BBM 413 Fundamentals of Image Processing

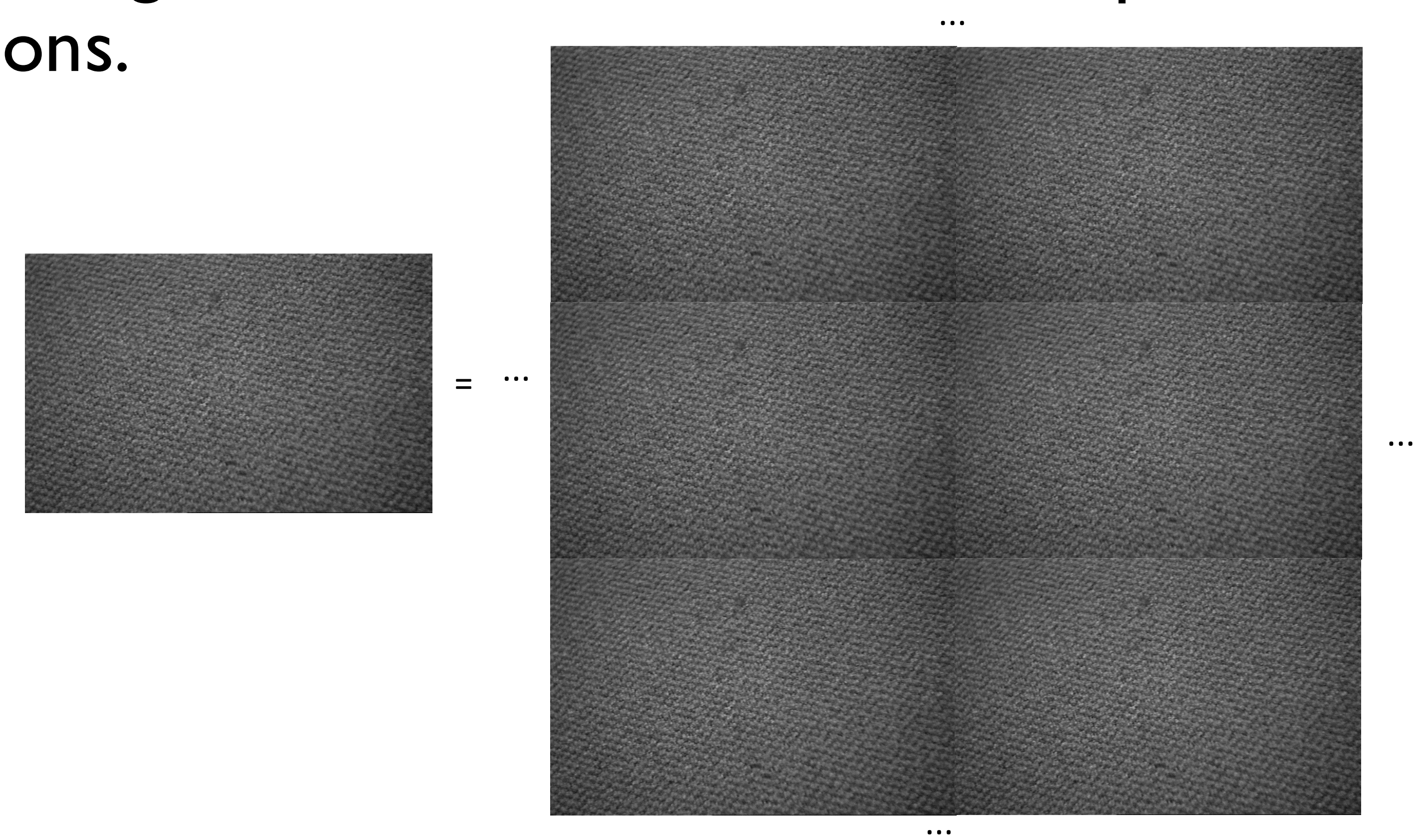
Erkut Erdem
Dept. of Computer Engineering
Hacettepe University

Frequency Domain Techniques – Part 2

Review – Frequency Domain Techniques

- Thinking images in terms of frequency.

- Treat images as infinite-size, continuous periodic functions.



Review - Fourier Transform

We want to understand the frequency *w* of our signal. So, let's reparametrize the signal by *w* instead of *x*:

f(x) → Fourier Transform → ***F(w)***

For every *w* from 0 to inf, ***F(w)*** holds the amplitude *A* and phase *f* of the corresponding sine $A\sin(\omega x + \phi)$

- How can *F* hold both? Complex number trick!

$$F(\omega) = R(\omega) + iI(\omega)$$

$$A = \pm\sqrt{R(\omega)^2 + I(\omega)^2} \qquad \phi = \tan^{-1}\frac{I(\omega)}{R(\omega)}$$

We can always go back:

F(w) → Inverse Fourier Transform → ***f(x)***

Slide credit: A. Efros

Review - Fourier Transform

- Fourier transform stores the magnitude and phase at each frequency
  - Magnitude encodes how much signal there is at a particular frequency
  - Phase encodes spatial information (indirectly)
  - For mathematical convenience, this is often notated in terms of real and complex numbers

Amplitude: $A = \pm\sqrt{R(\omega)^2 + I(\omega)^2}$

Phase: $\phi = \tan^{-1}\frac{I(\omega)}{R(\omega)}$

Review - Discrete Fourier transform

- Forward transform

$$F(u,v) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)}$$

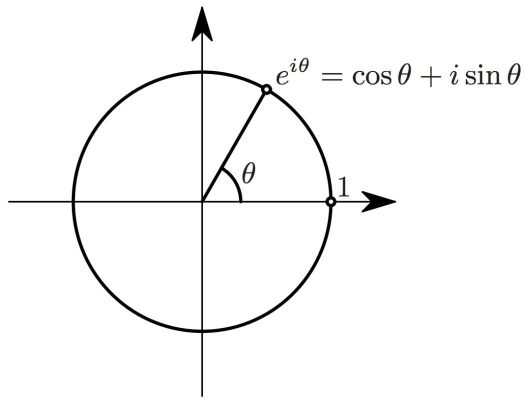
$$\text{for } u = 0,1,2,\ldots,M-1, v = 0,1,2,\ldots,N-1$$

- Inverse transform

$$f(x,y) = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi(ux/M+vy/N)}$$

$$\text{for } x = 0,1,2,\ldots,M-1, y = 0,1,2,\ldots,N-1$$

u, v : the transform or frequency variables
x, y : the spatial or image variables



Euler's definition of $e^{i\theta}$

Slide credit: B. Freeman and A. Torralba

Review - The Fourier Transform

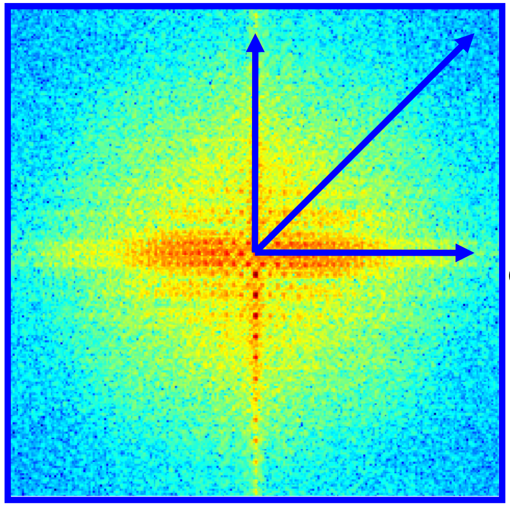
- Represent function on a new basis
  - Think of functions as vectors, with many components
  - We now apply a linear transformation to transform the basis
    - dot product with each basis element
- In the expression, u and v select the basis element, so a function of x and y becomes a function of u and v
- basis elements have the form $e^{-i2\pi(ux+vy)}$

Slide credit: S. Thrun

Review - The Fourier Transform

Vertical orientation

0



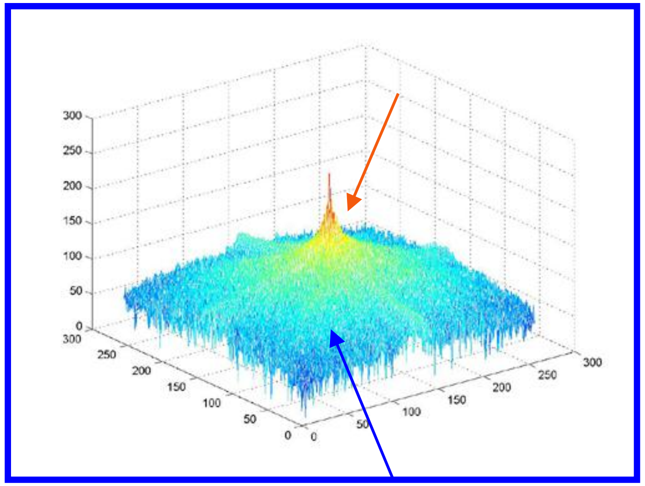
45 deg.

Horizontal orientation

0 f_{max}

fx in cycles/image

Low spatial frequencies



High spatial frequencies

Log power spectrum

Slide credit: B. Freeman and A. Torralba

Review - The Convolution Theorem

- The Fourier transform of the convolution of two functions is the product of their Fourier transforms

$$\mathrm{F}[g*h]=\mathrm{F}[g]\,\mathrm{F}[h]$$

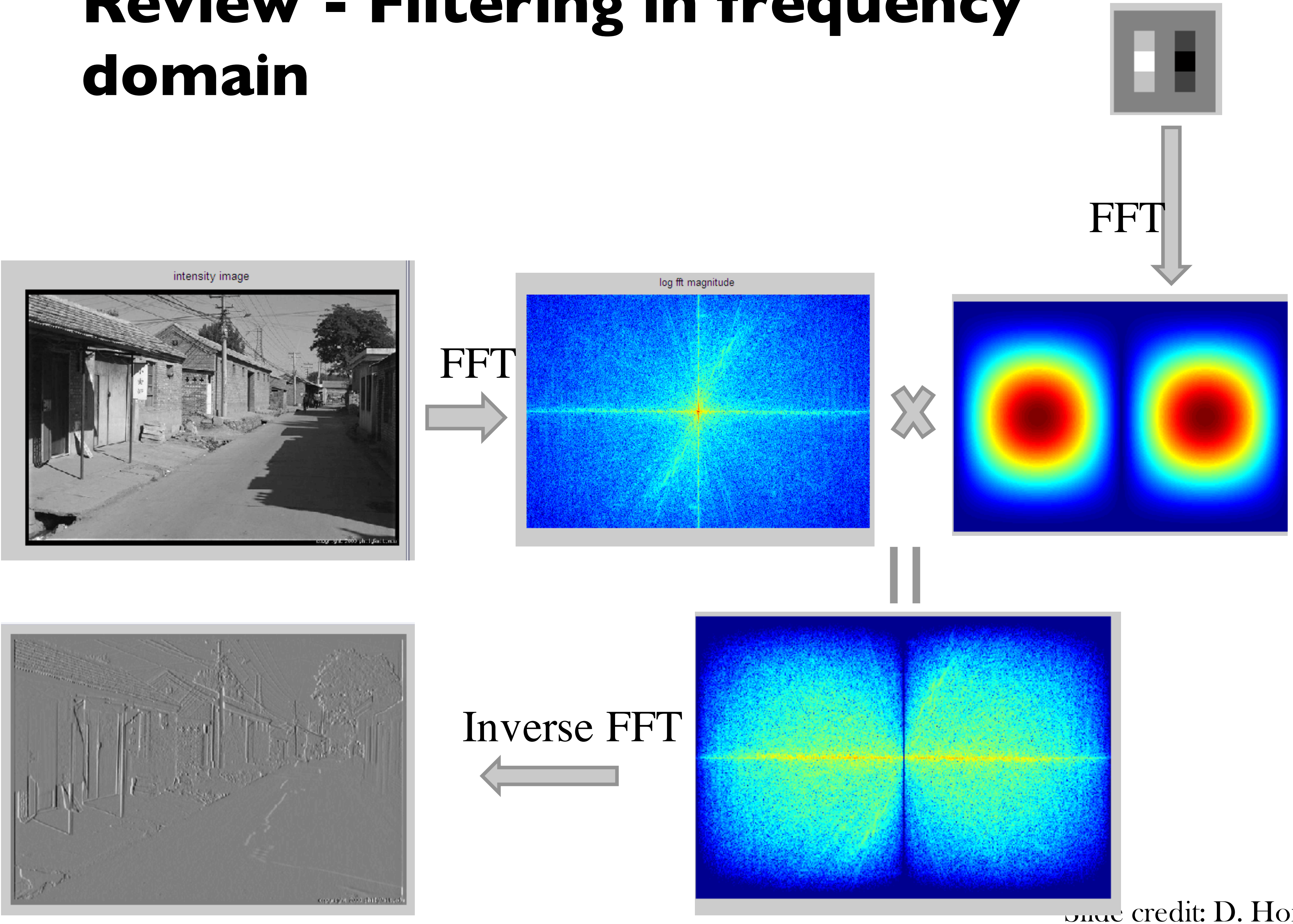
- The inverse Fourier transform of the product of two Fourier transforms is the convolution of the two inverse Fourier transforms

$$\mathrm{F}^{-1}[gh]=\mathrm{F}^{-1}[g]*\mathrm{F}^{-1}[h]$$

- **Convolution** in spatial domain is equivalent to **multiplication** in frequency domain!

Slide credit: A. Efros

Review - Filtering in frequency domain



Slide credit: D. Hoiem

Today

- Sampling
- Gabor wavelets, Steerable filters

Today

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Sampling

Why does a lower resolution image still make sense to us?
What do we lose?

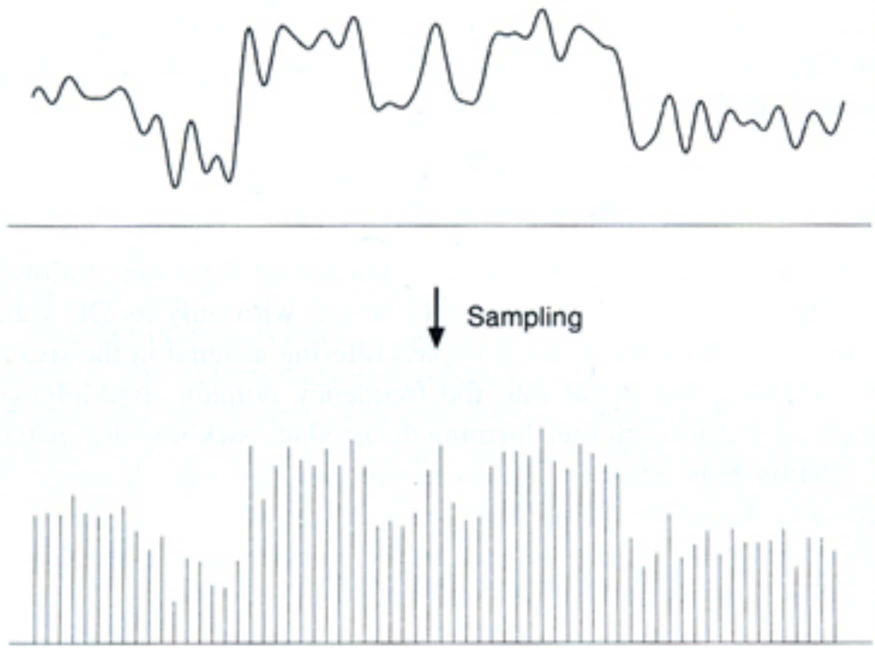
Image: http://www.flickr.com/photos/igorms/136916757/

Slide credit: D. Hoiem

Sampled representations

- How to store and compute with continuous functions?
- Common scheme for representation: samples

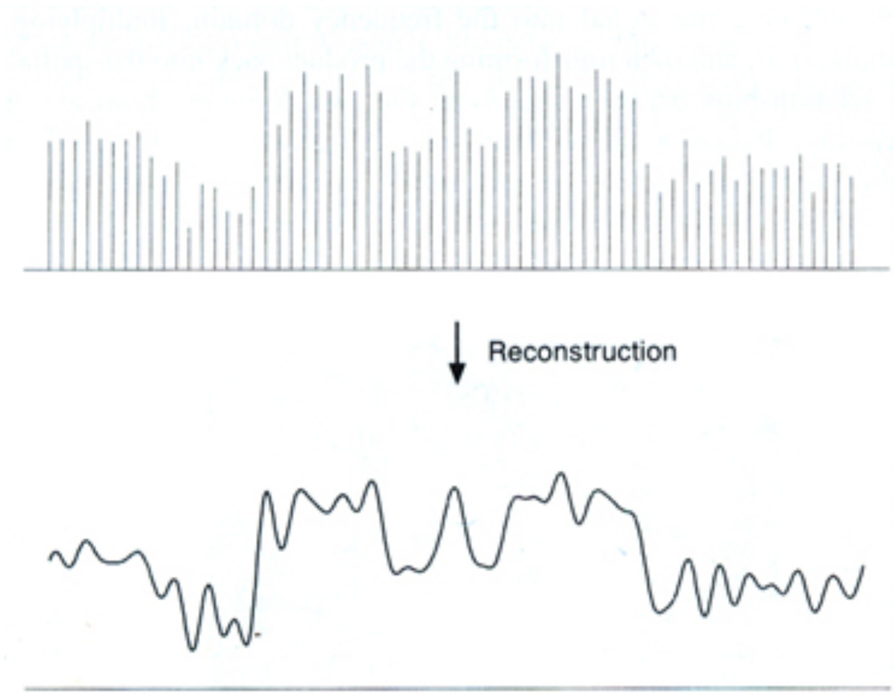
– write down the function's values at many points



Slide credit: S. Marschner

Reconstruction

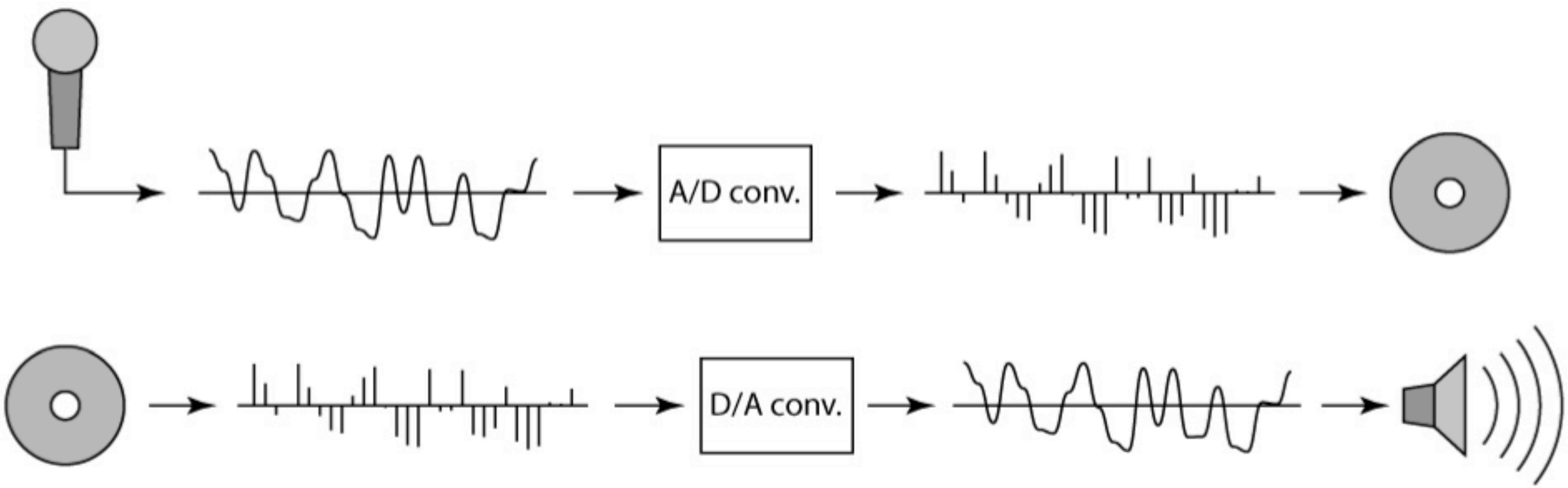
- Making samples back into a continuous function
  - for output (need realizable method)
  - for analysis or processing (need mathematical method)
  - amounts to "guessing" what the function did in between



Slide credit: S. Marschner

Sampling in digital audio

- Recording: sound to analog to samples to disc
- Playback: disc to samples to analog to sound again
  – how can we be sure we are filling in the gaps correctly?

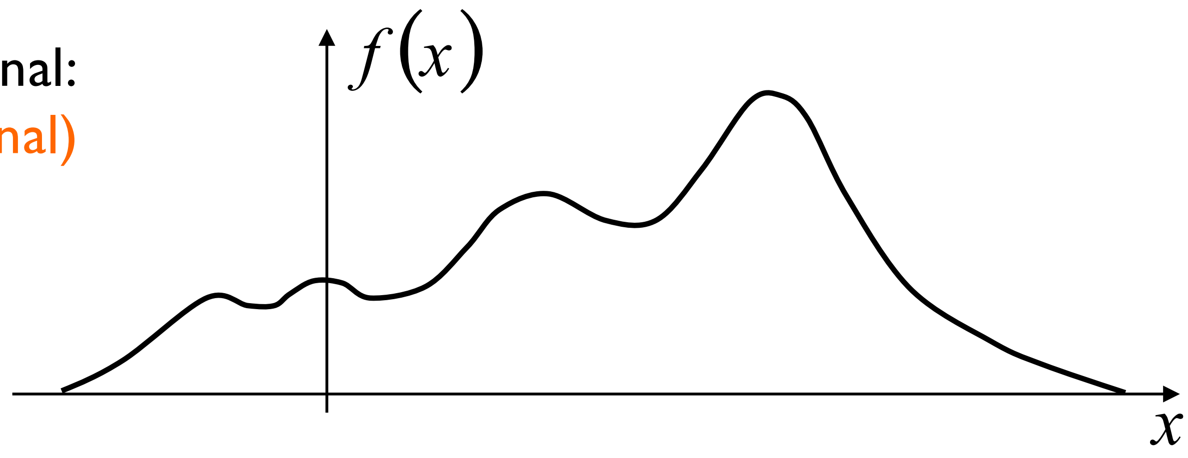


Slide credit: S. Marschner

Sampling Theorem

Continuous signal:

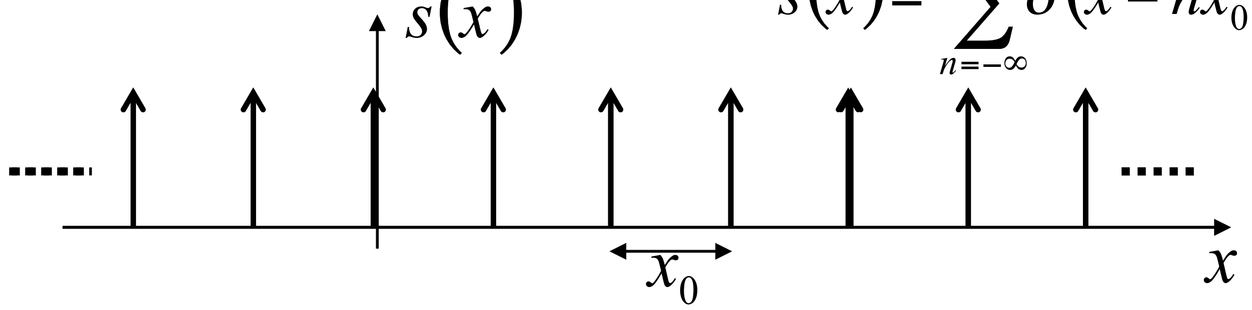
(Real world signal)



Shah function (Impulse train):

(What the image measures)

$$s(x) = \sum_{n=-\infty}^{\infty} \delta(x - nx_0)$$



Sampled function:

$$f_s(x) = f(x)s(x) = f(x)\sum_{n=-\infty}^{\infty} \delta(x - nx_0)$$

Slide credit: S. Narasimhan

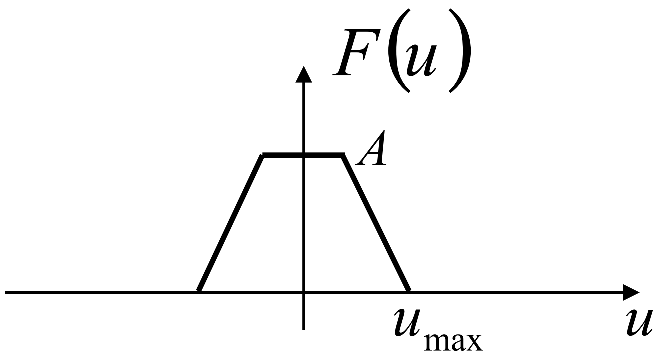
Sampling Theorem

Sampled function:

$$f_s(x) = f(x)s(x) = f(x)\sum_{n=-\infty}^{\infty}\delta(x - nx_0)$$

Sampling frequency $\frac{1}{x_0}$

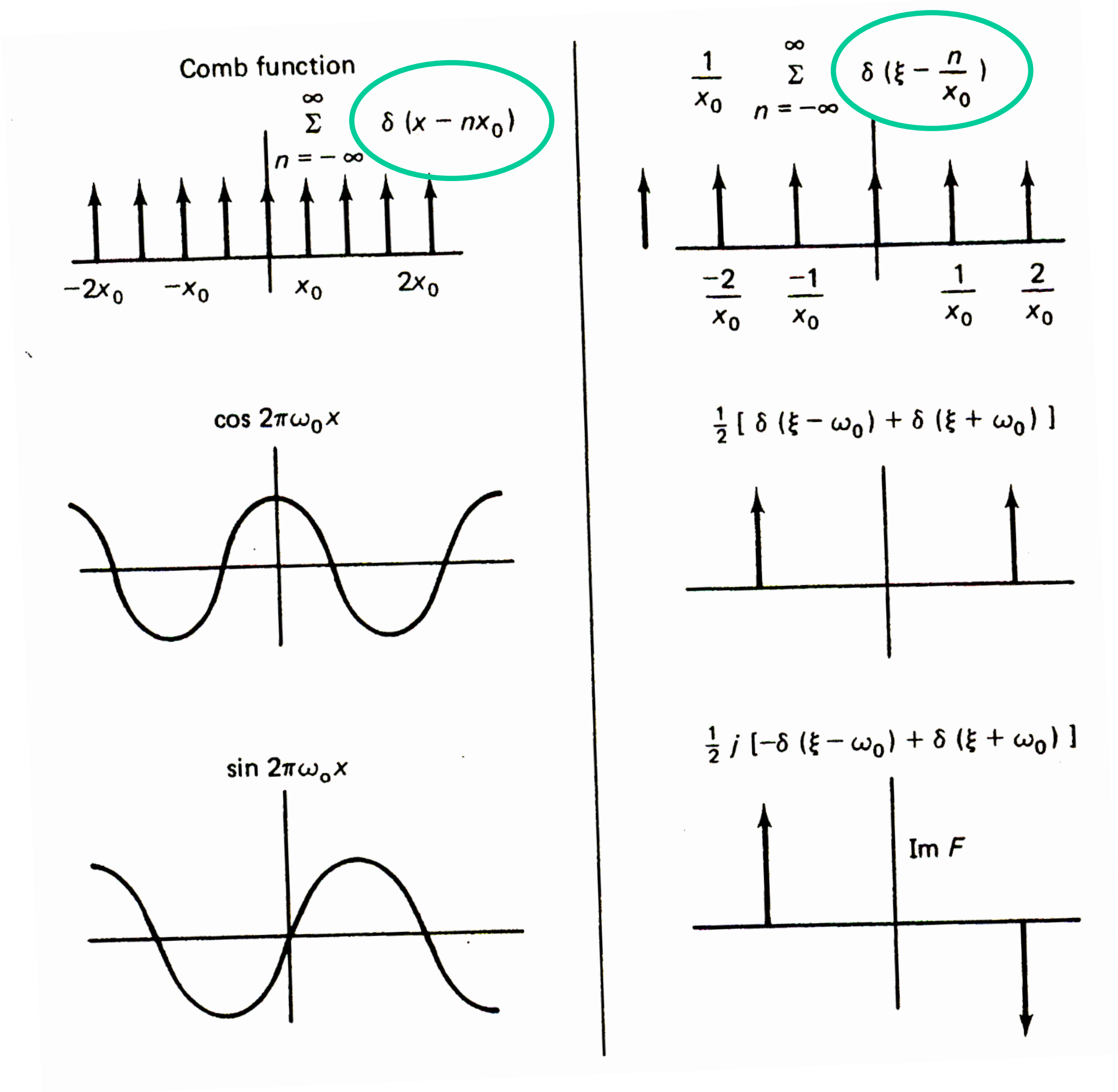
$$F_S(u) = F(u) * S(u) = F(u) * \frac{1}{x_0}\sum_{n=-\infty}^{\infty}\delta\left(u - \frac{n}{x_0}\right)$$



Slide credit: S. Narasimhan

Fourier Transform Pairs

FT of an "impulse train" is an impulse train!



Slide credit: S. Narasimhan

Note that these are derived using angular frequency (e^{-iux})

Sampling Theorem

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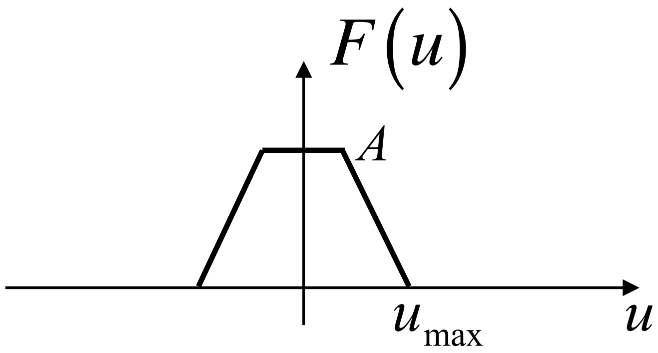

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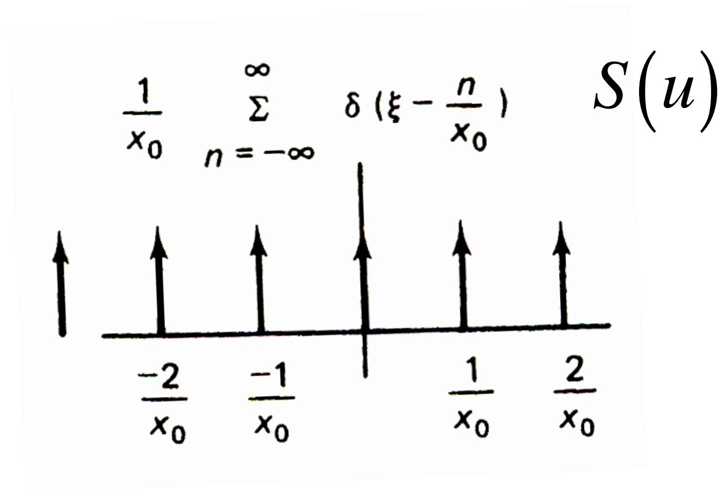

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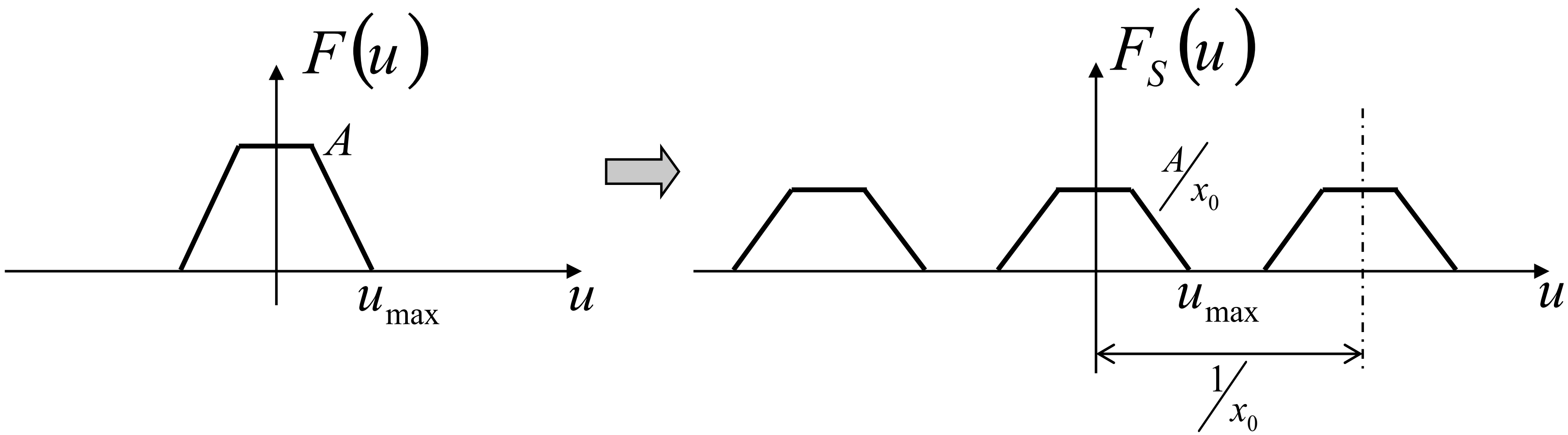
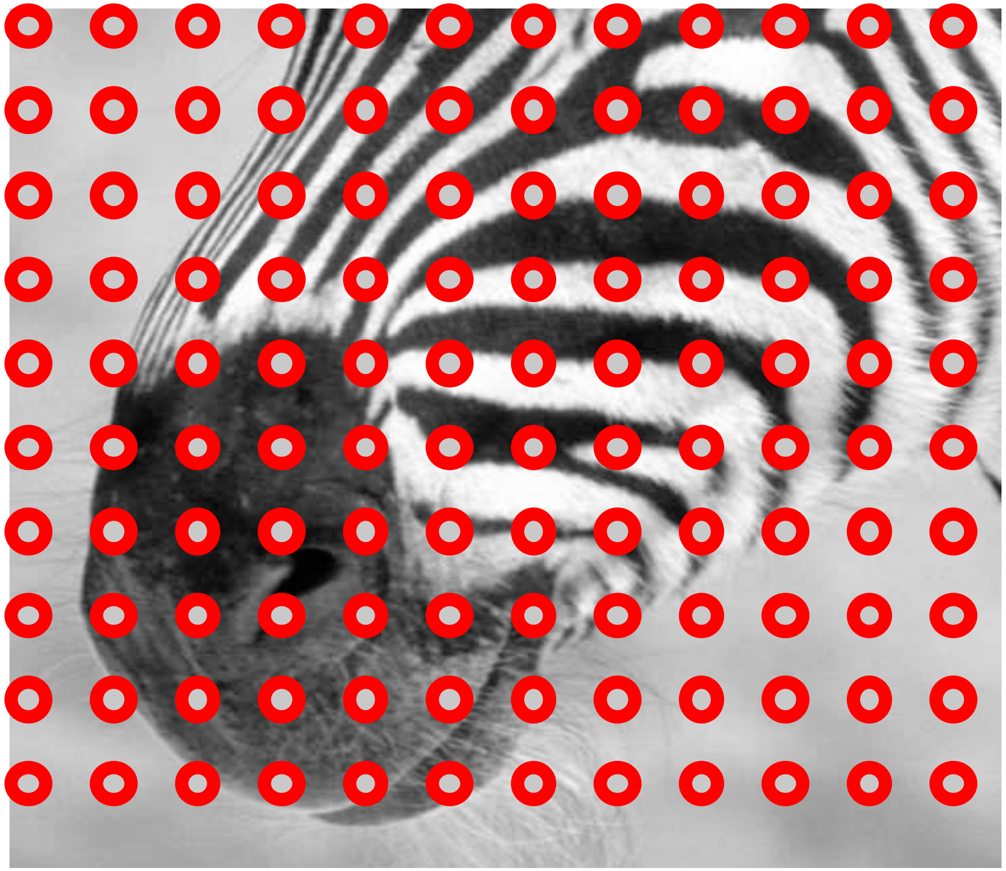


Slide credit: S. Narasimhan

Subsampling by a factor of 2

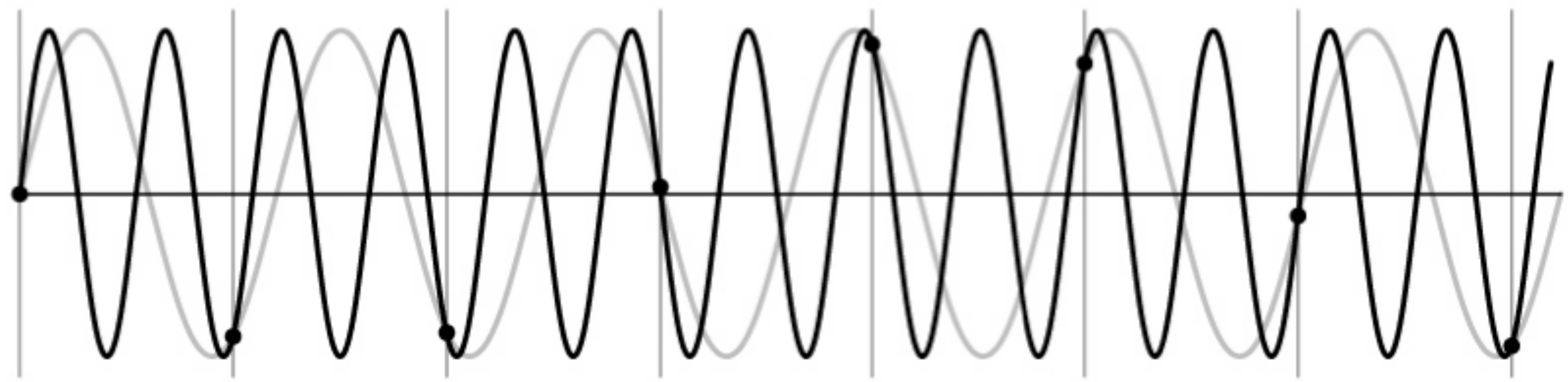


Throw away every other row and column
to create a 1/2 size image

Slide credit: D. Hoiem

Undersampling

- What if we "missed" things between the samples?
- Simple example: undersampling a sine wave
  - unsurprising result: information is lost
  - surprising result: indistinguishable from lower frequency
  - also was always indistinguishable from higher frequencies
  - *aliasing*: signals "traveling in disguise" as other frequencies



Slide credit: S. Marschner

Aliasing problem

- Sub-sampling may be dangerous….
- Characteristic errors may appear:
  - "Wagon wheels rolling the wrong way in movies"
  - "Checkerboards disintegrate in ray tracing"
  - "Striped shirts look funny on color television"

Slide credit: D. Forsyth

Moire patterns in real-world images. Here are comparison images by Dave Etchells of [Imaging Resource] using the Canon D60 (with an antialias filter) and the Sigma SD-9 (which has no antialias filter). The bands below the fur in the image at right are the kinds of artifacts that appear in images when no antialias filter is used. Sigma chose to eliminate the filter to get more sharpness, but the resulting apparent detail may or may not reflect features in the image.

Slide credit: N. Kumar

More examples

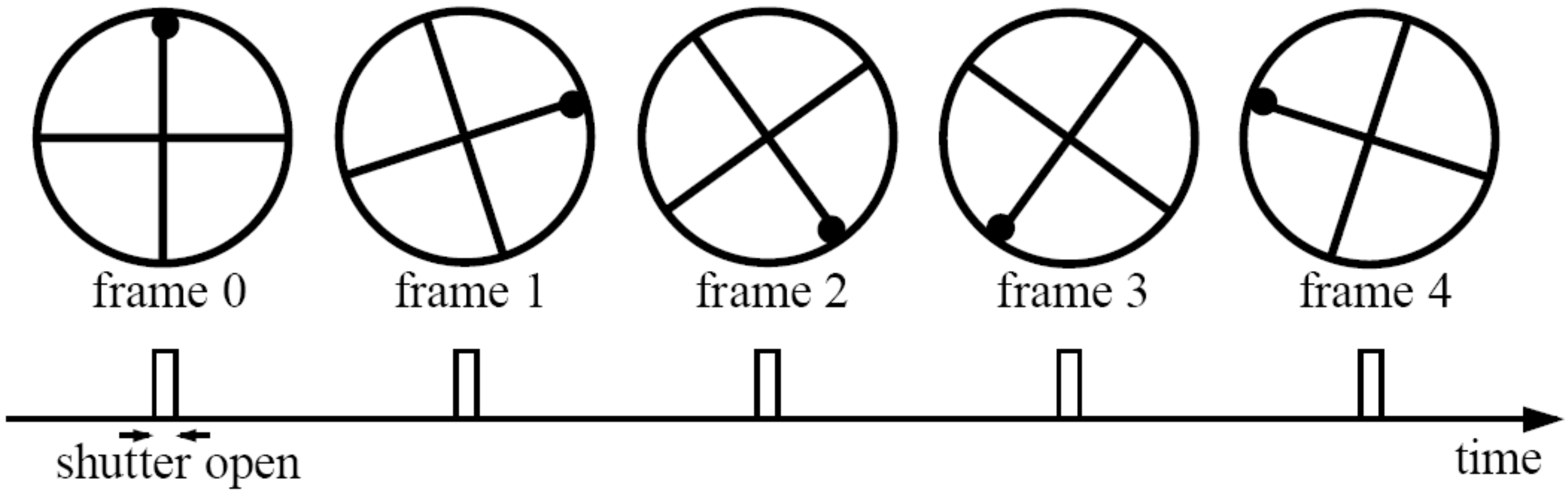
Check out Moire patterns
on the web.

Slide credit: A. Farhadi

Aliasing in video

Imagine a spoked wheel moving to the right (rotating clockwise). Mark wheel with dot so we can see what's happening.

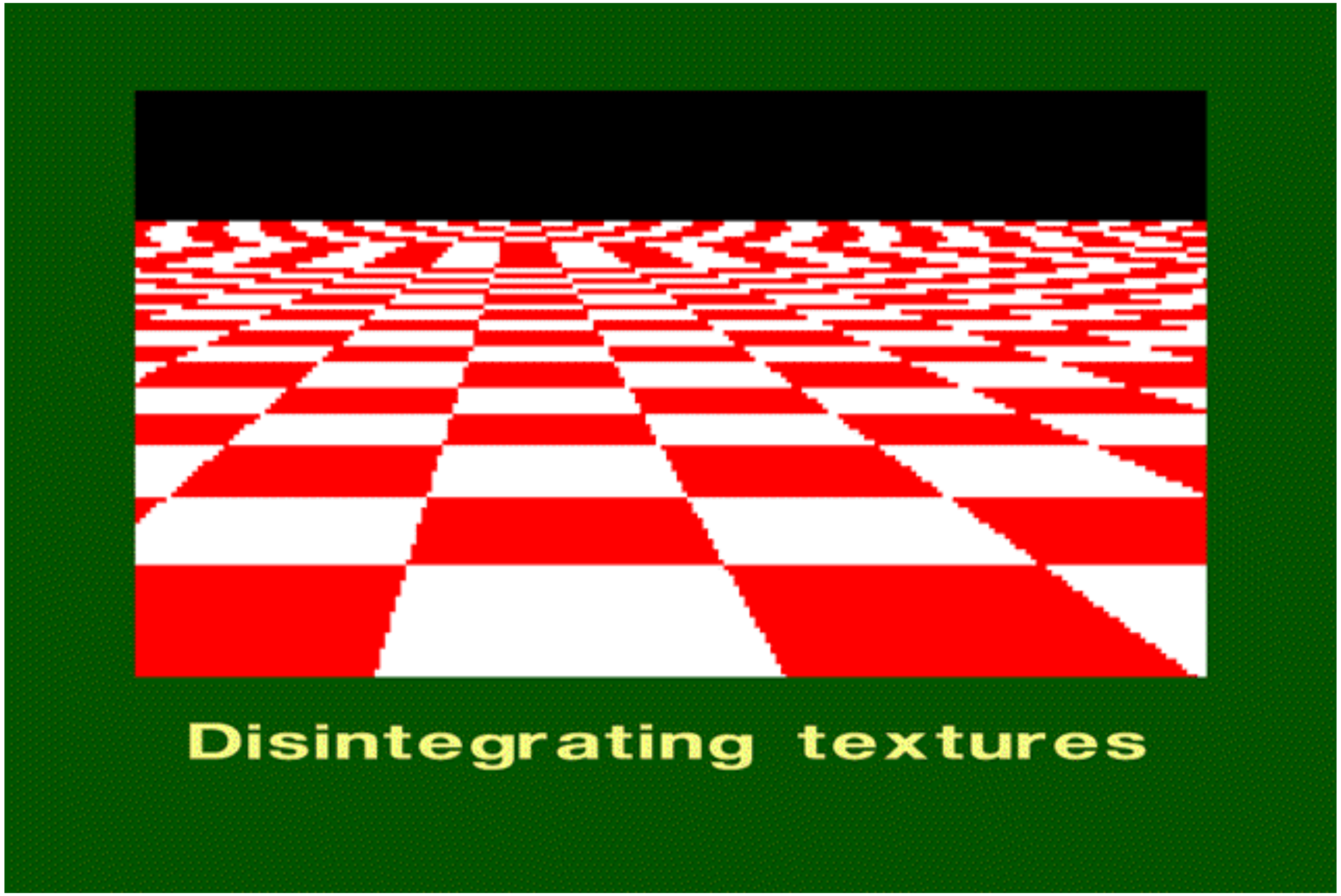
If camera shutter is only open for a fraction of a frame time (frame time = 1/30 sec. for video, 1/24 sec. for film):



Without dot, wheel appears to be rotating slowly backwards! (counterclockwise)

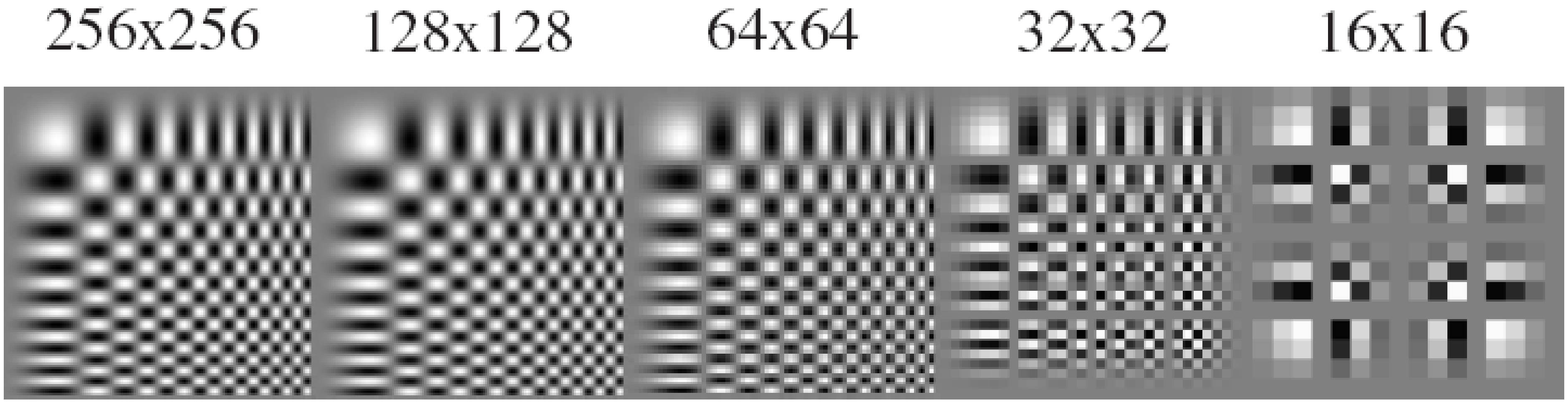
Slide credit: S. Seitz

Aliasing in graphics



Slide credit: A. Efros

Sampling and aliasing



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Sampling Theorem

Sampled function:

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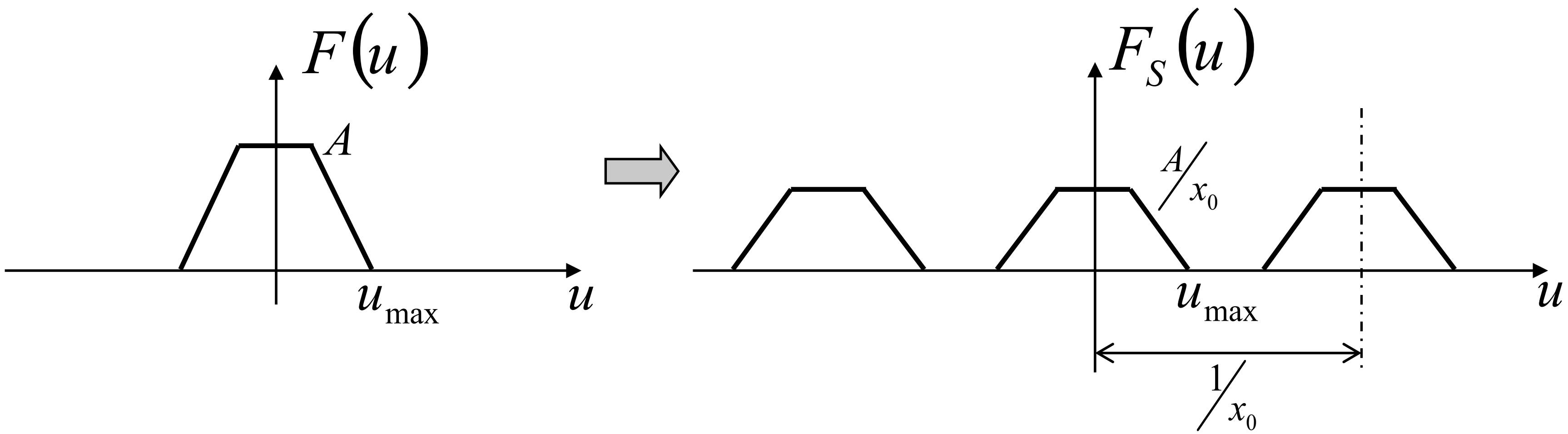
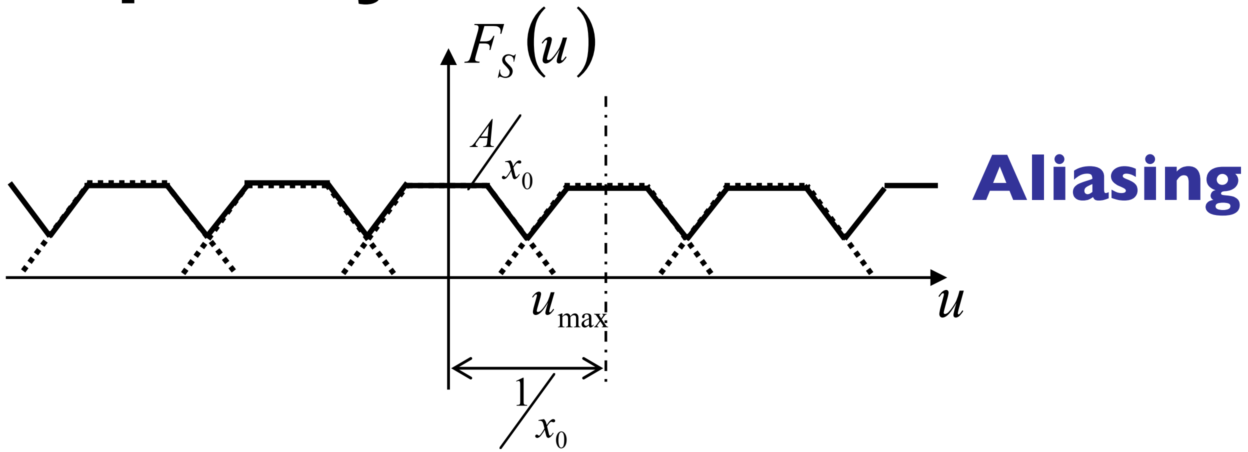


Slide credit: S. Narasimhan

Nyquist Frequency

If $u_{max} > \frac{1}{2x_0}$



When can we recover $F(u)$ from $F_S(u)$?

> Only if $u_{max} \leq \frac{1}{2x_0}$ (Nyquist Frequency)

We can use

$$C(u) = \begin{cases} x_0 & |u| < 1/2x_0 \\ 0 & \text{otherwise} \end{cases}$$

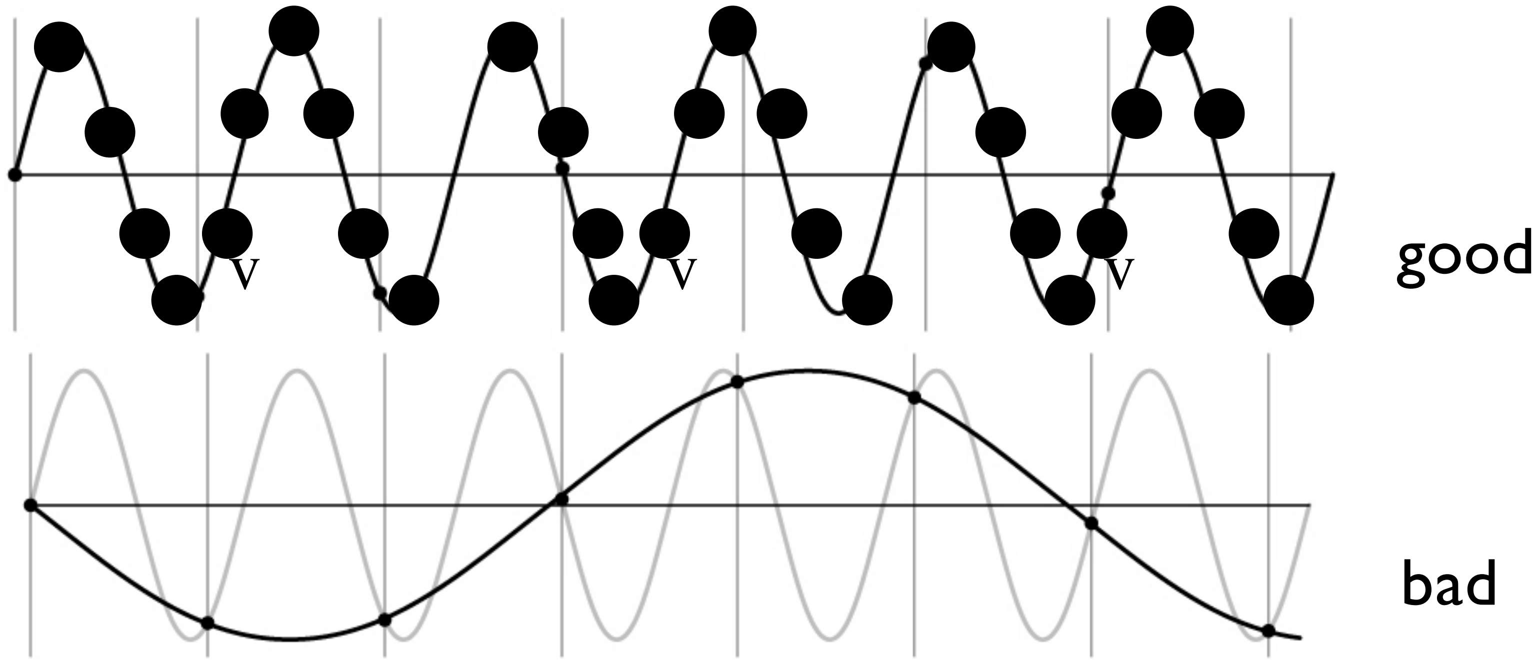
Then $F(u) = F_S(u)C(u)$ and $f(x) = \mathrm{IFT}[F(u)]$

> Sampling frequency must be greater than $2u_{max}$

Slide credit: S. Narasimhan

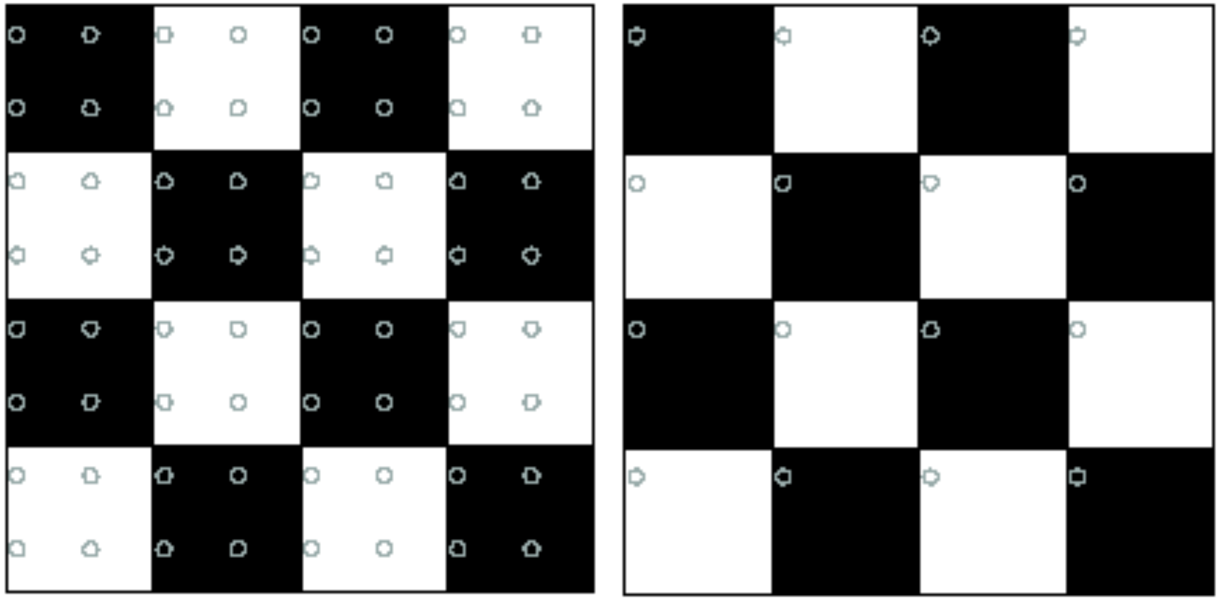
Nyquist-Shannon Sampling Theorem

- When sampling a signal at discrete intervals, the sampling frequency must be $\geq 2 \times f_{max}$
- f_{max} = max frequency of the input signal
- This will allows to reconstruct the original perfectly from the sampled version

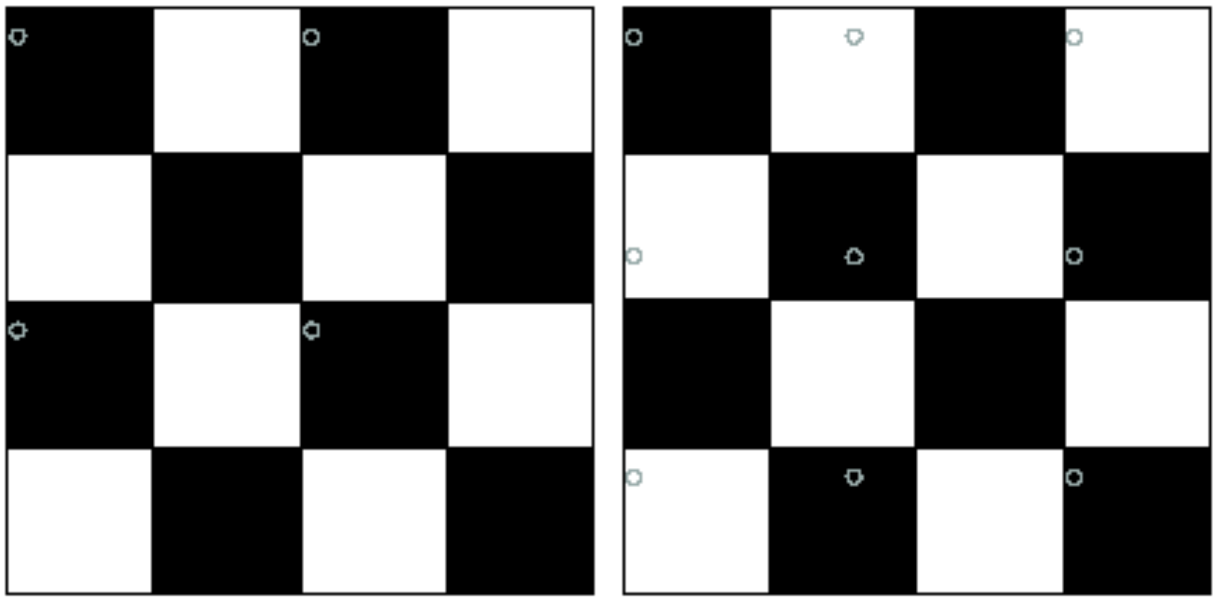


Slide credit: D. Hoiem

2D example



Good sampling



Bad sampling

Slide credit: N. Kumar

Anti-aliasing

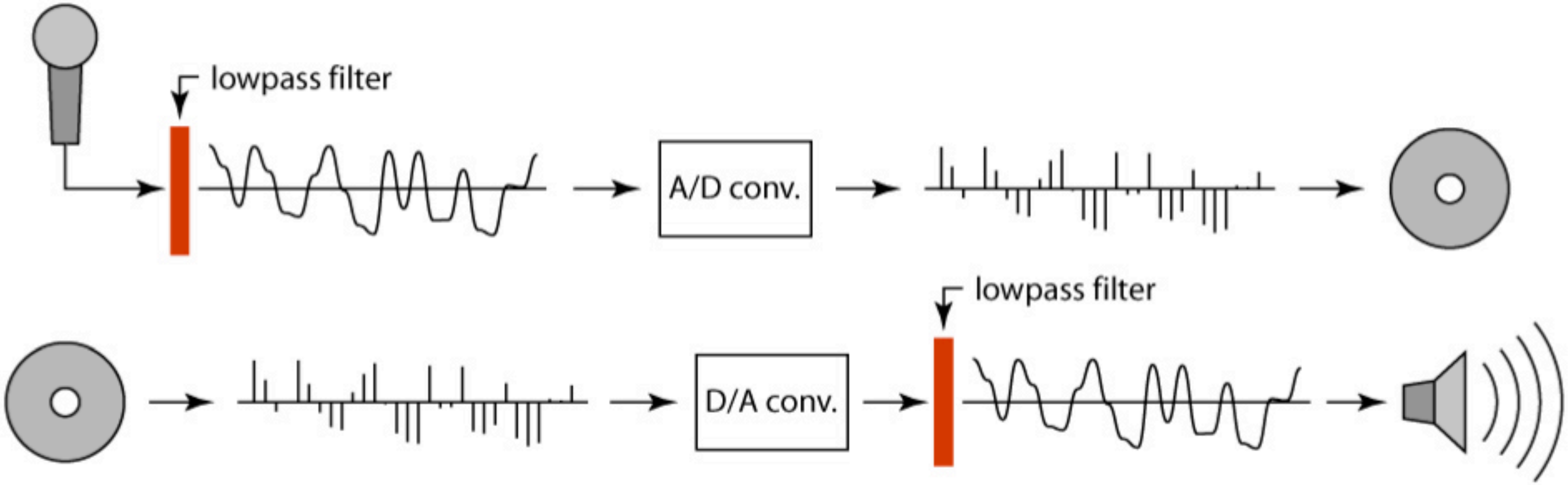
Solutions:

- Sample more often
- Get rid of all frequencies that are greater than half the new sampling frequency
  - Will lose information
  - But it's better than aliasing
  - Apply a smoothing filter

Slide credit: D. Hoiem

Preventing aliasing

- Introduce lowpass filters:
  - remove high frequencies leaving only safe, low frequencies
  - choose lowest frequency in reconstruction (disambiguate)



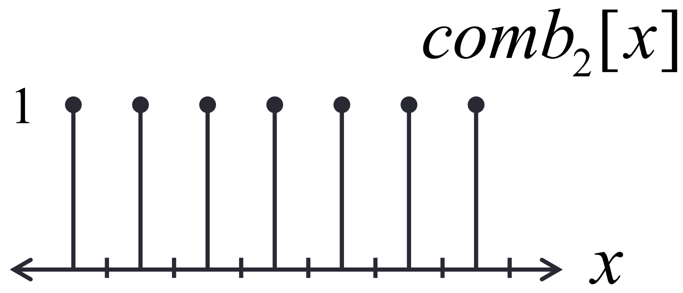
Slide credit: S. Marschner

Impulse Train

- Define a *comb* function (impulse train) in 1D as follows

$$comb_M[x] = \sum_{k=-\infty}^{\infty} \delta[x - kM]$$

where *M* is an integer



Slide credit: B. K. Gunturk

Impulse Train in 2D *(bed of nails)*

$$comb_{M,N}(x,y) \triangleq \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \delta\left(x-kM,\, y-lN\right)$$

- Fourier Transform of an impulse train is also an impulse train:

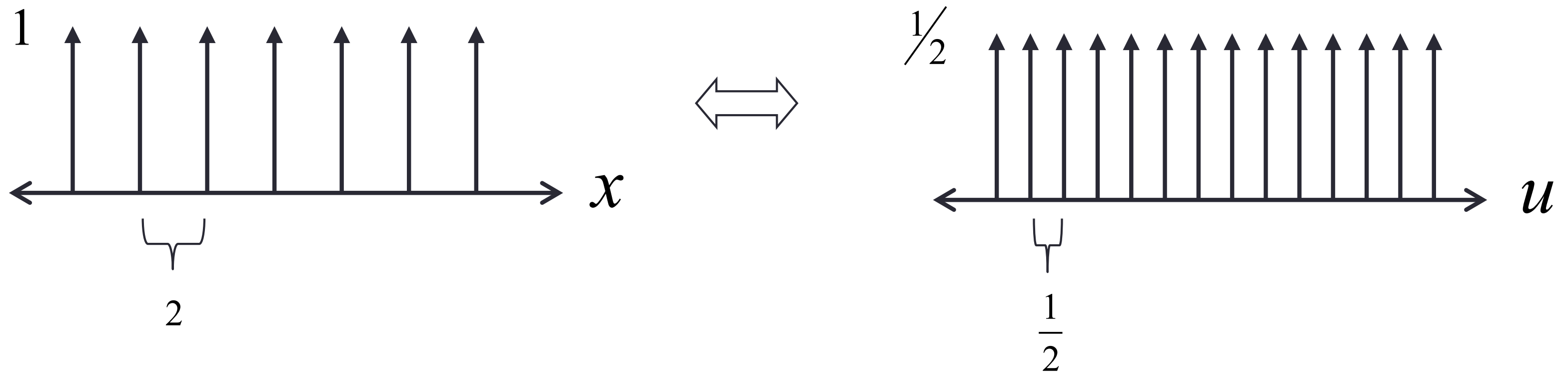
$$\underbrace{\sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \delta\left(x-kM,\, y-lN\right)}_{comb_{M,N}(x,y)} \Leftrightarrow \frac{1}{MN} \underbrace{\sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \delta\left(u-\frac{k}{M},\, v-\frac{l}{N}\right)}_{comb_{\frac{1}{M},\frac{1}{N}}(u,v)}$$

As the comb samples get further apart,
the spectrum samples get closer together!

Slide credit: B. K. Gunturk

Impulse Train in 1D

$$comb_2(x) \qquad \Longleftrightarrow \qquad \frac{1}{2}comb_{\frac{1}{2}}(u)$$

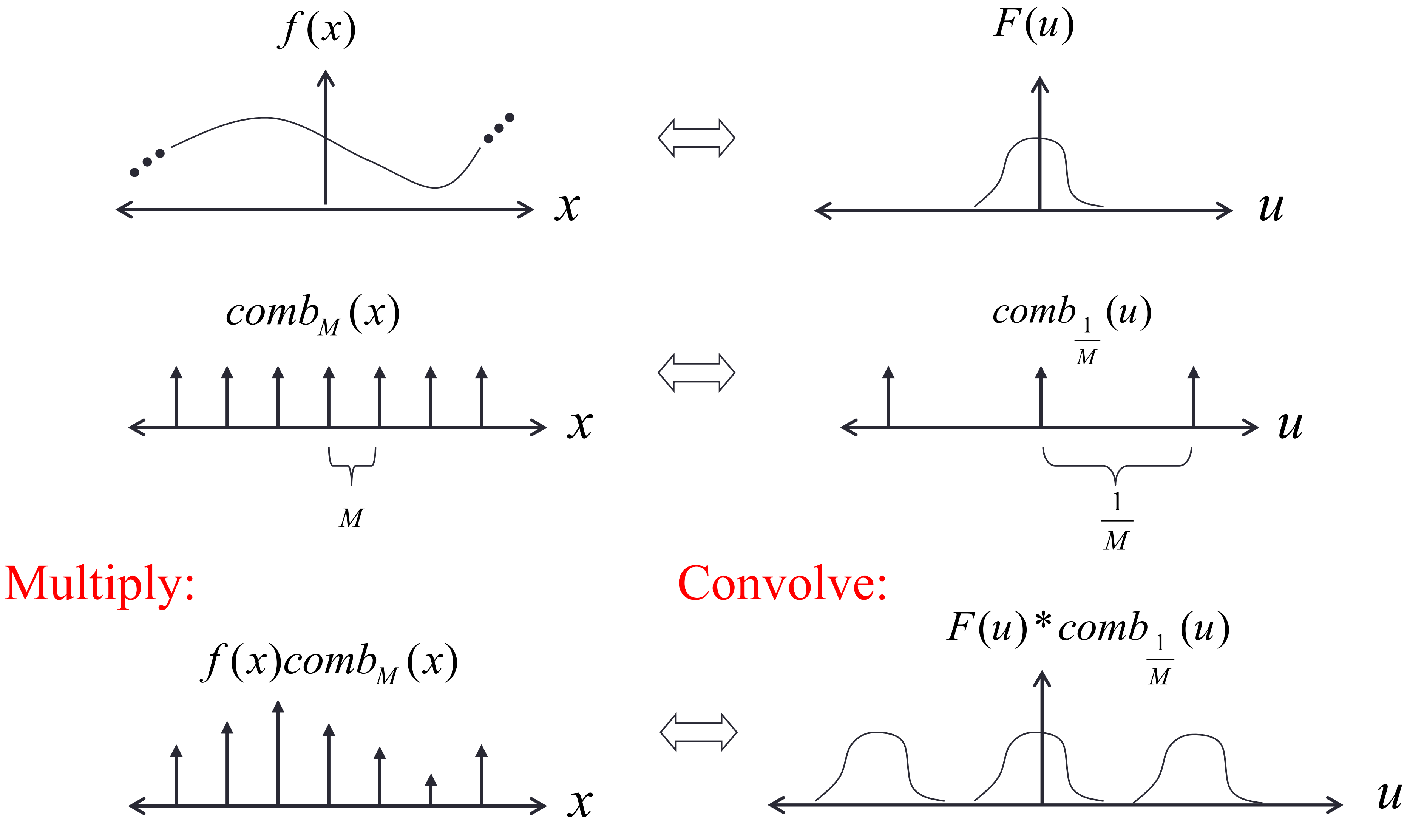


- *Remember:*

Scaling $\quad f(ax) \qquad \frac{1}{|a|}F\left(\frac{u}{a}\right)$

Slide credit: B. K. Gunturk

Sampling low frequency signal



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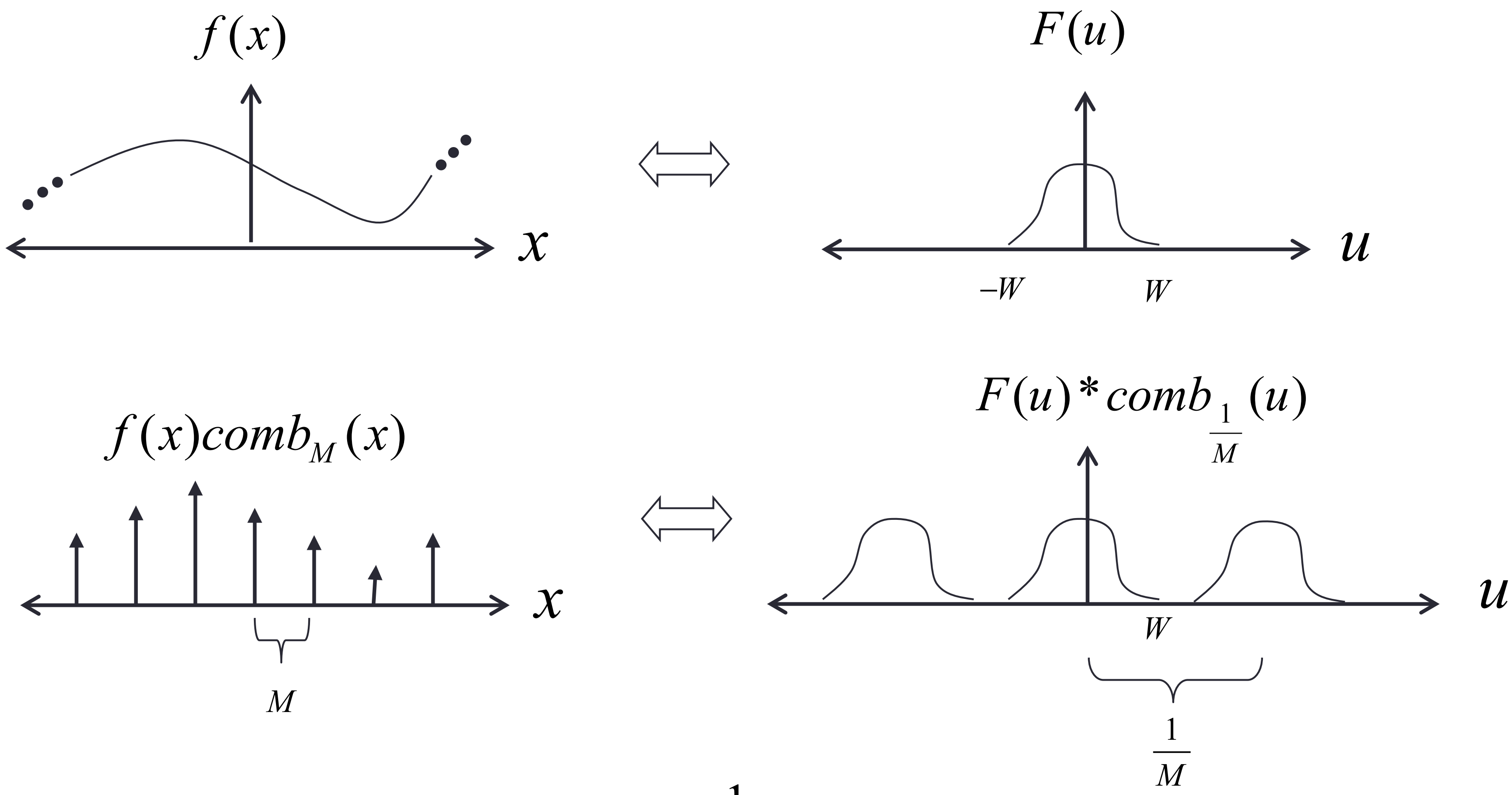
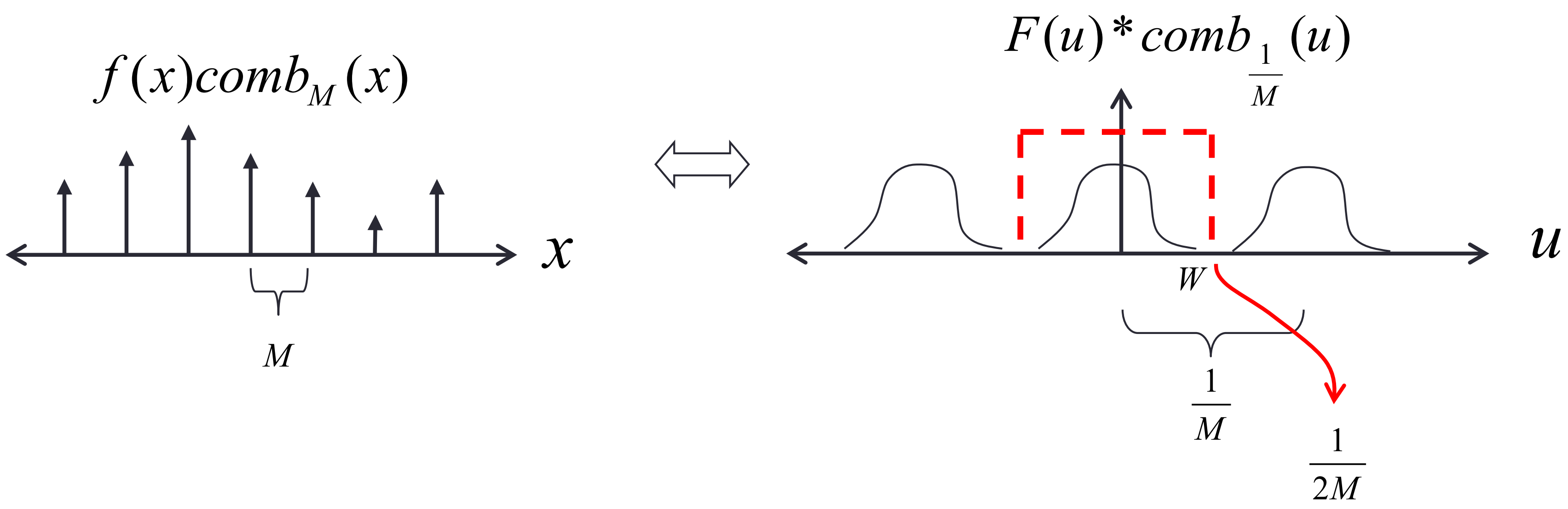


"No problem" if $\frac{1}{M} > 2W$

Slide credit: B. K. Gunturk

Sampling low frequency signal

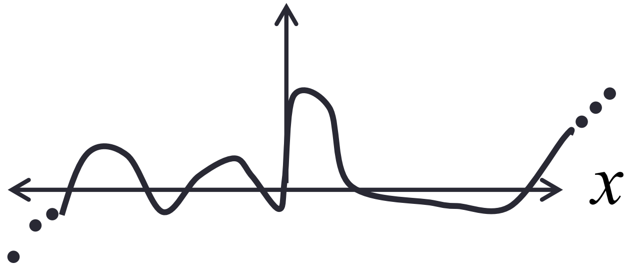


If there is no overlap, the original signal can be recovered from its samples by low-pass filtering.

Slide credit: B. K. Gunturk

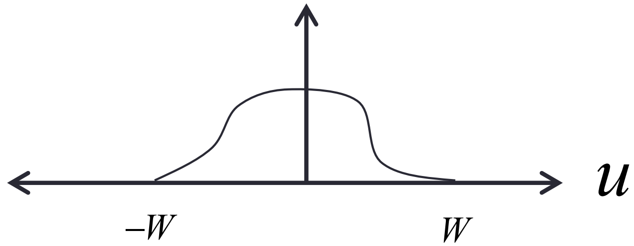
Sampling high frequency signal

$f(x)$



$\Leftrightarrow$

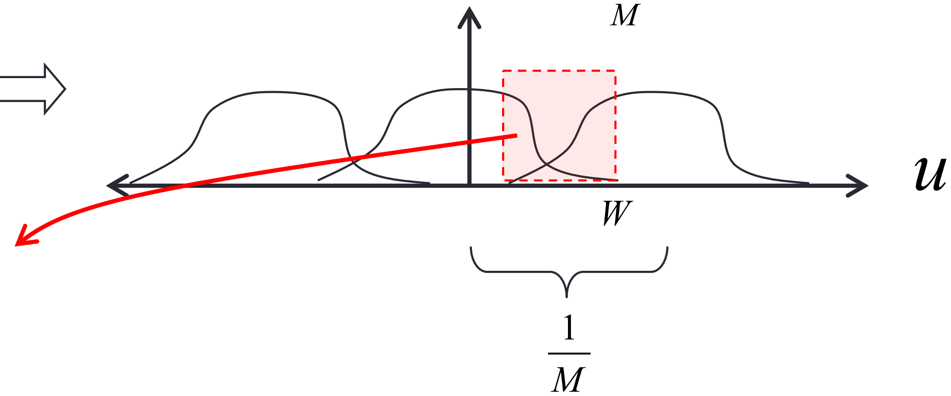
$F(u)$



$f(x)comb_M(x)$

$\Leftrightarrow$

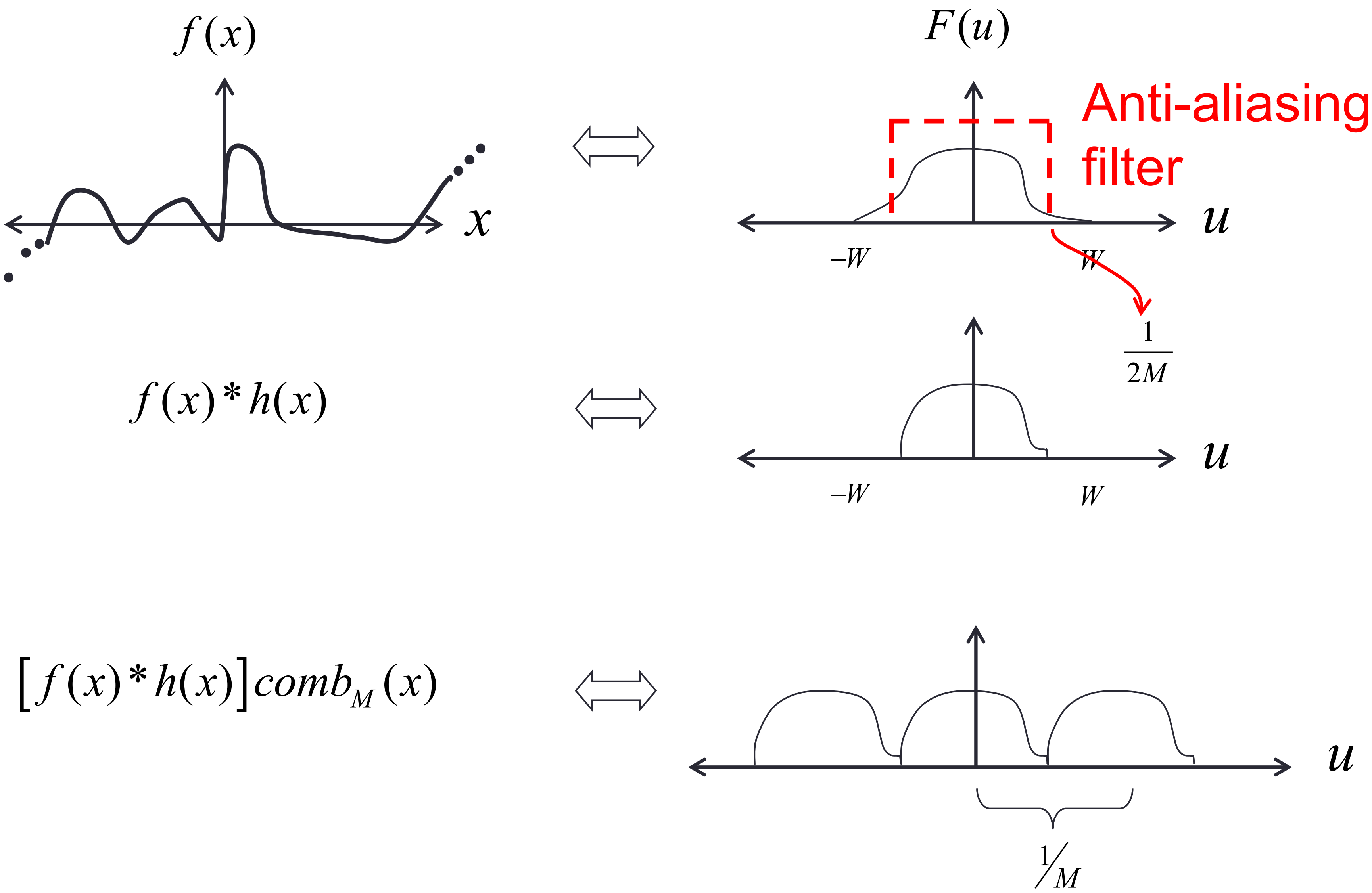
$F(u) * comb_{\frac{1}{M}}(u)$



Overlap: The high frequency energy is folded over into low frequency. It is “aliasing” as lower frequency energy. And you cannot fix it once it has happened.

Slide credit: B. K. Gunturk

Sampling high frequency signal

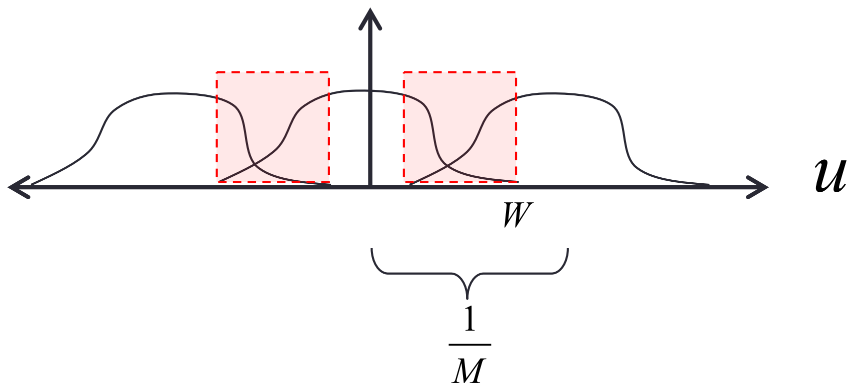


Slide credit: B. K. Gunturk

Sampling high frequency signal

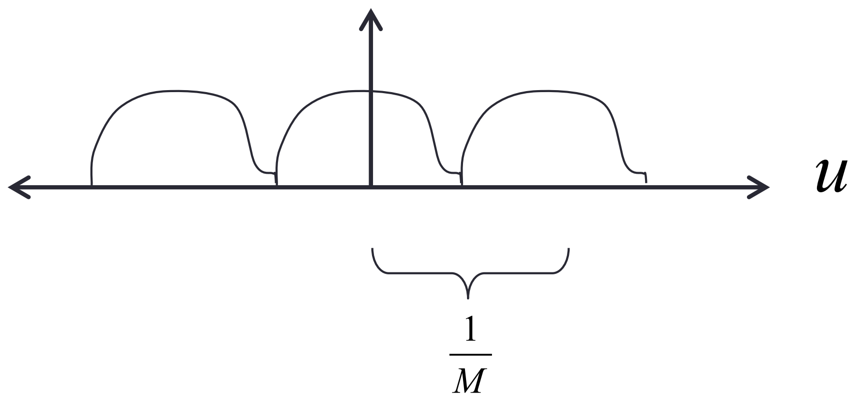
- Without anti-aliasing filter:

$$f(x)comb_M(x) \quad \Longleftrightarrow$$



- With anti-aliasing filter:

$$[f(x) * h(x)]comb_M(x) \quad \Longleftrightarrow$$



Slide credit: B. K. Gunturk

Sampling high frequency signal

- Without anti-aliasing filter:

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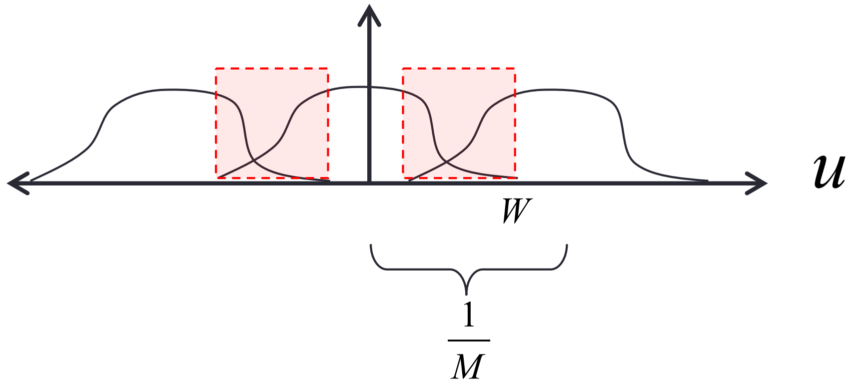
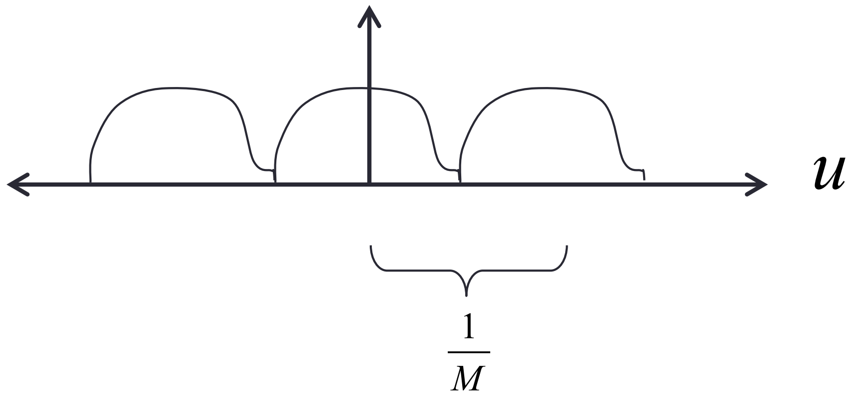


- With anti-aliasing filter:

$$[f(x) * h(x)]comb_M(x) \Leftrightarrow$$



B.K. Gunturk

Algorithm for downsampling by factor of 2

1. Start with image(h, w)
2. Apply low-pass filter

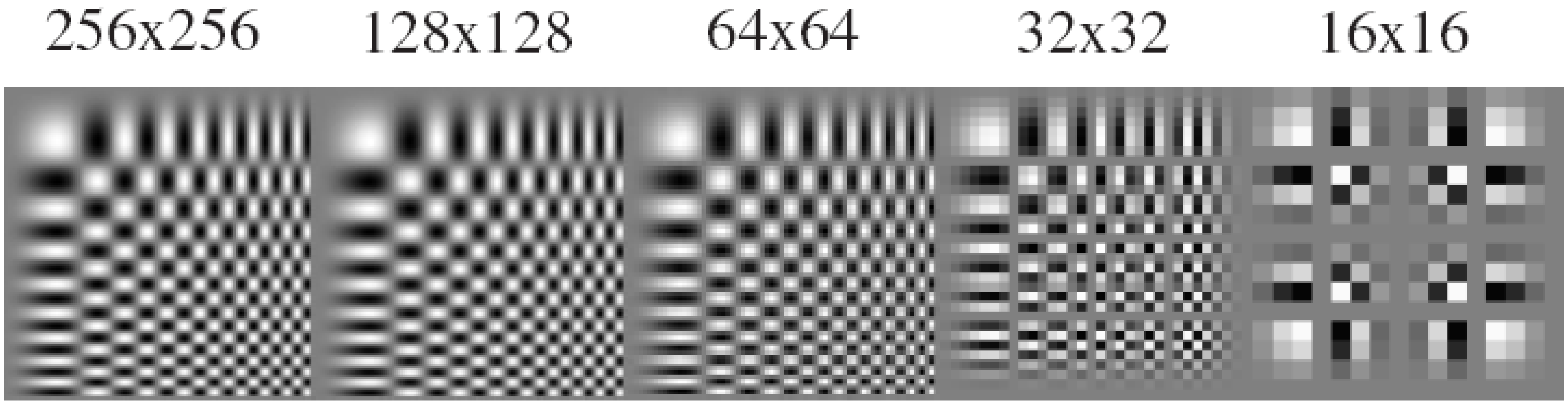
   ```
   im_blur = imfilter(image, fspecial('gaussian', 7, 1))
   ```

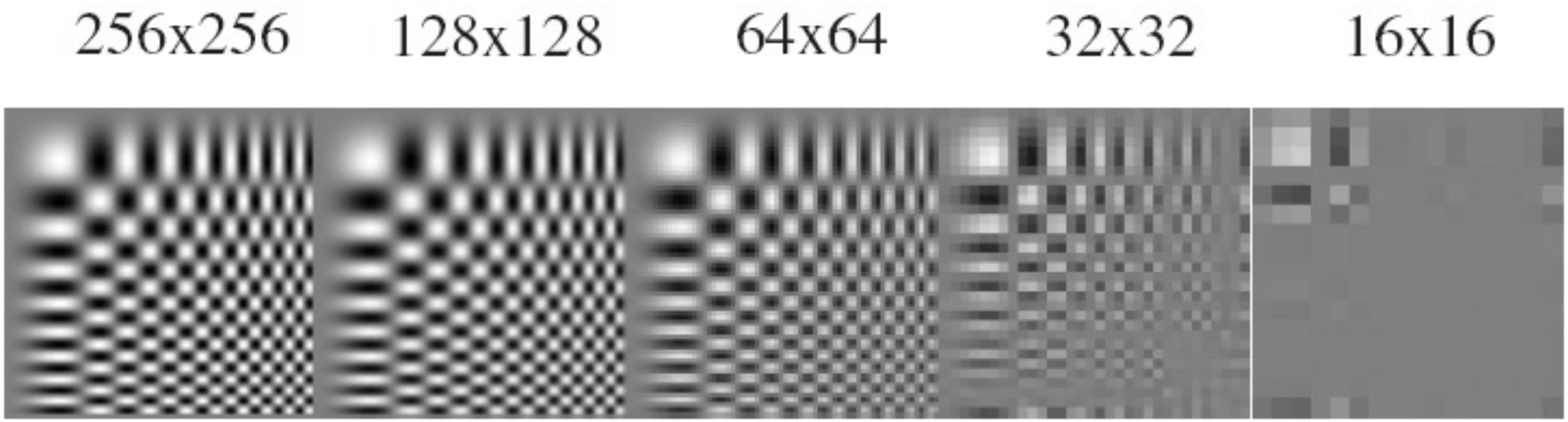
3. Sample every other pixel

   ```
   im_small = im_blur(1:2:end, 1:2:end);
   ```

Slide credit: D. Hoiem

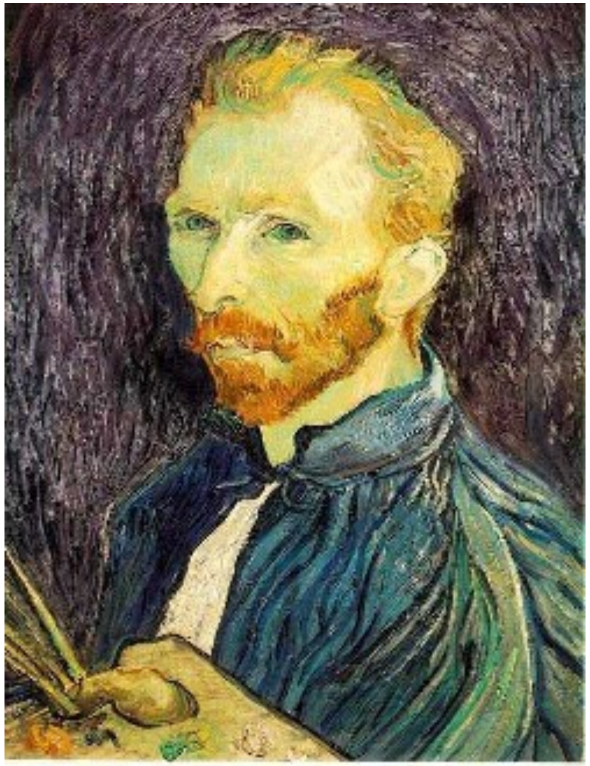
Anti-aliasing





Slide credit: Forsyth and Ponce

Subsampling without pre-filtering



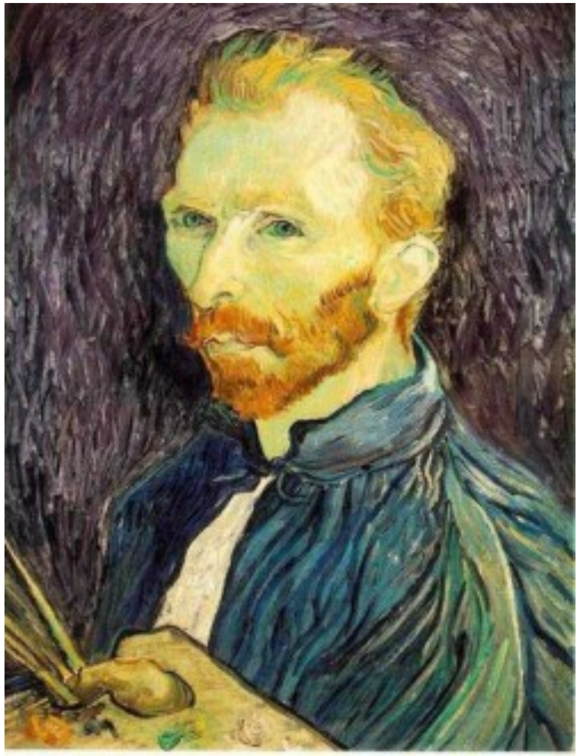
1/2

1/4 (2x zoom)

1/8 (4x zoom)

Slide credit: S. Seitz

Subsampling with Gaussian pre-filtering



Gaussian 1/2

G 1/4

G 1/8

Slide credit: S. Seitz

[Philip Greenspun]

1000 pixel width

Slide credit: S. Marschner

[Philip Greenspun]

by dropping pixels

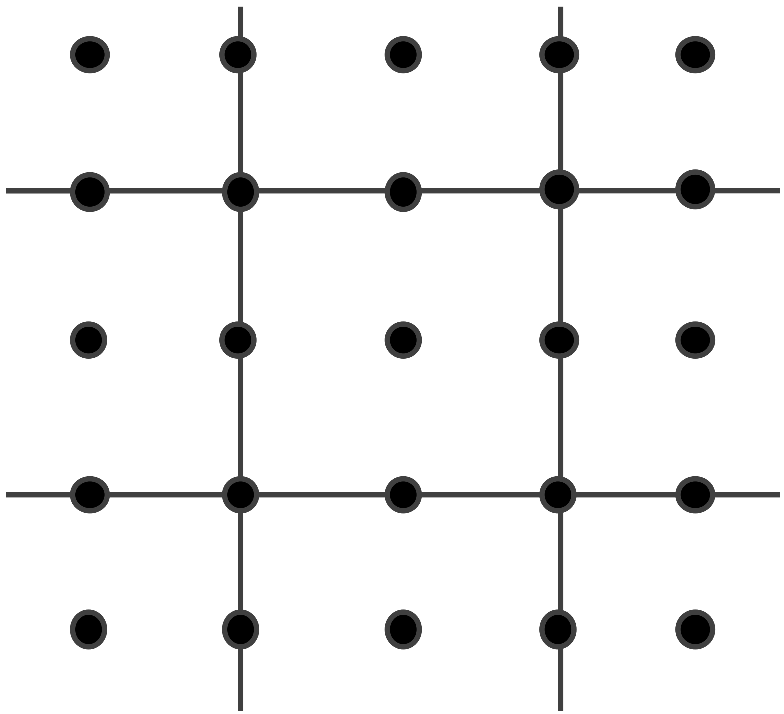
gaussian filter

250 pixel width

Slide credit: S. Marschner

Up-sampling

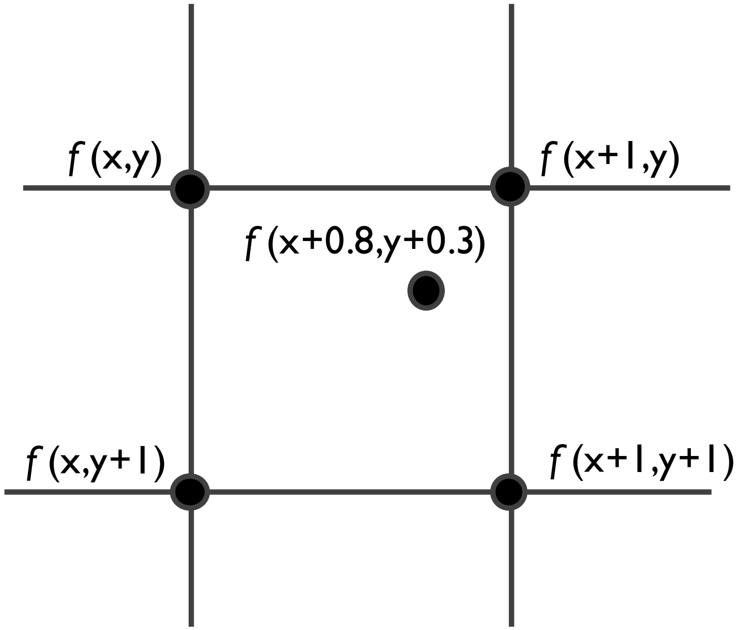
How do we compute the values of pixels at fractional positions?



Slide credit: A. Farhadi

Up-sampling

How do we compute the values of pixels at fractional positions?



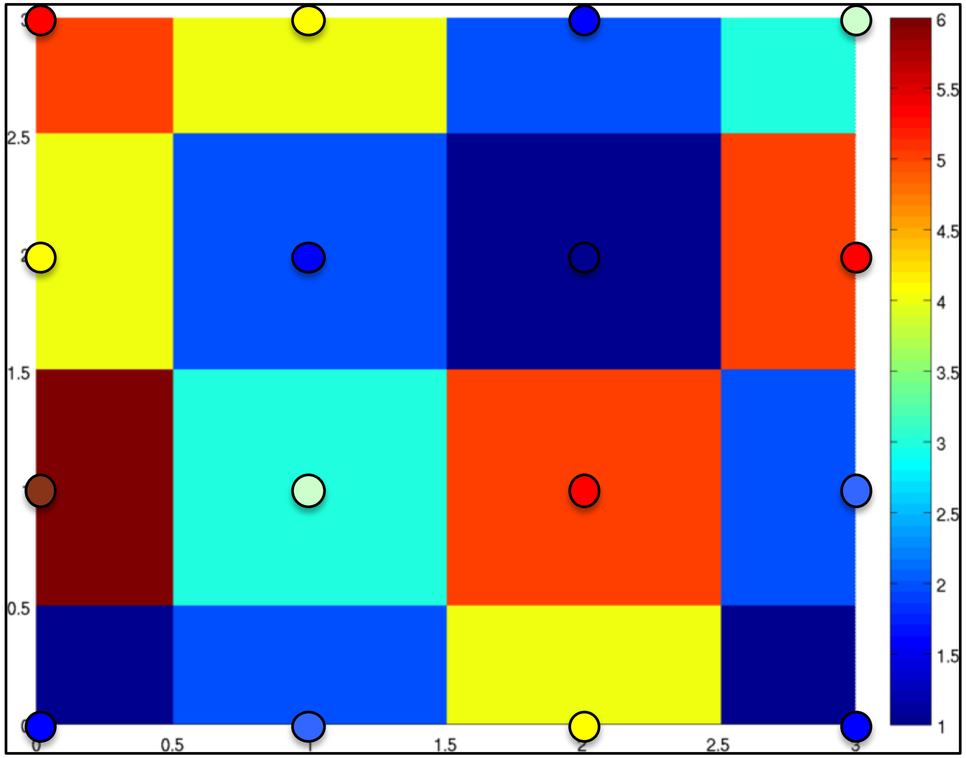
Bilinear sampling:

$$
\begin{aligned}
f(x + a, y + b) = \; & (1 - a)(1 - b)\, f(x, y) + \\
& a(1 - b)\, f(x + 1, y) + \\
& (1 - a)b\, f(x, y + 1) + \\
& ab\, f(x + 1, y + 1)
\end{aligned}
$$

Bicubic sampling fits a higher order function using a larger area of support.

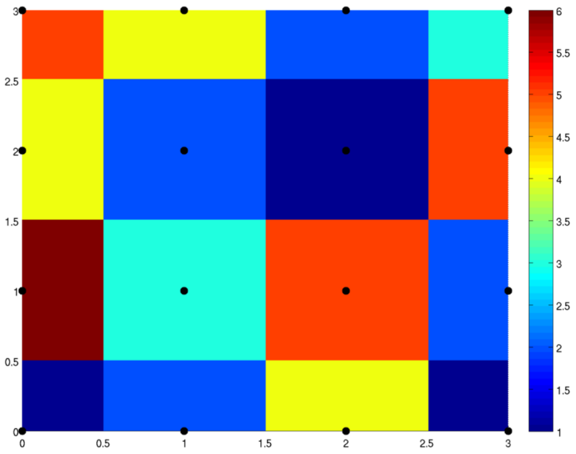
Slide credit: A. Farhadi

Up-sampling Methods

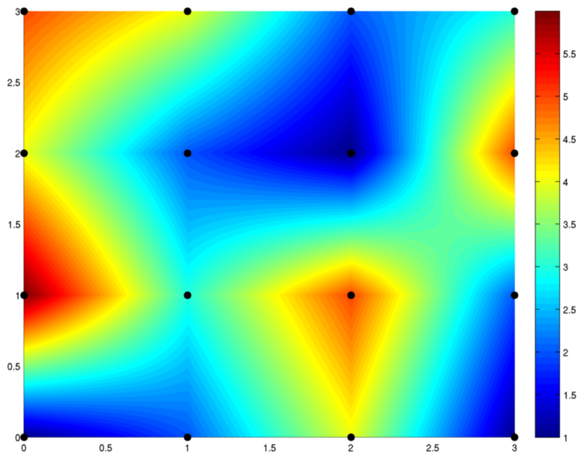


Slide credit: A. Farhadi

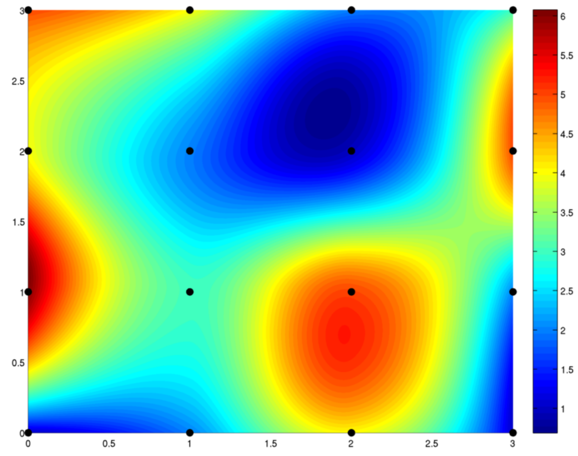
Up-sampling



Nearest neighbor



Bilinear



Bicubic

Slide credit: A. Farhadi

Up-sampling

Nearest neighbor

Bilinear

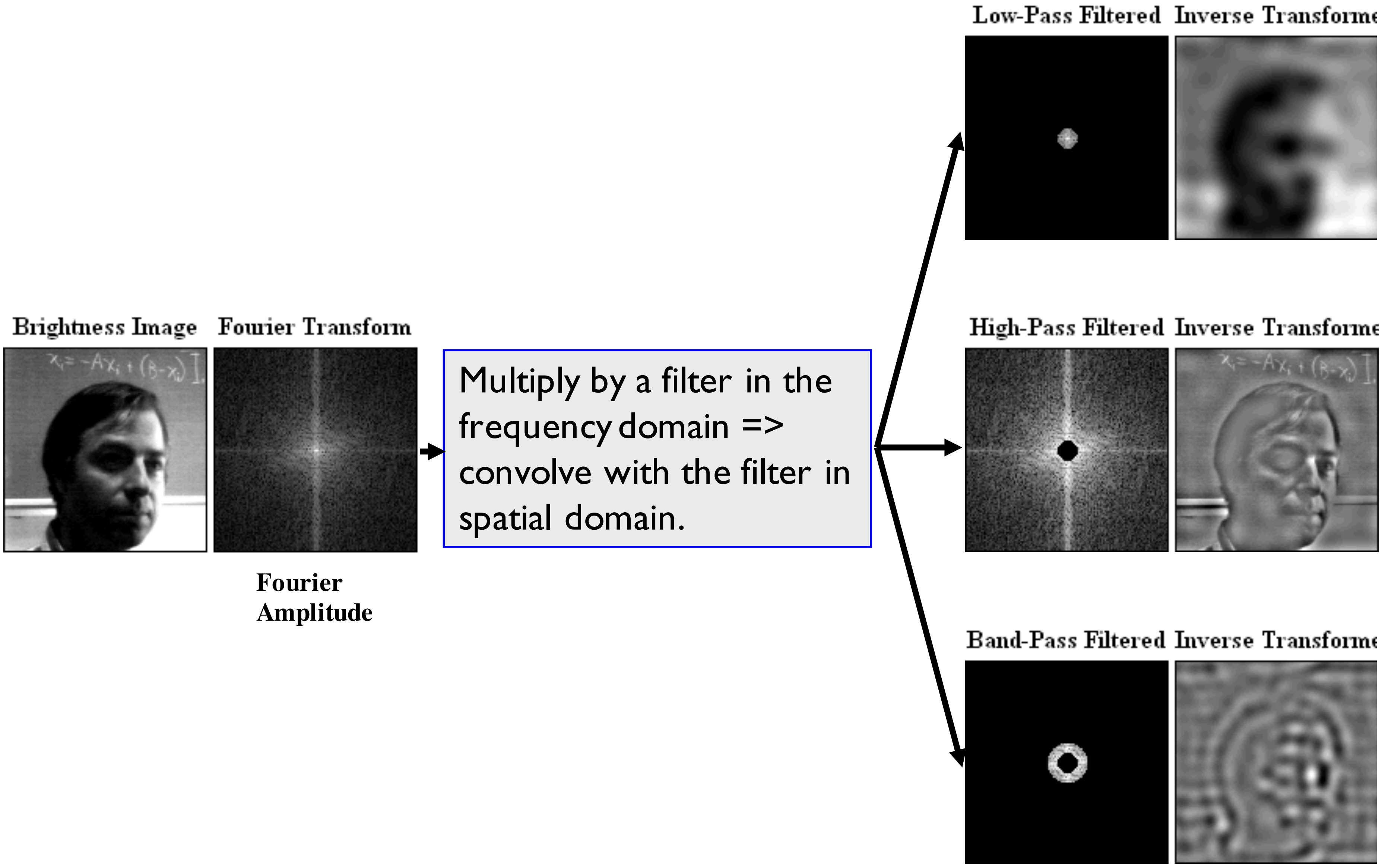
Bicubic

Slide credit: A. Farhadi

Today

- Sampling
- Gabor wavelets, Steerable filters

Fourier Filtering



Images from Steve Lehar http://cns-alumni.bu.edu/~slehar An Intuitive Explanation of Fourier Theory

Slide credit: S. Thrun

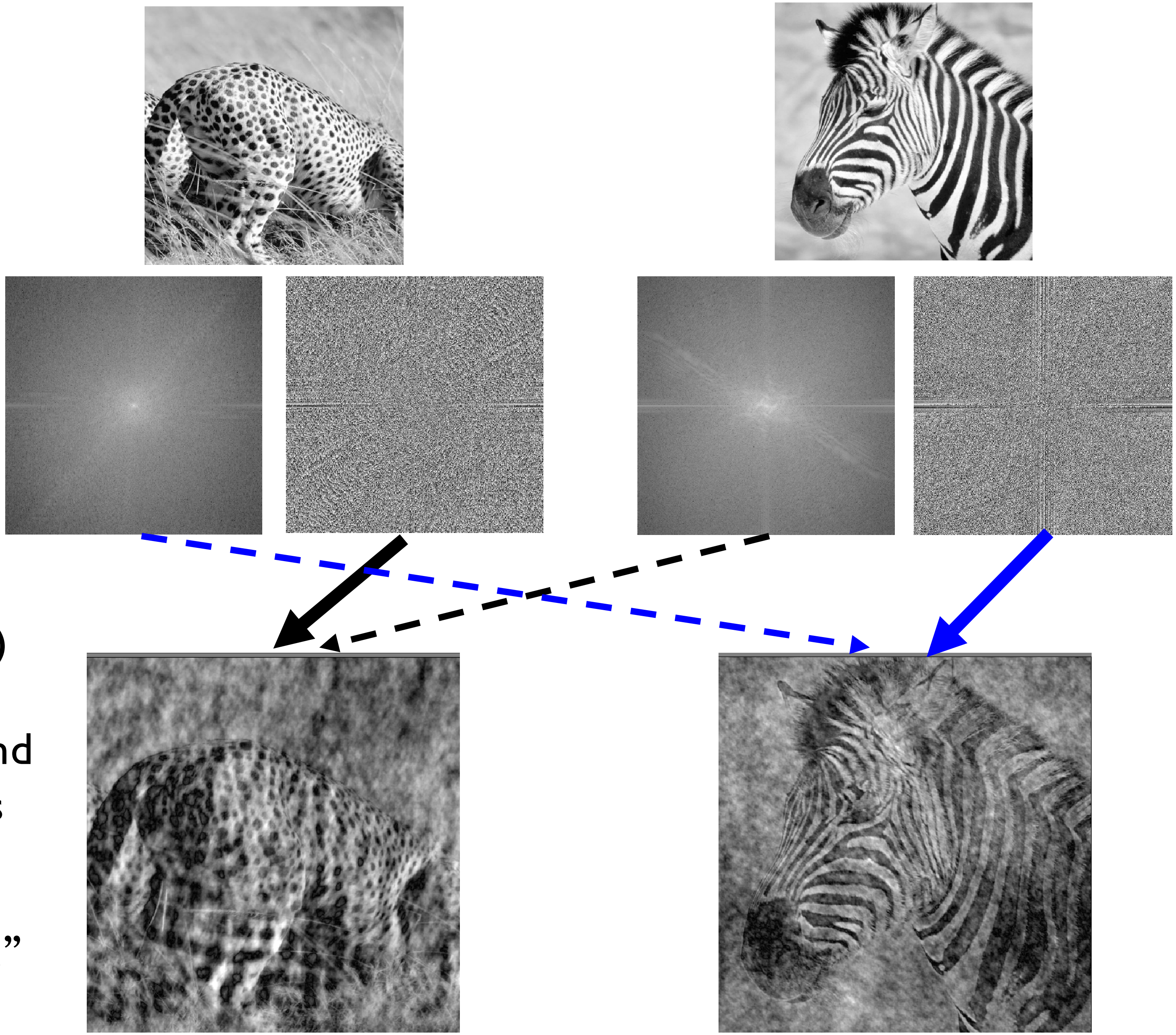
Phase Caries More Information

Raw Images:

Magnitude and Phase:

Reconstruct (inverse FFT) mixing the magnitude and phase images

Phase “Wins”



Slide credit: S. Thrun

What is a good representation for image analysis?

- Fourier transform domain tells you “what” (textural properties), but not “where”.
- Pixel domain representation tells you “where” (pixel location), but not “what”.
- Want an image representation that gives you a local description of image events—what is happening where.

Slide credit: B. Freeman and A. Torralba

Analyzing local image structures

Too much

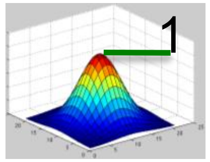
Too little

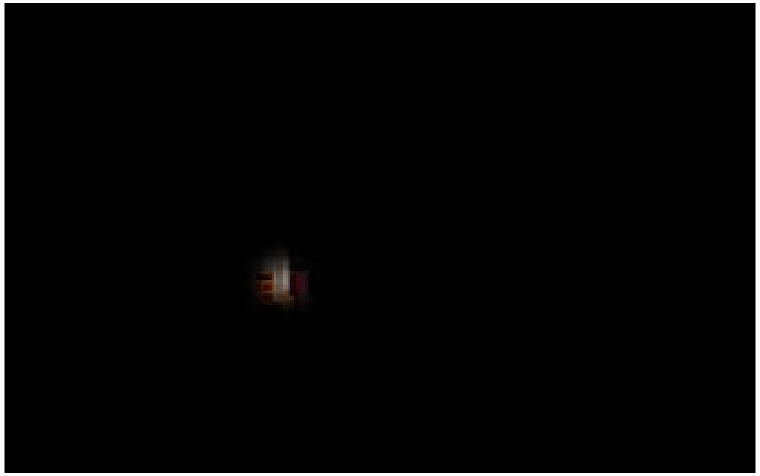
Slide credit: B. Freeman and A. Torralba

The image through the Gaussian window

Too much

$$h(x,y) = e^{-\frac{x^2+y^2}{2\sigma^2}}$$



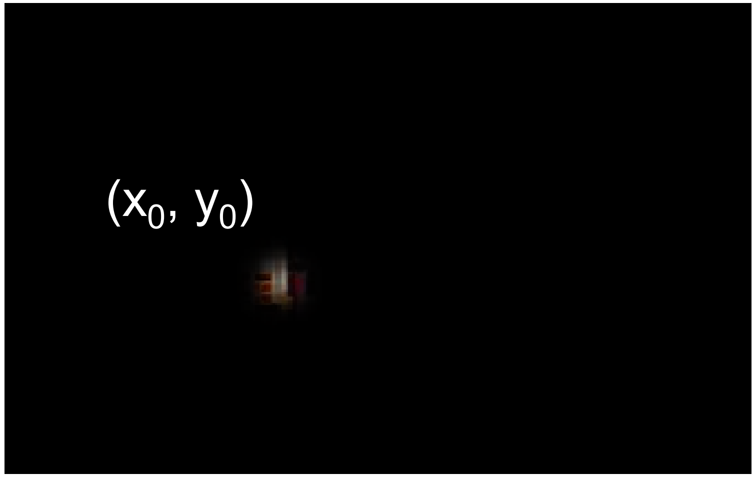


Too little

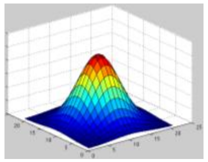
Probably still too little…
…but hard enough for now

Slide credit: B. Freeman and A. Torralba

Analysis of local frequency



$$h(x,y;x_0,y_0) = e^{-\frac{(x-x_o)^2+(y-y_o)^2}{2\sigma^2}}$$



Fourier basis:

$$e^{j2\pi u_0 x}$$

Gabor wavelet:

$$\psi(x,y) = e^{-\frac{x^2+y^2}{2\sigma^2}} e^{j2\pi u_0 x}$$

We can look at the real and imaginary parts:

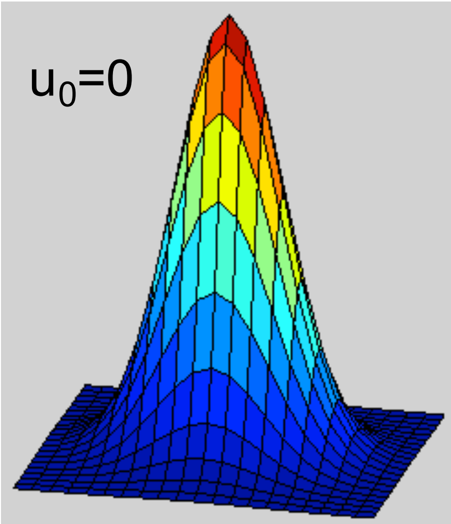
$$\psi_c(x,y) = e^{-\frac{x^2+y^2}{2\sigma^2}} \cos(2\pi u_0 x)$$

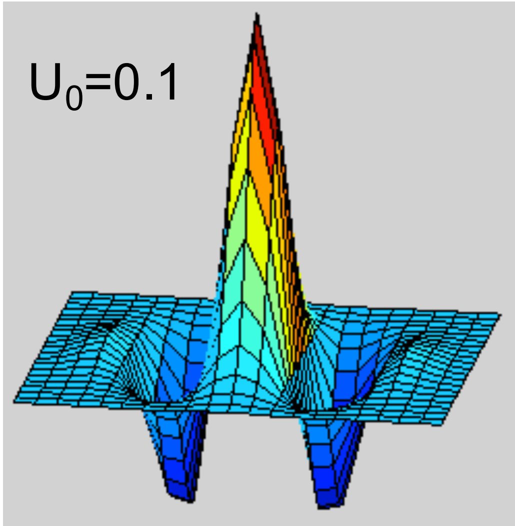
$$\psi_s(x,y) = e^{-\frac{x^2+y^2}{2\sigma^2}} \sin(2\pi u_0 x)$$

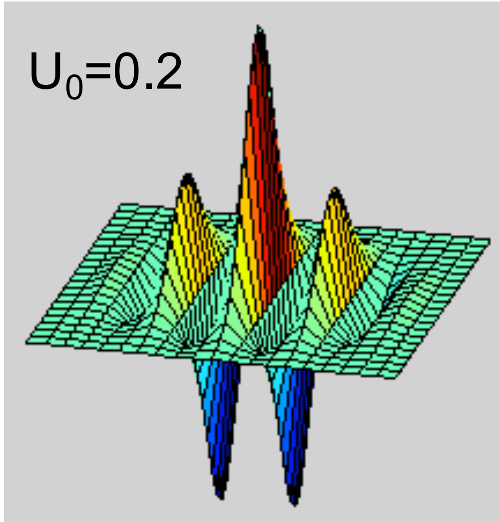
Slide credit: B. Freeman and A. Torralba

Gabor wavelets

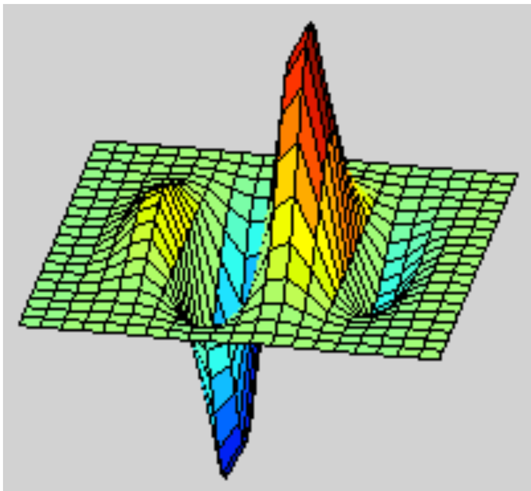
$$\psi_c(x,y) = e^{-\frac{x^2+y^2}{2\sigma^2}} \cos(2\pi u_0 x)$$

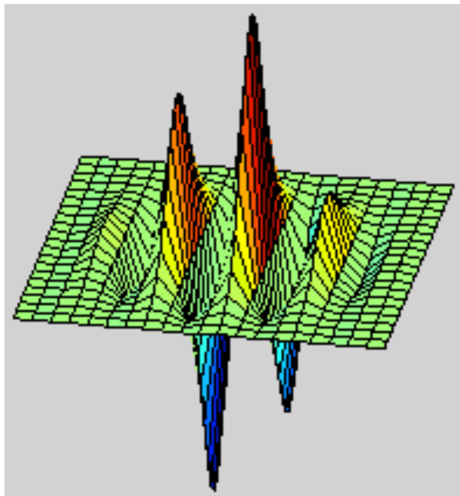






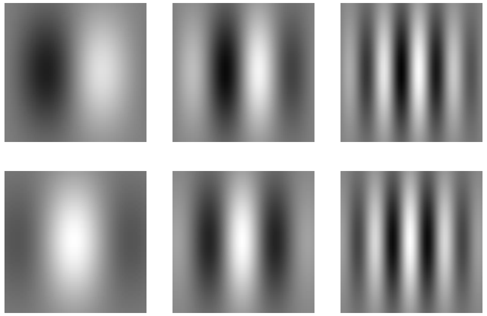
$$\psi_s(x,y) = e^{-\frac{x^2+y^2}{2\sigma^2}} \sin(2\pi u_0 x)$$



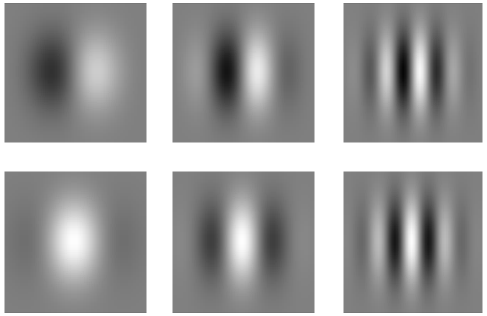


Slide credit: B. Freeman and A. Torralba

Gabor filters

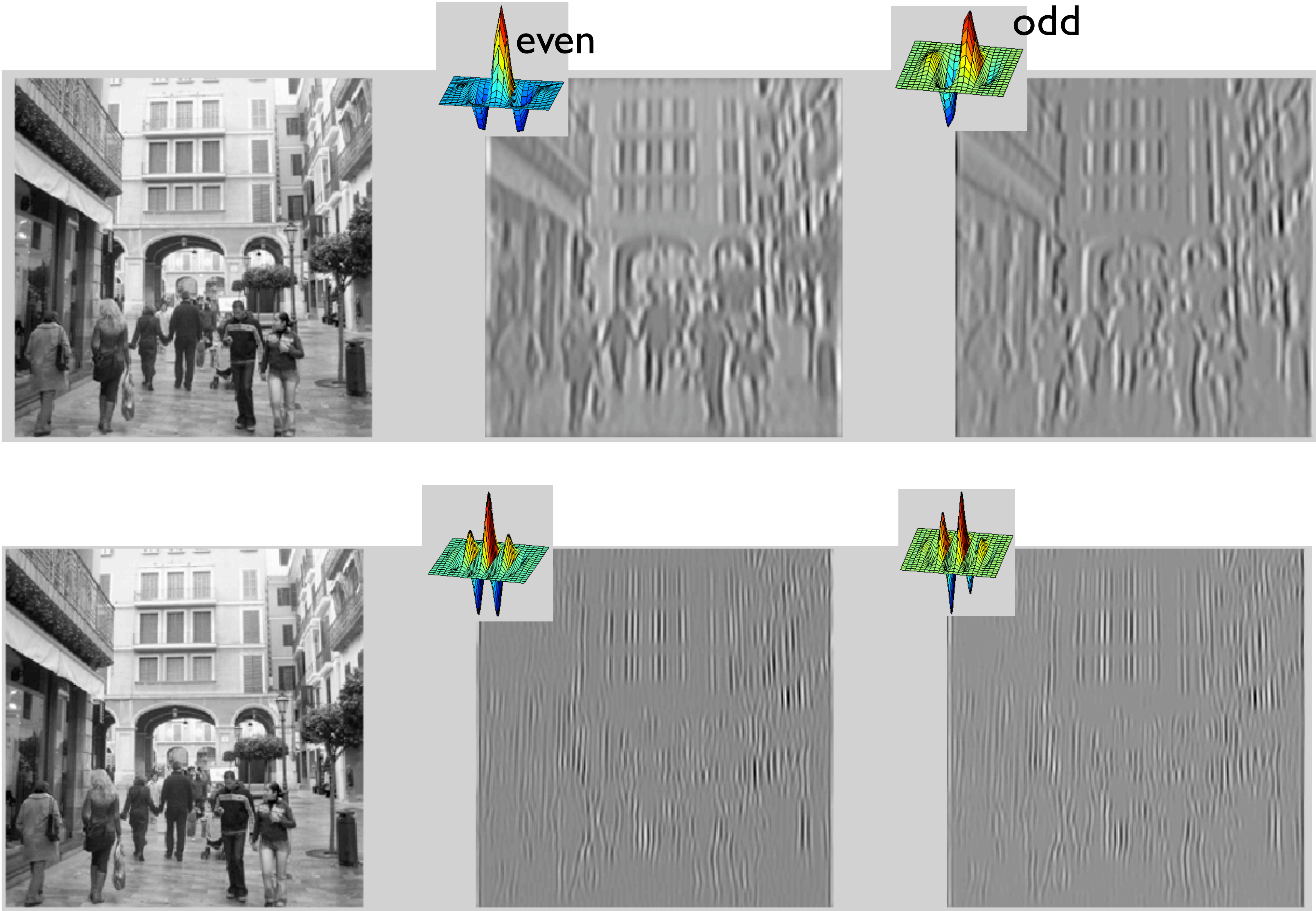


Gabor filters at different scales and spatial frequencies



Top row shows anti-symmetric (or odd) filters; these are good for detecting odd-phase structures like edges.
Bottom row shows the symmetric (or even) filters, good for detecting line phase contours.

Slide credit: B. Freeman and A. Torralba


even
odd

John Daugman, 1988

IEEE TRANSACTIONS ON ACOUSTICS, SPEECH, AND SIGNAL PROCESSING, VOL. 36, NO. 7, JULY 1988

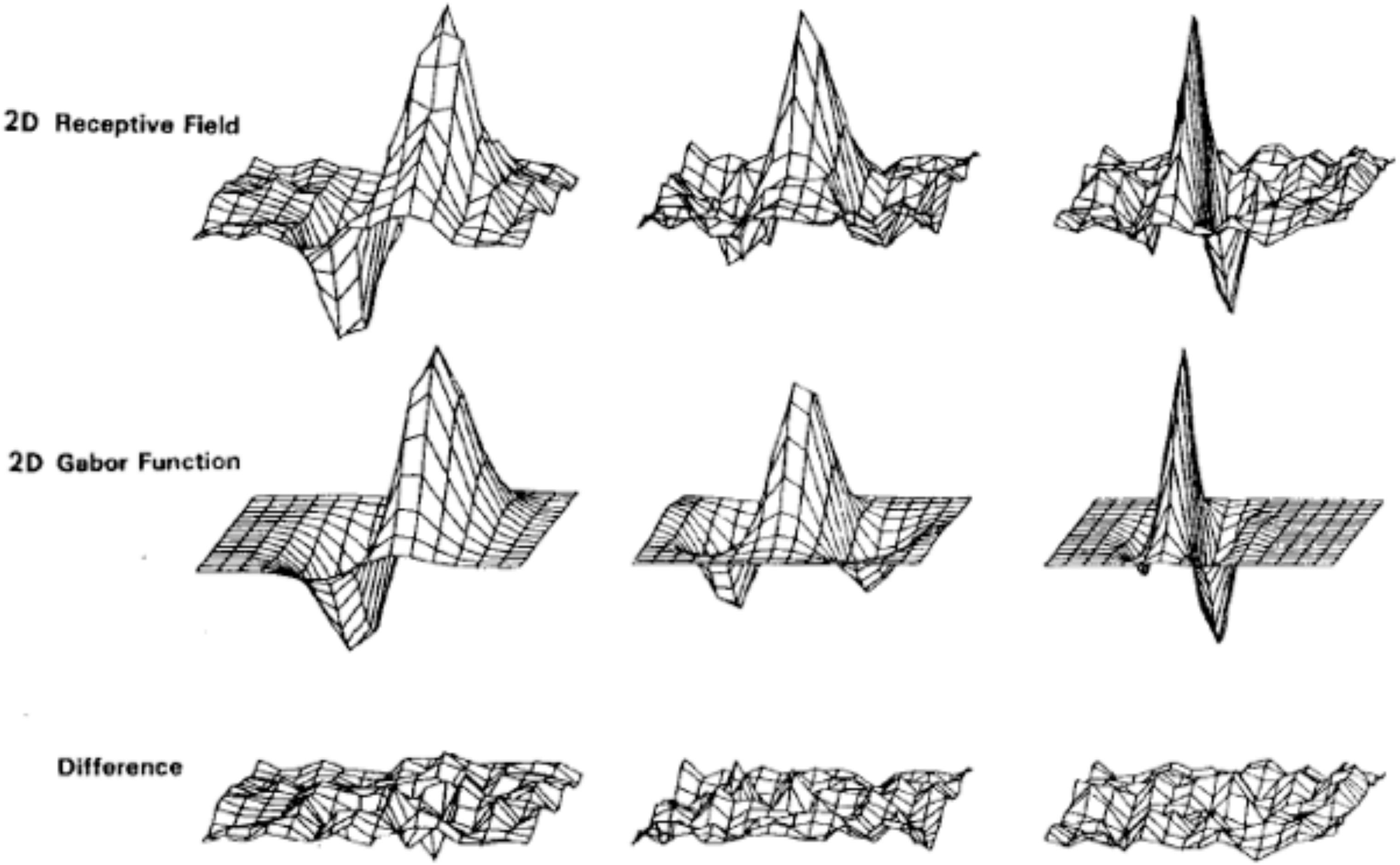


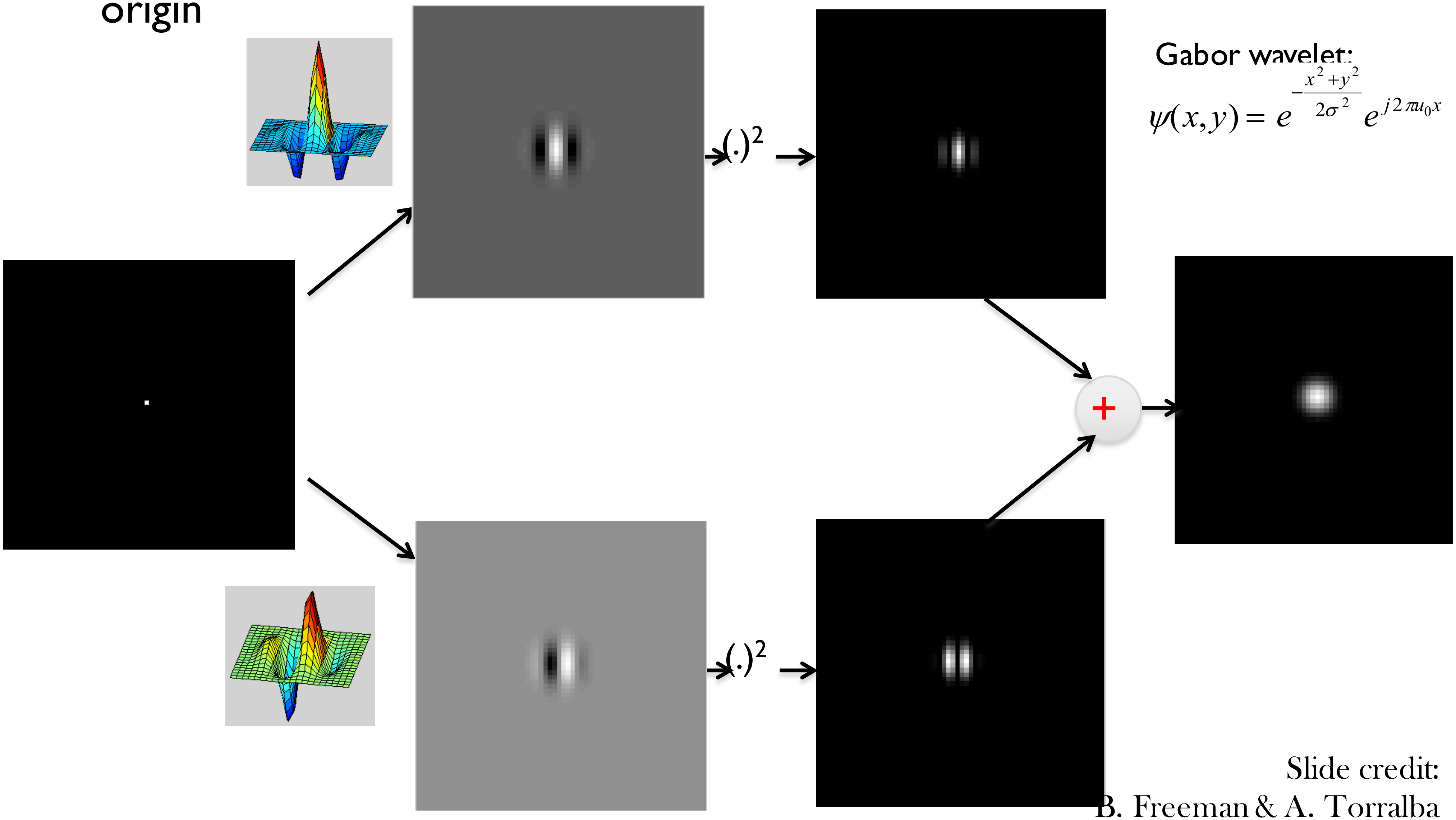
Fig. 5. Top row: illustrations of empirical 2-D receptive field profiles measured by J. P. Jones and L. A. Palmer (personal communication) in simple cells of the cat visual cortex. Middle row: best-fitting 2-D Gabor elementary function for each neuron, described by (10). Bottom row: residual error of the fit, indistinguishable from random error in the Chi-squared sense for 97 percent of the cells studied.

66

Slide credit: B. Freeman and A. Torralba

Quadrature filter pairs

- A quadrature filter is a complex filter whose real part is related to its imaginary part via a Hilbert transform along a particular axis through the origin

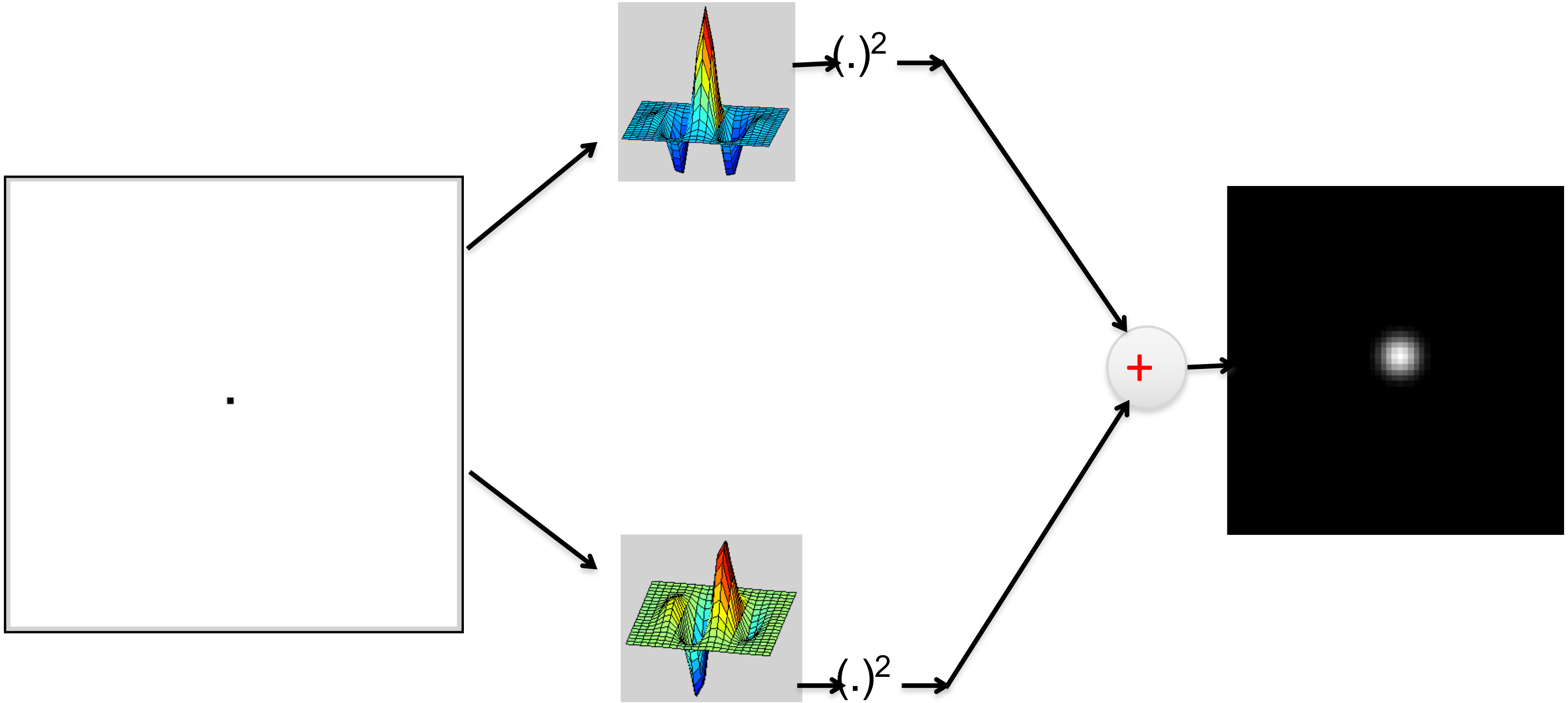


Gabor wavelet:

$$\psi(x,y) = e^{-\frac{x^2+y^2}{2\sigma^2}} e^{j2\pi u_0 x}$$

Slide credit:
B. Freeman & A. Torralba

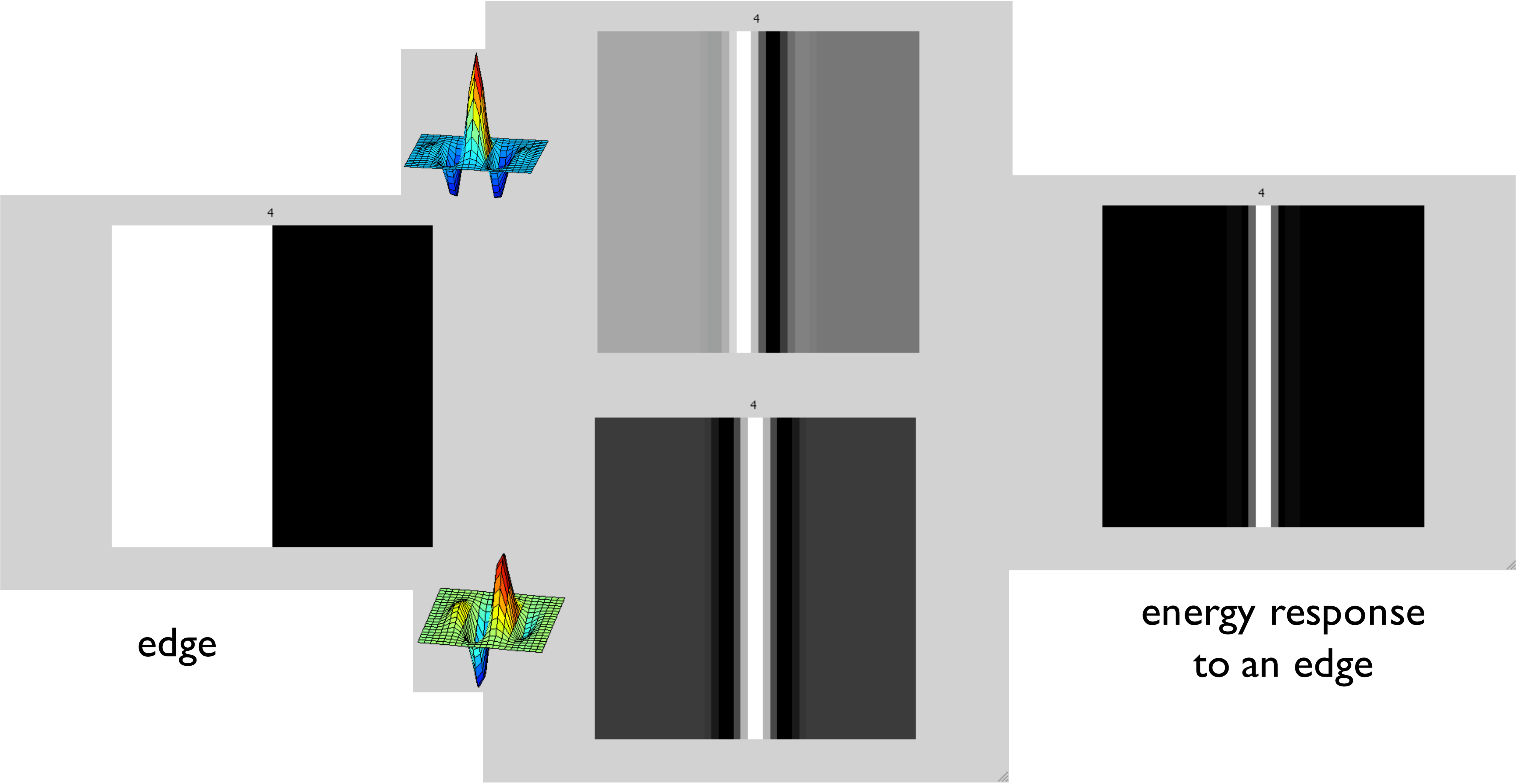
Quadrature filter pairs



Contrast invariance! (same energy response for white dot on black background as for a black dot on a white background).

Slide credit: B. Freeman and A. Torralba

Quadrature filter pairs

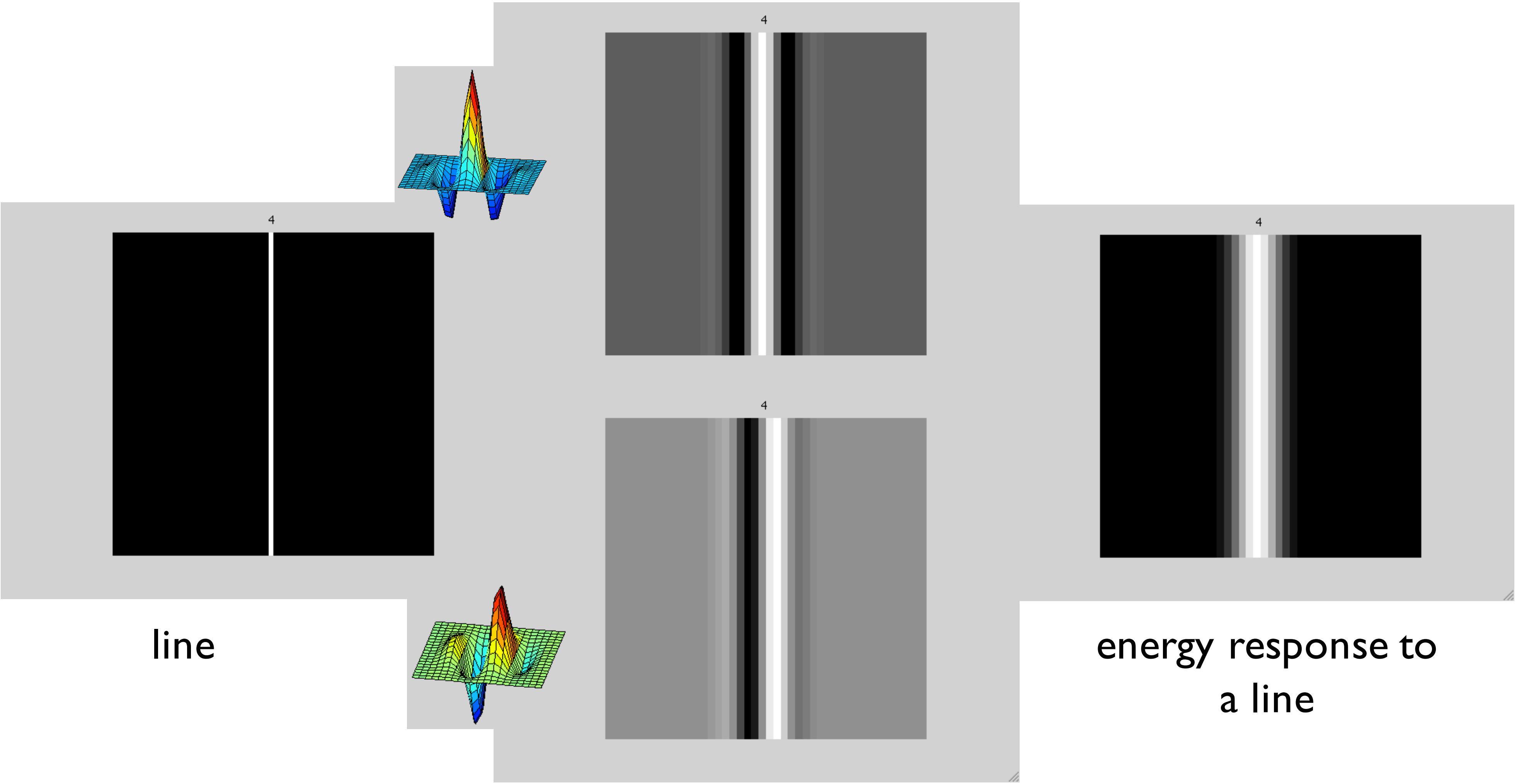


edge

energy response
to an edge

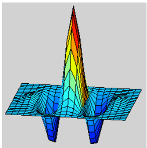
Slide credit: B. Freeman and A. Torralba

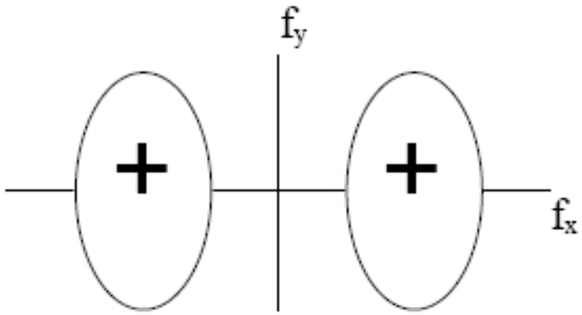
Quadrature filter pairs



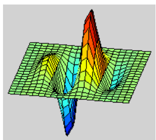
Slide credit: B. Freeman and A. Torralba

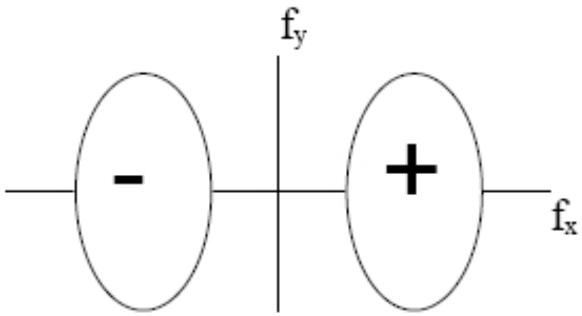
How quadrature pair filters work





(a) Frequency response of even filter, G
(real)



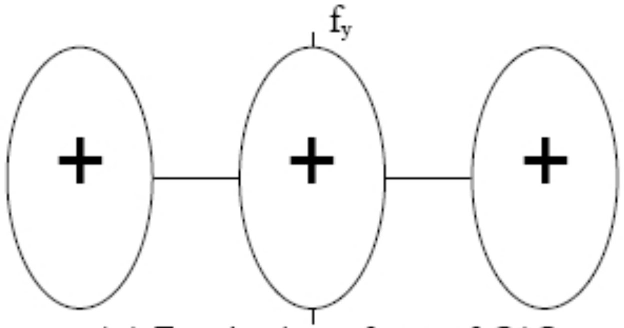


(b) Frequency response of odd filter, H
(imaginary)

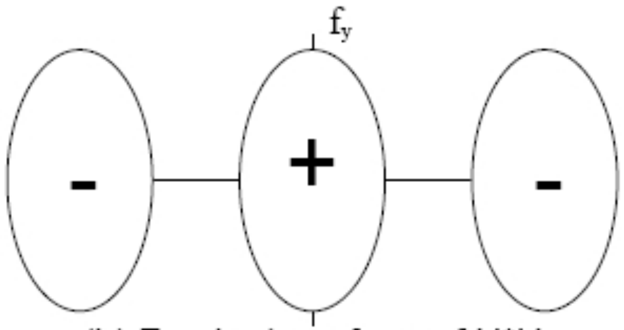
Figure 3-5: Frequency content of two bandpass filters in quadrature. (a) even phase filter, called G in text, and (b) odd phase filter, H. Plus and minus sign illustrate relative sign of regions in the frequency domain. See Fig. 3-6 for calculation of the frequency content of the energy measure derived from these two filters.

Slide credit: B. Freeman and A. Torralba

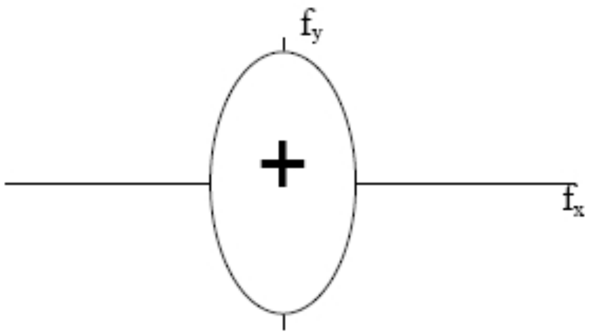
How quadrature pair filters work



(a) Fourier transform of G*G



(b) Fourier transform of H*H



(c) Fourier transform of G*G + H*H

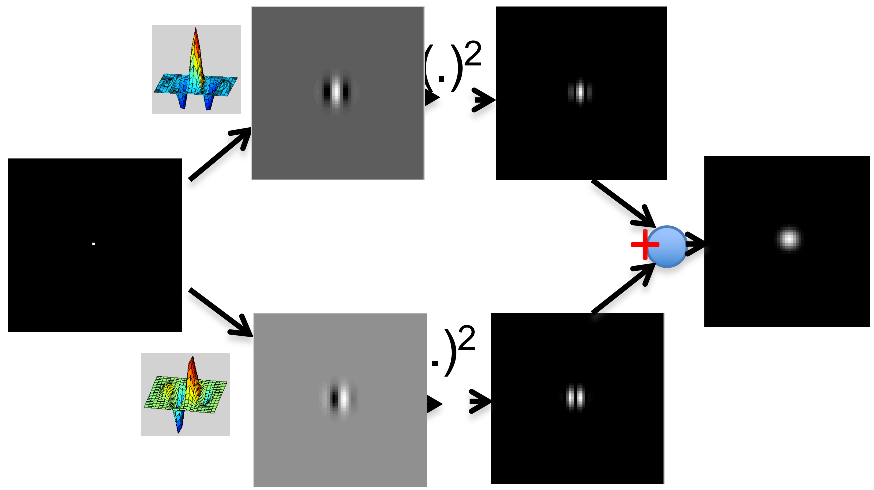
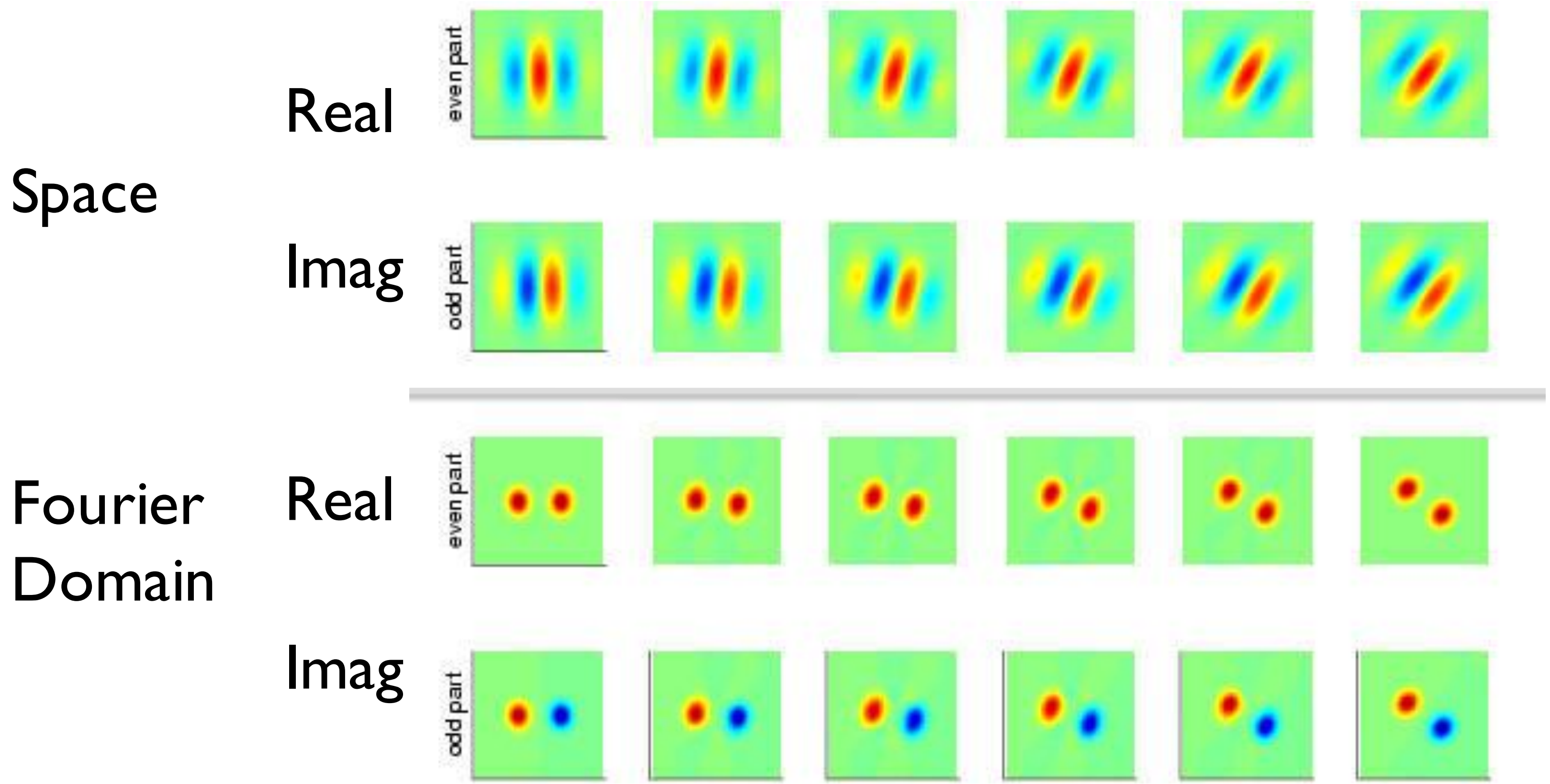


Figure 3-6: Derivation of energy measure frequency content for the filters of Fig. 3-5. (a) Fourier transform of $G * G$. (b) Fourier transform of $H * H$. Each squared response has 3 lobes in the frequency domain, arising from convolution of the frequency domain responses. The center lobe is modulated down in frequency while the two outer lobes are modulated up. (There are two sign changes which combine to give the signs shown in (b). To convolve H with itself, we flip it in f_x and f_y, which interchanges the $+$ and $-$ lobes of Fig. 3-5 (b). Then we slide it over an unflipped version of itself, and integrate the product of the two. That operation will give positive outer lobes, and a negative inner lobe. However, H has an imaginary frequency response, so multiplying it by itself gives an extra factor of -1, which yields the signs shown in (b)). (c) Fourier transform of the energy measure, $G * G + H * H$. The high frequency lobes cancel, leaving only the baseband spectrum, which has been demodulated in frequency from the original bandpass response. This spectrum is proportional to the sum of the auto-correlation functions of either lobe of Fig. 3-5 (a) and either lobe of Fig. 3-5 (b).

Slide credit: B. Freeman and A. Torralba

Oriented Filters

- Gabor wavelet: $$\psi(x,y) = e^{-\frac{x^2+y^2}{2\sigma^2}} e^{j2\pi u_0 x}$$
- Tuning filter orientation: $$x' = \cos(\alpha)x + \sin(\alpha)y$$ $$y' = -\sin(\alpha)x + \cos(\alpha)y$$



Slide credit: B. Freeman and A. Torralba

Simple example

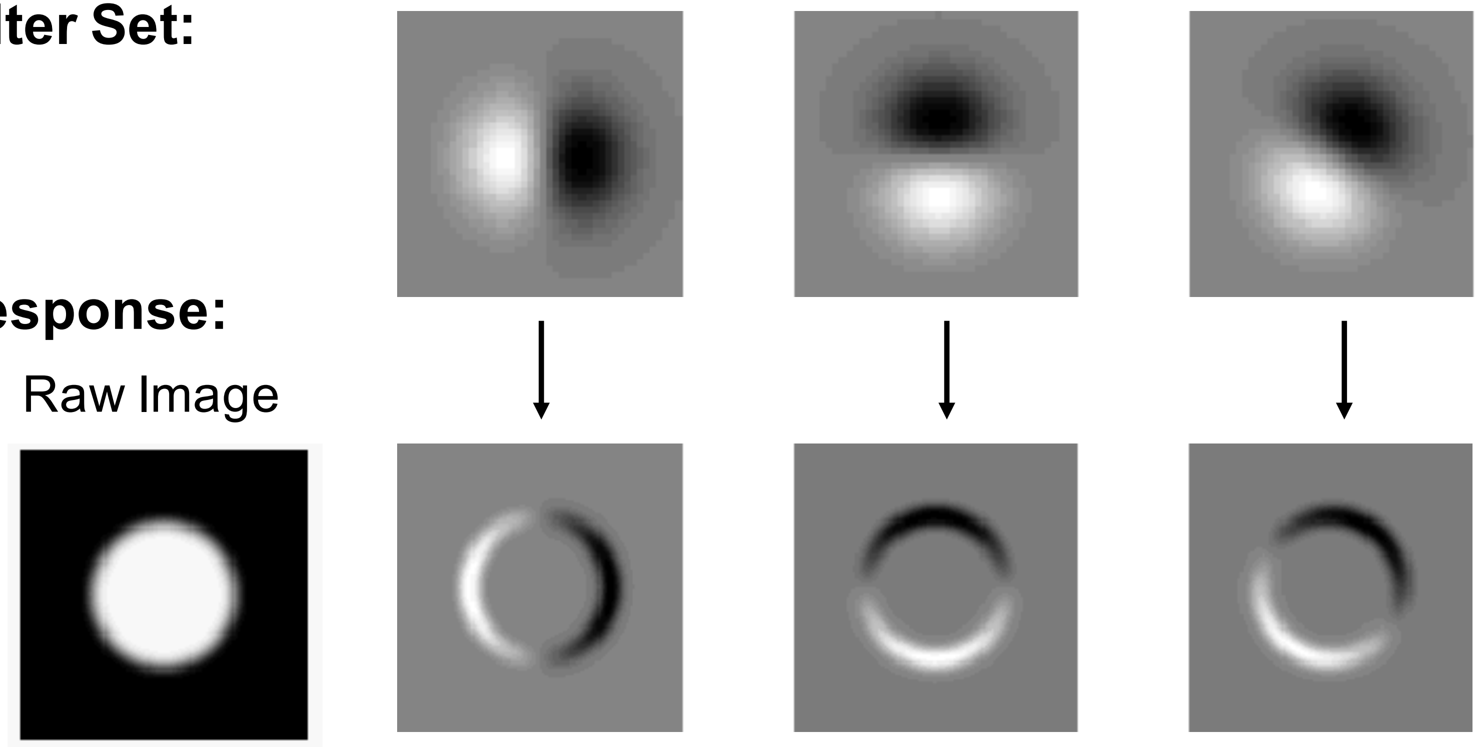
"Steerability"-- the ability to synthesize a filter of any orientation from a linear combination of filters at fixed orientations.

$$G_\theta^1 = \cos(\theta) G_0^1 + \sin(\theta) G_{90}^1$$

0°　　90°　　Synthesized 30°

Filter Set:

Response:



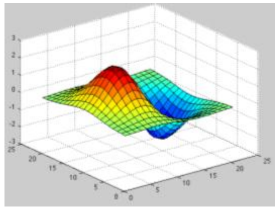
Taken from:
W. Freeman, T. Adelson, "The Design and Use of Sterrable Filters", IEEE Trans. Patt, Anal. and Machine Intell., vol 13, #9, pp 891-900, Sept 1991

Slide credit: B. Freeman and A. Torralba

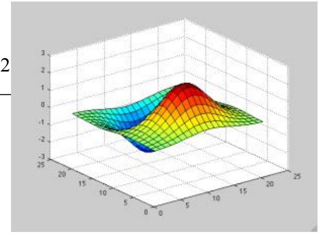
Steerable filters

Derivatives of a Gaussian:

$$h_x(x,y) = \frac{\partial h(x,y)}{\partial x} = \frac{-x}{2\pi\sigma^4} e^{-\frac{x^2+y^2}{2\sigma^2}}$$



$$h_y(x,y) = \frac{\partial h(x,y)}{\partial y} = \frac{-y}{2\pi\sigma^4} e^{-\frac{x^2+y^2}{2\sigma^2}}$$

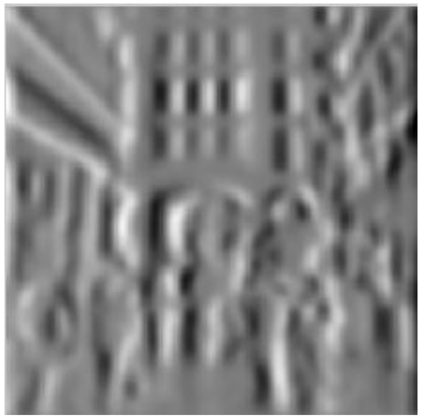


An arbitrary orientation can be computed as a linear combination of those two basis functions:

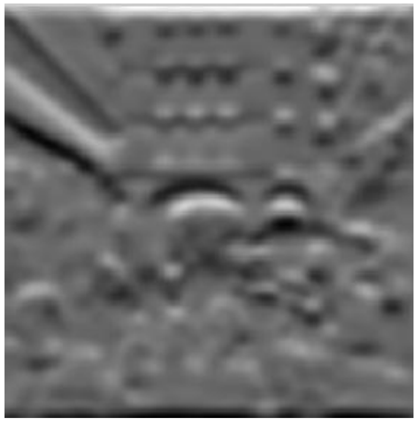
$$h_\alpha(x,y) = \cos(\alpha) h_x(x,y) + \sin(\alpha) h_y(x,y)$$

The representation is "shiftable" on orientation: We can interpolate any other orientation from a finite set of basis functions.

cos(〈)



+sin(〈)



=

Freeman & Adelson, 1992

Slide credit: B. Freeman and A. Torralba

Steerable filters

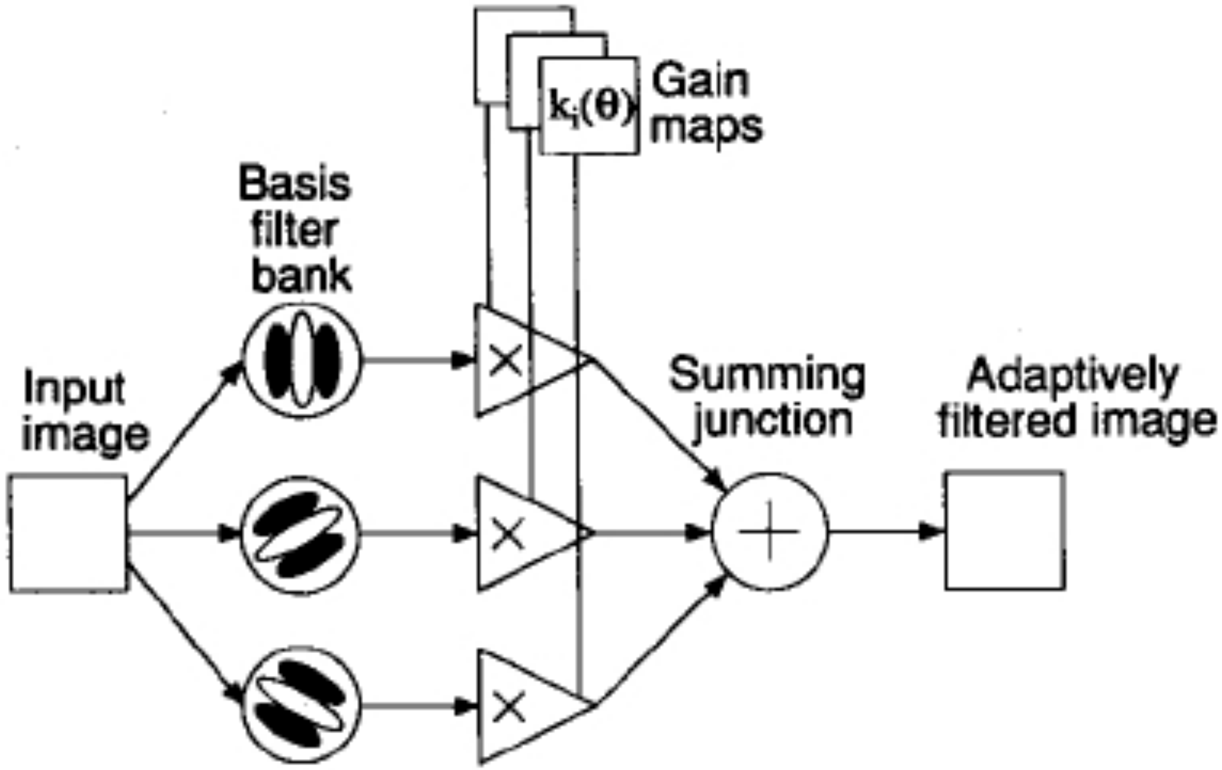


Fig. 3. Steerable filter system block diagram. A bank of dedicated filters process the image. Their outputs are multiplied by a set of gain maps that adaptively control the orientation of the synthesized filter.

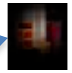
Slide credit: B. Freeman and A. Torralba

Local image representations

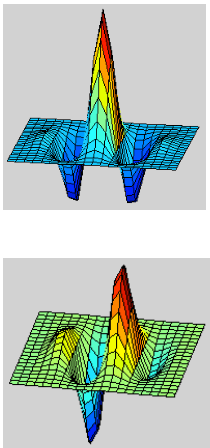
A pixel

[r,g,b]

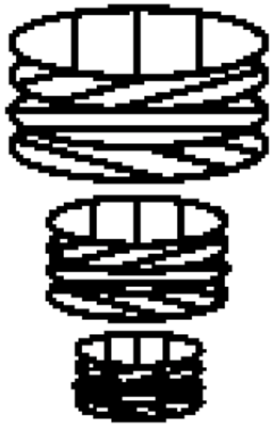
An image patch



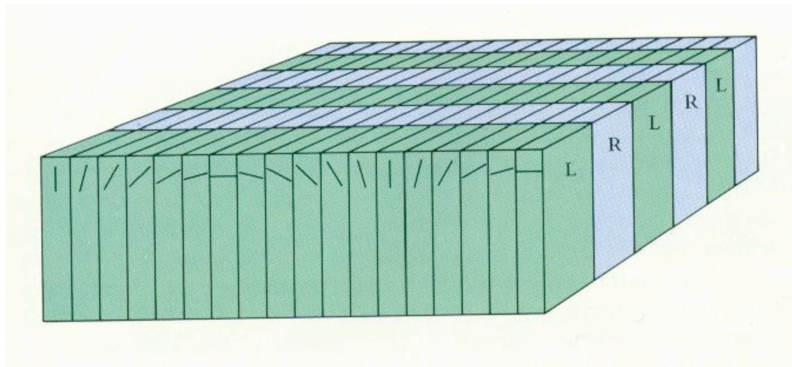
Gabor filter
pair in quadrature



Gabor jet



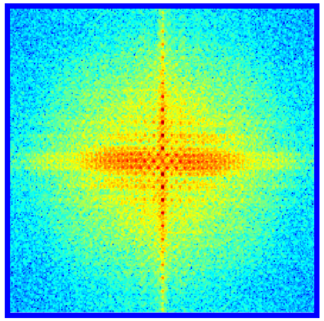
V1 sketch:
hypercolumns



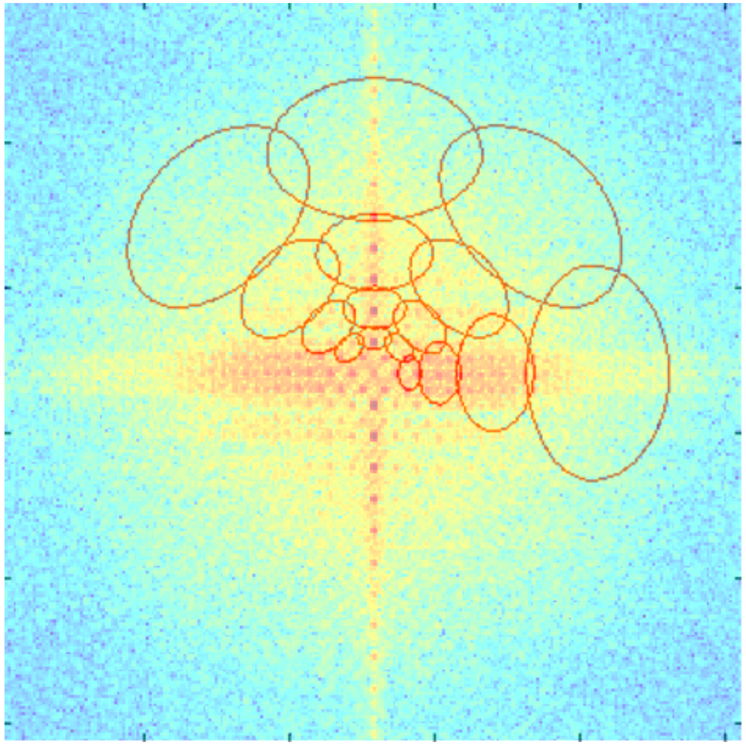
J.G.Daugman, "Two dimensional spectral analysis of cortical receptive field profiles," Vision Res., vol.20.pp.847-856.1980

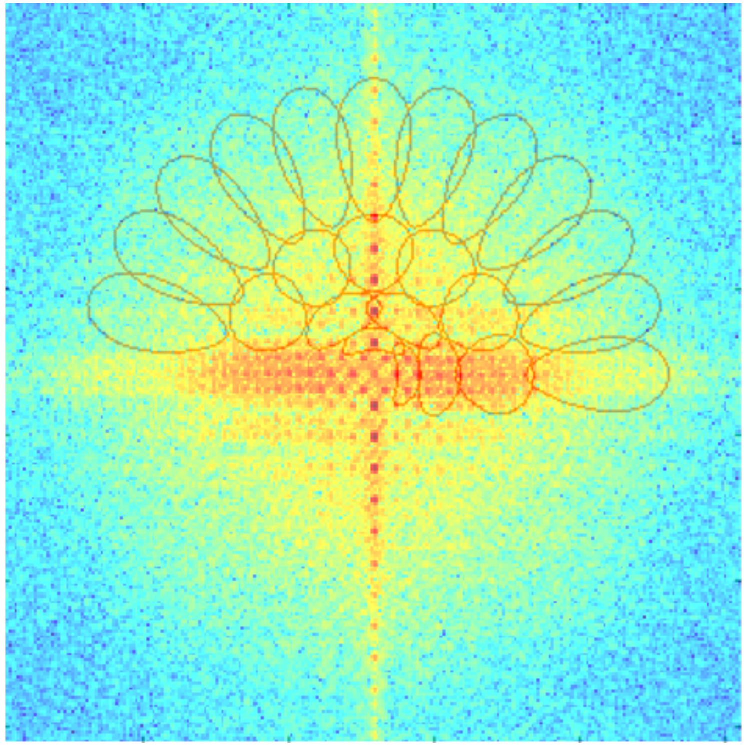
L. Wiskott, J-M. Fellous, N. Kuiger, C. Malsburg, "Face Recognition by Elastic Bunch Graph Matching", *IEEE Transactions on Pattern Analysis and Machine Intelligence*, vol.19(7), July 1997, pp. 775-779.

Slide credit: B. Freeman and A. Torralba



Gabor Filter Bank





or = [4 4 4 4]; or = [12 6 3 2];

Not for image reconstruction. It does NOT cover the entire space!

Slide credit: B. Freeman and A. Torralba

Summary

- Sampling
- Gabor wavelets, Steerable filters

Next week

- Image pyramids