Linear Diffusion

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1 Linear Diffusion

The linear diffusion (heat) equation is the oldest and best investigated PDE method in image processing. Let $f(x)$ denote a grayscale (noisy) input image and $u(x,t)$ be initialized with $u(x,0) = u^0(x) = f(x)$. Then, the linear diffusion process can be defined by the equation

$$(1) \quad \frac{\partial u}{\partial t} = \nabla \cdot (\nabla u) = \nabla^2 u$$

where $\nabla \cdot$ denotes the divergence operator. Thus, the equation is:

$$(2) \quad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \ .$$

The diffusion process can be seen as an evolution process with an artificial time variable t denoting the *diffusion time* where the input image is smoothed at a constant rate in all directions. Starting from the initial image $u^0(x)$, the evolving images $u(x,t)$ under the governed equation represent the successively smoothed versions of the initial input image $f(x)$, and thus create a *scale space* representation of the given image f, with $t > 0$ being the scale. As we move to coarser scales, the evolving images become more and more simplified since the diffusion process removes the image structures at finer scales. Figures 1 and 2 show example scale space representations sampled at different diffusion times for two different images. In fact, the notion of scale is an essential part of early visual processing, where the main task is to separate the image into relevant and irrelevant parts.

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Figure 1: Linear diffusion results for different diffusion times.

It is shown that the solution of the linear diffusion equation with the given initial condition $u(x,0) = f(x)$ for a specific diffusion time T is equivalent to the convolution of the input image $f(x)$ with the Gaussian kernel $G_\sigma(x)$ with standard deviation $\sigma = \sqrt{2T}$ [2, 3, 6]. Thus, linear diffusion can be regarded as a low-pass filter. The correspondence between the diffusion time variable t and the standard deviation σ clearly depicts the effect of t on the evolving images. The higher the value of t, the higher the value of σ, and the more smooth the image becomes. This relation also provides the following explicit solution to (1):

$$u(x,T) = \left(G_{\sqrt{2T}} * f\right)(x) \qquad \text{with} \qquad G_\sigma(x) = \frac{1}{2\pi\sigma^2} exp\left(-\frac{|x|^2}{2\sigma^2}\right). \tag{3}$$

Numerical Implementation

Since we deal with digital images, solving the linear diffusion equation requires discretization in both spatial and time coordinates. Central differences are the typical choices for the spatial derivatives:

$$\frac{d^2 u_{i,j}}{dx^2} \approx \frac{u_{i+h_x,j} - 2u_{i,j} + u_{i-h_x,j}}{h_x^2}, \qquad \frac{d^2 u_{i,j}}{dy^2} \approx \frac{u_{i,j+h_y} - 2u_{i,j} + u_{i,j-h_y}}{h_y^2} \tag{4}$$

where $u_{i,j}$ denotes the gray value or the brightness of the evolving image at pixel location (i,j).

The values of h_x and h_y are generally set to 1 as digital images are discretized on a regular pixel grid. For the remainder of this thesis, we take $h_x = h_y = 1$. This leads to

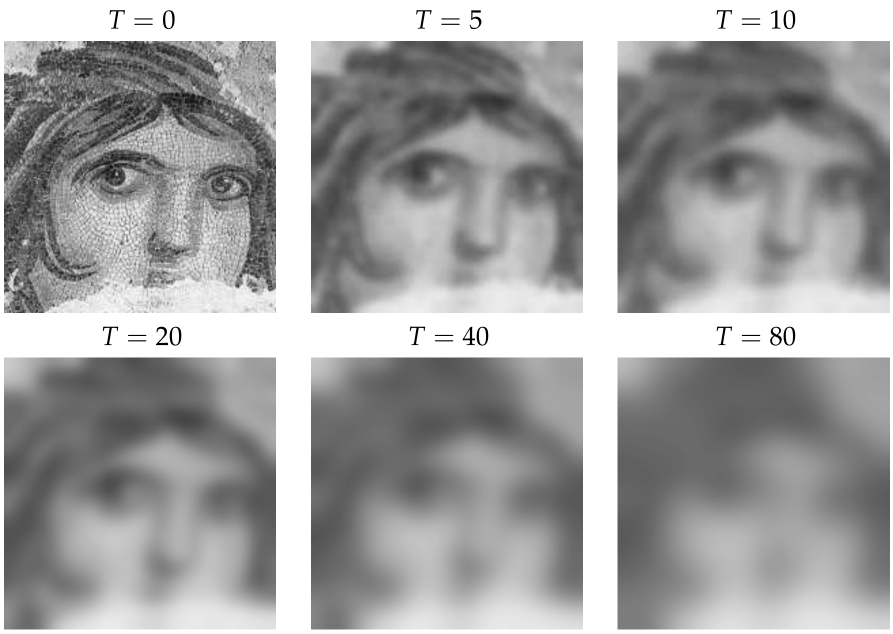

Figure 2: Linear diffusion results for different diffusion times.

the following space-discrete equation for (1):

$$(5) \qquad \frac{du_{i,j}}{dt} = u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} \; .$$

The straightforward approach to solve (5) is to consider an iterative scheme with an explicit time discretization, where homogeneous Neumann boundary condition is imposed along the image boundary

$$(6) \qquad \frac{u^{k+1}_{i,j} - u^k_{i,j}}{\Delta t} = u^k_{i+1,j} + u^k_{i-1,j} + u^k_{i,j+1} + u^k_{i,j-1} - 4u^k_{i,j}$$

where Δt is the time step, and u^k represents the restored image u at iteration k. Numerical stability condition for the discrete scheme requires that $\Delta t \leq 0.25$.

Relation Between Variational Regularization and Diffusion Equations

Interestingly, there is a strong relation between variational regularization methods and diffusion equations [4]. The variational regularization methods formulate smoothing process as a functional minimization via which a noise-free approximation of a given image is to be estimated. Most of these formulations assume an additive noise model

$$(7) \qquad f(x) = u(x) + n(x)$$

where $f(x)$ and $u(x)$ respectively denote the given noisy image and the desired denoised image, and $n(x)$ represents the additive noise.

Consider the Tikhonov energy functional [5] as an illustrative example:

$$E(u) = \int_{\Omega} \left((u - f)^2 + \alpha |\nabla u|^2 \right) dx \tag{8}$$

where

- $\Omega \subset \mathbf{R}^2$ is connected, bounded, open subset representing the image domain,
- f is an image defined on Ω,
- u is the smooth approximation of f,
- $\alpha > 0$ is the scale parameter.

The first term in $E(u)$ is the *data fidelity* term that penalizes the deviations between u and f, and thus forces the restored image to be close to the original image. The second term is called the *regularization* or *smoothness* term which penalizes the high gradients, and gives preference to smooth approximations. The relative importance of these two terms are defined by the scale parameter α.

The minimizing function u formally satisfies the Euler-Lagrange equation

$$(u - f) - \alpha \nabla^2 u = 0 \tag{9}$$

with the Neumann boundary condition $\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0$.

It is possible to rewrite (9) as

$$\frac{u - u^0}{\alpha} = \nabla^2 u \qquad \text{with} \qquad u^0 = f, \tag{10}$$

which may be regarded as an implicit time discretization of the linear diffusion equation (1) where a single time step ($T = \alpha$) is used. Note that diffusion time (scale selection) problem is not really eliminated by the variational regularization, it is replaced with a new parameter α that determines the strength of the smoothness prior.

The main drawback of linear diffusion filtering is that the smoothing process does not consider information regarding important image features such as edges. It follows that same amount of smoothing to be applied at every image location. As a result, the diffusion process does smooth not only noise, but also image edges.

2 Appendix - The Calculus of Variations

The calculus of variations[1]. aims at finding the extrema of expressions that depend on functions rather than parameters, which is called functionals.

Consider a functional of the form:

$$E(f) = \int F(x, f, f')dx \ . \tag{11}$$

If F is differentiable, then the calculus of variations says that the minimizer of $E(f)$ satisfies the following Euler-Langrange equation:

$$F_f - \frac{d}{dx}F_{f'} = 0 \ . \tag{12}$$

Now, consider the integrand contains higher-order derivatives:

$$E(f) = \int F(x, f, f', f'', \dots)dx \ . \tag{13}$$

In this case, the Euler-Lagrange equation becomes:

$$F_f - \frac{d}{dx}F_{f'} + \frac{d^2}{dx^2}F_{f''} - \dots = 0 \ . \tag{14}$$

In more general terms, if the integrand depends on several functions $f_1(x)$,$f_2(x)$,$\dots$:

$$E(f) = \int F(x, f_1, f_2, \dots, f_1', f_2', \dots)dx \ , \tag{15}$$

then, the minimizer of $E(f_1, f_2, \dots)$ satisfies the following set of PDEs:

$$F_{f_i} - \frac{d}{dx}F_{f_i'} = 0 \ . \tag{16}$$

Next, consider that the functional is defined over two independent variables x and y:

$$E(f) = \int F(x, y, f, f_x, f_y)dxdy \ . \tag{17}$$

Then, the minimizer of $E(f)$ satisfies following PDE:

$$F_f - \frac{\partial}{\partial x}F_{f_x} - \frac{\partial}{\partial y}F_{f_y} = 0 \ . \tag{18}$$

Consider, finally the case

$$E(f) = \int F(x, y, u, v, u_x, v_x)dx \ . \tag{19}$$

[1] This section is a brief summary of Appendix A.6 of Horn's Robot Vision. Please refer to [1] for the details.

This leads to the following coupled PDEs:

$$
\begin{aligned}
&F_u - \frac{d}{dx} F_{u_x} = 0 \, , \\
&F_v - \frac{d}{dx} F_{v_x} = 0 \, .
\end{aligned} \tag{20}
$$

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