Photo by Unsplash user @foodiesfeed

BBM406
Fundamentals of Machine Learning

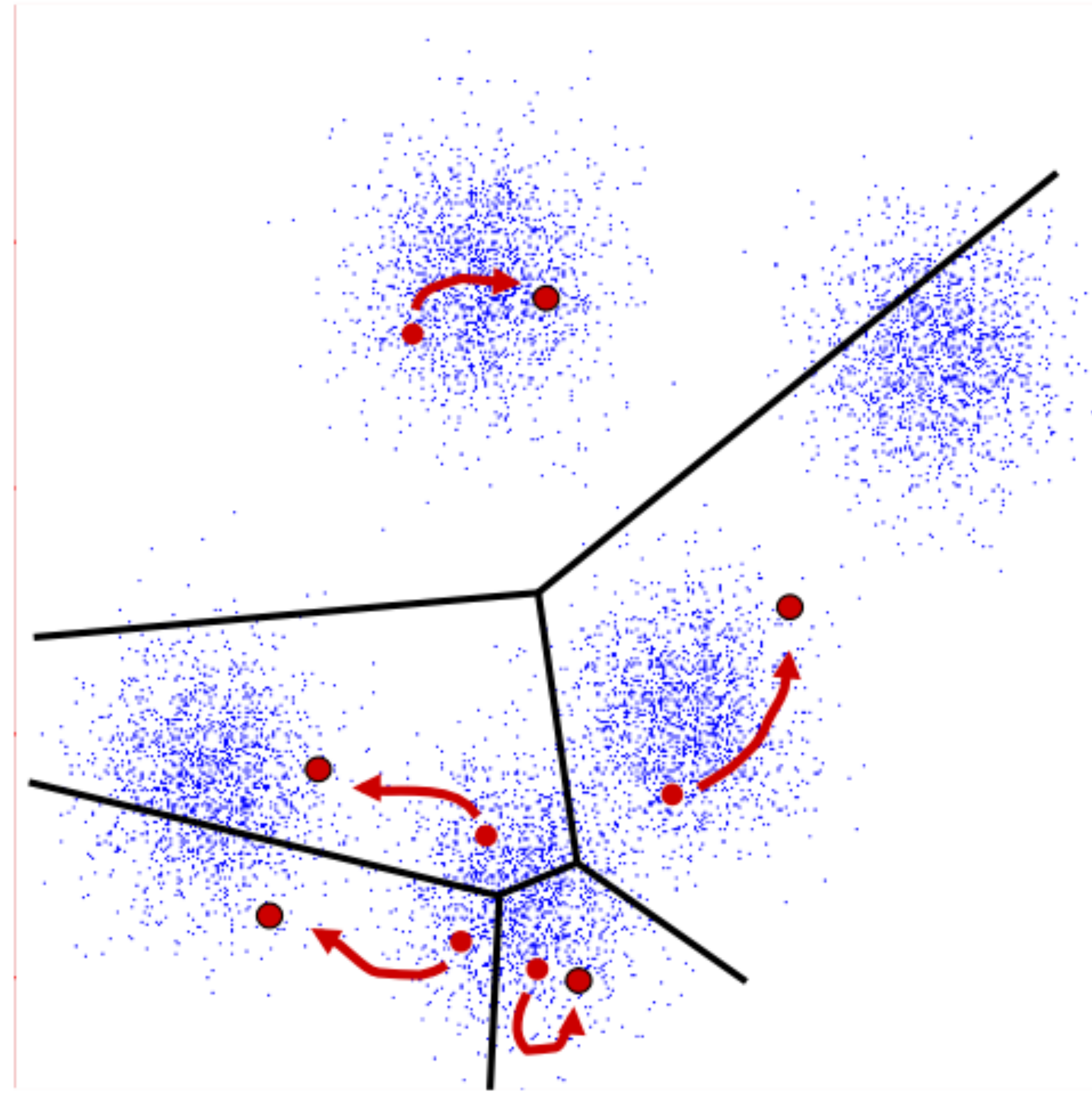
Lecture 22:
K-Means Example Applications
Spectral clustering
Hierarchical clustering
What is a good clustering?

HACETTEPE UNIVERSITY COMPUTER VISION LAB

Erkut Erdem // Hacettepe University // Spring 2021

Last time… **K-Means**

- An iterative clustering algorithm
  - Initialize: Pick K random points as cluster centers (means)
  - Alternate:
    - Assign data instances to closest mean
    - Assign each mean to the average of its assigned points
  - Stop when no points' assignments change



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Today

- K-Means Example Applications
- Spectral clustering
- Hierarchical clustering

K-Means
Example Applications

Example: K-Means for Segmentation

K=2

Goal of Segmentation is to partition an image into regions each of which has reasonably homogenous visual appearance.

Original

slide by David Sontag

Example: K-Means for Segmentation

K=2

K=3

Original

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Example: K-Means for Segmentation

K=2 K=3 K=10 Original







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Example: Vector quantization

FIGURE 14.9. *Sir Ronald A. Fisher (1890 – 1962) was one of the founders of modern day statistics, to whom we owe maximum-likelihood, sufficiency, and many other fundamental concepts. The image on the left is a* 1024×1024 *grayscale image at 8 bits per pixel. The center image is the result of* 2×2 *block VQ, using 200 code vectors, with a compression rate of 1.9 bits/pixel. The right image uses only four code vectors, with a compression rate of 0.50 bits/pixel*

[Figure from Hastie *et al.* book]

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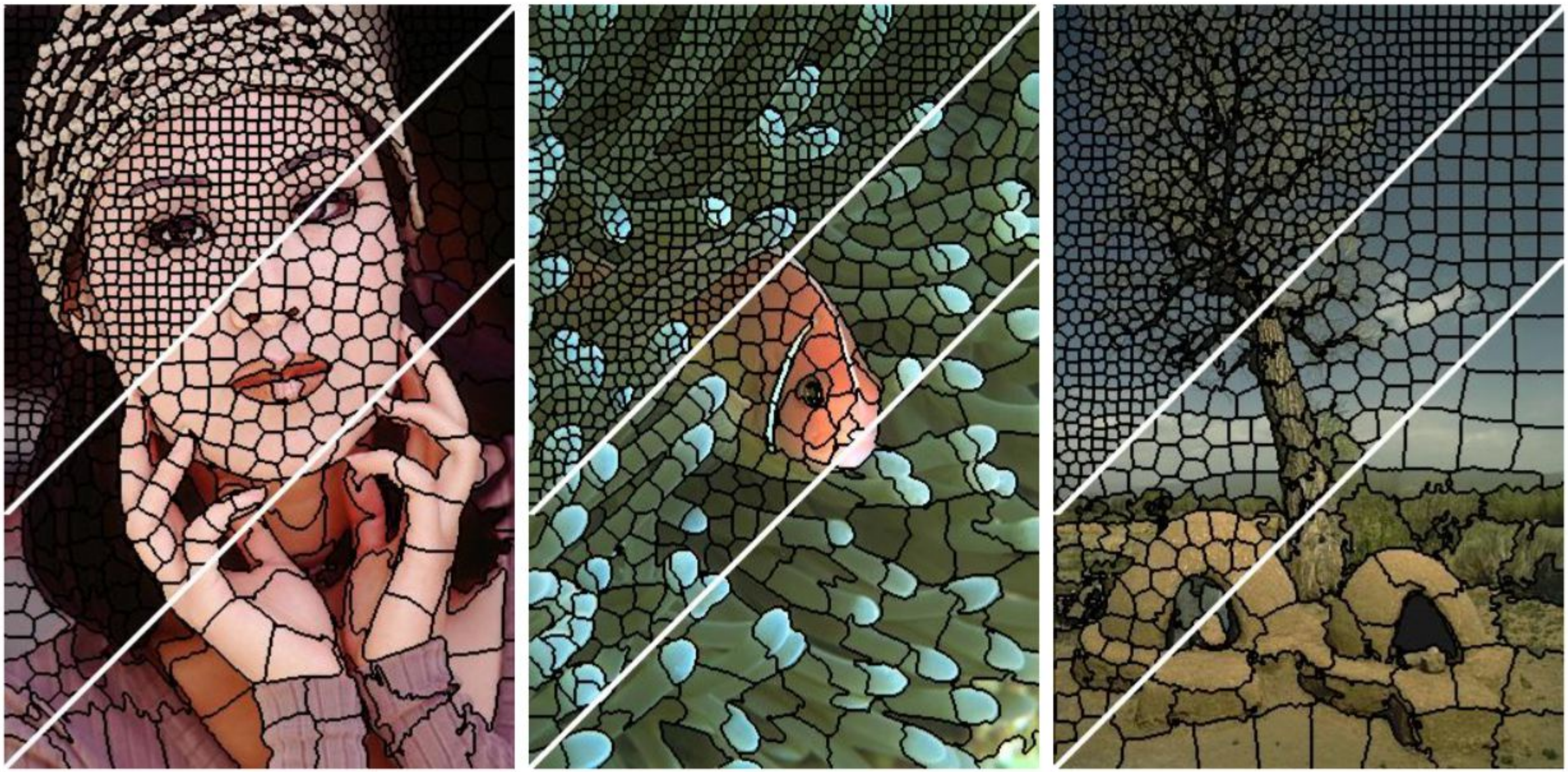
Example: Simple Linear Iterative Clustering (SLIC) superpixels



$$\Psi(x, y) = \begin{bmatrix} \lambda x \\ \lambda y \\ I(x, y) \end{bmatrix}$$

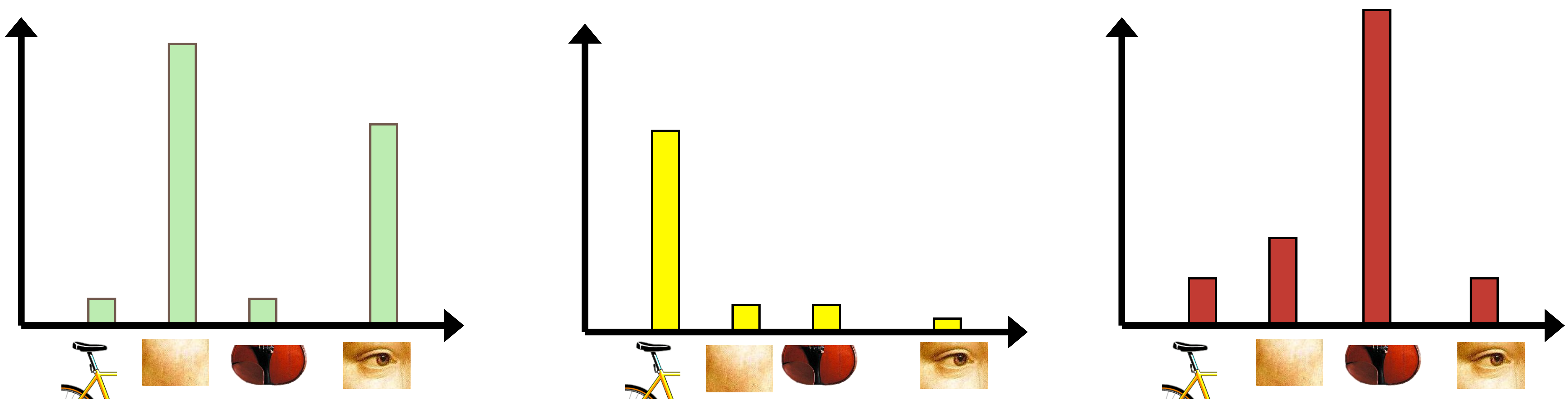
λ: spatial regularization parameter

R. Achanta, A. Shaji, K. Smith, A. Lucchi, P. Fua, and S. Susstrunk SLIC Superpixels Compared to State-of-the-art Superpixel Methods, IEEE T-PAMI, 2012

Bag of Words model

aardvark	0
about	2
all	2
Africa	1
apple	0
anxious	0
...	
gas	1
...	
oil	1
…	
Zaire	0

slide by Carlos Guestrin



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Object → **Bag of ‘words’**

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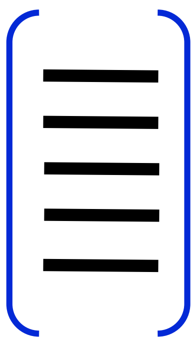
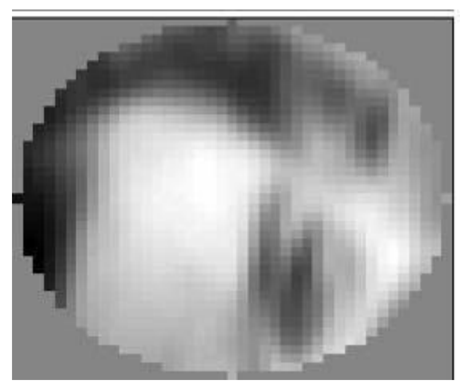
Interest Point Features

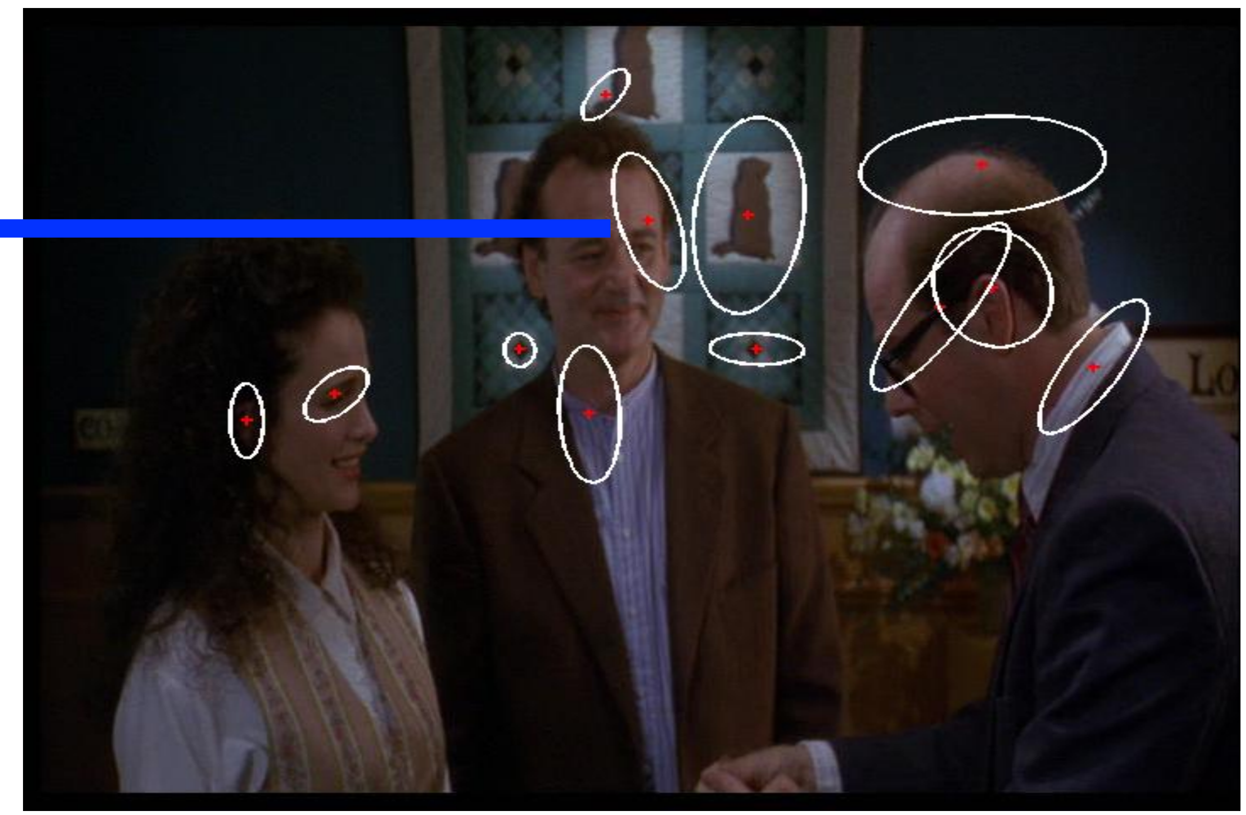


Compute SIFT descriptor

[Lowe'99]



Normalize patch



Detect patches

[Mikojaczyk and Schmid '02]

[Matas et al. '02]

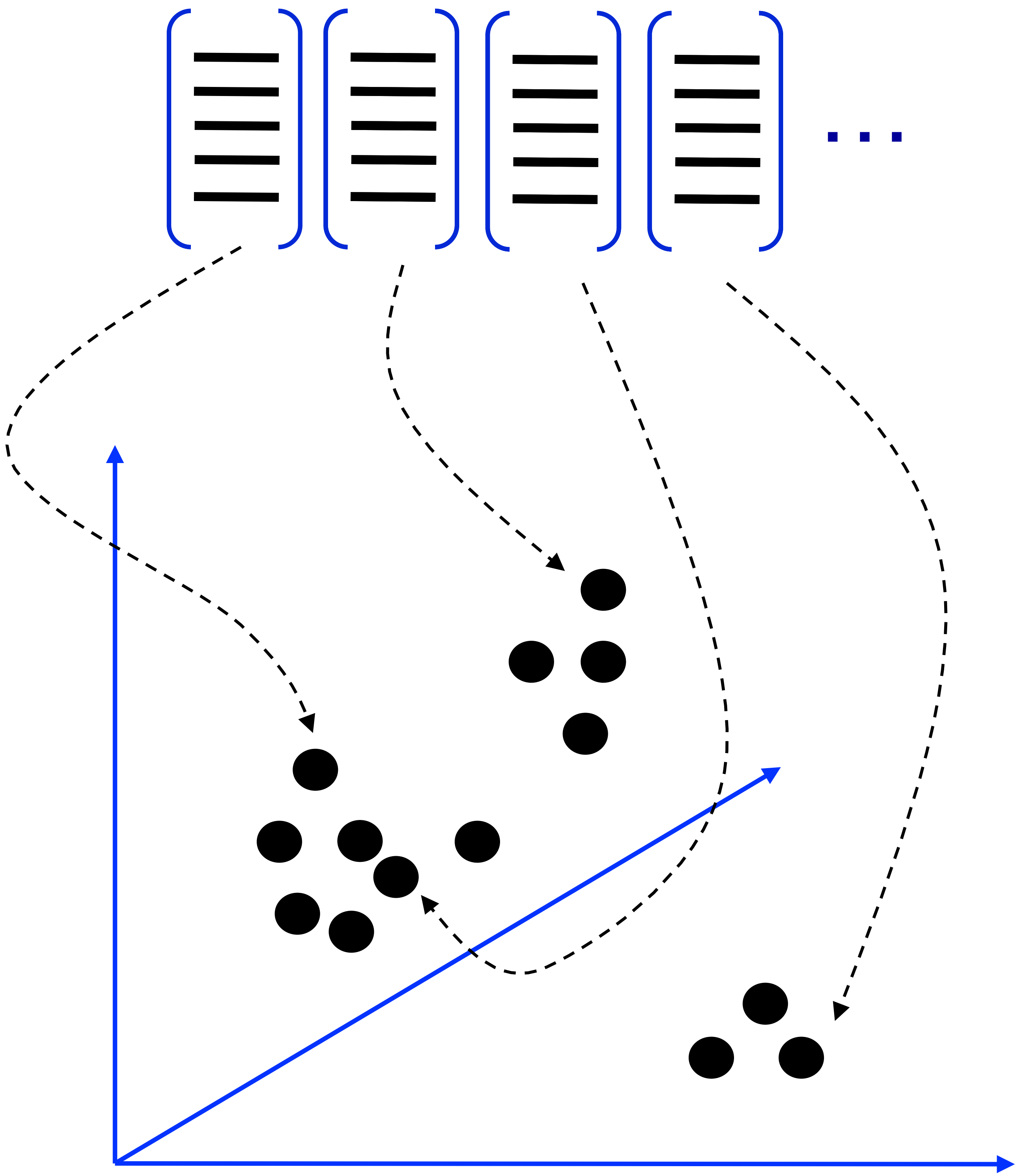
[Sivic et al. '03]

slide by Josef Sivic

Patch Features

slide by Josef Sivic

Dictionary Formation



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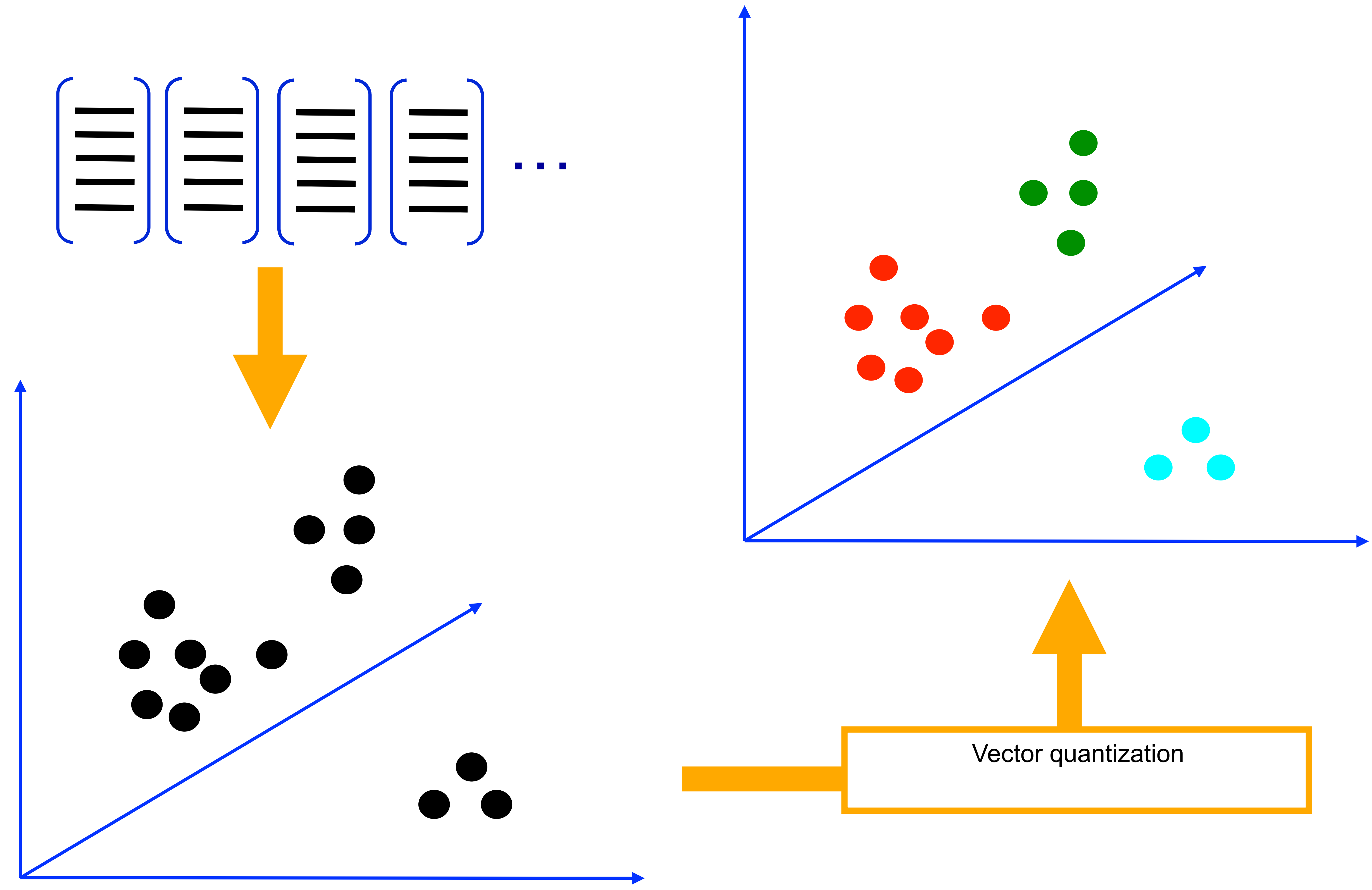
Clustering (usually K-means)



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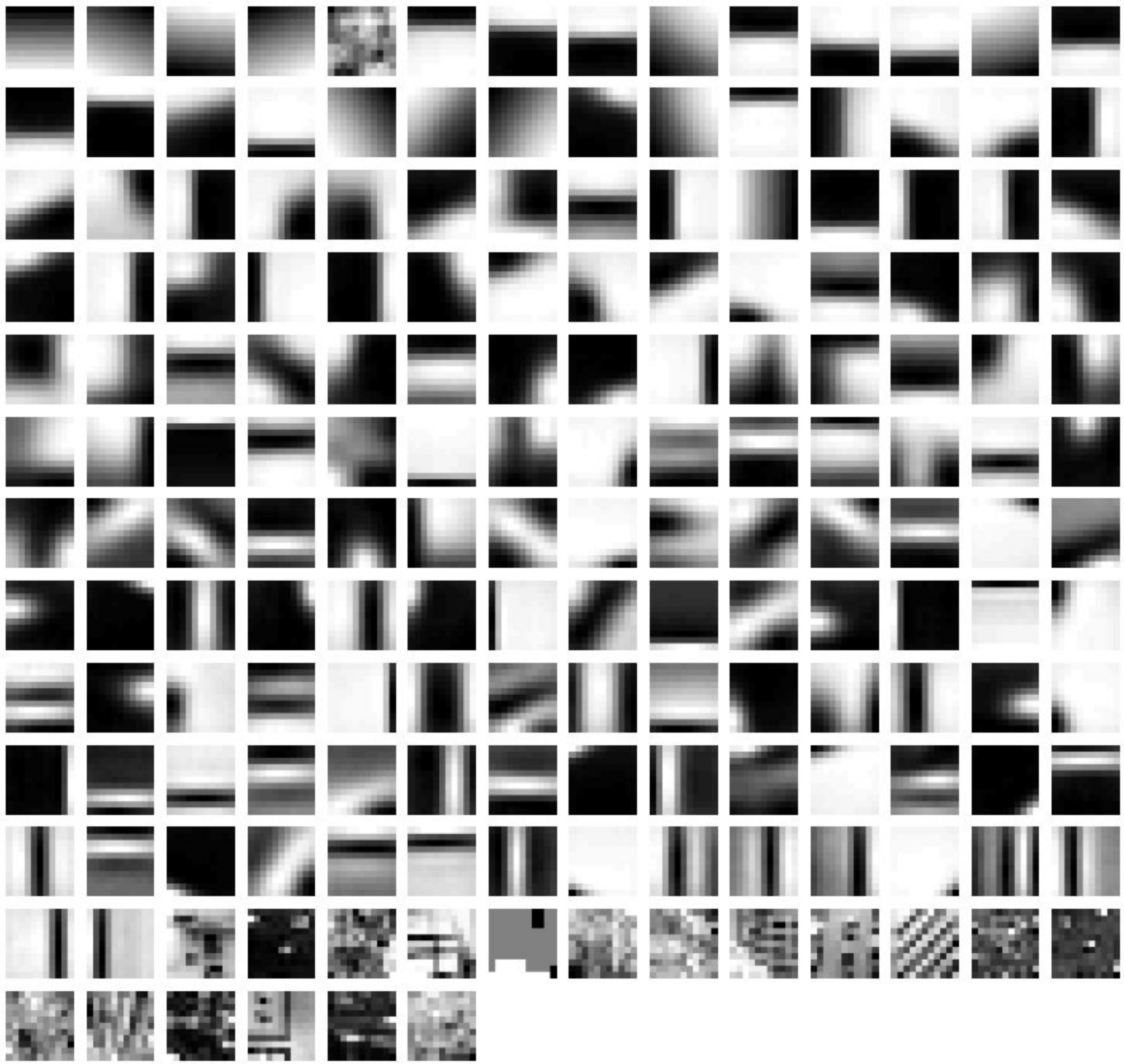
Clustered Image Patches



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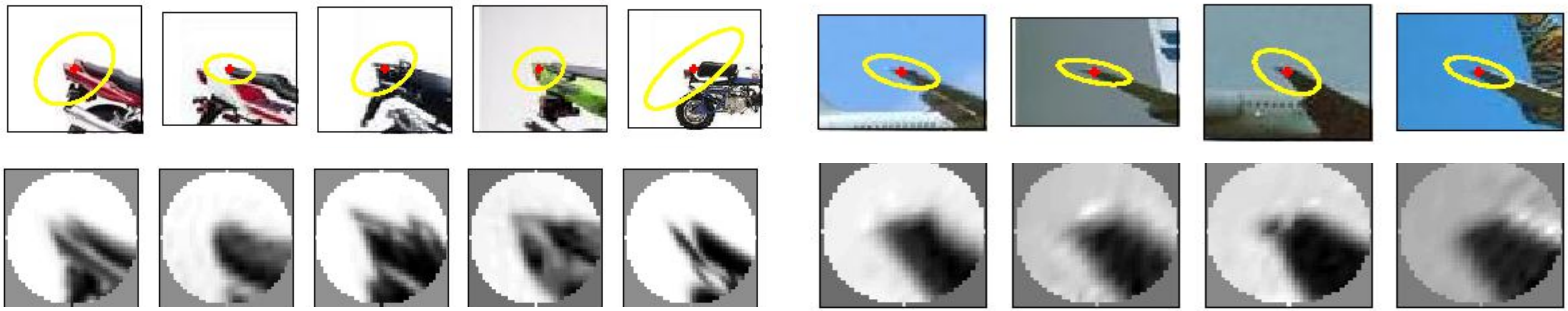
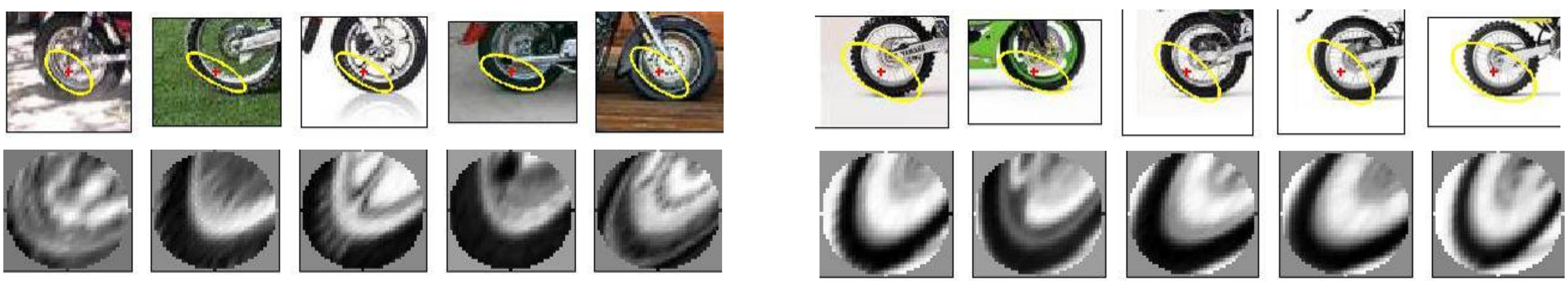
Visual synonyms and polysemy



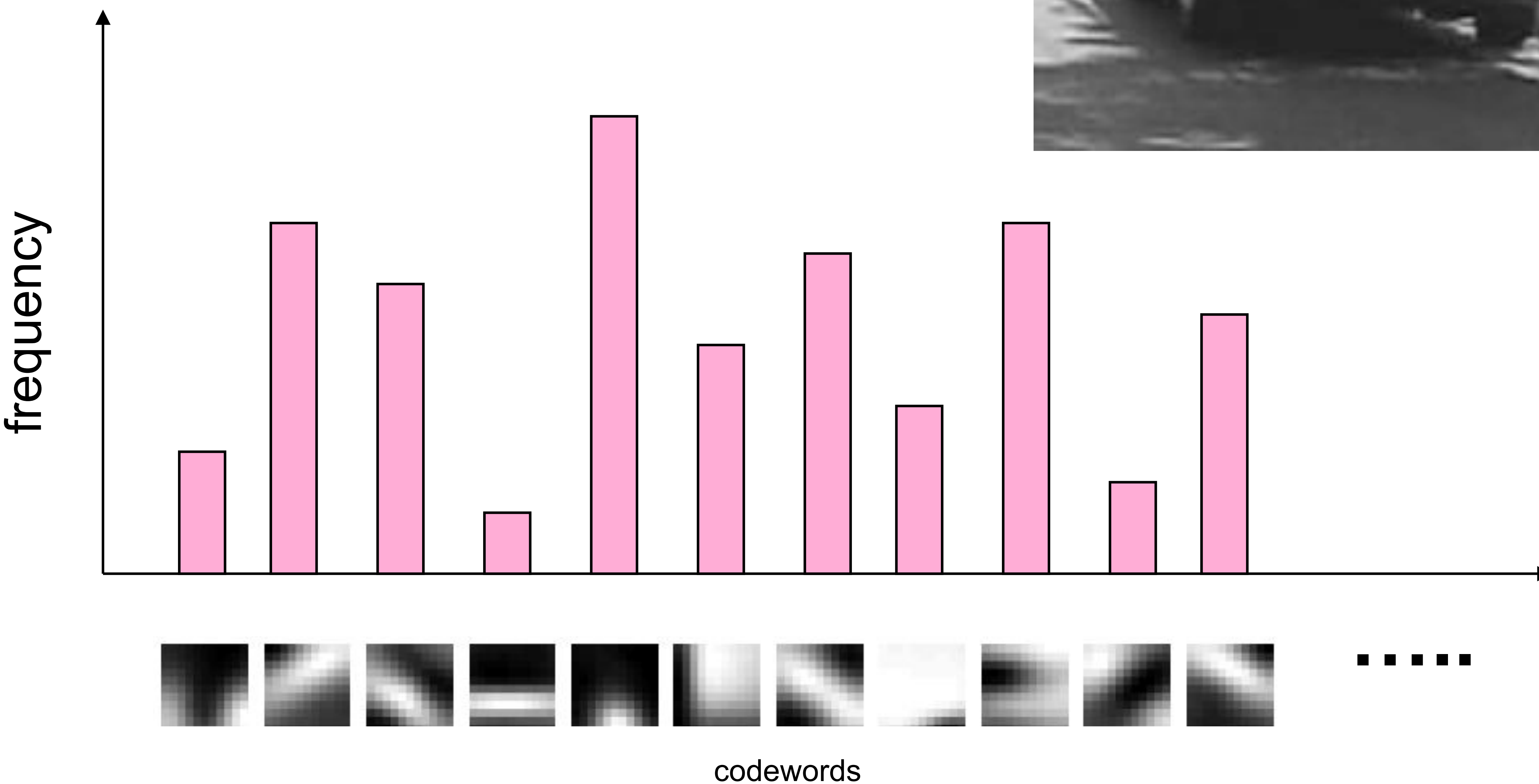
Visual Polysemy. Single visual word occurring on different (but locally similar) parts on different object categories.



Visual Synonyms. Two different visual words representing a similar part of an object (wheel of a motorbike).

slide by Andrew Zisserman

Image Representation



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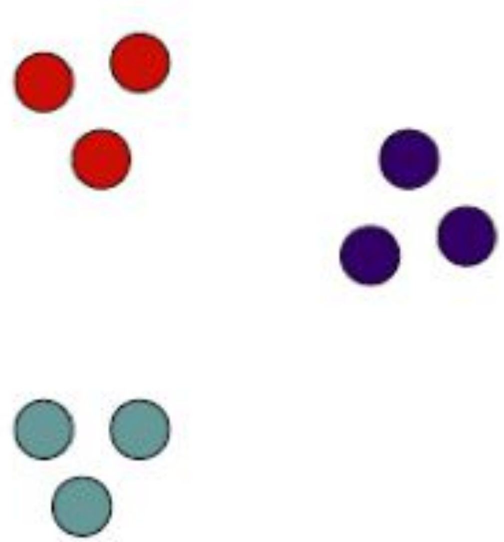
Spectral clustering

Graph-Theoretic Clustering

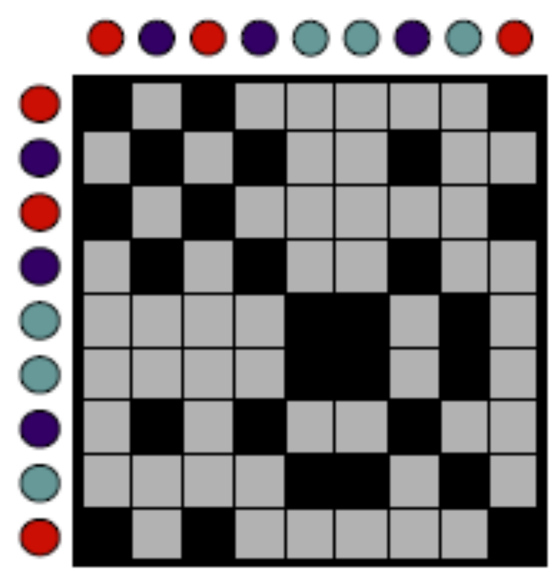
Goal: Given data points $x_1, \ldots, x_n$ and similarities $W(x_i, x_j)$, partition the data into groups so that points in a group are similar and points in different groups are dissimilar.

Similarity Graph: G(V,E,W)

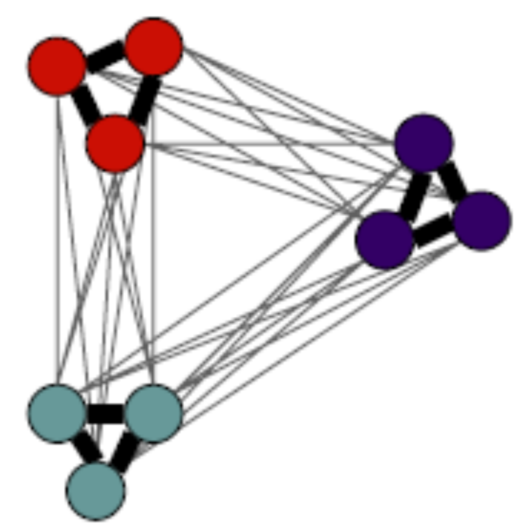
V – Vertices (Data points)
E – Edge if similarity > 0
W - Edge weights (similarities)



Data



Similarities

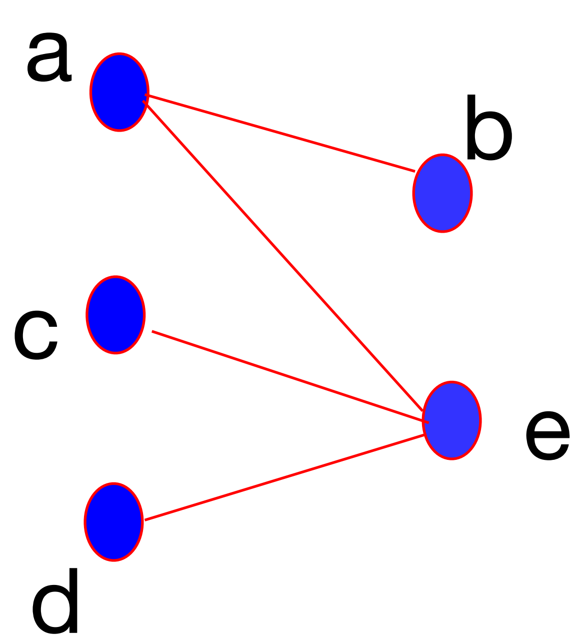


Similarity graph

Partition the graph so that edges within a group have large weights and edges across groups have small weights.

slide by Aarti Singh

Graphs Representations

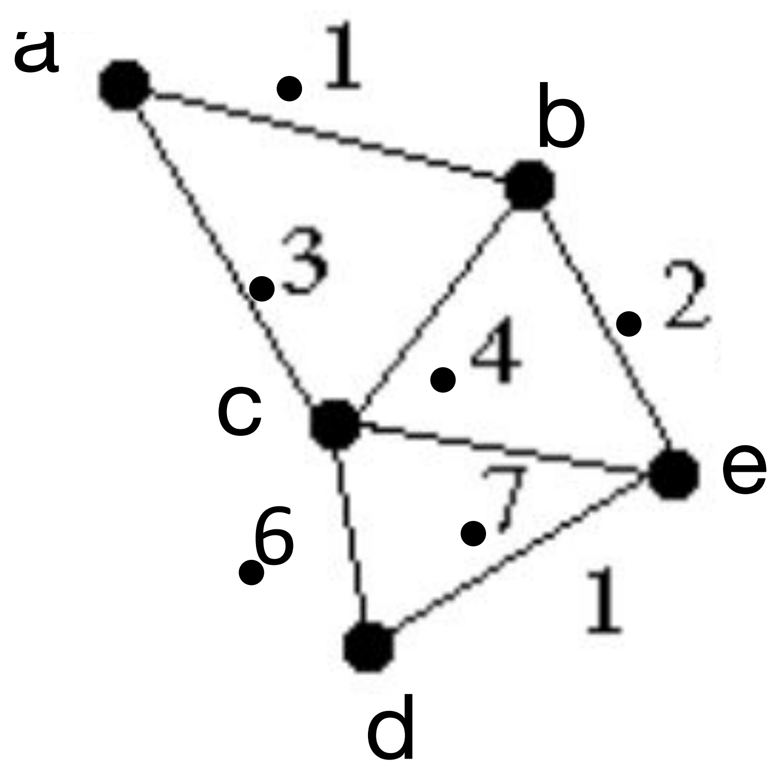


$$
\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array}
\begin{array}{c}
\begin{array}{ccccc} a & b & c & d & e \end{array} \\
\begin{bmatrix}
0 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 1 & 0
\end{bmatrix}
\end{array}
$$

Adjacency Matrix

slide by Bill Freeman and Antonio Torralba

A Weighted Graph and its Representation



Affinity Matrix

$$W = \begin{bmatrix} 1 & .1 & .3 & 0 & 0 \\ .1 & 1 & .4 & 0 & .2 \\ .3 & .4 & 1 & .6 & .7 \\ 0 & 0 & .6 & 1 & 1 \\ 0 & .2 & .7 & 1 & 1 \end{bmatrix}$$

W_{ij} : probability that i &j belong to the same cluster

slide by Bill Freeman and Antonio Torralba

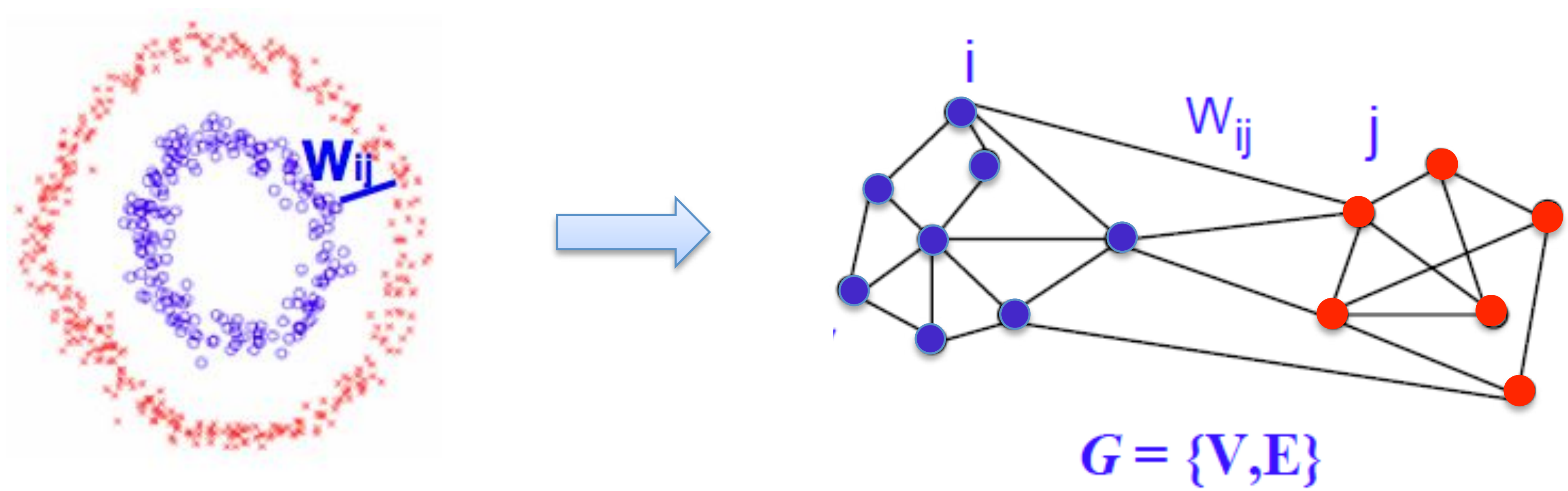
Similarity graph construction

- Similarity Graphs: Model local neighborhood relations between data points
- E.g. epsilon-NN

$$W_{ij} = \begin{cases} 1 & \|x_i - x_j\| \leq \epsilon \\ 0 & \text{otherwise} \end{cases}$$

Controls size of neighborhood

or mutual k-NN graph (W_{ij} = 1 if x_i or x_j is k nearest neighbor of the other)



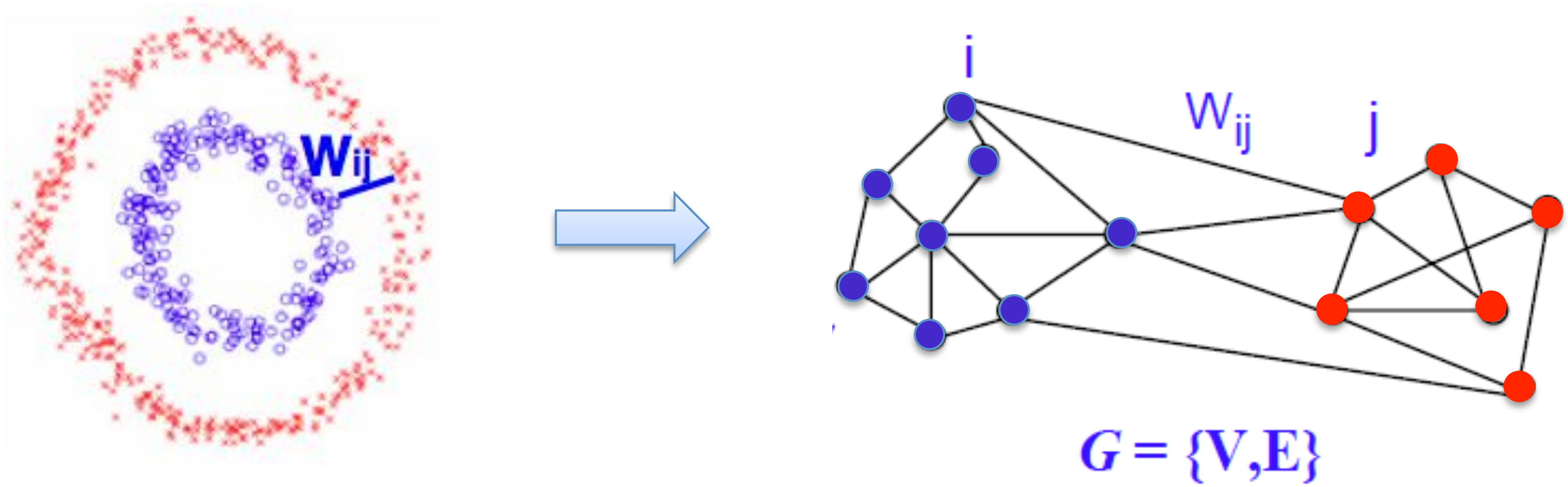
slide by Aarti Singh

Similarity graph construction

- Similarity Graphs: Model local neighborhood relations between data points
- E.g. Gaussian kernel similarity function

$$W_{ij} = e^{\frac{\|x_i - x_j\|^2}{2\sigma^2}}$$

Controls size of neighborhood



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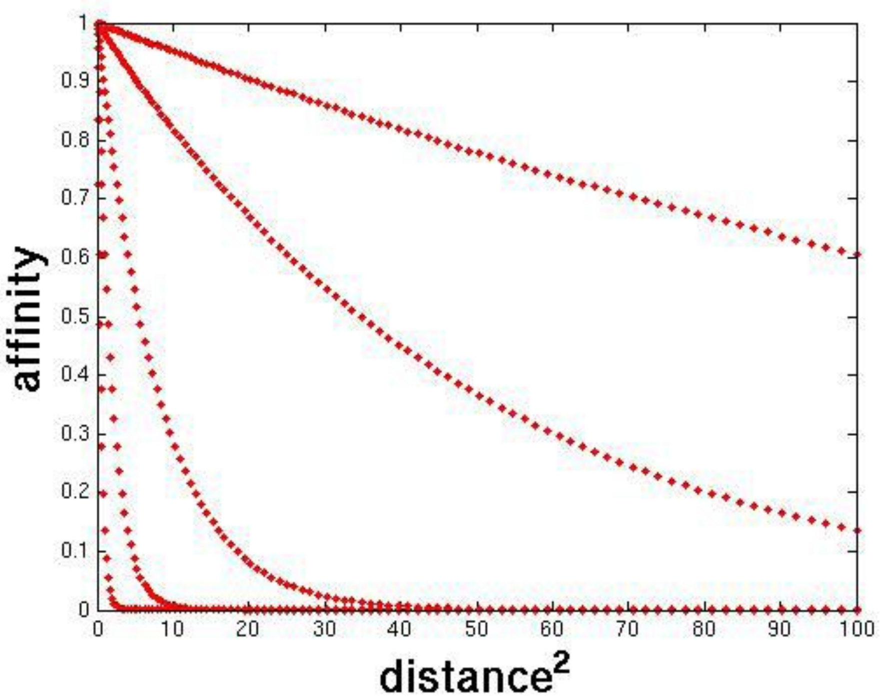
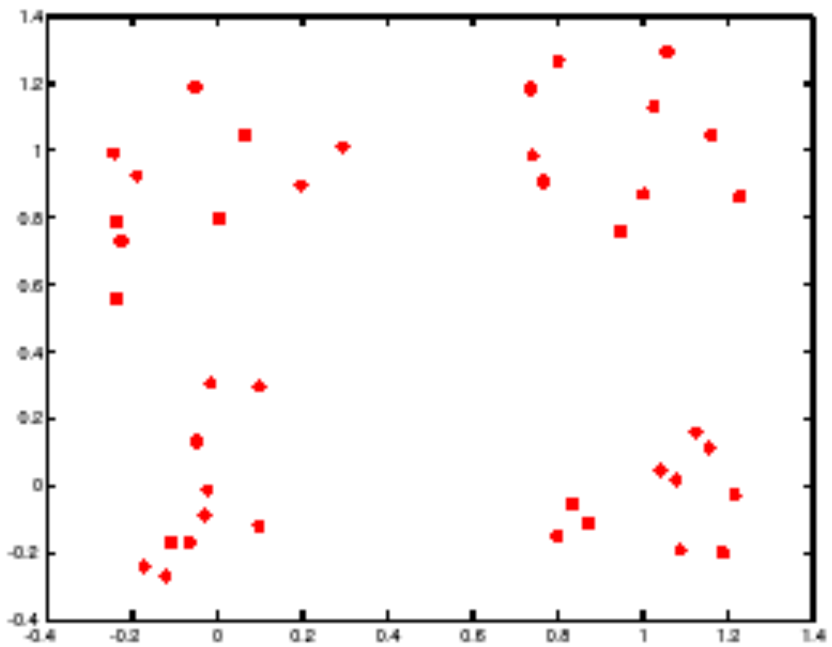
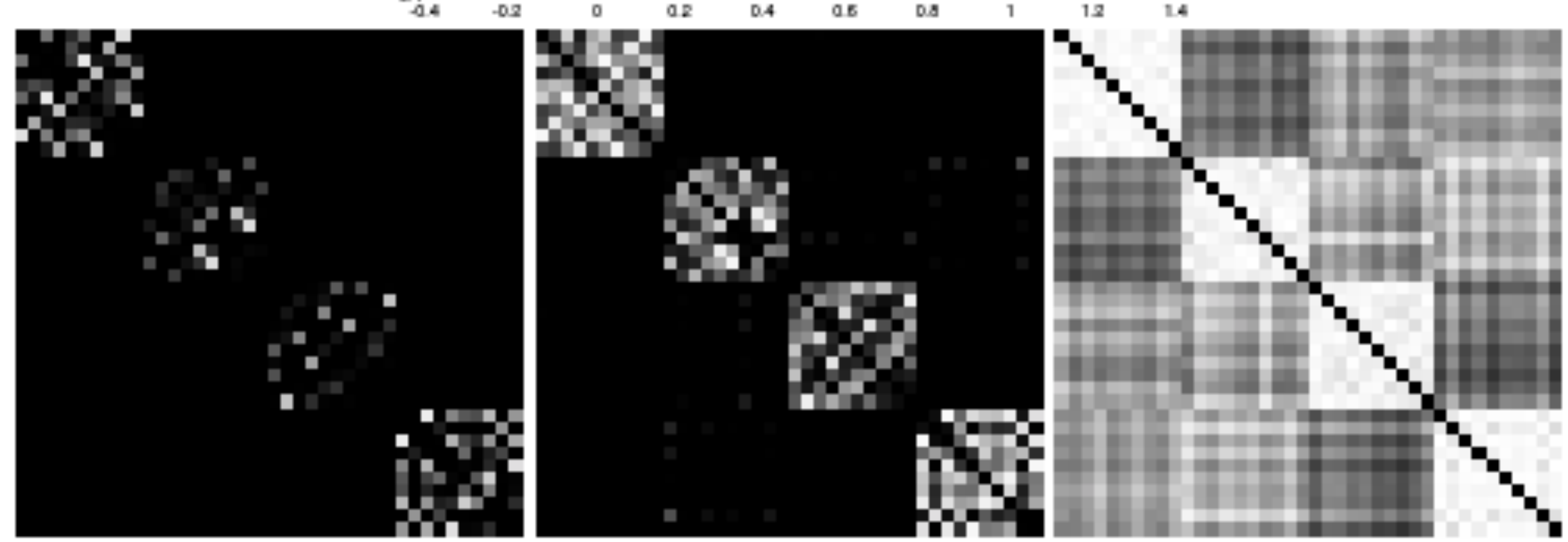
Scale affects affinity

- Small σ: group only nearby points
- Large σ: group far-away points







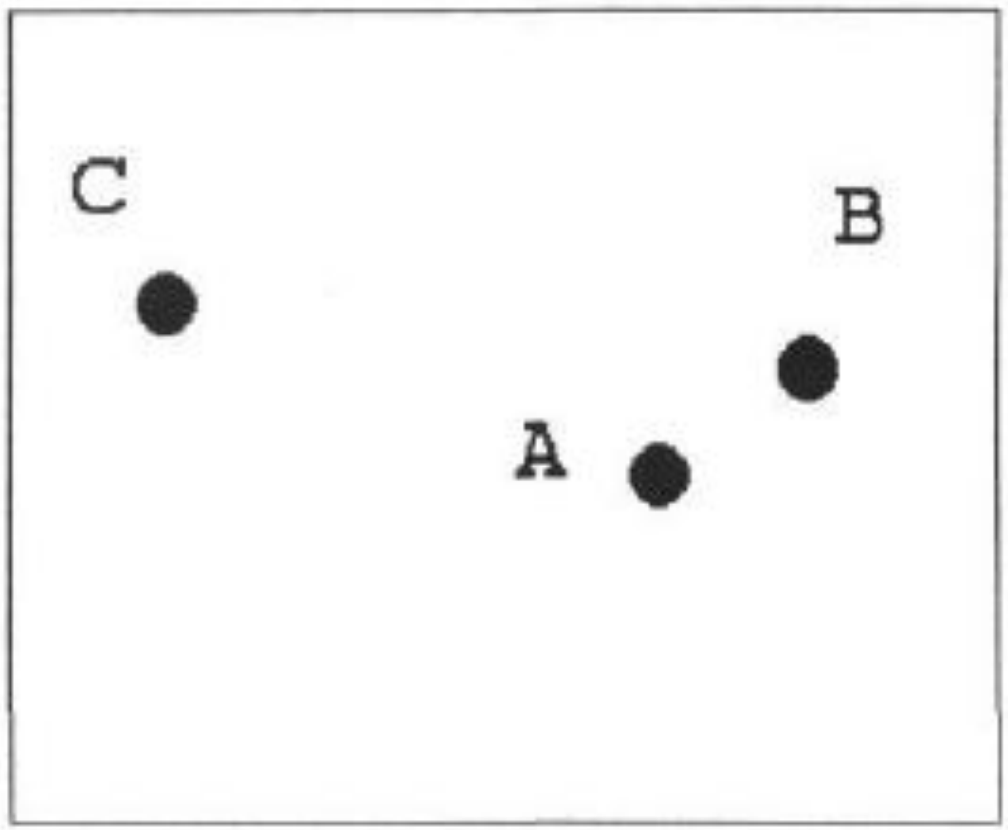
slide by Svetlana Lazebnik

Feature grouping by "relocalisation" of eigenvectors of the proximity matrix

British Machine Vision Conference, pp. 103-108, 1990

Guy L. Scott
Robotics Research Group
Department of Engineering Science
University of Oxford

H. Christopher Longuet-Higgins
University of Sussex
Falmer
Brighton



Three points in feature space

$$W_{ij} = \exp(-\| z_i - z_j \|^2 / s^2)$$

With an appropriate s

W=

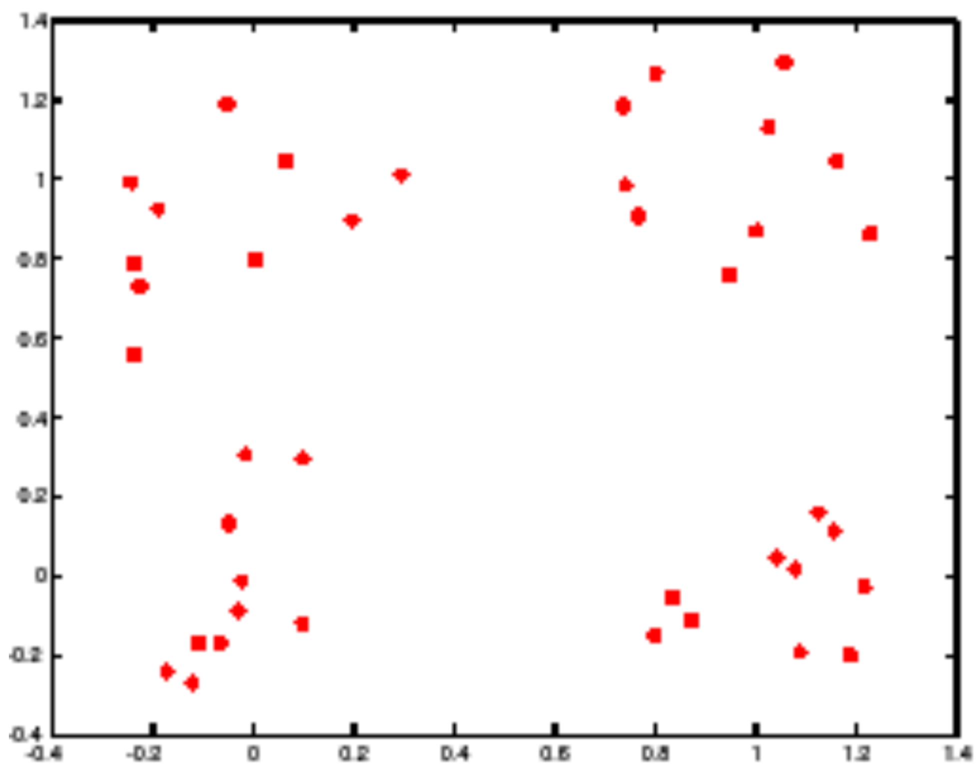
	A	B	C
A	1.00	0.63	0.03
B	0.63	1.00	0.0
C	0.03	0.0	1.00

The eigenvectors of W are:

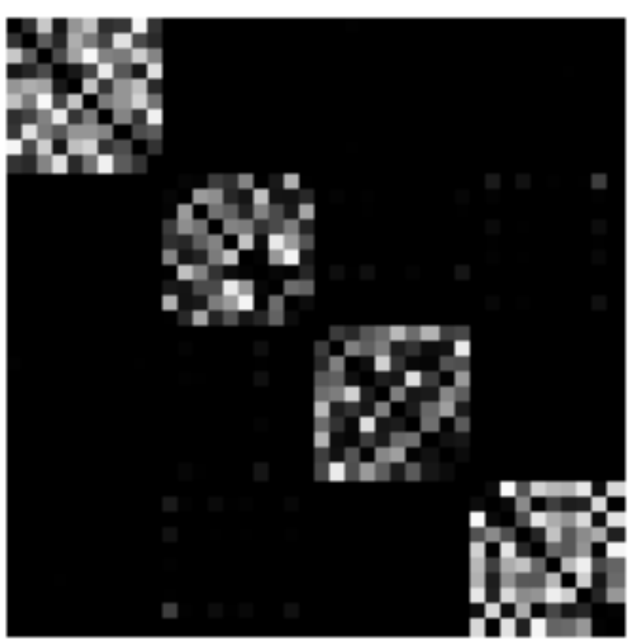
	E_1	E_2	E_3
Eigenvalues	1.63	1.00	0.37
A	-0.71	-0.01	0.71
B	-0.71	-0.05	-0.71
C	-0.04	1.00	-0.03

The first 2 eigenvectors group the points as desired…

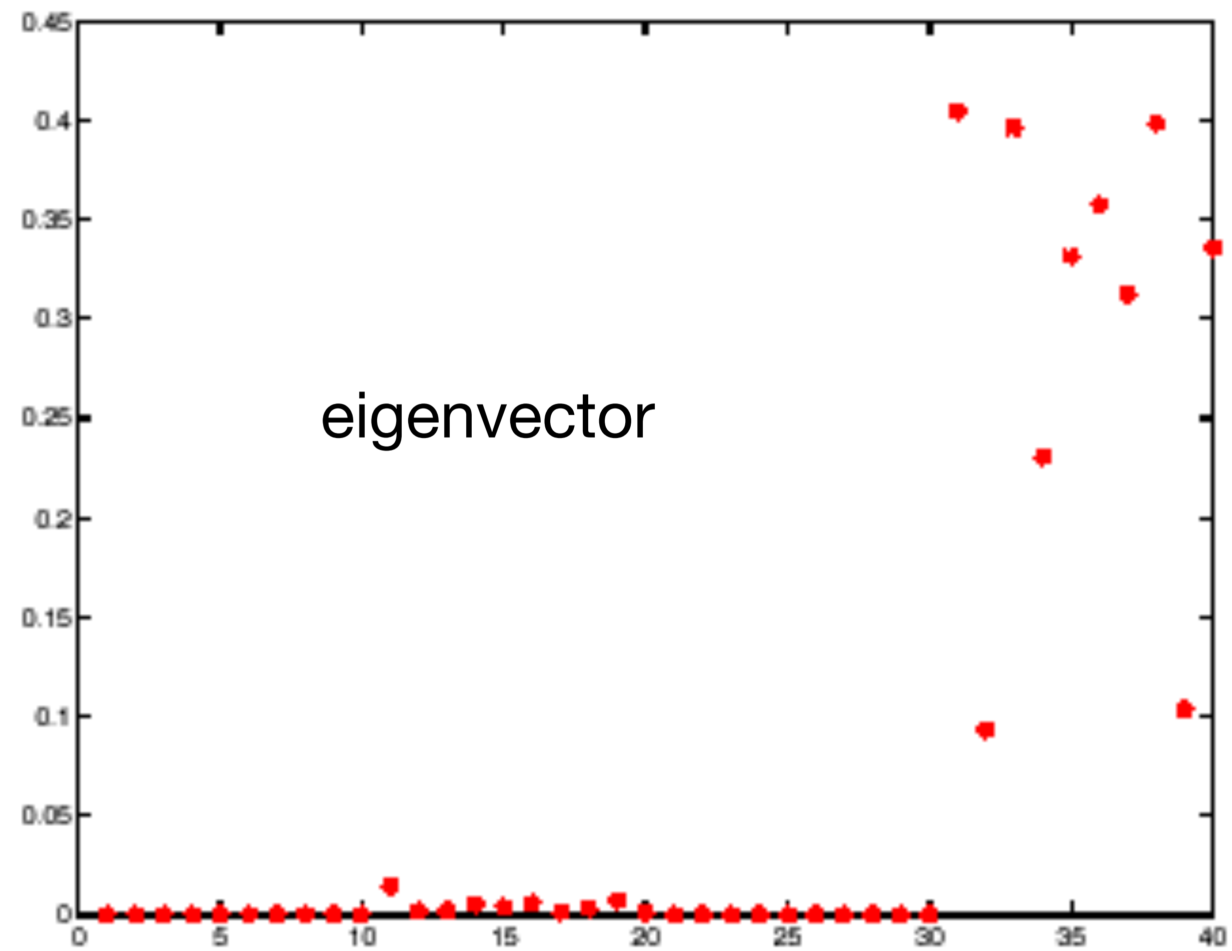
Example eigenvector



points



Affinity matrix



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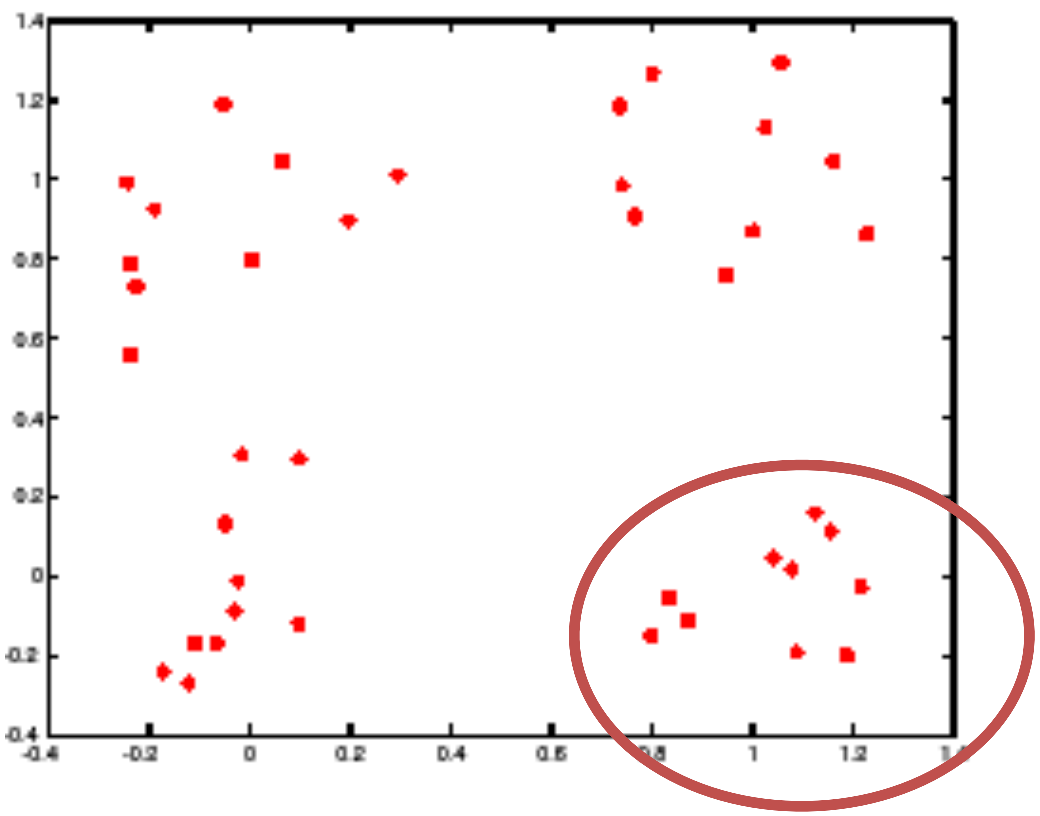
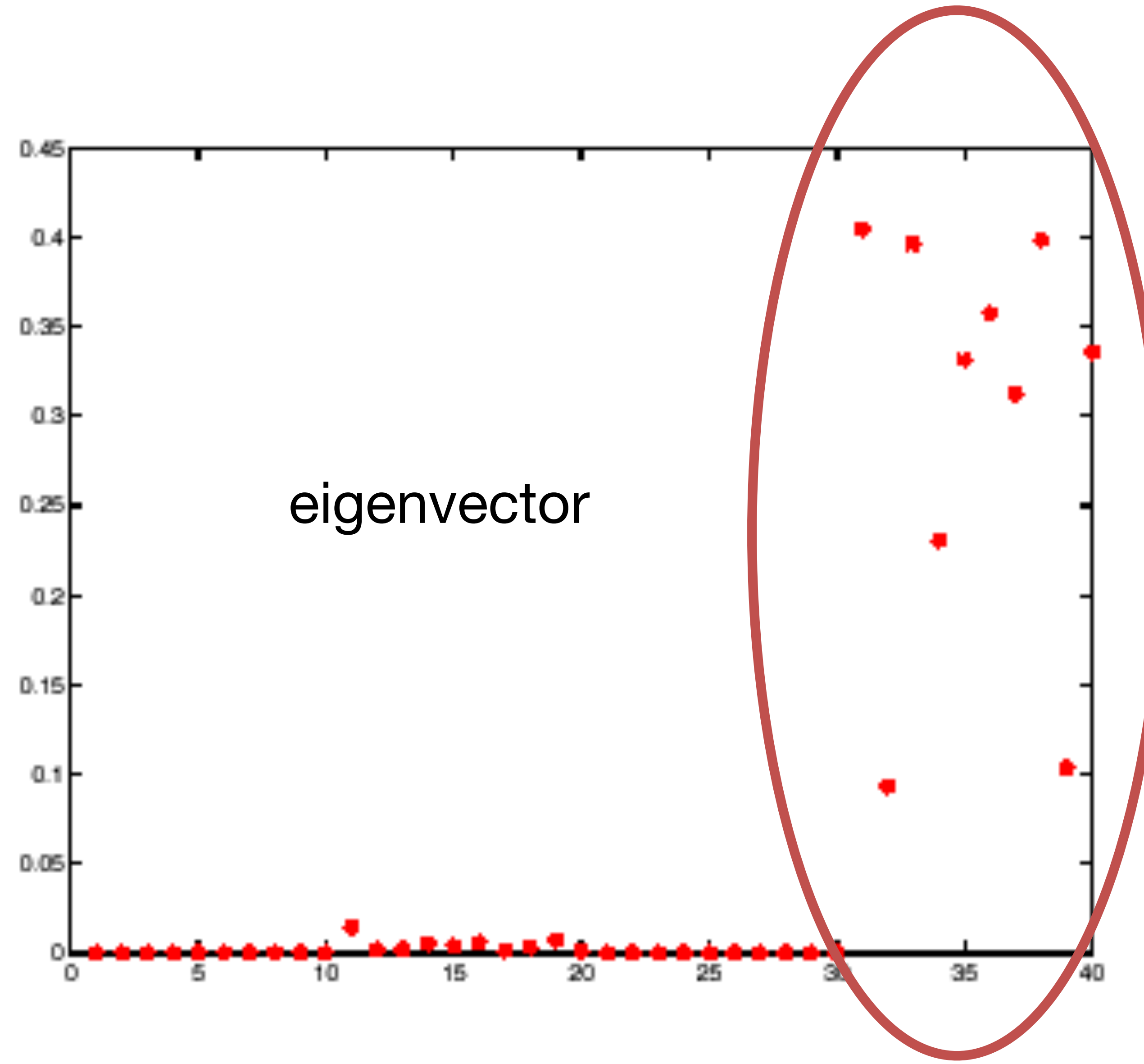
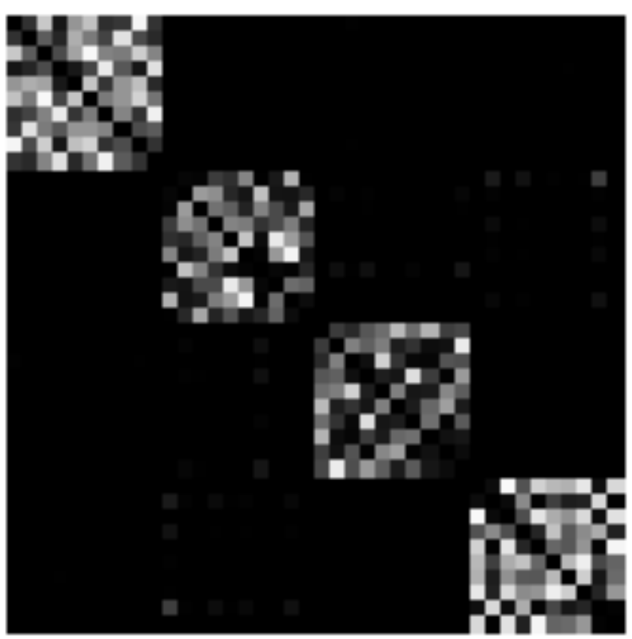
Example eigenvector



points



eigenvector



Affinity matrix

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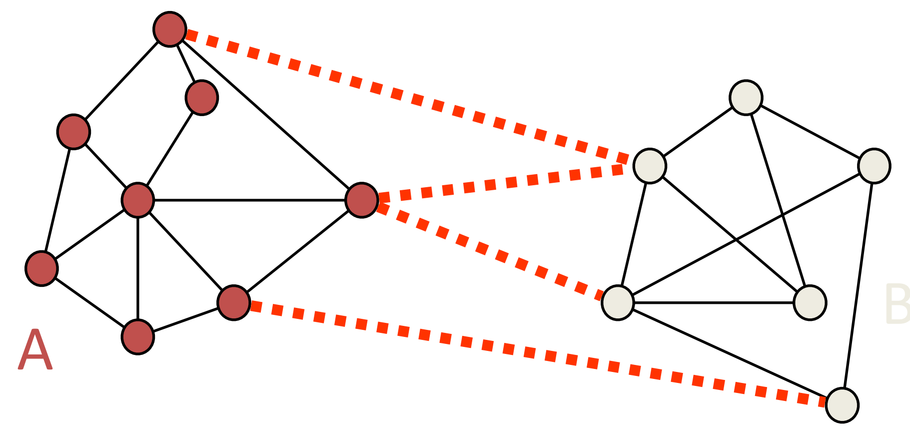
Graph cut

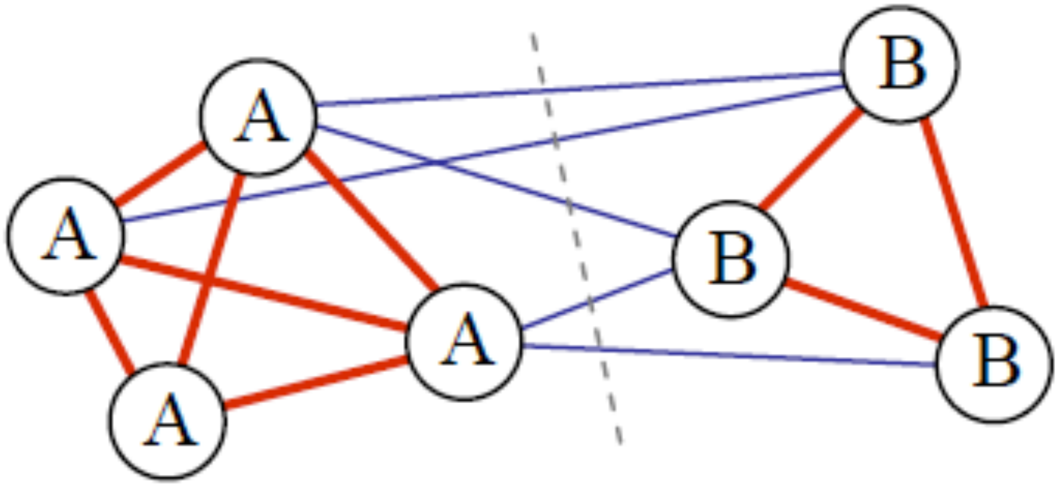


- Set of edges whose removal makes a graph disconnected
- Cost of a cut: sum of weights of cut edges
- A graph cut gives us a partition (clustering)
  - What is a "good" graph cut and how do we find one?

slide by Steven Seitz

Minimum cut

- A cut of a graph *G* is the set of edges *S* such that removal of *S* from *G* disconnects *G*.



Cut: sum of the weight of the cut edges:

$$cut(\mathrm{A},\mathrm{B}) = \sum_{u \in \mathrm{A}, v \in \mathrm{B}} \mathrm{W}(u, v),$$

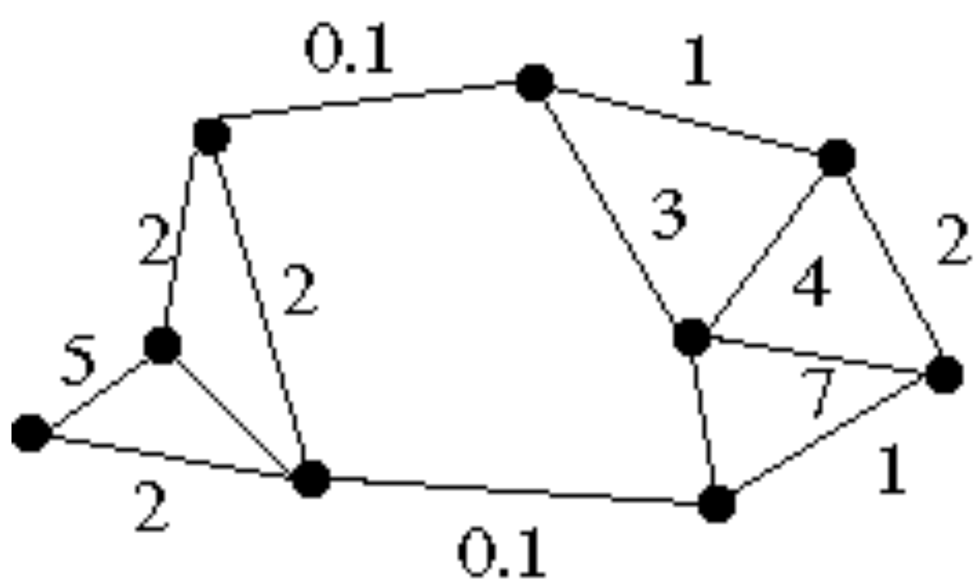
with $\mathrm{A} \cap \mathrm{B} = \emptyset$

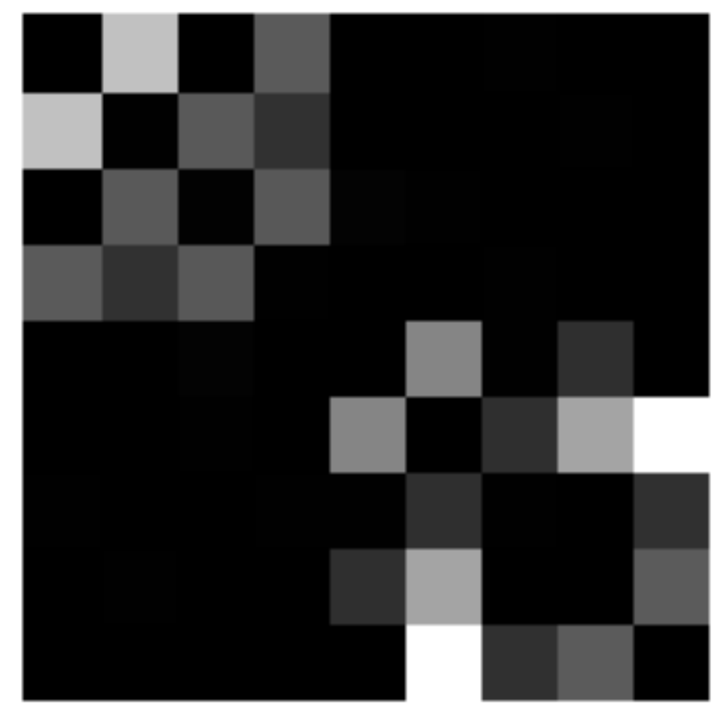
slide by Bill Freeman and Antonio Torralba

Minimum cut

- We can do clustering by finding the minimum cut in a graph
  - Efficient algorithms exist for doing this

Minimum cut example



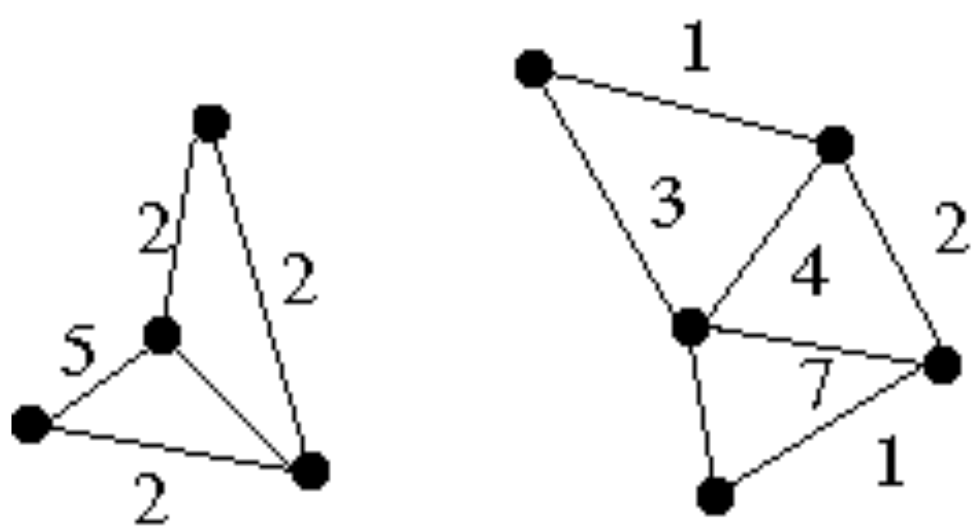


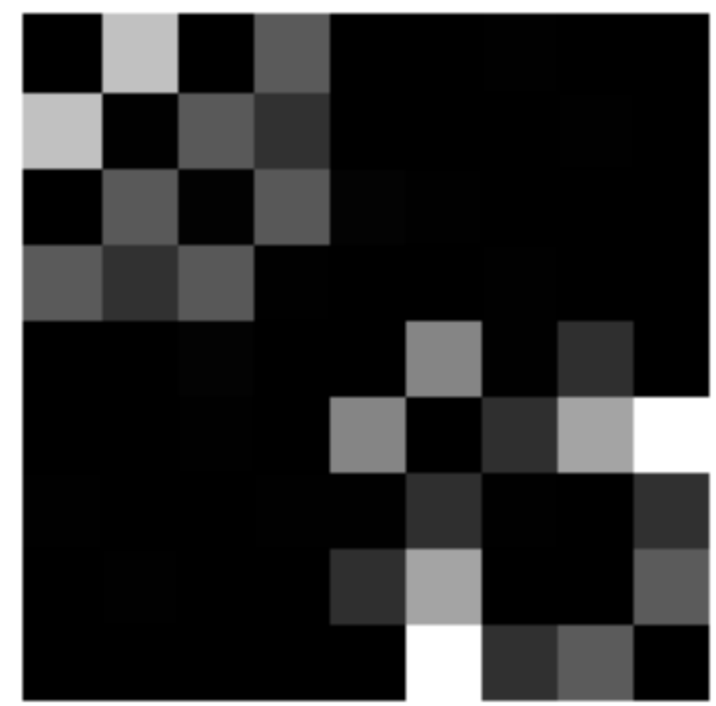
slide by Svetlana Lazebnik

Minimum cut

- We can do segmentation by finding the minimum cut in a graph
  - Efficient algorithms exist for doing this

Minimum cut example





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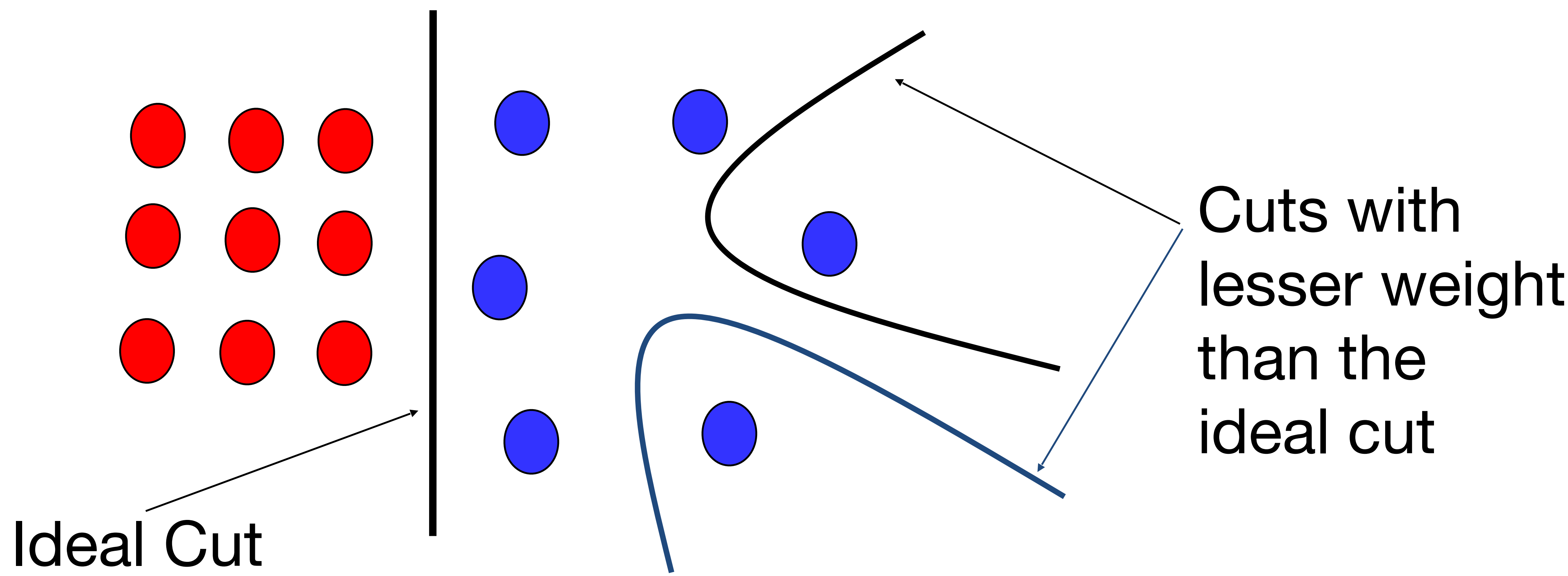
Drawbacks of Minimum cut

- Weight of cut is directly proportional to the number of edges in the cut.



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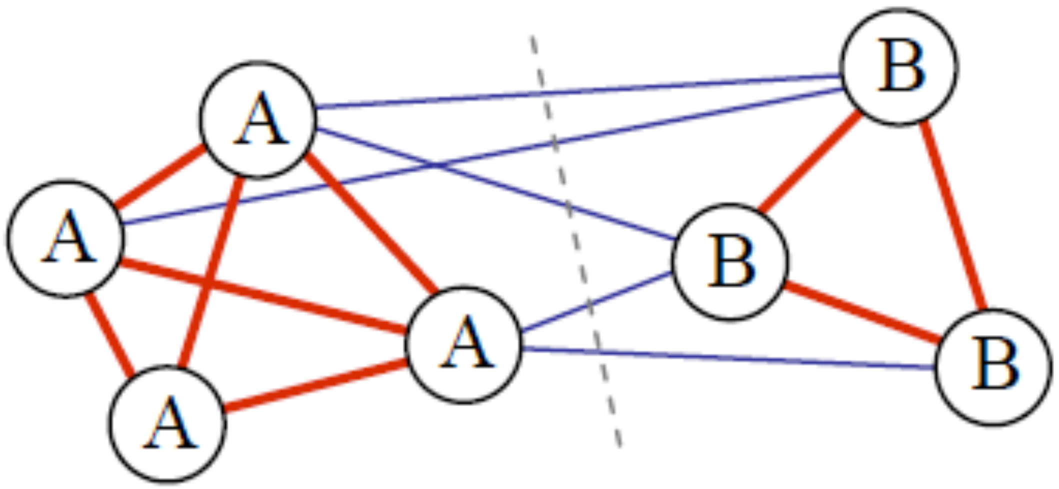
* Slide from Khurram Hassan-Shafique CAP5415 Computer Vision 2003

Normalized cuts

Write graph as V, one cluster as A and the other as B



$$\text{Ncut}(A,B) = \frac{\text{cut}(A,B)}{\text{assoc}(A,V)} + \frac{\text{cut}(A,B)}{\text{assoc}(B,V)}$$

cut(A,B) is sum of weights with one end in A and one end in B

$$cut(\mathrm{A},\mathrm{B}) = \sum_{u \in \mathrm{A}, v \in \mathrm{B}} \mathrm{W}(u,v),$$

with $\mathrm{A} \cap \mathrm{B} = \varnothing$

assoc(A,V) is sum of all edges with one end in A.

$$assoc(\mathrm{A},\mathrm{B}) = \sum_{u \in \mathrm{A}, v \in \mathrm{B}} \mathrm{W}(u,v)$$

A and B not necessarily disjoint

slide by Bill Freeman and Antonio Torralba

J. Shi and J. Malik. [Normalized cuts and image segmentation.]() PAMI 2000

Normalized cut

- Let W be the adjacency matrix of the graph
- Let D be the diagonal matrix with diagonal entries $D(i, i) = \Sigma_j W(i, j)$
- Then the normalized cut cost can be written as

$$\frac{y^T (D - W) y}{y^T D y}$$

D-W: Graph Laplacian

where y is an indicator vector whose value should be 1 in the i-th position if the i-th feature point belongs to A and a negative constant otherwise

slide by Svetlana Lazebnik

J. Shi and J. Malik. [Normalized cuts and image segmentation.]() PAMI 2000

Normalized cut

- Finding the exact minimum of the normalized cut cost is NP-complete, but if we *relax* y to take on arbitrary values, then we can minimize the relaxed cost by solving the **generalized eigenvalue problem** $(D - W)y = \lambda Dy$
- The solution y is given by the generalized eigenvector corresponding to the second smallest eigenvalue, aka the Fiedler vector
- Intuitively, the i-th entry of y can be viewed as a “soft” indication of the component membership of the i-th feature
  - Can use 0 or median value of the entries as the splitting point (threshold), or find threshold that minimizes the Ncut cost

slide by Svetlana Lazebnik

J. Shi and J. Malik. [Normalized cuts and image segmentation.](#) PAMI 2000

Normalized cut algorithm

1. Given an image or image sequence, set up a weighted graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$, and set the weight on the edge connecting two nodes being a measure of the similarity between the two nodes.

2. Solve $(\mathbf{D} - \mathbf{W})\boldsymbol{x} = \lambda \mathbf{D}\boldsymbol{x}$ for eigenvectors with the smallest eigenvalues.

3. Use the eigenvector with second smallest eigenvalue to bipartition the graph.

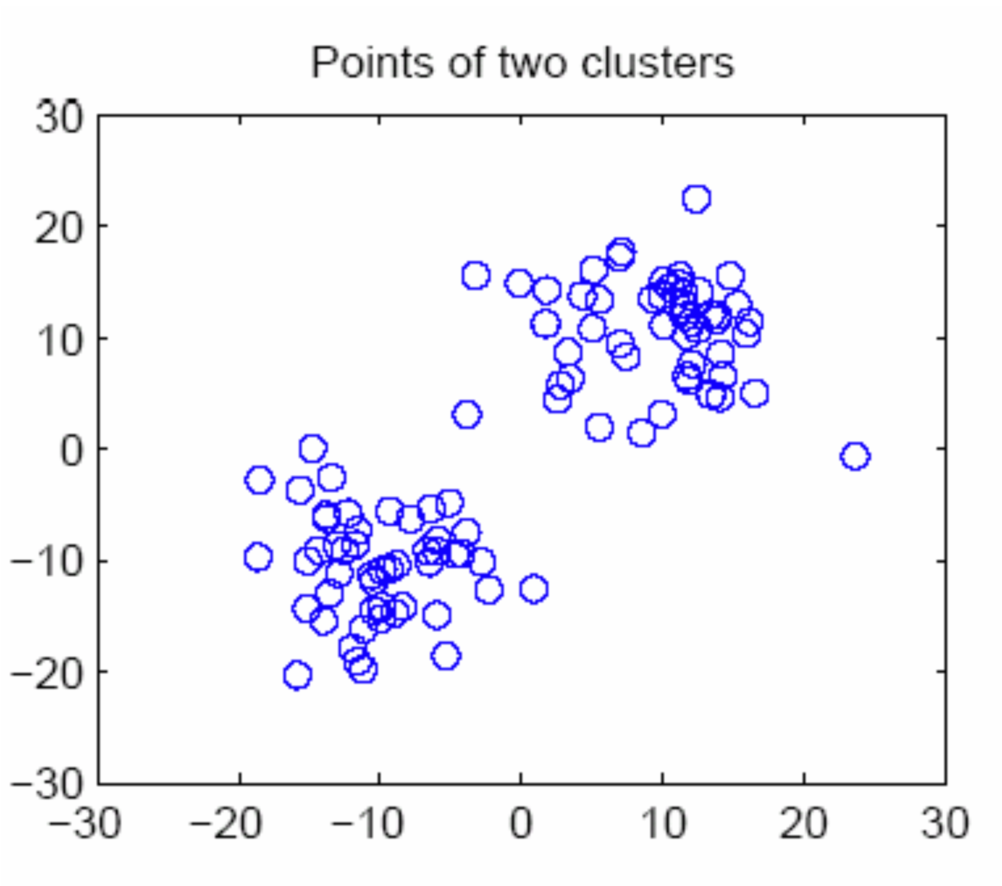
4. Decide if the current partition should be sub-divided, and recursively repartition the segmented parts if necessary.

slide by Bill Freeman and Antonio Torralba

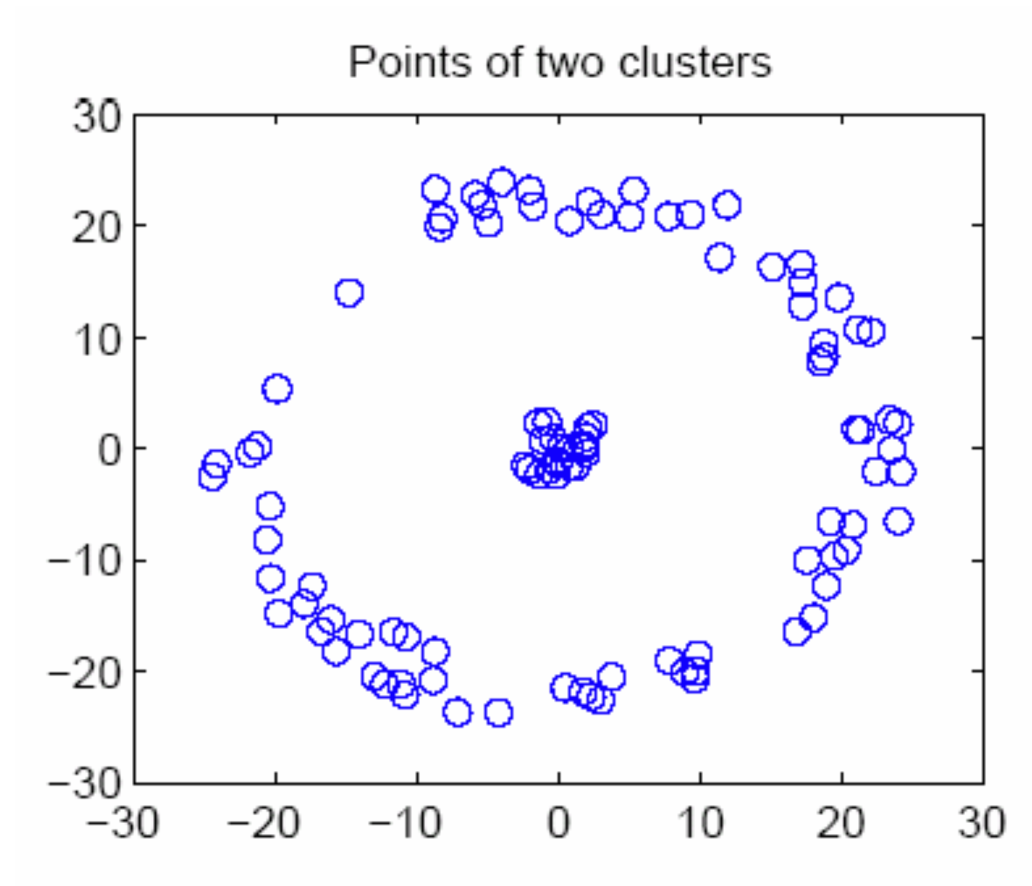
J. Shi and J. Malik. Normalized cuts and image segmentation. PAMI 2000

K-Means vs. Spectral Clustering

- Applying k-means to Laplacian eigenvectors allows us to find cluster with non-convex boundaries.



Both perform same

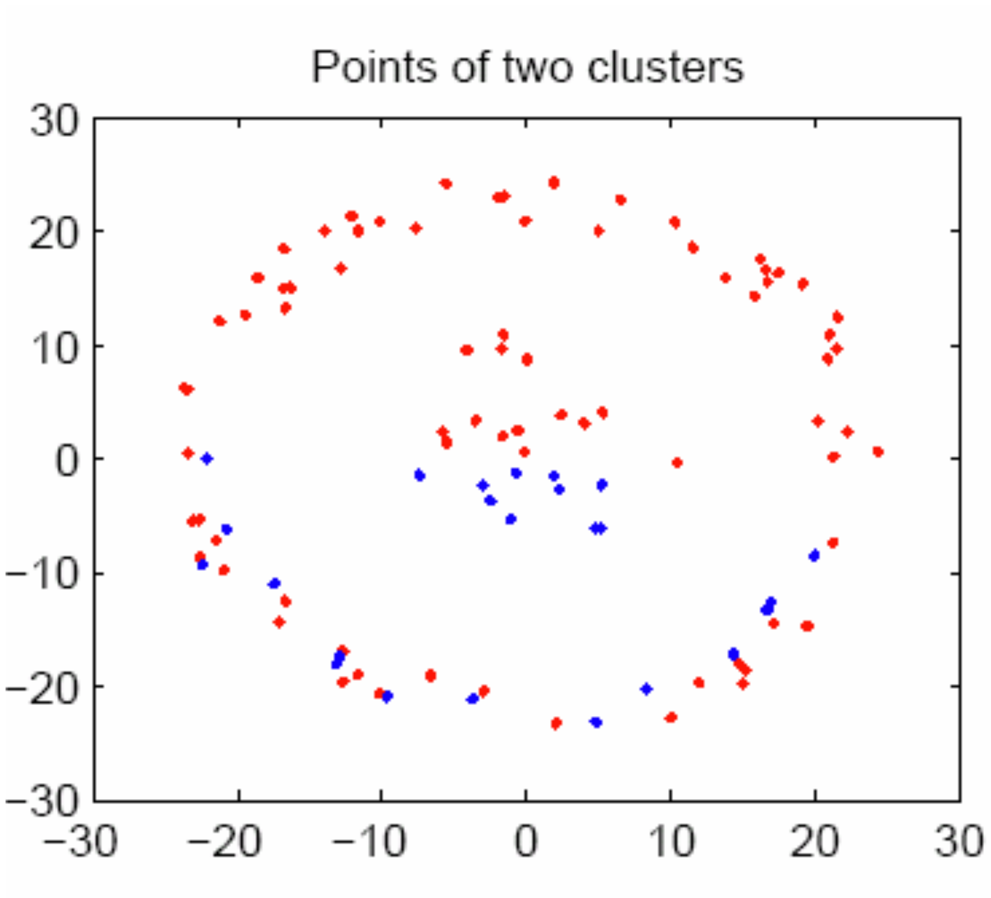


Spectral clustering is superior

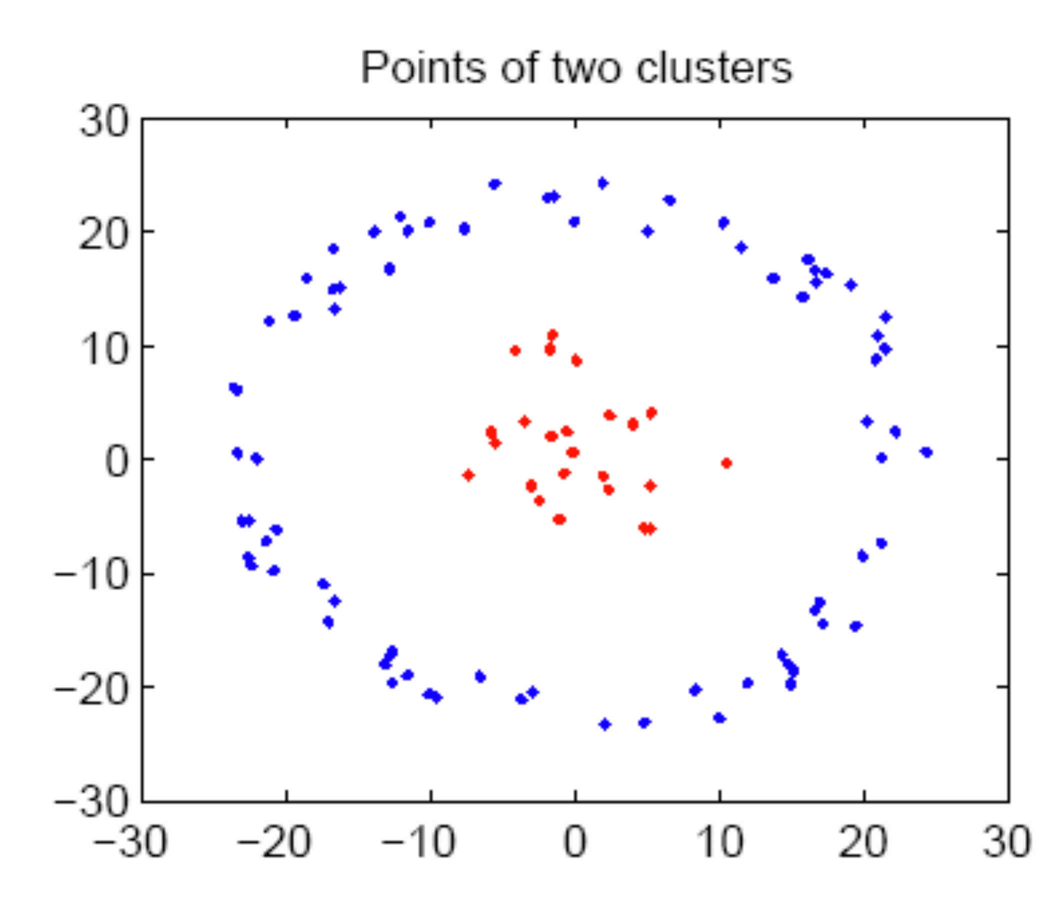
slide by Aarti Singh

K-Means vs. Spectral Clustering

- Applying k-means to Laplacian eigenvectors allows us to find cluster with non-convex boundaries.



k-means output



Spectral clustering output

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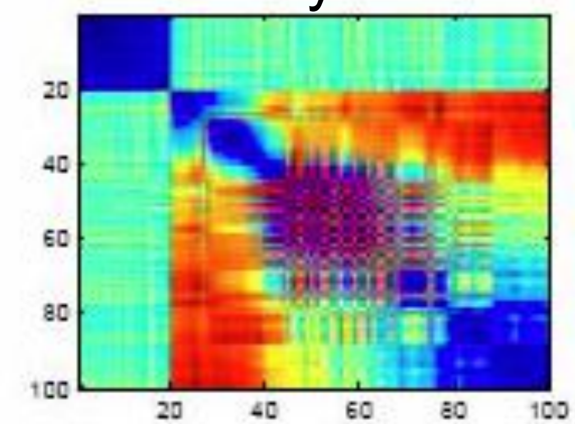
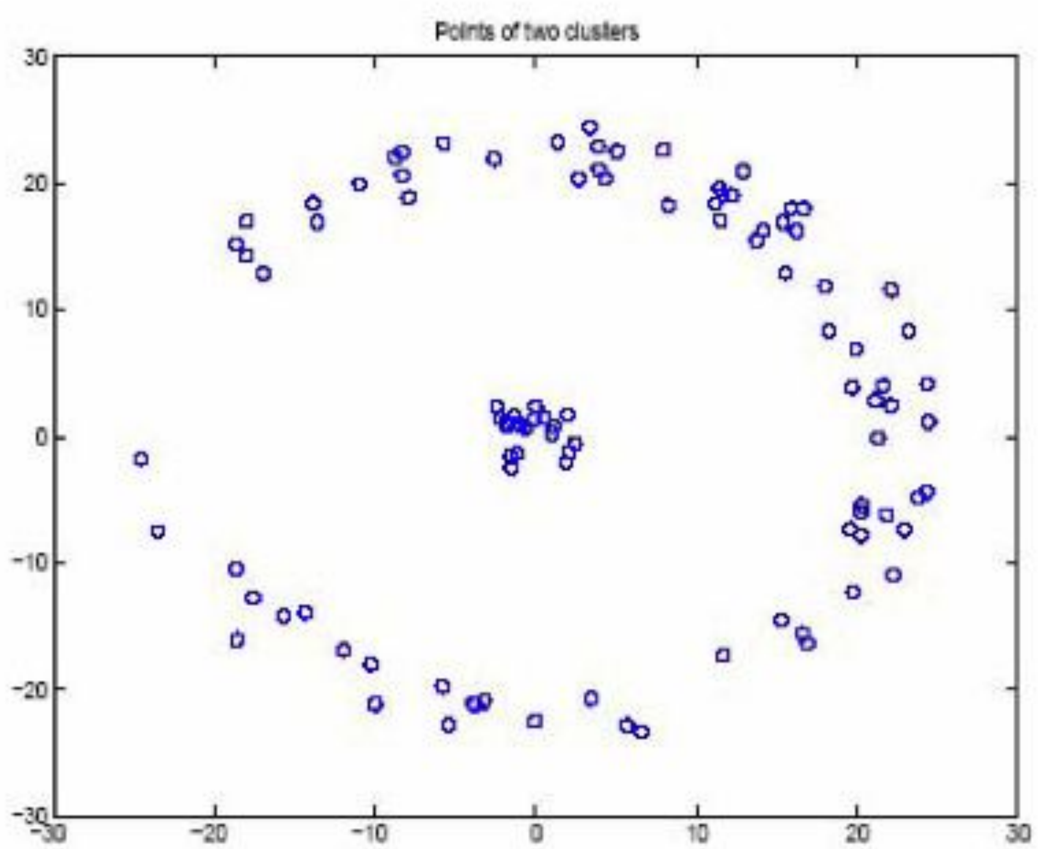
K-Means vs. Spectral Clustering

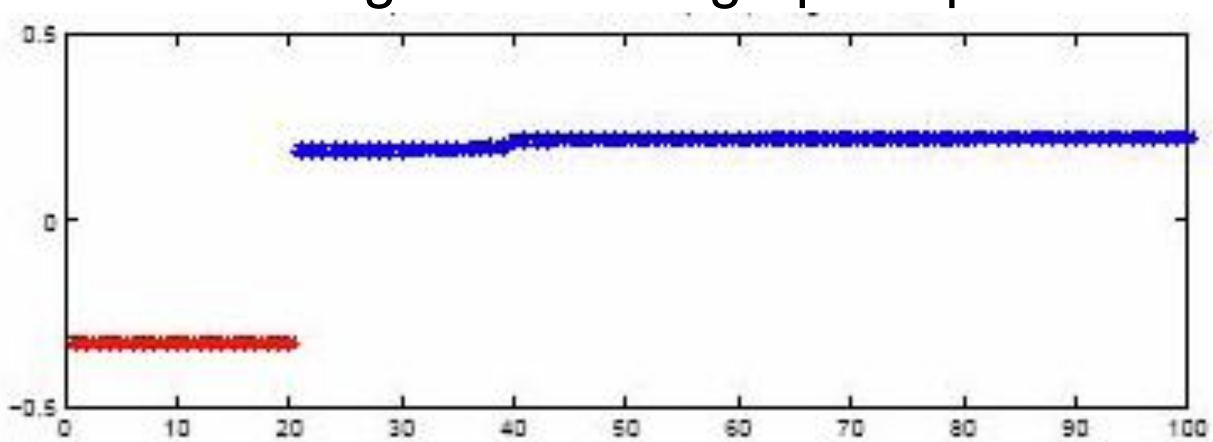
- Applying k-means to Laplacian eigenvectors allows us to find cluster with non-convex boundaries.

Similarity matrix





Second eigenvector of graph Laplacian



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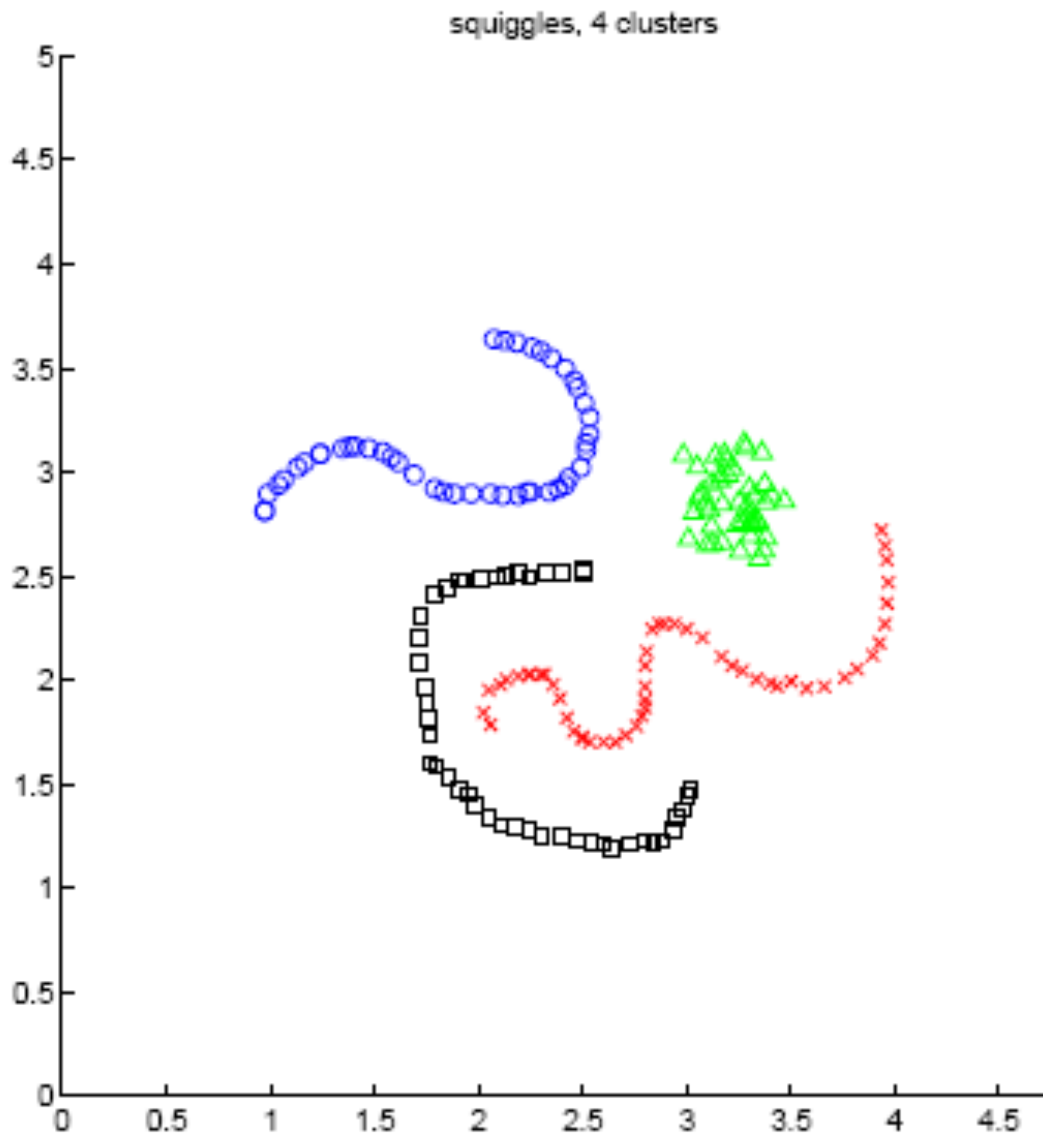
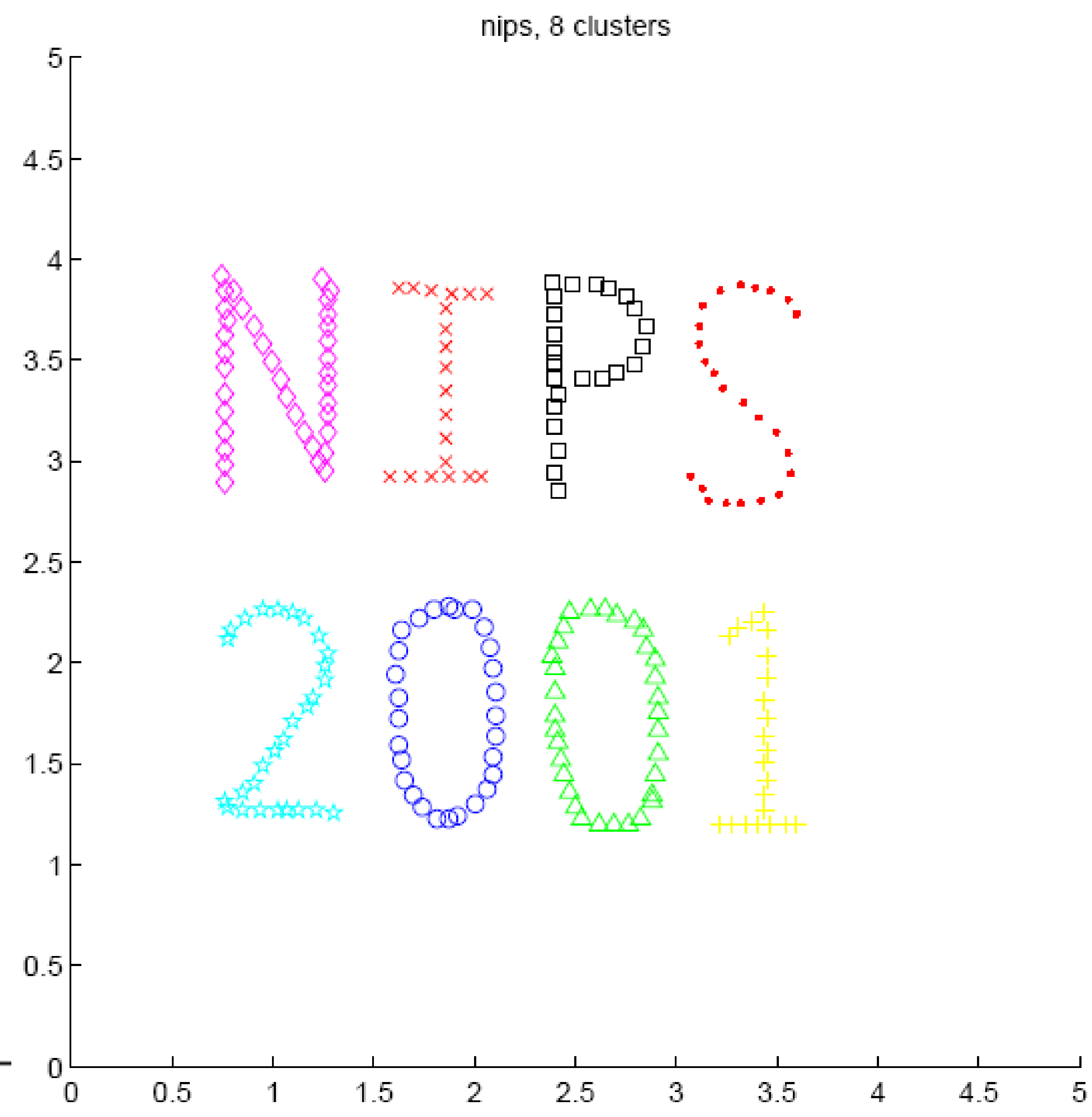
Examples





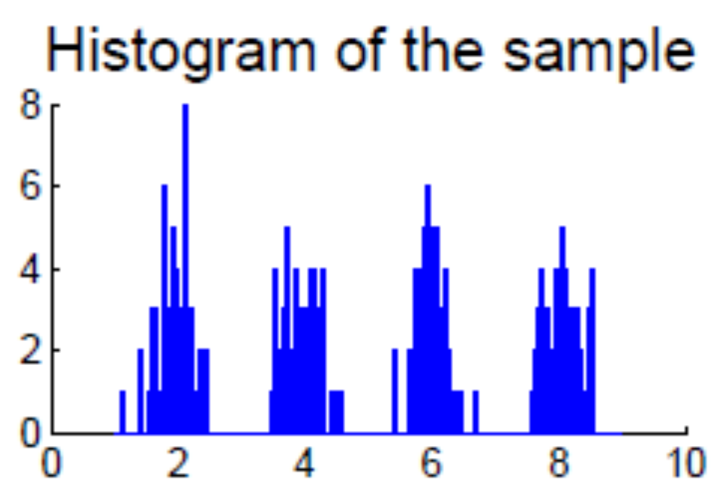
[Ng et al., 2001]

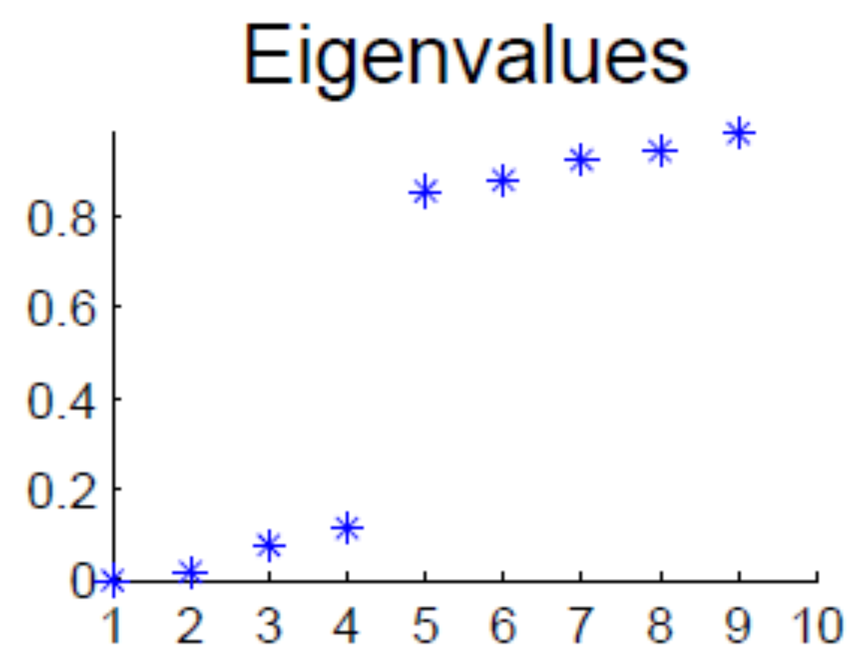
slide by Aarti Singh

Some Issues

- Choice of number of clusters k
  - Most stable clustering is usually given by the value of k that maximizes the eigengap (difference between consecutive eigenvalues)

$$\Delta_k = \left|\lambda - \lambda_{k-1}\right|$$



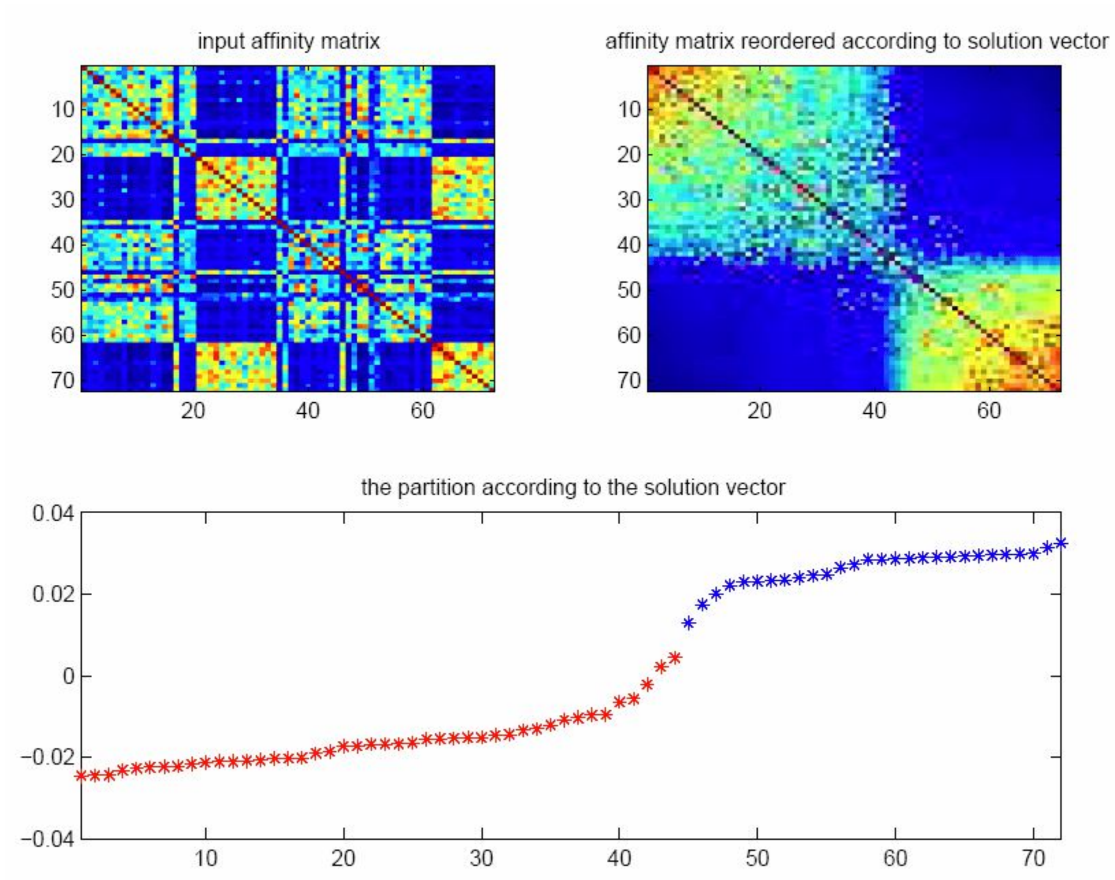


slide by Aarti Singh

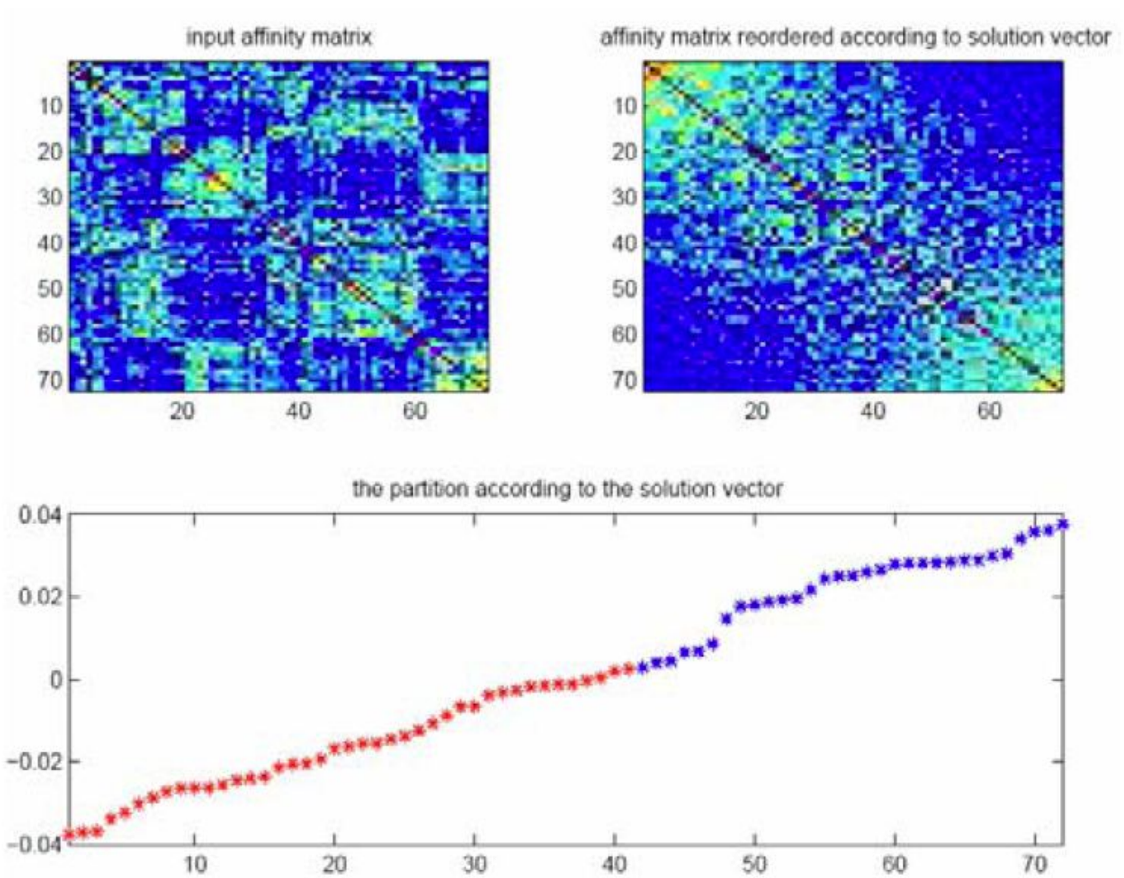
Some Issues

- Choice of number of clusters k
- Choice of similarity
  - Choice of kernel

    for Gaussian kernels, choice of σ



Good similarity measure



Poor similarity measure

slide by Aarti Singh

Some Issues

- Choice of number of clusters k

- Choice of similarity
  - Choice of kernel

  for Gaussian kernels, choice of σ

- Choice of clustering method
  - k-way vs. recursive 2-way

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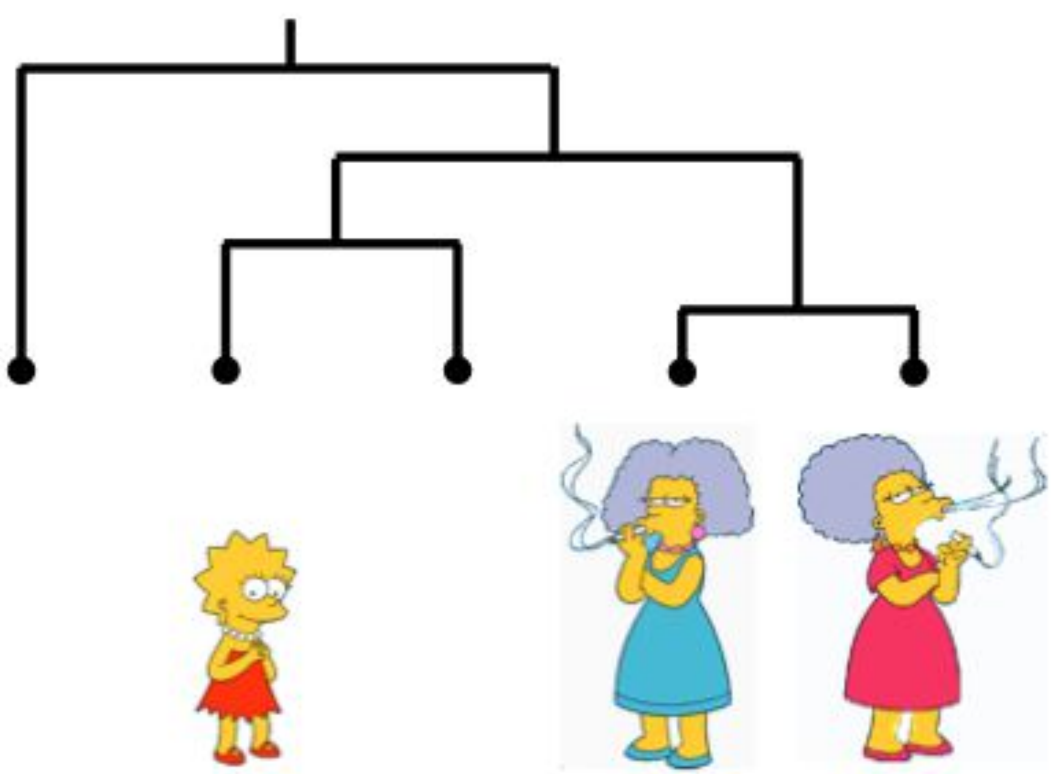
Hierarchical clustering

Hierarchical Clustering

- Bottom-Up (agglomerative): Starting with each item in its own cluster, find the best pair to merge into a new cluster. Repeat until all clusters are fused together.



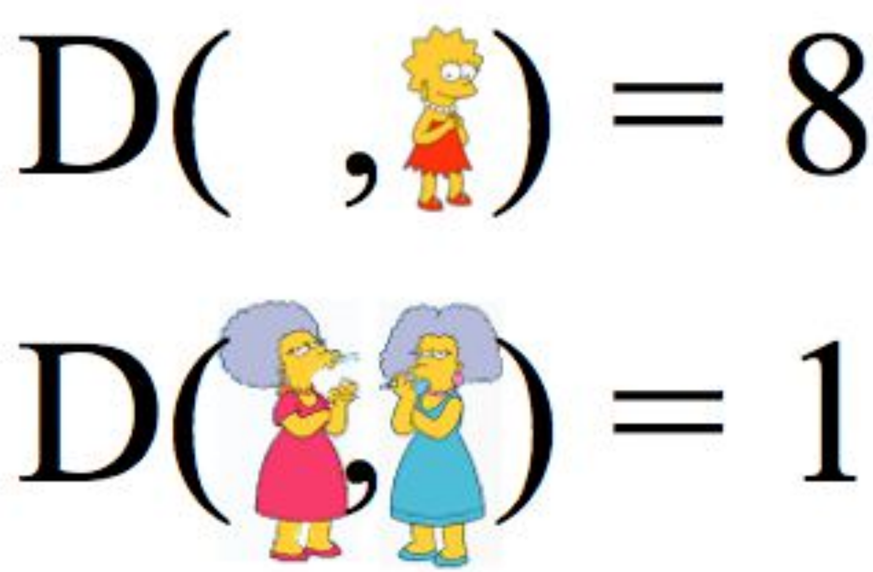
- The number of dendrograms with
  n leafs = (2n -3)!/[(2(n -2)) (n -2)!]

Number of leafs	Number of possible Dendrongrams
2	1
3	3
4	15
5	105
…	…
10	34,459,425

slide by Andrew Moore

We begin with a distance matrix which contains the distances between every pair of objects in our dataset

0	8	8	7	7
	0	2	4	4
		0	3	3
			0	1
				0



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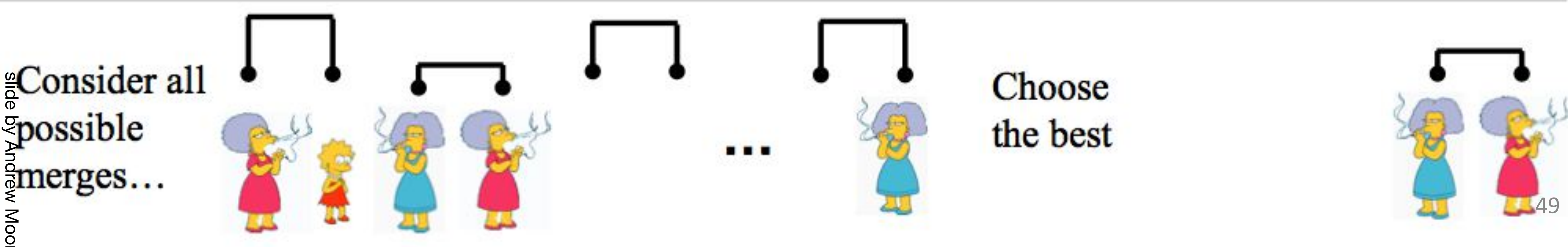
Bottom-Up (agglomerative):

Start with each item in its own cluster, find the best pair to merge into a new cluster. Repeat until all clusters are fused together.



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Bottom-Up (agglomerative):

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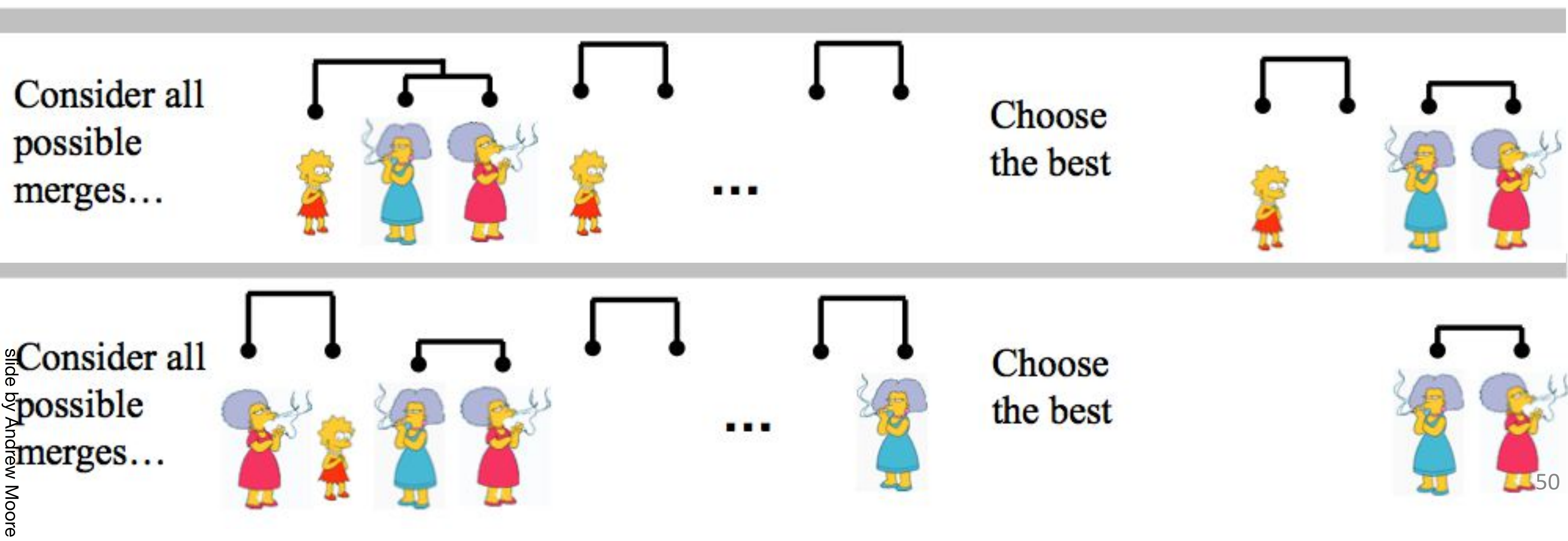

slide by Andrew Moore

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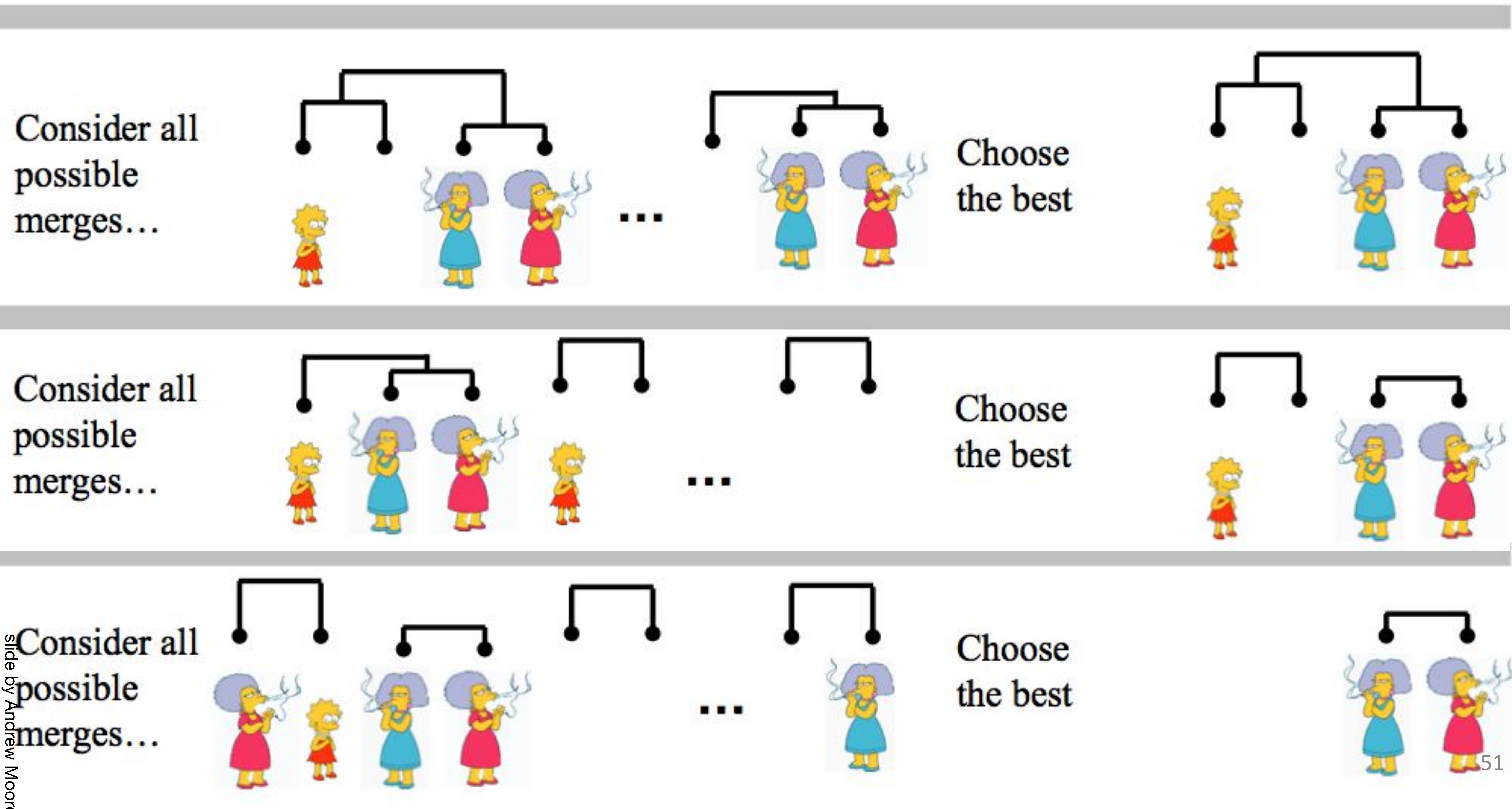

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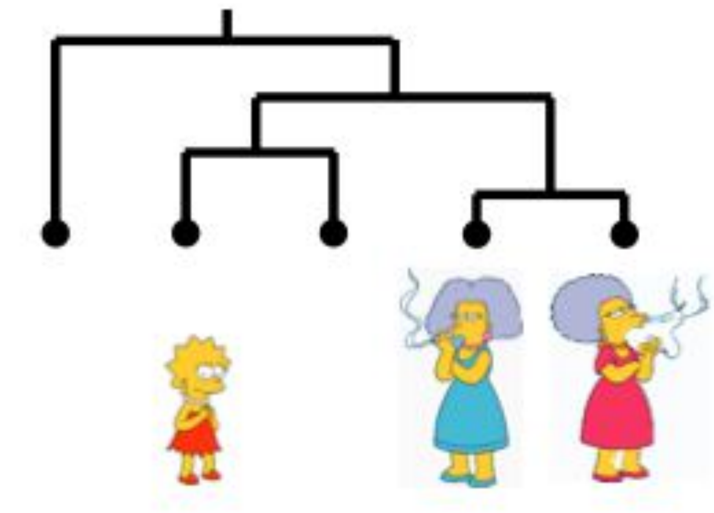
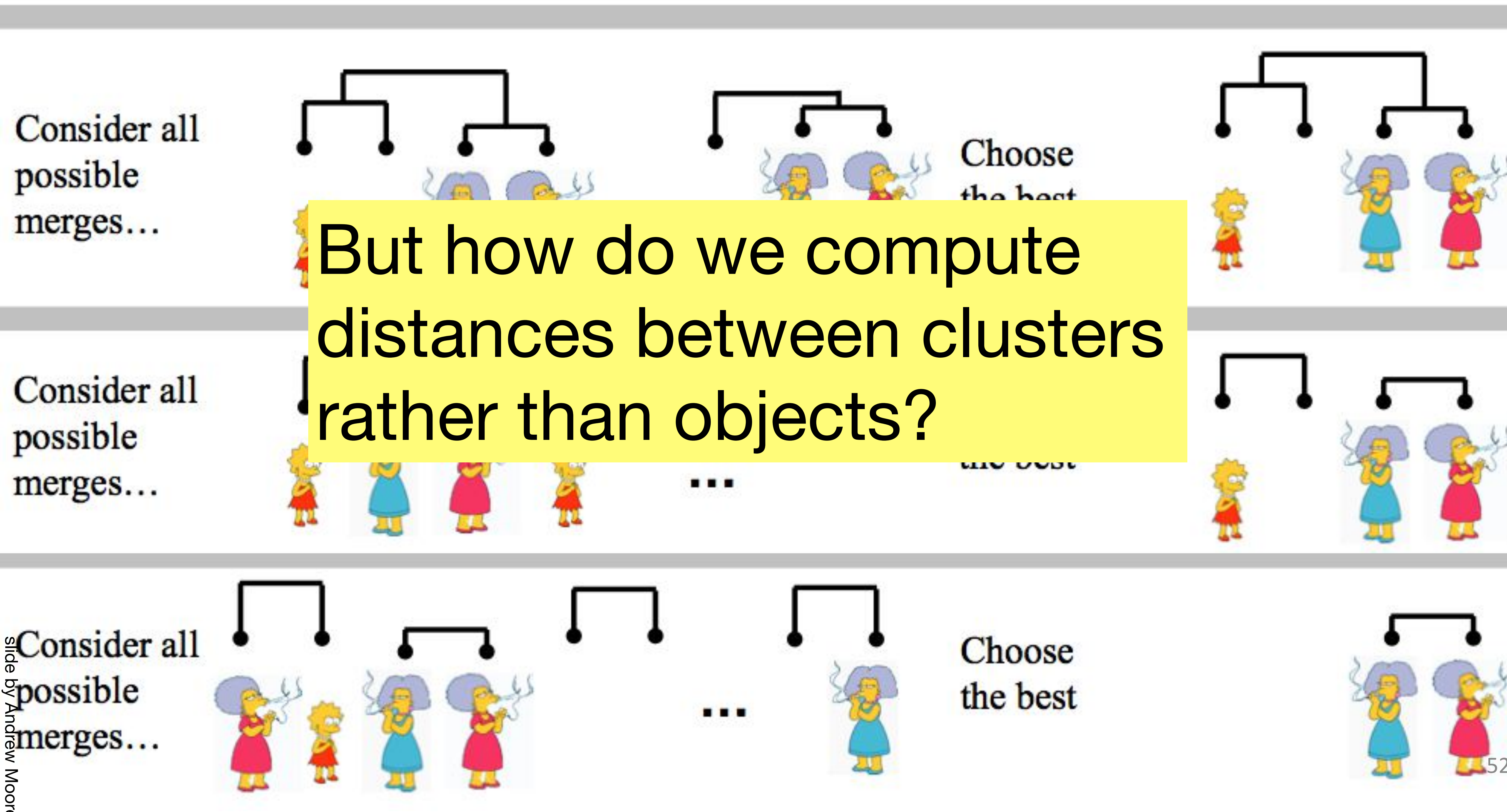
Bottom-Up (agglomerative):

Start with each item in its own cluster, find the best pair to merge into a new cluster. Repeat until all clusters are fused together.





But how do we compute distances between clusters rather than objects?

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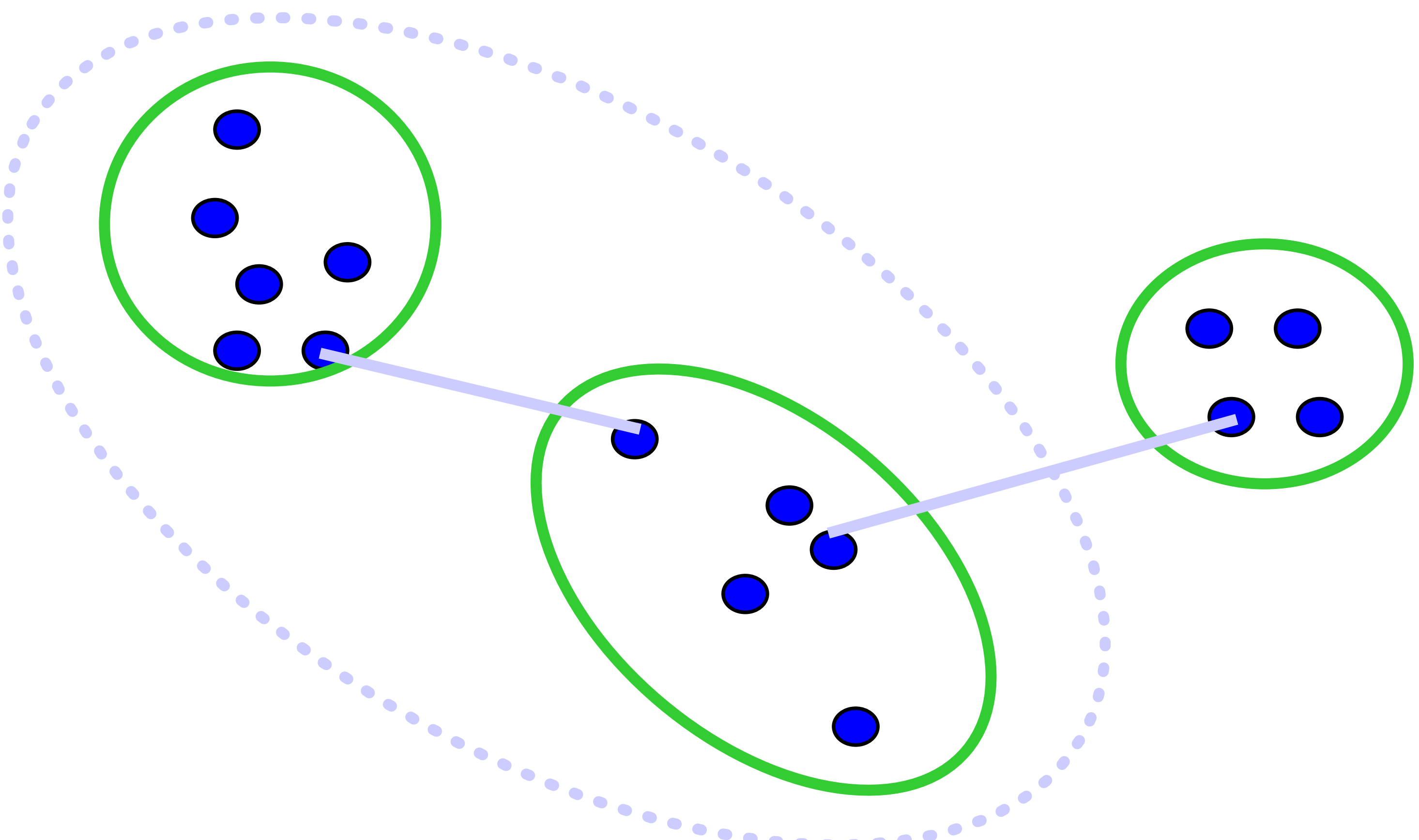
Computing distance between clusters: **Single Link**

- Cluster distance = distance of two closest members in each class

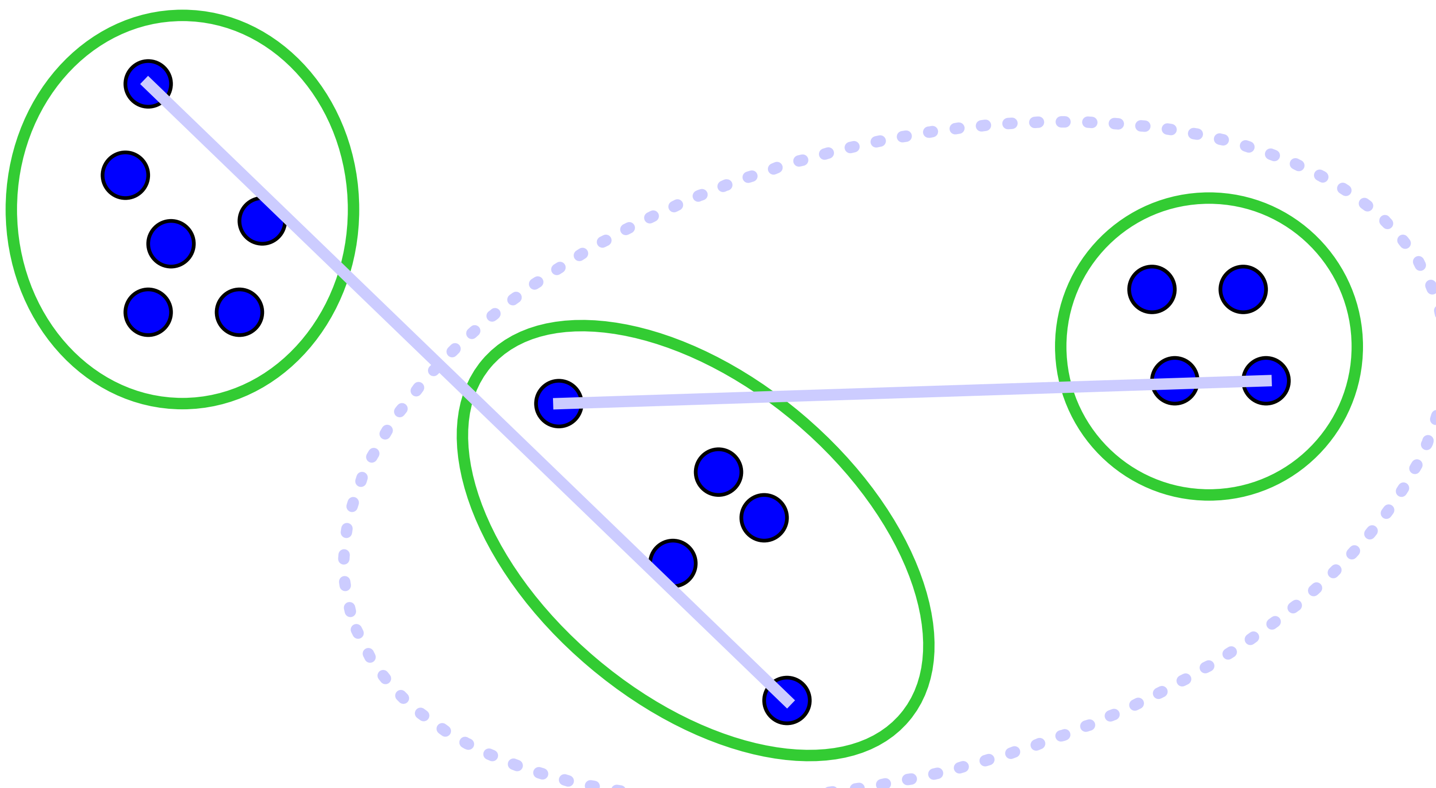


- Potentially long and skinny clusters

Computing distance between clusters: **Complete Link**

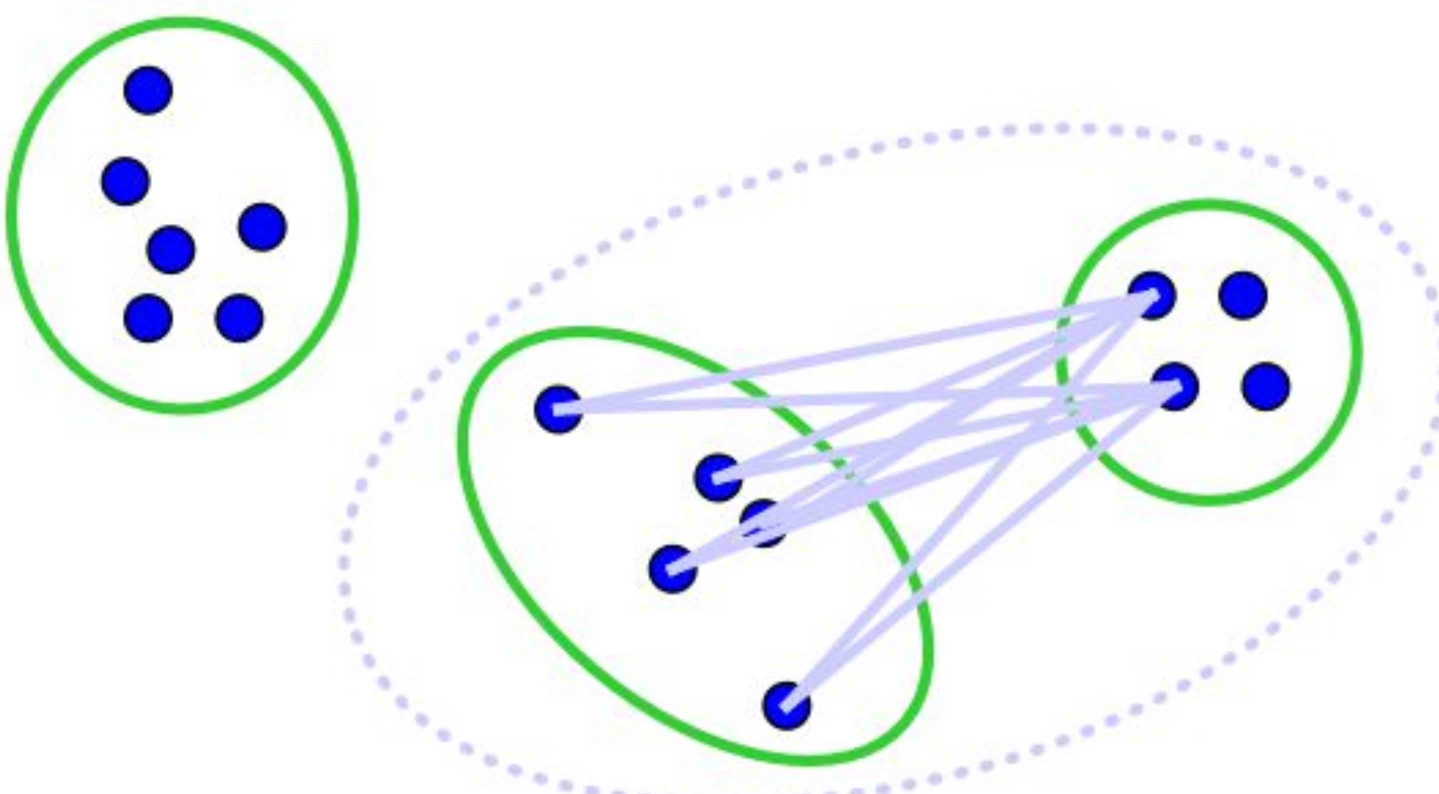
- Cluster distance = distance of two farthest members in each class

- Tight clusters



Computing distance between clusters: **Average Link**

- Cluster distance = average distance of all pairs



- The most widely used measure
- Robust against noise

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Agglomerative Clustering

Good

- Simple to implement, widespread application
- Clusters have adaptive shapes
- Provides a hierarchy of clusters

Bad

- May have imbalanced clusters
- Still have to choose number of clusters or threshold
  - **silhouette coefficient**
- Need to use an “ultrametric” to get a meaningful hierarchy

slide by Derek Hoiem