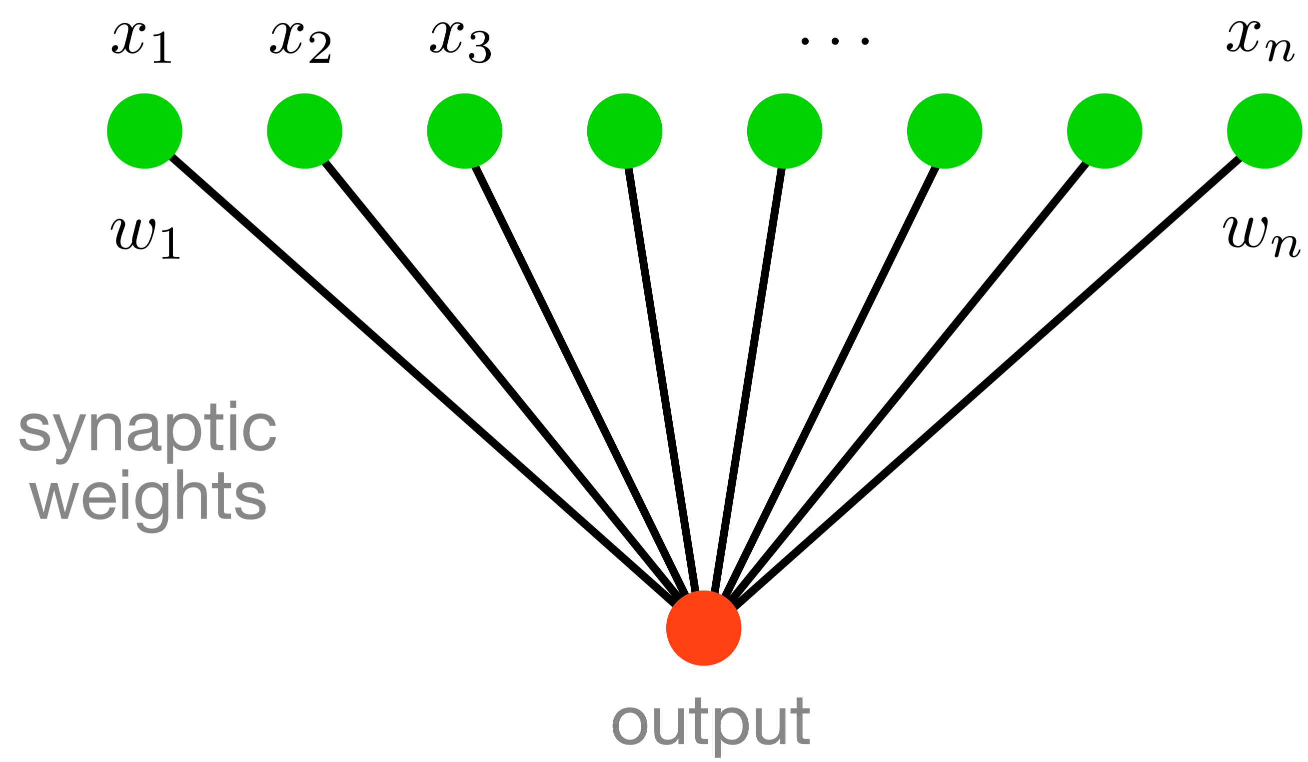
# Perceptron

- Weighted linear combination
- Nonlinear decision function
- Linear offset (bias)




$$f(x) = \sigma\left(\langle w, x\rangle + b\right)$$

- Linear separating hyperplanes (spam/ham, novel/typical, click/no click)
- Learning
  Estimating the parameters w and b

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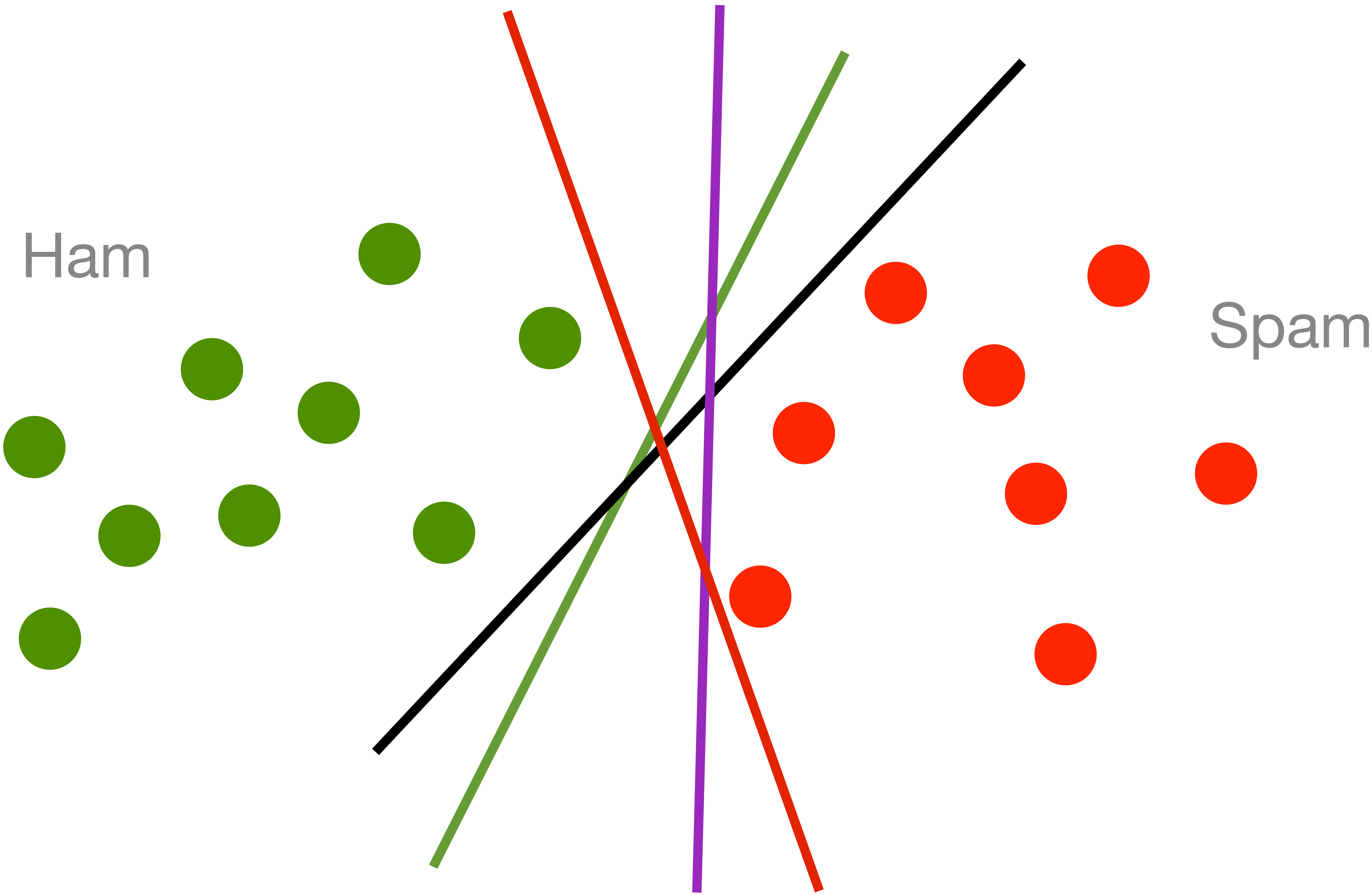
# Perceptron

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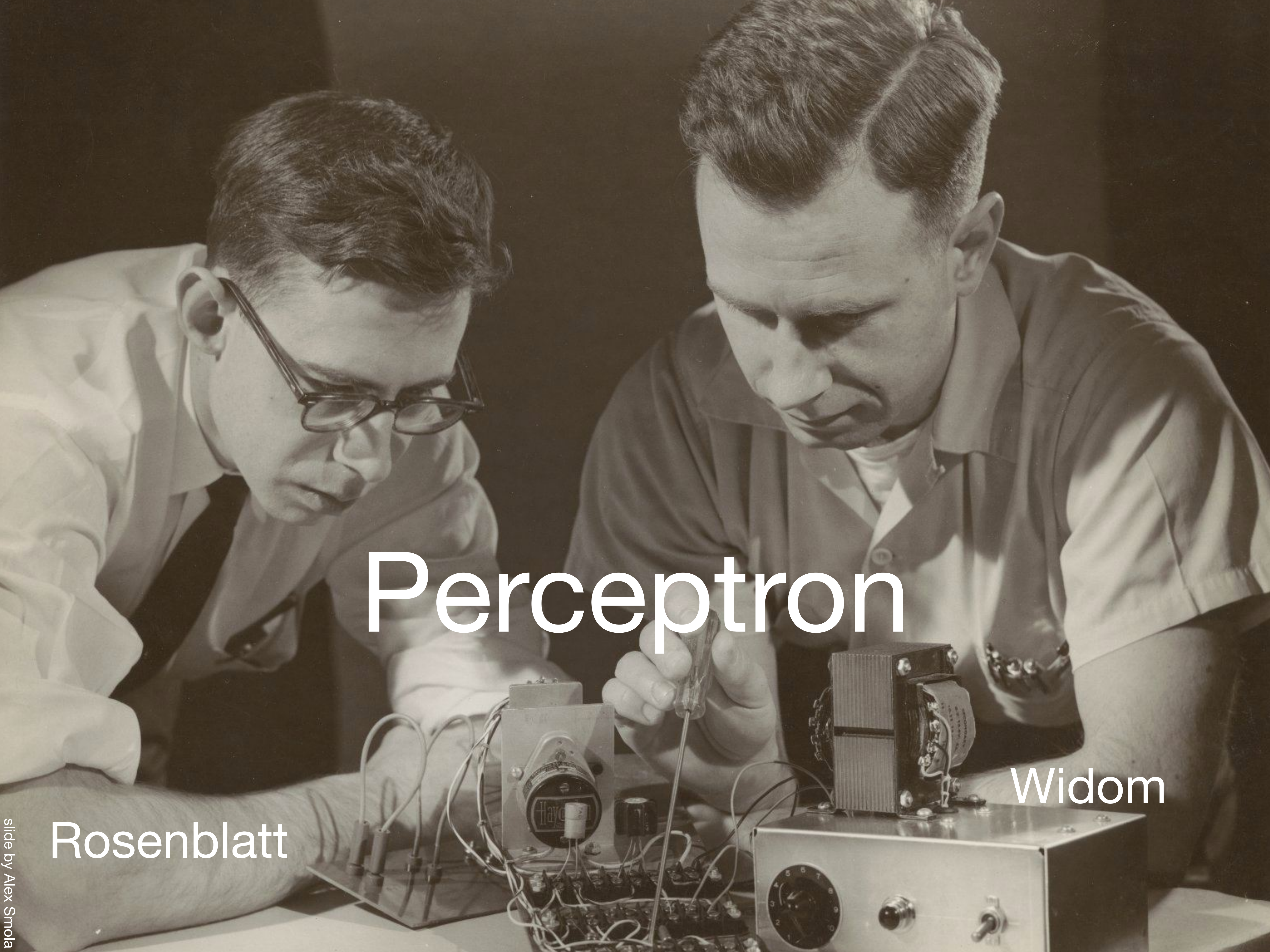
Perceptron
Widom
Rosenblatt
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# The Perceptron

$$
\begin{array}{l}
\textbf{initialize } w = 0 \text{ and } b = 0 \\
\textbf{repeat} \\
\quad \textbf{if } y_i \left[\langle w, x_i \rangle + b\right] \leq 0 \textbf{ then} \\
\qquad w \leftarrow w + y_i x_i \text{ and } b \leftarrow b + y_i \\
\quad \textbf{end if} \\
\textbf{until} \text{ all classified correctly}
\end{array}
$$

- Nothing happens if classified correctly
- Weight vector is linear combination $w = \sum_{i \in I} y_i x_i$
- Classifier is linear combination of inner products $f(x) = \sum_{i \in I} y_i \langle x_i, x \rangle + b$

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# Convergence Theorem

- If there exists some $(w^*, b^*)$ with unit length and
$$y_i \left[\langle x_i, w^* \rangle + b^*\right] \geq \rho \text{ for all } i$$
then the perceptron converges to a linear separator after a number of steps bounded by
$$\left(b^{*2} + 1\right)\left(r^2 + 1\right)\rho^{-2} \text{ where } \|x_i\| \leq r$$

- Dimensionality independent
- Order independent (i.e. also worst case)
- Scales with 'difficulty' of problem

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# Consequences

- Only need to store errors.
  This gives a compression bound for perceptron.
- Stochastic gradient descent on hinge loss
  $$l(x_i, y_i, w, b) = \max(0, 1 - y_i [\langle w, x_i \rangle + b])$$
- Fails with noisy data

do NOT train your avatar with perceptrons

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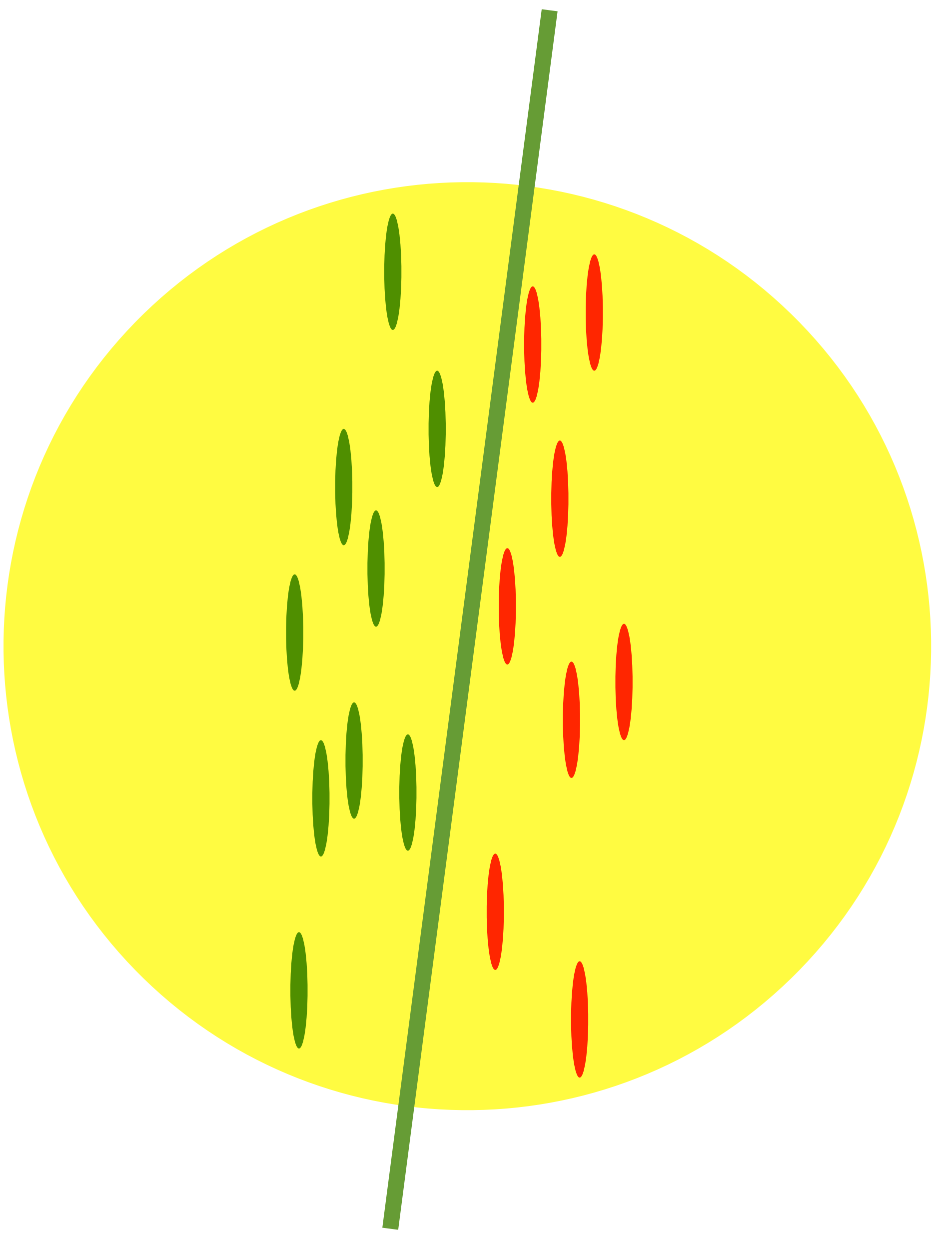
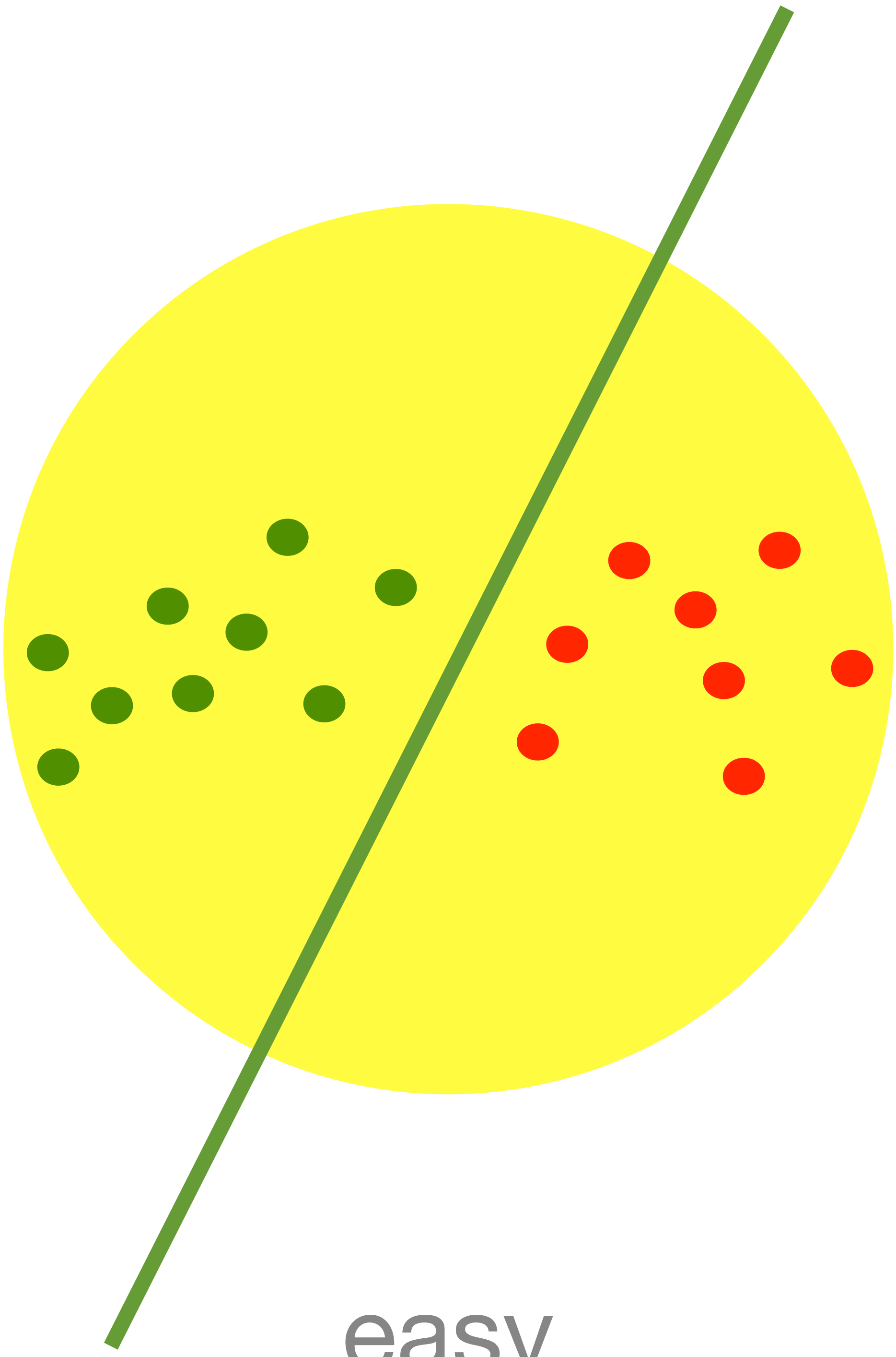
# Hardness: margin vs. size



hard



easy

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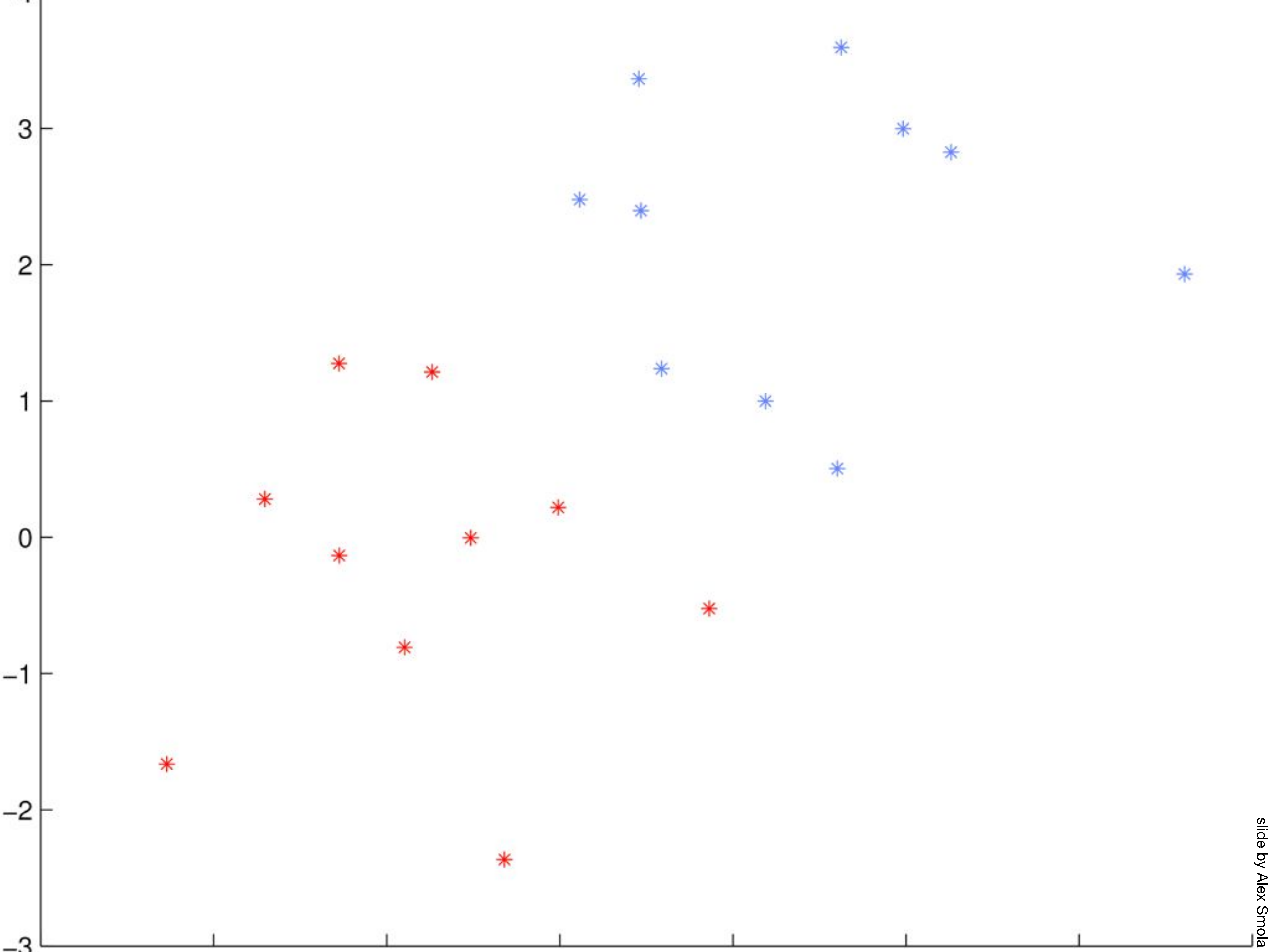
3
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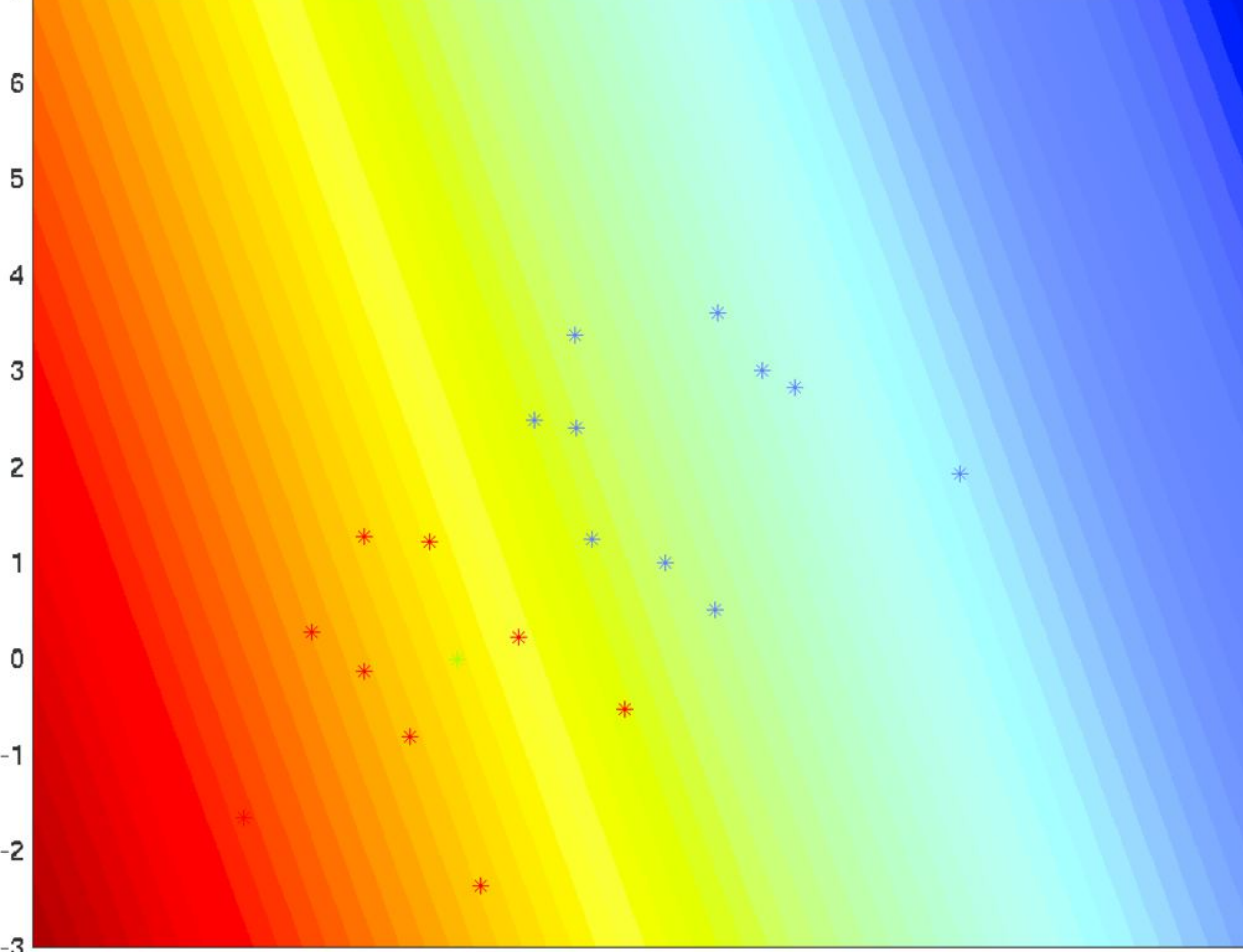

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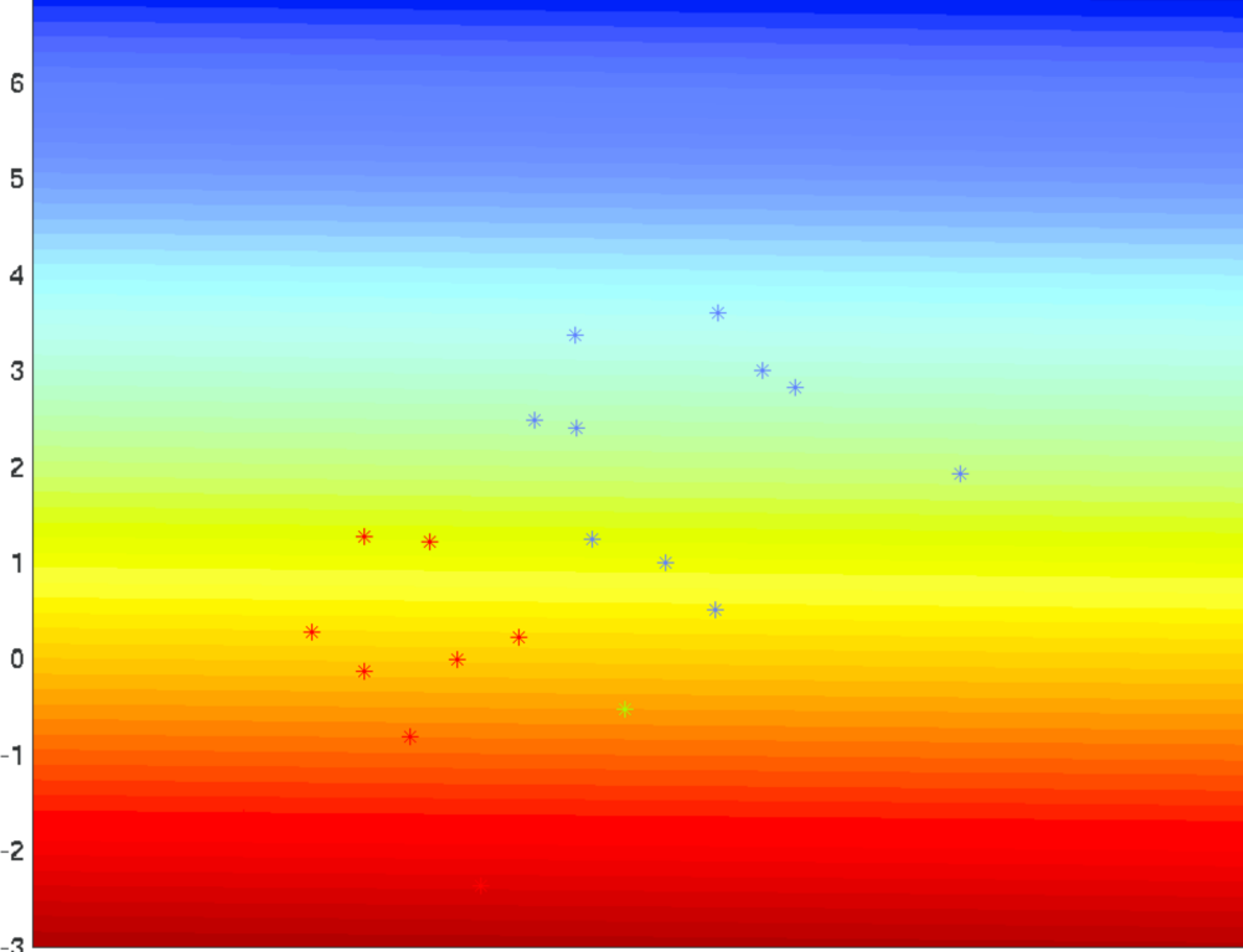


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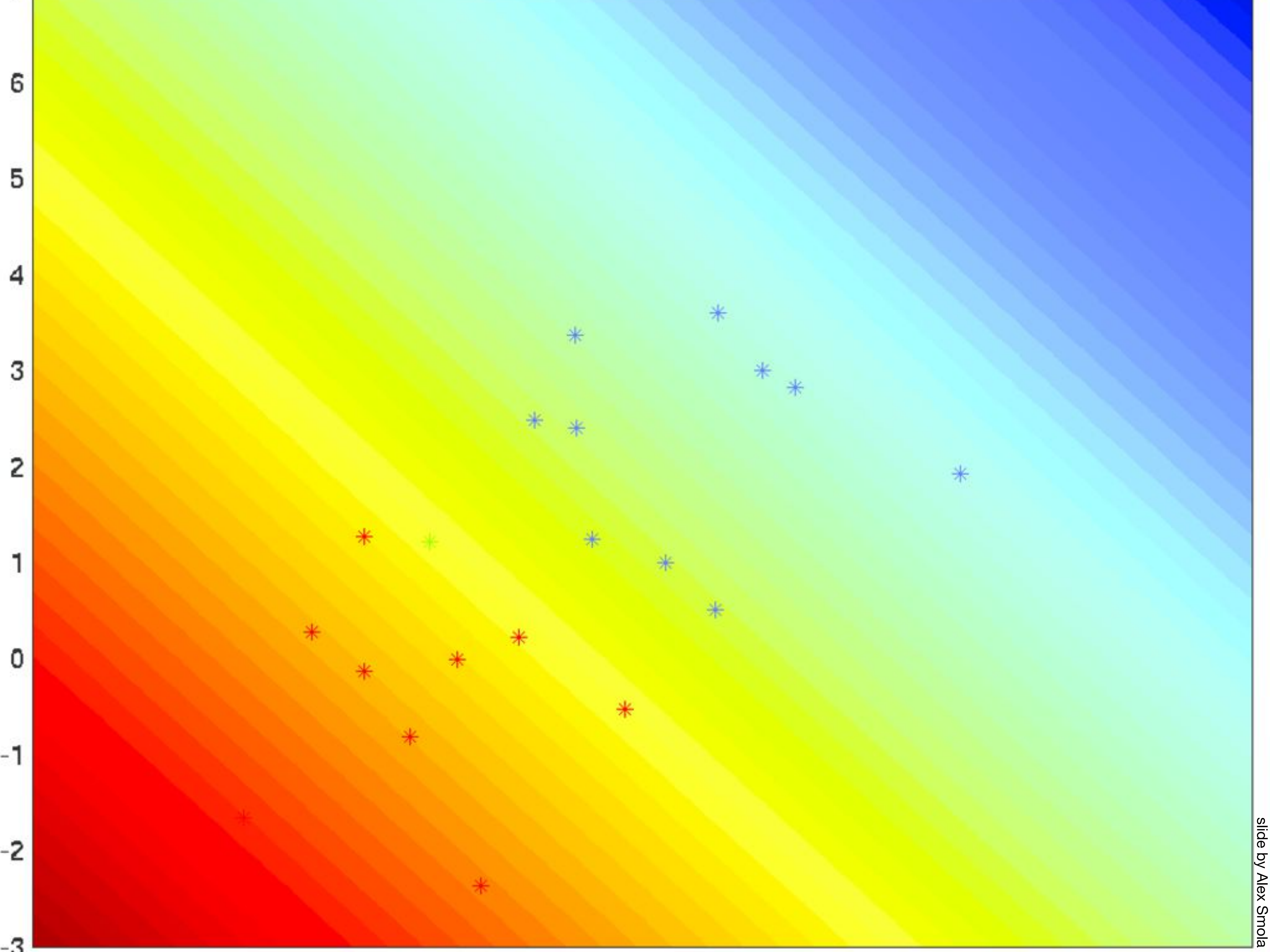
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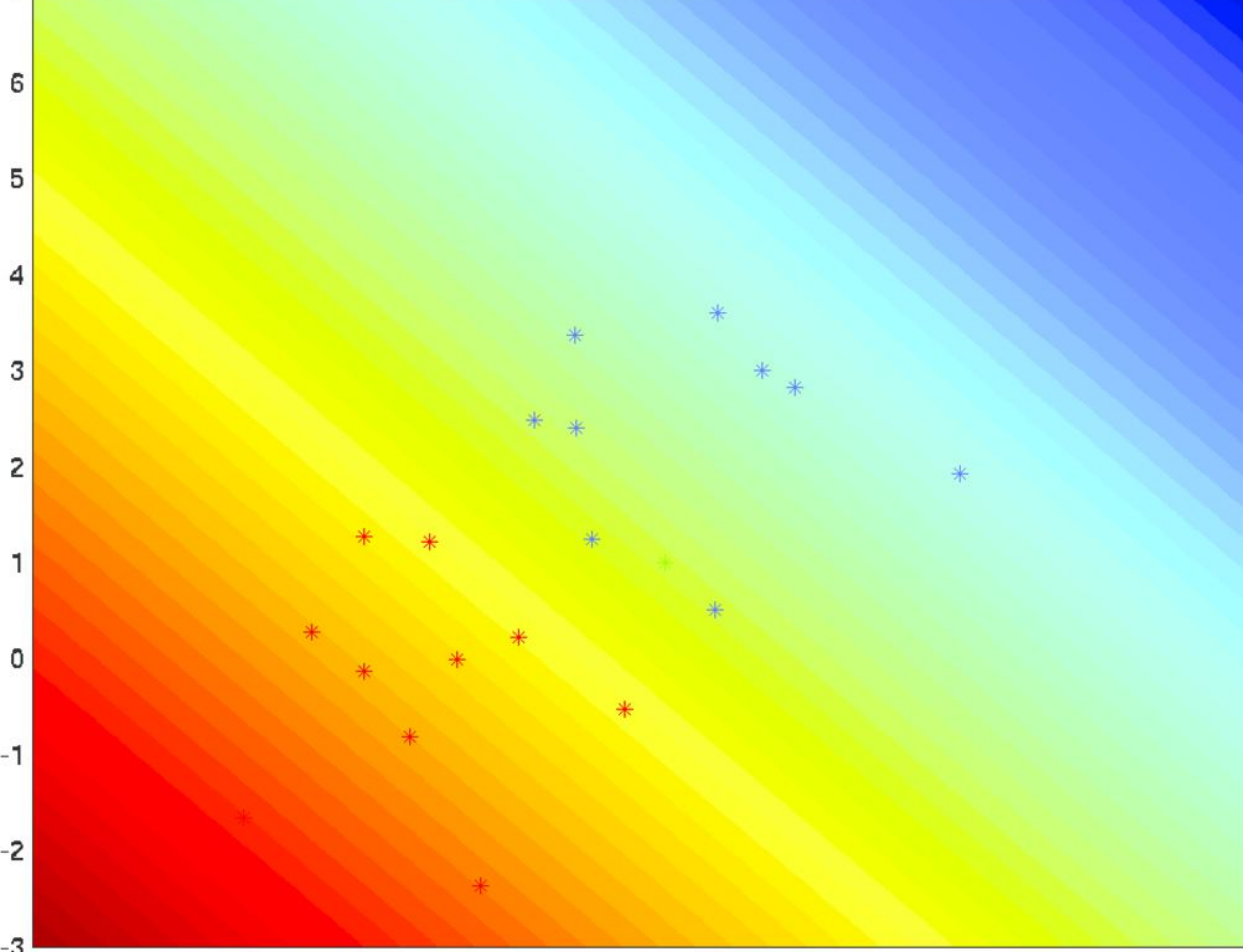
slide by Alex Smola



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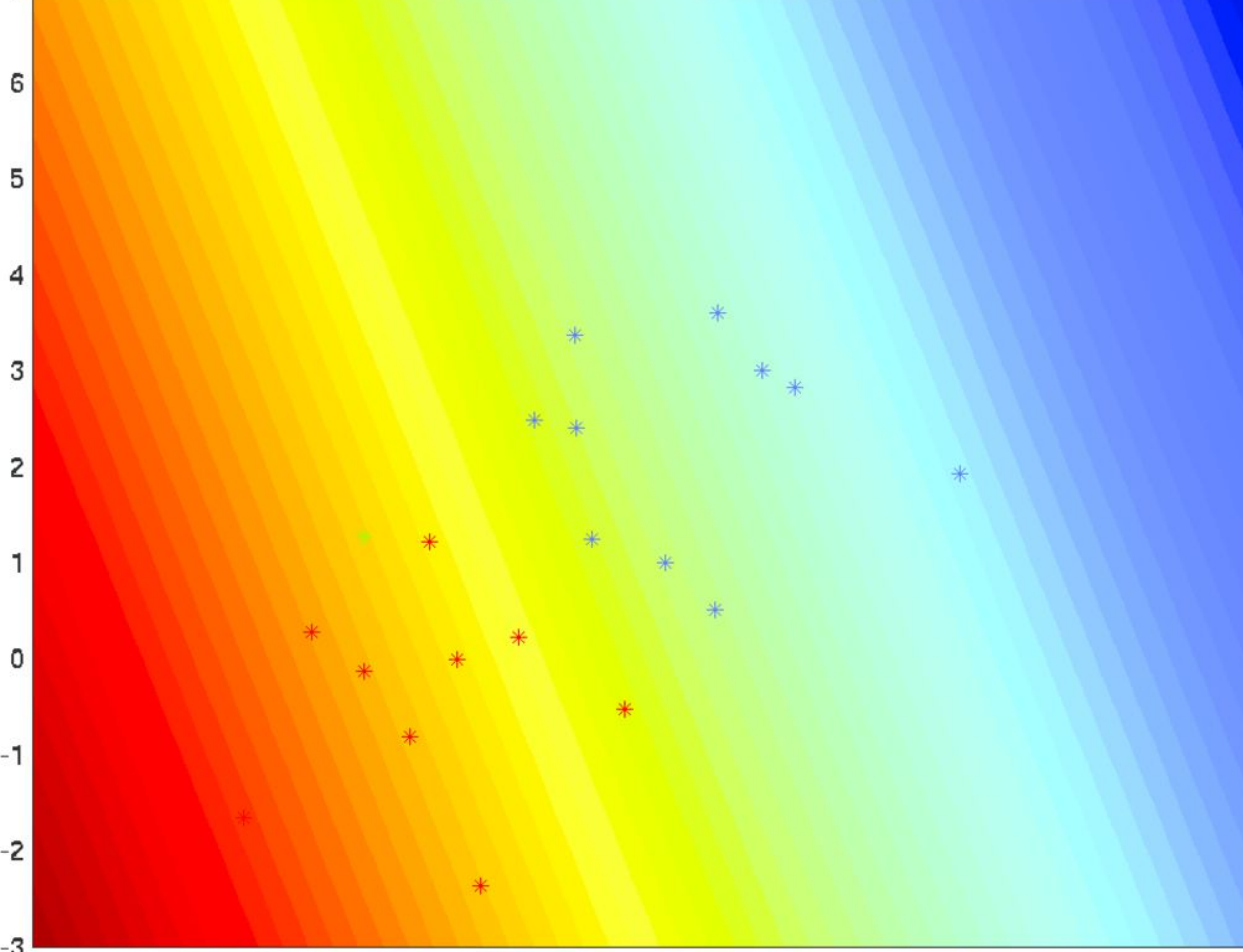


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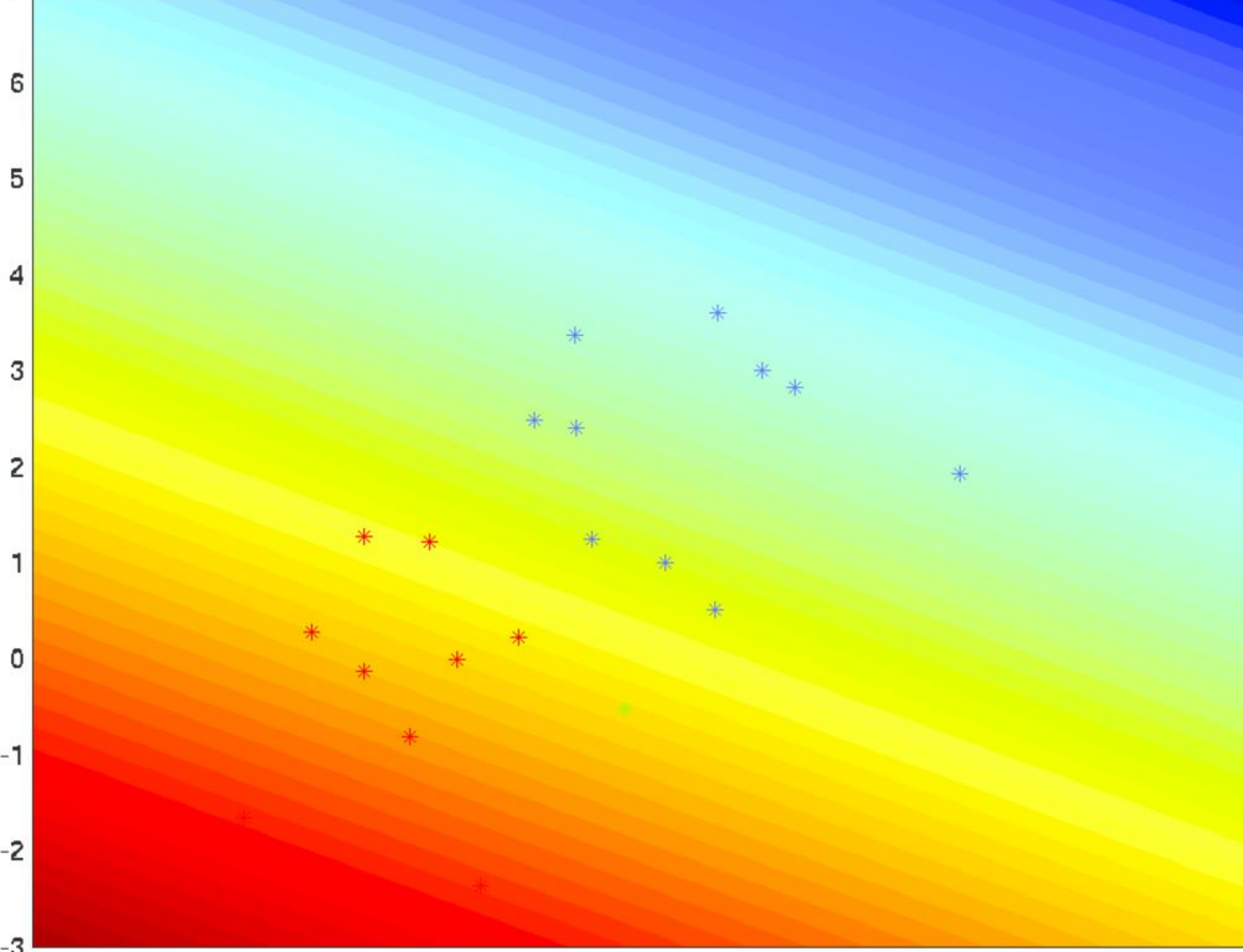


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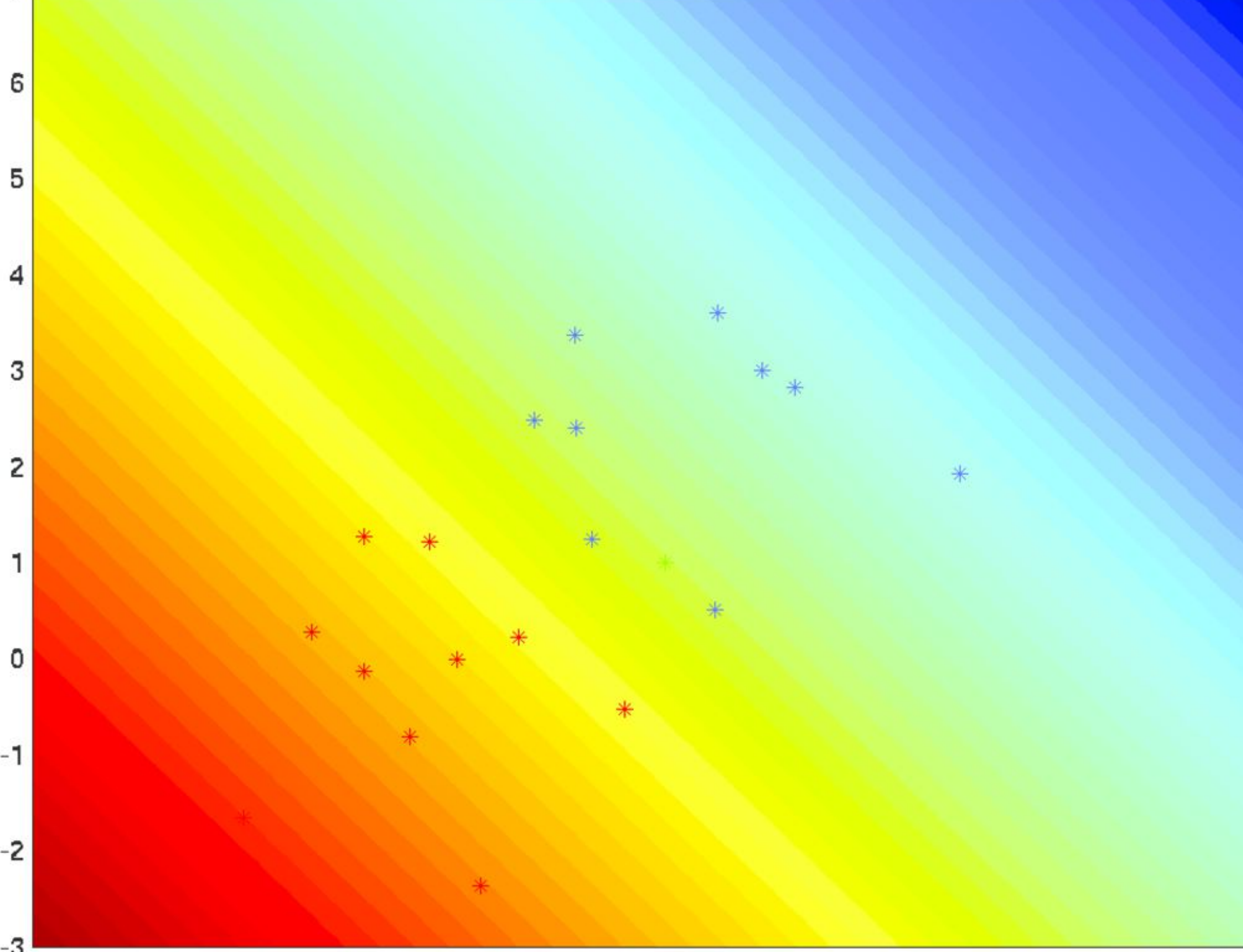


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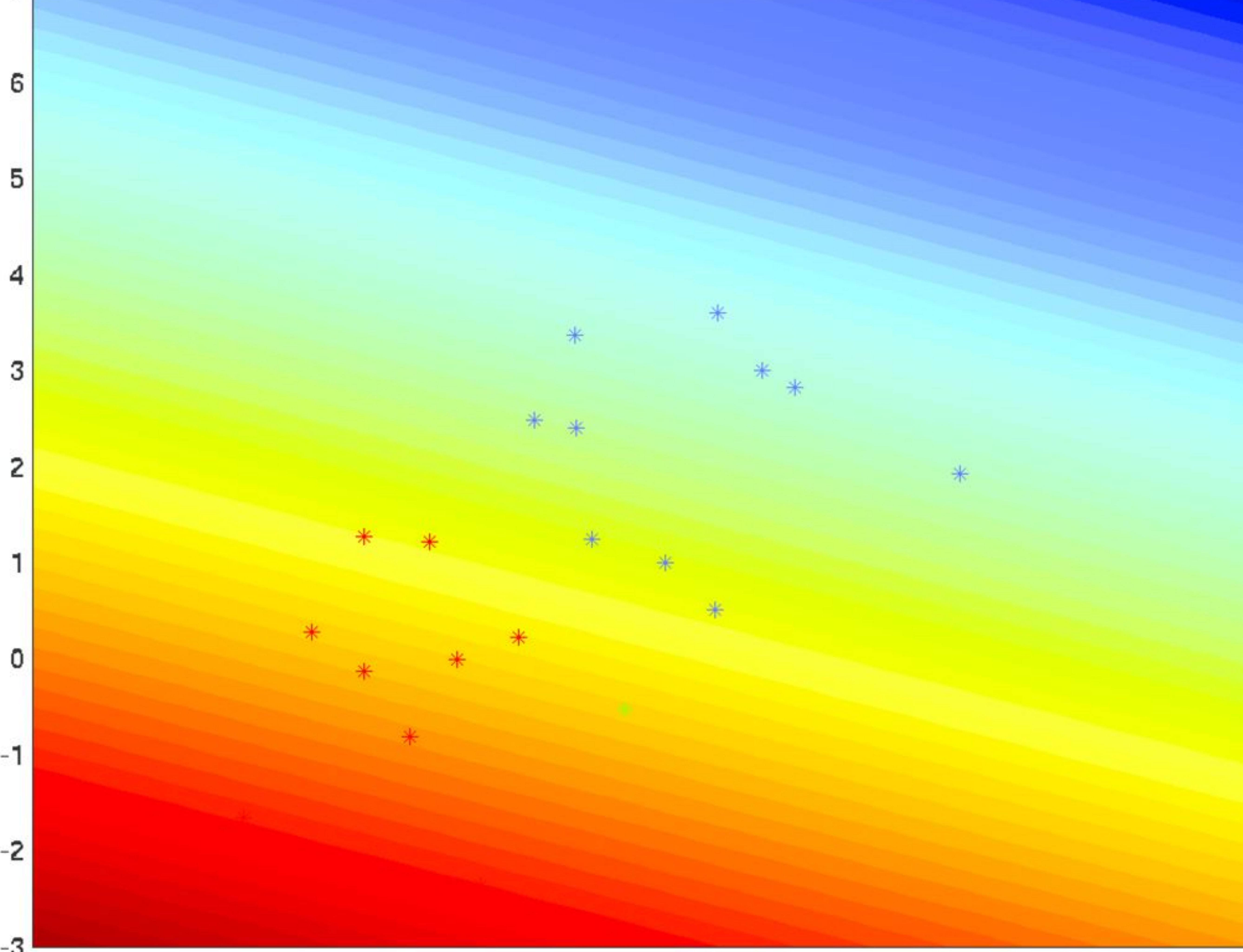


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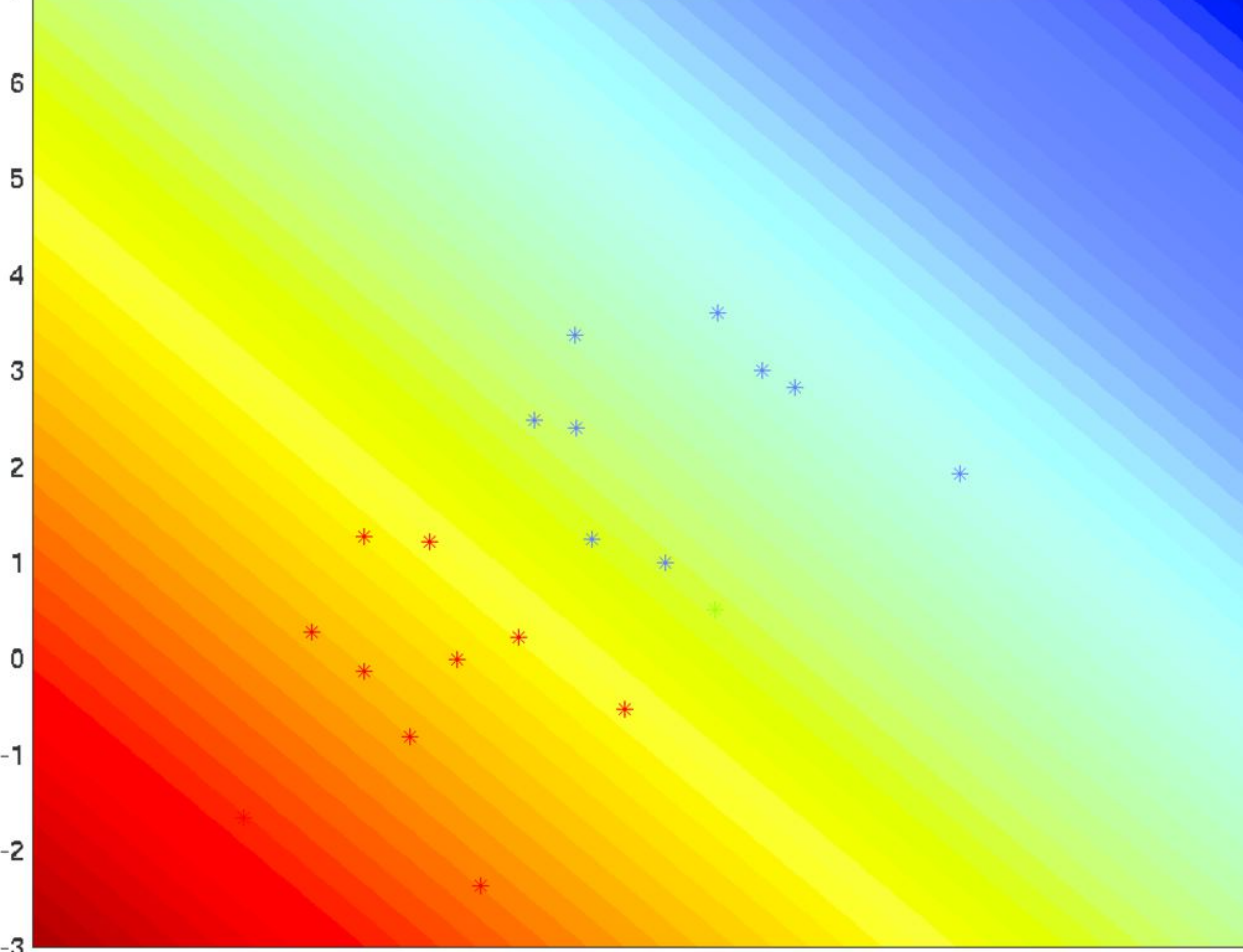


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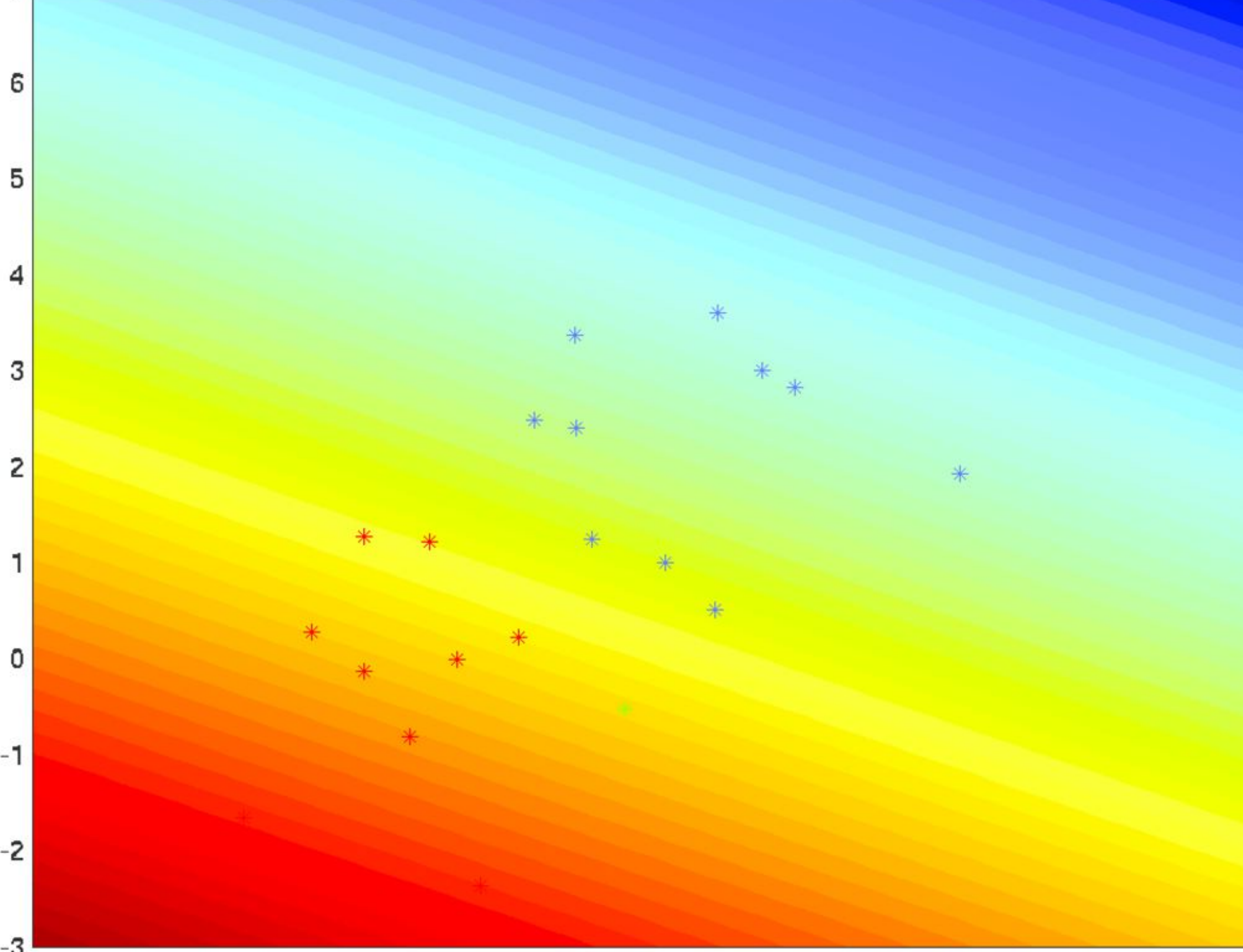


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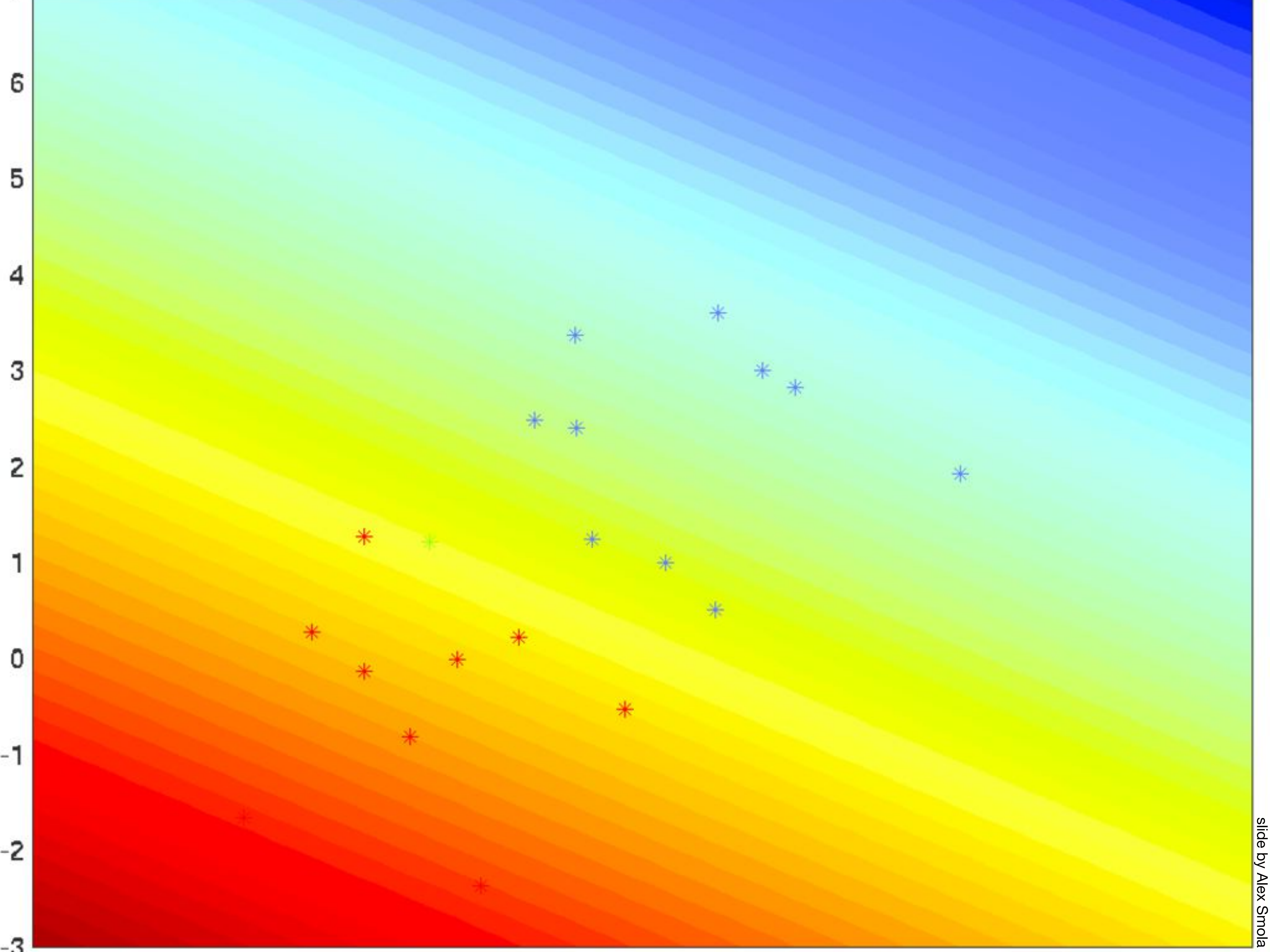

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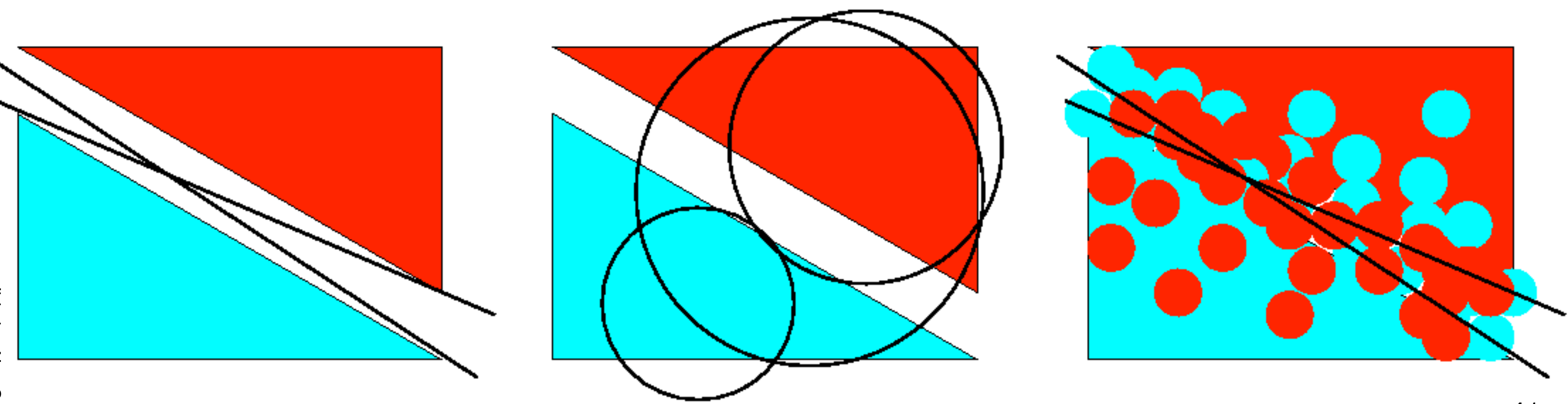
-3
slide by Alex Smola

# Concepts & version space

- Realizable concepts
  - Some function exists that can separate data and is included in the concept space
  - For perceptron - data is linearly separable
- Unrealizable concept
  - Data not separable
  - We don't have a suitable function class (often hard to distinguish)



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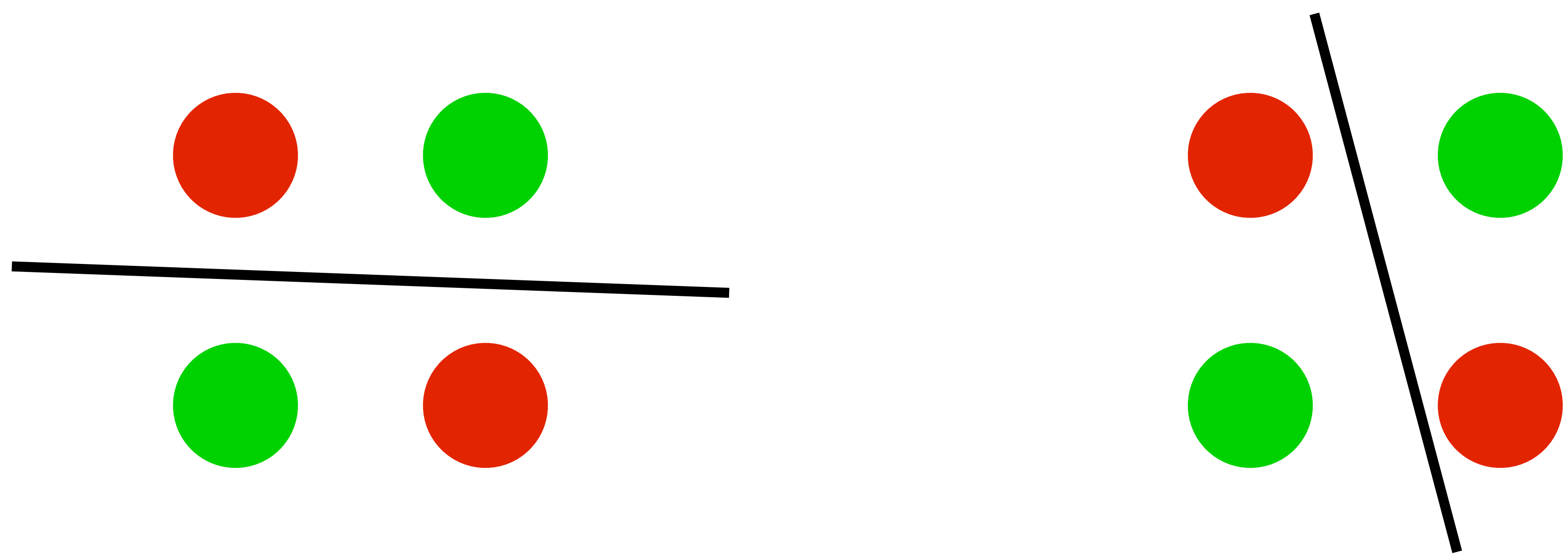
# Minimum error separation



- XOR - not linearly separable
- Nonlinear separation is trivial
- Caveat (Minsky & Papert)
  Finding the minimum error linear separator is NP hard (this killed Neural Networks in the 70s).

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# Nonlinear Features

- Regression
  We got nonlinear functions by preprocessing
- Perceptron
  - Map data into feature space $x \to \phi(x)$
  - Solve problem in this space
  - Query replace $\langle x, x' \rangle$ by $\langle \phi(x), \phi(x') \rangle$ for code
- Feature Perceptron
  - Solution in span of $\phi(x_i)$

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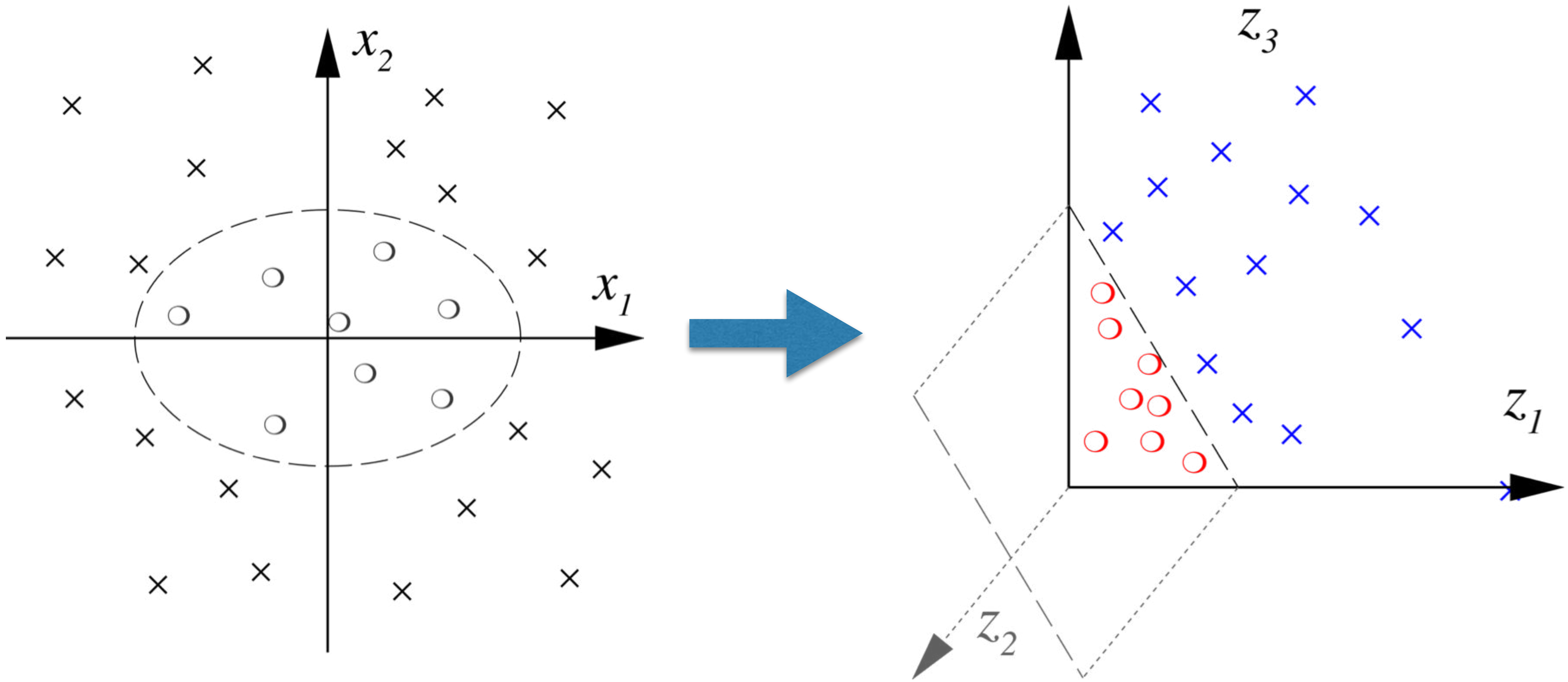
# Quadratic Features




- Separating surfaces are Circles, hyperbolae, parabolae

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# Constructing Features (very naive OCR system)

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 0 |
|---|---|---|---|---|---|---|---|---|---|---|
| Loops | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 2 | 1 | 1 |
| 3 Joints | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 |
| 4 Joints | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 |
| Angles | 0 | 1 | 1 | 1 | 1 | 0 | 1 | 0 | 0 | 0 |
| Ink | 1 | 2 | 2 | 2 | 2 | 2 | 1 | 3 | 2 | 2 |

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# Feature Engineering for Spam Filtering

- bag of words
- pairs of words
- date & time
- recipient path
- IP number
- sender
- encoding
- links
- ... secret sauce ...

```
Delivered-To: alex.smola@gmail.com
Received: by 10.216.47.73 with SMTP id s51cs361171web;
        Tue, 3 Jan 2012 14:17:53 -0800 (PST)
Received: by 10.213.17.145 with SMTP id s17mr2519891eba.147.1325629071725;
        Tue, 03 Jan 2012 14:17:51 -0800 (PST)
Return-Path: <alex+caf_=alex.smola=gmail.com@smola.org>
Received: from mail-ey0-f175.google.com (mail-ey0-f175.google.com [209.85.215.175])
        by mx.google.com with ESMTPS id n4si29264232eef.57.2012.01.03.14.17.51
        (version=TLSv1/SSLv3 cipher=OTHER);
        Tue, 03 Jan 2012 14:17:51 -0800 (PST)
Received-SPF: neutral (google.com: 209.85.215.175 is neither permitted nor denied by best
guess record for domain of alex+caf_=alex.smola=gmail.com@smola.org) client-
ip=209.85.215.175;
Authentication-Results: mx.google.com; spf=neutral (google.com: 209.85.215.175 is neither
permitted nor denied by best guess record for domain of
alex+caf_=alex.smola=gmail.com@smola.org)
smtp.mail=alex+caf_=alex.smola=gmail.com@smola.org; dkim=pass (test mode)
header.i=@googlemail.com
Received: by eaal1 with SMTP id l1so15092746eaa.6
        for <alex.smola@gmail.com>; Tue, 03 Jan 2012 14:17:51 -0800 (PST)
Received: by 10.205.135.18 with SMTP id ie18mr5325064bkc.72.1325629071362;
        Tue, 03 Jan 2012 14:17:51 -0800 (PST)
X-Forwarded-To: alex.smola@gmail.com
X-Forwarded-For: alex@smola.org alex.smola@gmail.com
Delivered-To: alex@smola.org
Received: by 10.204.65.198 with SMTP id k6cs206093bki;
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From: Tim Althoff <althoff@eecs.berkeley.edu>
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# More feature engineering

- Two Interlocking Spirals
  Transform the data into a radial and angular part
  $$(x_1, x_2) = (r \sin\phi, r\cos\phi)$$
- Handwritten Japanese Character Recognition
  - Break down the images into strokes and recognize it
  - Lookup based on stroke order
- Medical Diagnosis
  - Physician's comments
  - Blood status / ECG / height / weight / temperature ...
  - Medical knowledge
- Preprocessing
  - Zero mean, unit variance to fix scale issue (e.g. weight vs. income)
  - Probability integral transform (inverse CDF) as alternative

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# The Perceptron on features

initialize $w, b = 0$
repeat
  Pick $(x_i, y_i)$ from data
  if $y_i(w \cdot \Phi(x_i) + b) \leq 0$ then

$$\begin{aligned} w' &= w + y_i\Phi(x_i) \\ b' &= b + y_i \end{aligned}$$

until $y_i(w \cdot \Phi(x_i) + b) > 0$ for all $i$

- Nothing happens if classified correctly
- Weight vector is linear combination $w = \sum_{i \in I} y_i \phi(x_i)$
- Classifier is linear combination of inner products $f(x) = \sum_{i \in I} y_i \langle \phi(x_i), \phi(x) \rangle + b$

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# Problems

- Problems
  - Need domain expert (e.g. Chinese OCR)
  - Often expensive to compute
  - Difficult to transfer engineering knowledge
- Shotgun Solution
  - Compute many features
  - Hope that this contains good ones
  - Do this efficiently

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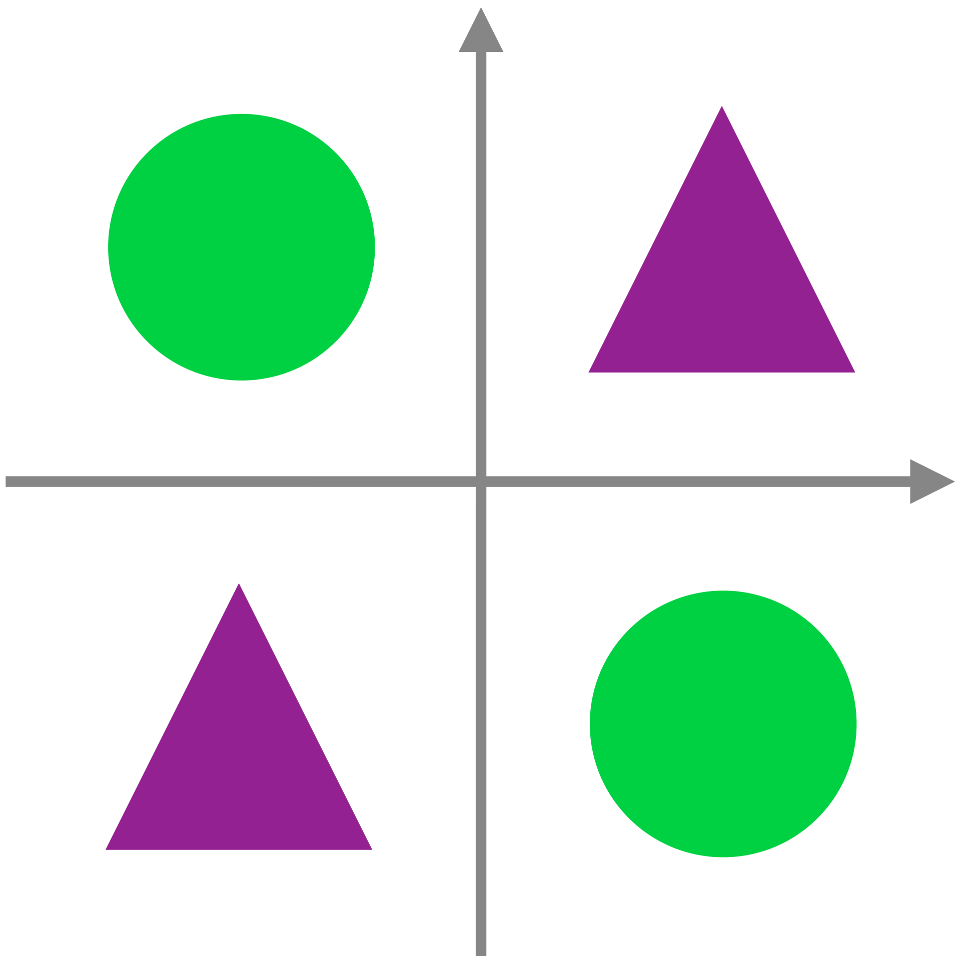
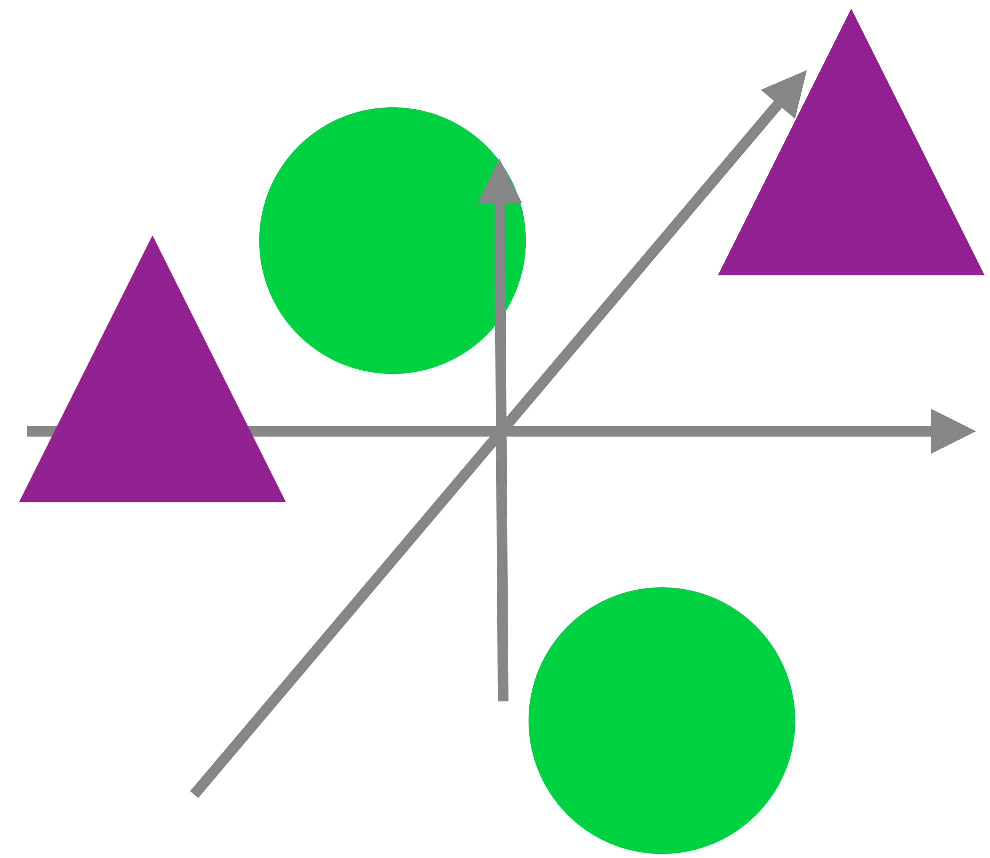
# Solving XOR



$(x_1, x_2)$



$(x_1, x_2, x_1 x_2)$

- XOR not linearly separable
- Mapping into 3 dimensions makes it easily solvable

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# Next Lecture:

Multi-layer Perceptron