BIL 717
Image Processing

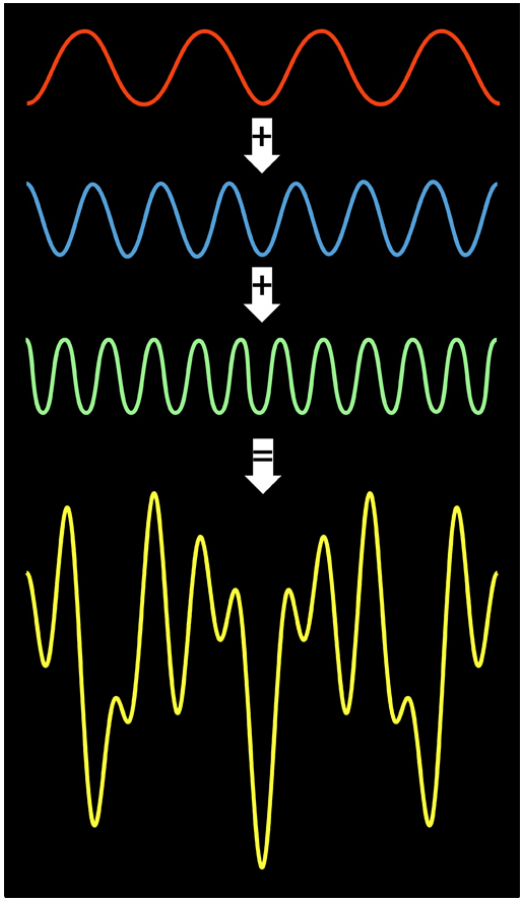
Mar. 12, 2013

Erkut Erdem
Dept. of Computer Engineering
Hacettepe University

Nonlinear Filtering

Review - Signals

- A signal is composed of low and high frequency components



low frequency components: smooth / piecewise smooth

Neighboring pixels have similar brightness values

You're within a region

high frequency components: oscillatory

Neighboring pixels have different brightness values

You're either at the edges or noise points

Review - Linear Diffusion

- The linear diffusion (heat) equation is the oldest and best investigated PDE method in image processing.
- Let $f(x)$ denote a grayscale (noisy) input image and $u(x, t)$ be initialized with $u(x,0) = u^0(x) = f(x)$.
- The linear diffusion process can be defined by the equation:

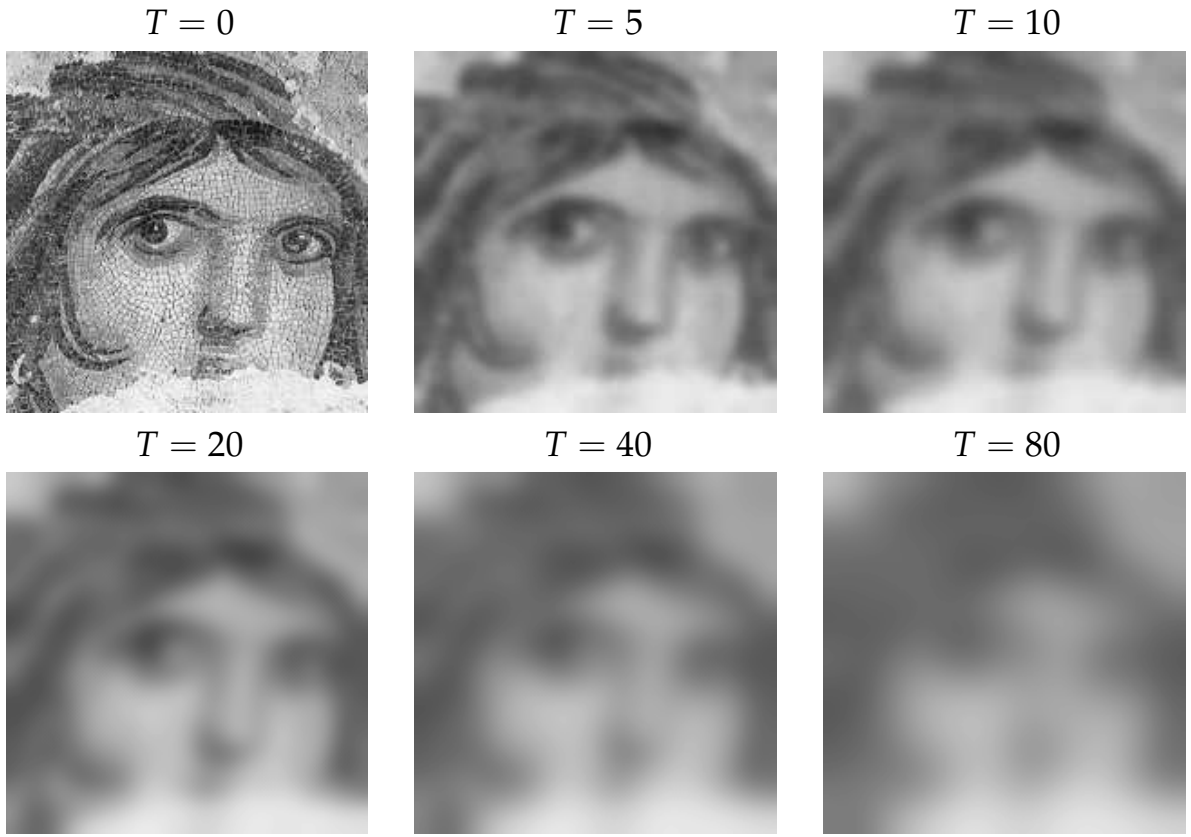
$$\frac{\partial u}{\partial t} = \nabla \cdot (\nabla u) = \nabla^2 u$$

where $\nabla \cdot$ denotes the divergence operator. Thus,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

Review - Linear Diffusion (cont'd.)

- As we move to coarser scales,
  - the evolving images become more and more simplified since the diffusion process removes the image structures at finer scales.



Review - Linear Diffusion and Gaussian Filtering

- The solution of the linear diffusion can be explicitly estimated as:

$$u(x,T) = \left(G_{\sqrt{2T}} * f\right)(x)$$

with $G_\sigma(x) = \frac{1}{2\pi\sigma^2} exp\left(-\frac{|x|^2}{2\sigma^2}\right)$

- Solution of the linear diffusion equation is equivalent to a proper convolution of the input image with the Gaussian kernel $G_\sigma(x)$ with standard deviation $\sigma = \sqrt{2T}$
- The higher the value of T, the higher the value of σ, and the more smooth the image becomes.

Review - Numerical Implementation

- Solving the linear diffusion equation requires discretization in both spatial and time coordinates.
- Central differences for the spatial derivatives:

$$\frac{d^2 u_{i,j}}{dx^2} \approx \frac{u_{i+h_x,j} - 2u_{i,j} + u_{i-h_x,j}}{h_x^2}$$

$$\frac{d^2 u_{i,j}}{dy^2} \approx \frac{u_{i,j+h_y} - 2u_{i,j} + u_{i,j-h_y}}{h_y^2}$$

where $u_{i,j}$ denotes the gray value or the brightness of the evolving image at pixel location (i, j).

- We take $h_x = h_y = 1$ for a regular grid.

Review - Numerical Implementation (cont'd.)

- Original model:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

- Space discrete version:

$$\frac{du_{i,j}}{dt} = u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}$$

- Space-time discrete version:

$$\frac{u_{i,j}^{k+1} - u_{i,j}^{k}}{\Delta t} = u_{i+1,j}^{k} + u_{i-1,j}^{k} + u_{i,j+1}^{k} + u_{i,j-1}^{k} - 4u_{i,j}^{k}$$

homogeneous Neumann boundary condition along the image boundary

$\Delta t \leq 0.25$ is required for numerical stability

Review - *Variational Regularization*

- The variational regularization models formulate smoothing process as a functional minimization via which a noise-free approximation of a given image is to be estimated.
- With an additive model, $f(x) = u(x) + n(x)$
  - *f(x): original image*
  - *u(x): smoothed image*
  - *n(x): noise component*
- An example: Tikhonov energy functional

$$E(u) = \int_{\Omega} \left((u - f)^2 + \alpha |\nabla u|^2 \right) dx$$

Review - Tikhonov energy functional

$$E(u) = \int_{\Omega} \underbrace{(u-f)^2}_{\text{data fidelity term}} + \underbrace{\alpha|\nabla u|^2}_{\text{regularization term}} \, dx$$

- $\Omega \subset \mathbf{R}^2$ is connected, bounded, open subset representing the image domain,
- f is an image defined on Ω,
- u is the smooth approximation of f,
- $\alpha > 0$ is the scale parameter.

Review - *Variational Regularization and Diffusion Equations*

- There is a strong relation between variational regularization methods and diffusion equations.
- The minimizing function *u* of the Tikhonov energy functional formally satisfies the Euler-Lagrange equation:

$$(u - f) - \alpha \nabla^2 u = 0$$

with the Neumann boundary condition $\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0$

Today – Nonlinear Diffusion

- use nonlinear PDEs to create a scale space representation
  - consists of gradually simplified images
  - some image features such as edges are maintained or even enhanced.
- Perona-Malik Type Nonlinear Diffusion (1990)
- Total Variation (TV) Regularization (1992)
- Weickert's Edge Enhancing Diffusion (1994)

Perona-Malik Type Nonlinear Diffusion

- The earliest nonlinear diffusion model proposed in image processing.
- called *anisotropic diffusion* by Perona and Malik.
- It uses a scalar-valued diffusivity.
- In fact, it is an isotropic nonhomogeneous equation.
  - A true example of anisotropic diffusion model: Weickert's Edge-enhancing diffusion (more later on)

Perona-Malik Type Nonlinear Diffusion

- The Perona-Malik equation is:

$$\frac{\partial u}{\partial t} = \nabla \cdot (g(|\nabla u|)\nabla u)$$

  with homogeneous Neumann boundary conditions and the initial condition $u0(x) = f(x)$, f denoting the input image.

- Constant diffusion coefficient of linear equation is replaced with a smooth non-increasing diffusivity function g satisfying
  - $g(0) = 1$,
  - $g(s) \geq 0$,
  - $\lim_{s\to\infty} g(s) = 0$
- The diffusivities become variable in both space and time.

Perona-Malik Type Nonlinear Diffusion

- The Perona-Malik equation: $\frac{\partial u}{\partial t} = \nabla \cdot (g(|\nabla u|)\nabla u)$

- Two different choices for the diffusivity function:

$$\text{(1)} \quad g(s) = \frac{1}{1 + s^2/\lambda^2}$$

$$\text{(2)} \quad g(s) = e^{-\frac{s^2}{\lambda^2}}$$

- λ corresponds to a contrast parameter.
- What is the effect of the parameter λ ?

1D Analysis of Perona-Malik Diffusion

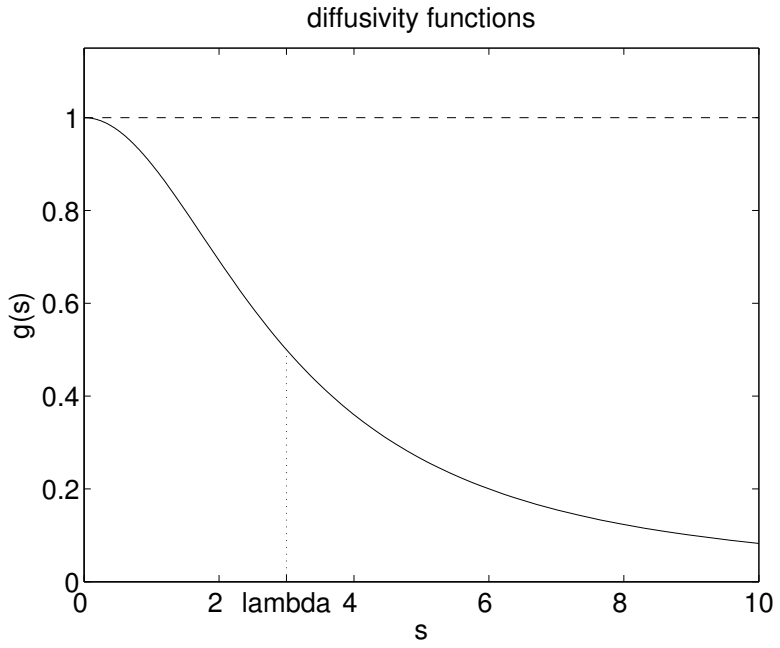
- 1D version to demonstrate the role of the contrast parameter
- For 1D case, the Perona-Malik equation is as follows:

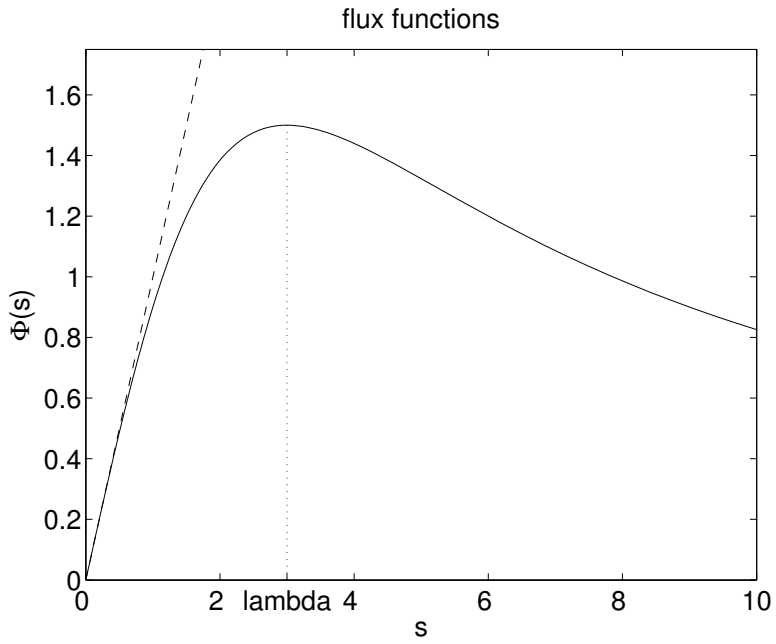
$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \underbrace{(g(|u_x|)u_x)}_{\Phi(u_x)} = \Phi'(u_x)u_{xx}$$

with $g(|u_x|) = \frac{1}{1+|u_x|^2/\lambda^2}$ or $g(|u_x|) = e^{-\frac{|u_x|^2}{\lambda^2}}$

1D Analysis of Perona-Malik Diffusion

- Diffusivities and the corresponding flux functions for the linear diffusion (*plotted in dashed line*) and the Perona-Malik type nonlinear diffusion (*plotted in solid line*).

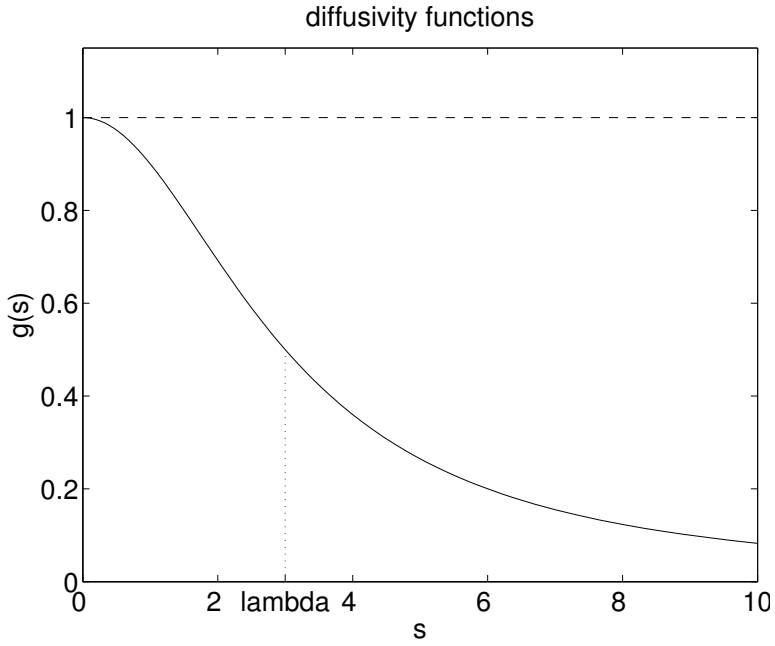


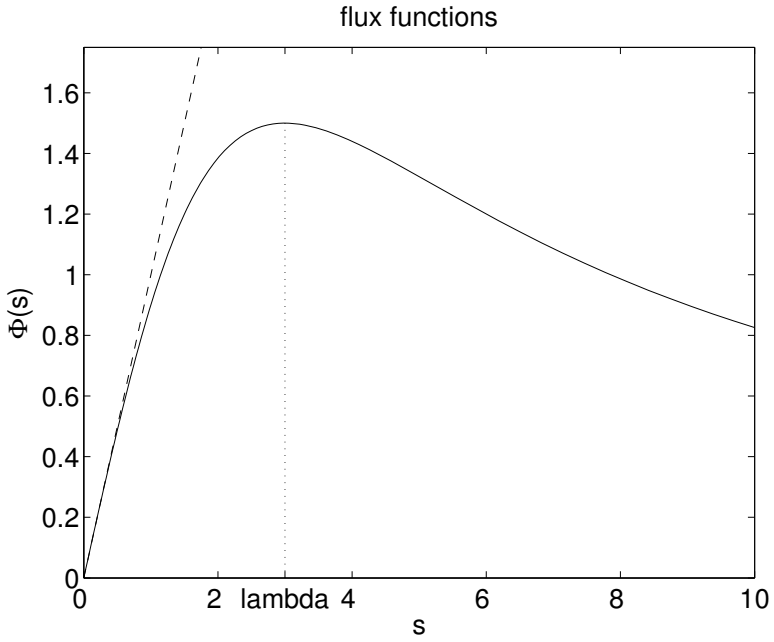


$$g(s) = \frac{1}{1+s^2/\lambda^2}$$

$$\lambda = 3$$

1D Analysis of Perona-Malik Diffusion



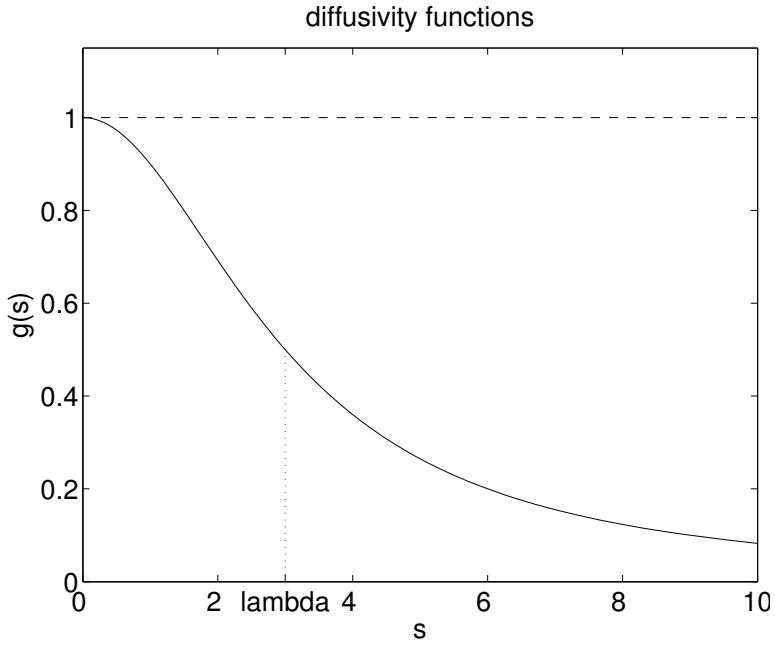


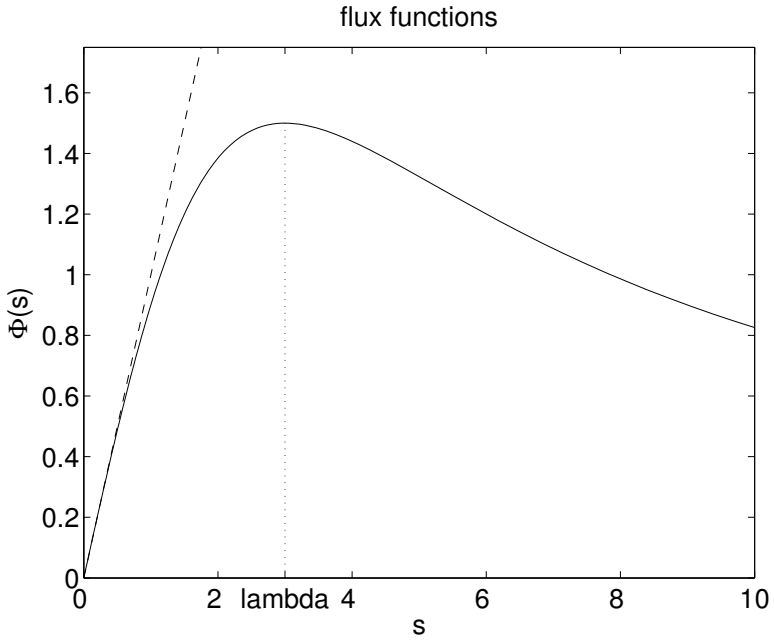
$$g(s) = \frac{1}{1+s^2/\lambda^2}$$

$$\lambda = 3$$

- For linear diffusion the diffusivity is constant ($g(s) = 1$), which results in a linearly increasing flux function.
- For linear diffusion all points, including the discontinuities, are smoothed equally.

1D Analysis of Perona-Malik Diffusion



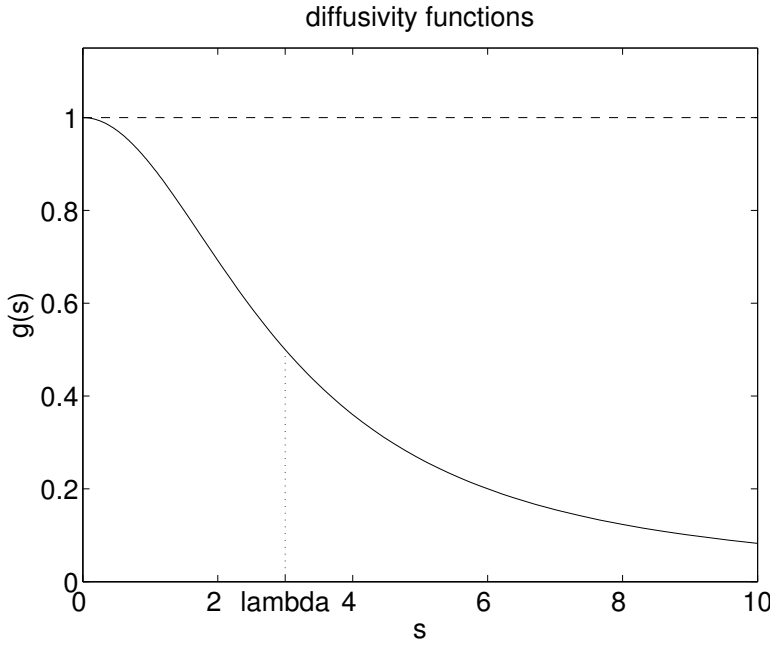


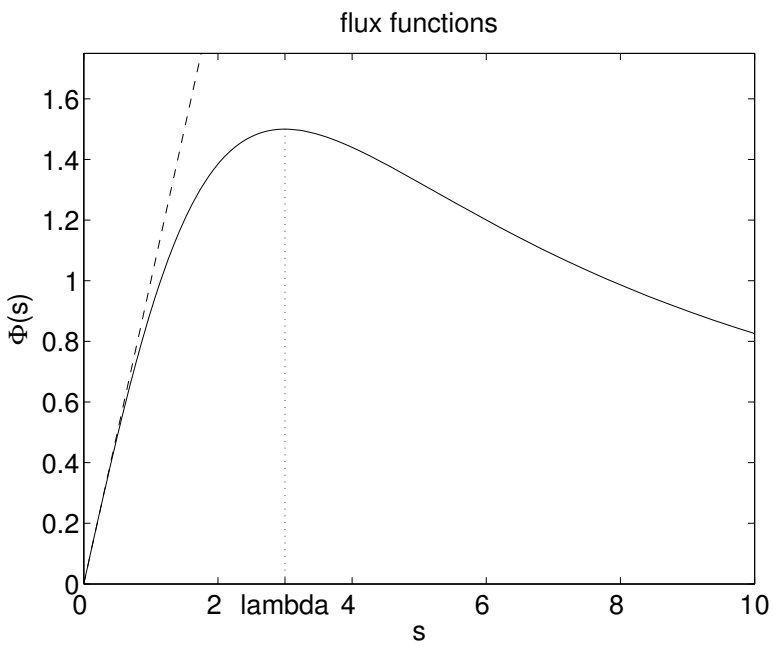
$$g(s) = \frac{1}{1+s^2/\lambda^2}$$

$$\lambda = 3$$

- For Perona-Malik, the diffusivity is variable and decreases as $|u_x|$ increases.
- The decay in diffusivity is particularly rapid after the contrast parameter λ.
- This leads to two different behaviors in the diffusion process.

1D Analysis of Perona-Malik Diffusion





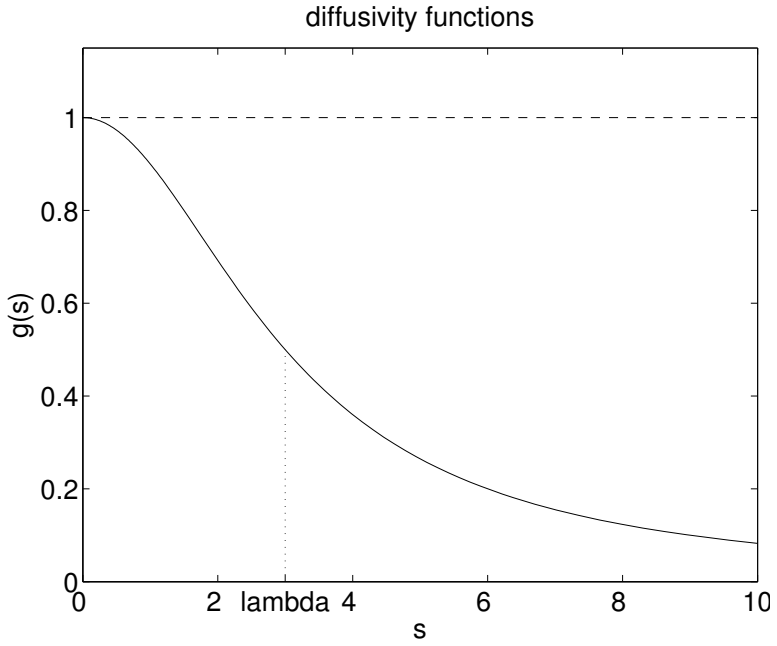
$$g(s) = \frac{1}{1+s^2/\lambda^2}$$

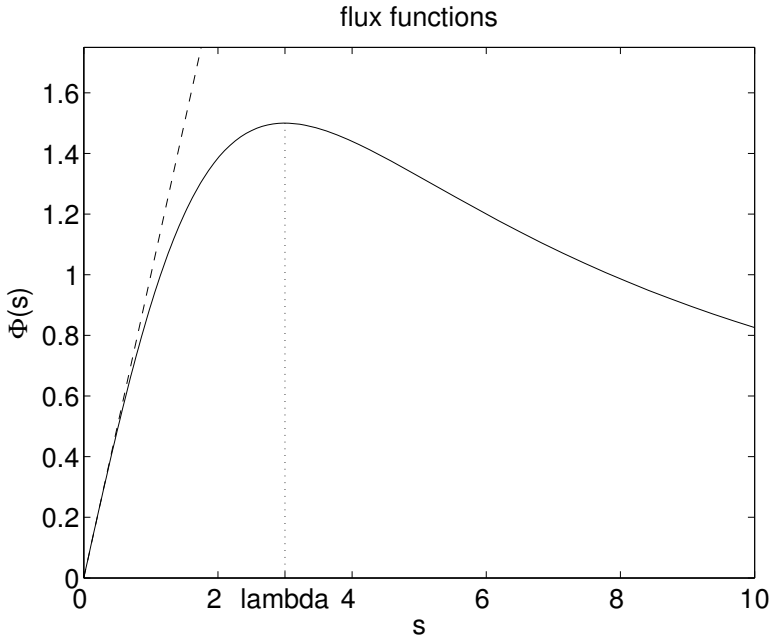
$$\lambda = 3$$

$$\frac{\partial u}{\partial t} = \Phi'(u_x)u_{xx}$$

- For the points where $|u_x| < \lambda$, $\Phi'(u_x) > 0$ we have lost in the material.
- For the points where $|u_x| > \lambda$, on the contrary, $\Phi'(u_x) < 0$ which generates an enhancement in the material.

1D Analysis of Perona-Malik Diffusion





$$g(s) = \frac{1}{1+s^2/\lambda^2}$$

$$\lambda = 3$$

$$\frac{\partial u}{\partial t} = \Phi'(u_x)u_{xx}$$

- Although the diffusivity is always nonnegative, one can observe both *forward* and *backward* diffusions during the smoothing.
- The contrast parameter λ separates the regions of forward diffusion from the regions of backward diffusion.

Perona-Malik Type Nonlinear Diffusion

- In 2D case, the diffusivities are reduced at the image locations where $|\nabla u|^2$ is large.
- As $|\nabla u|^2$ can be interpreted as a measure of edge likelihood, this means that the amount of smoothing is low along image edges.
- The contrast parameter λ specifies a measure that determines which edge points are to be preserved or blurred during the diffusion process.
- Even edges can be sharpened due to the local backward diffusion behavior as discussed for the 1D case.
- Since the backward diffusion is a well-known ill-posed process, this may cause an instability, the so-called *staircasing effect*.

Staircasing Effect

- Due to backward diffusion, a piece-wise smooth region in the original image evolves into many unintuitive piecewise constant regions.

Original noisy image

Perona-Malik Diffusion

- A possible solution to this drawback is to use regularized gradients in diffusivity computations.

Regularized Perona-Malik Model

- Replacing the diffusivities $g(|\nabla u|)$ with the regularized ones $g(|\nabla u_\sigma|)$ leads to the following equation:

$$\frac{\partial u}{\partial t} = \nabla \cdot (g(|\nabla u_\sigma|)\nabla u)$$

where $u_\sigma = G_\sigma * u$ represents a Gaussian-smoothed version of the image.

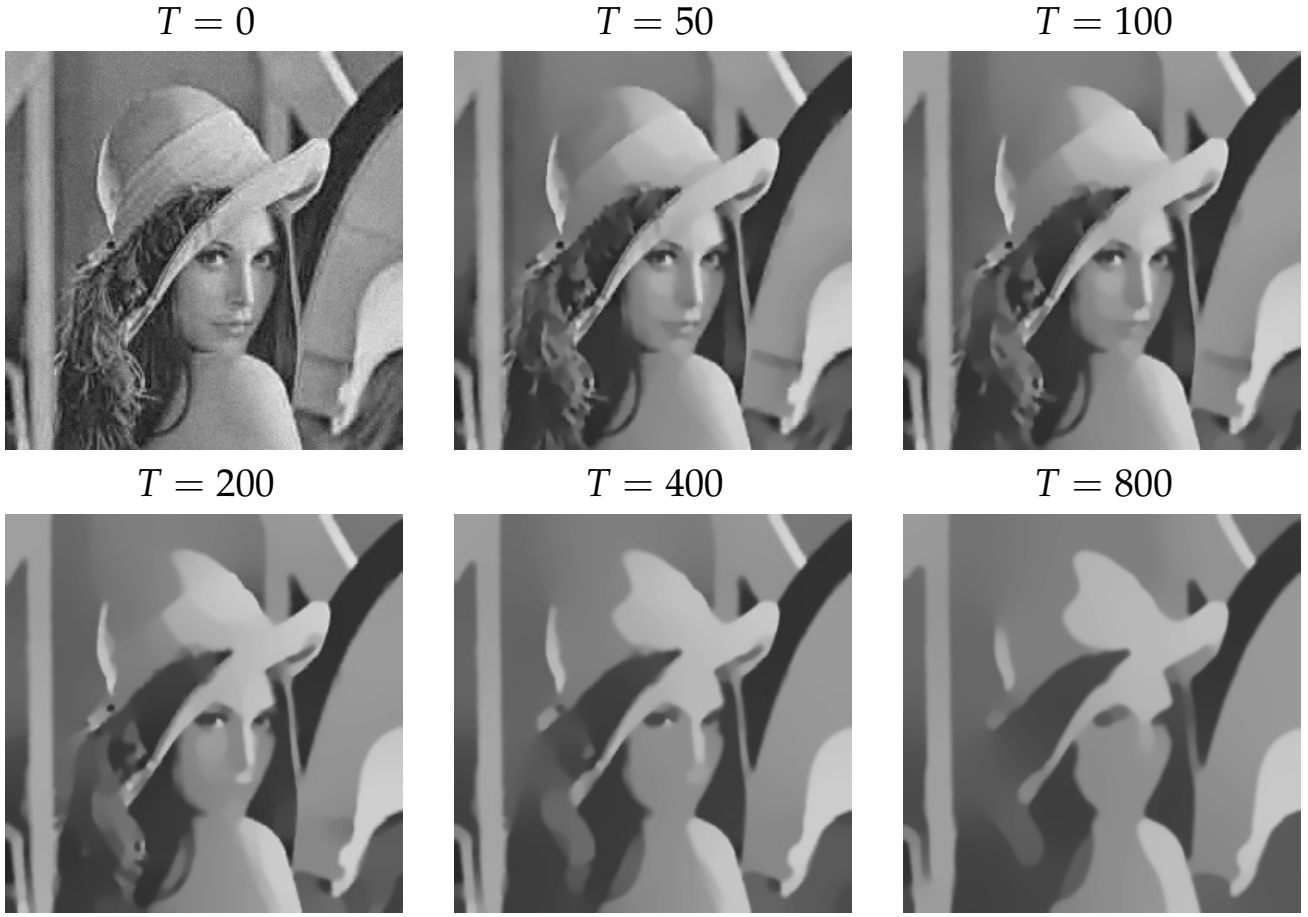
Original noisy image

Perona-Malik Diffusion

Regularized Perona-Malik Diffusion

Regularized Perona-Malik Model

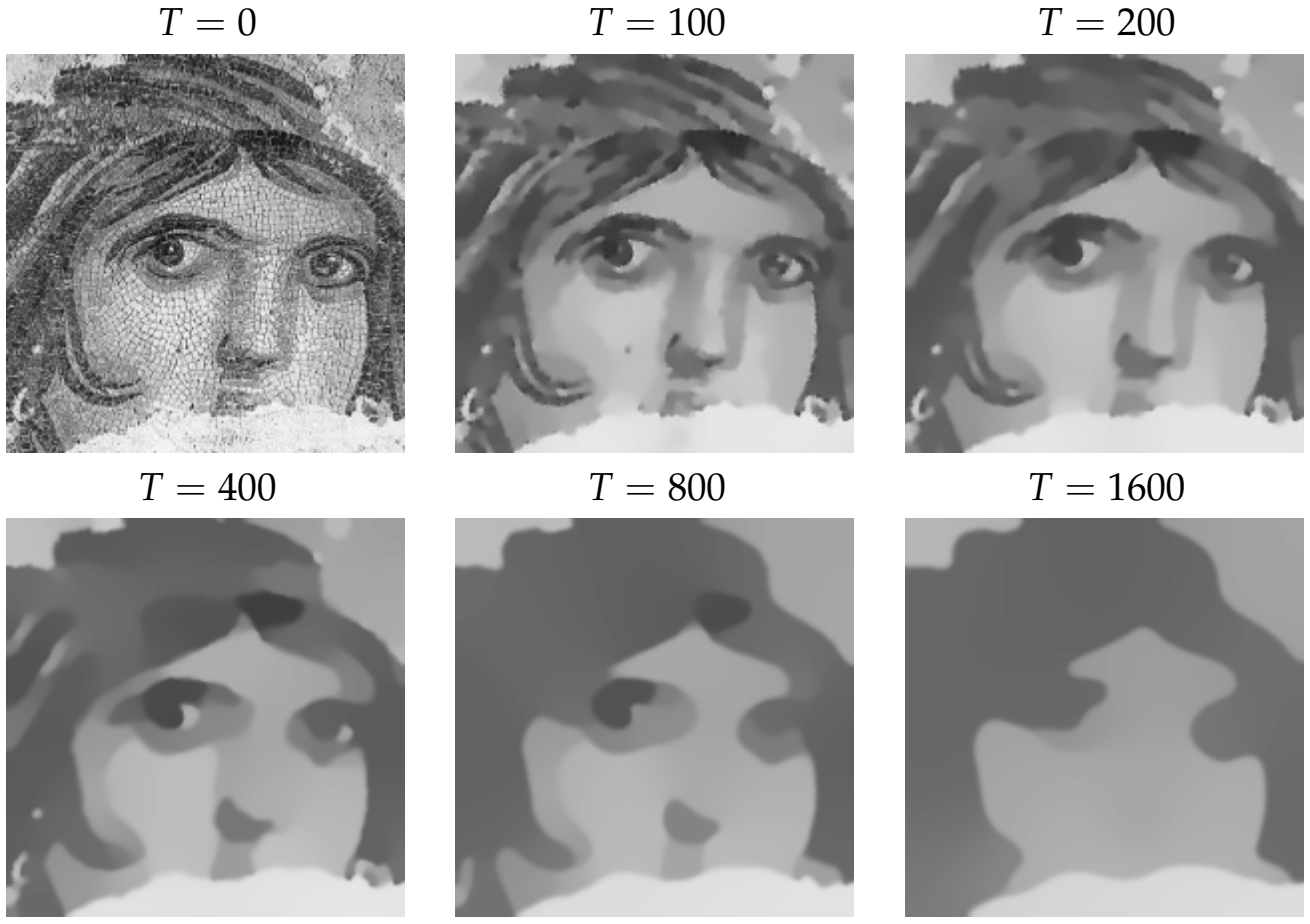
- Smoothing process diminishes noise while retaining or enhancing edges since it considers a kind of a priori edge knowledge



$(\lambda = 1, \sigma = 1)$

Regularized Perona-Malik Model

- Smoothing process diminishes noise while retaining or enhancing edges since it considers a kind of a priori edge knowledge



$(\lambda = 1, \sigma = 1)$

Numerical Implementation

- Central differences is used to approximate the gradient magnitude at a pixel (*i*, *j*) in the diffusivity estimation,

$g_{i,j} = g(|\nabla u_{i,j}|)$

$$|\nabla u_{i,j}| = \sqrt{\left(\frac{du_{i,j}}{dx}\right)^2 + \left(\frac{du_{i,j}}{dy}\right)^2}$$

$$\approx \sqrt{\left(\frac{u_{i+1,j} - u_{i-1,j}}{2}\right)^2 + \left(\frac{u_{i,j+1} - u_{i,j-1}}{2}\right)^2}$$

Numerical Implementation

- Original model:

$$\frac{\partial u}{\partial t} = \nabla \cdot (g(|\nabla u|)\nabla u)$$

- Space discrete version:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(g(|\nabla u|)u_x\right) + \frac{\partial}{\partial y}\left(g(|\nabla u|)u_y\right)$$

$$\begin{aligned}\frac{du_{i,j}}{dt} &= g_{i+\frac{1}{2},j}\cdot\left(u_{i+1,j}-u_{i,j}\right) - g_{i-\frac{1}{2},j}\cdot\left(u_{i,j}-u_{i-1,j}\right)\\ &+ g_{i,j+\frac{1}{2}}\cdot\left(u_{i,j+1}-u_{i,j}\right) - g_{i,j-\frac{1}{2}}\cdot\left(u_{i,j}-u_{i,j-1}\right)\end{aligned}$$

Numerical Implementation

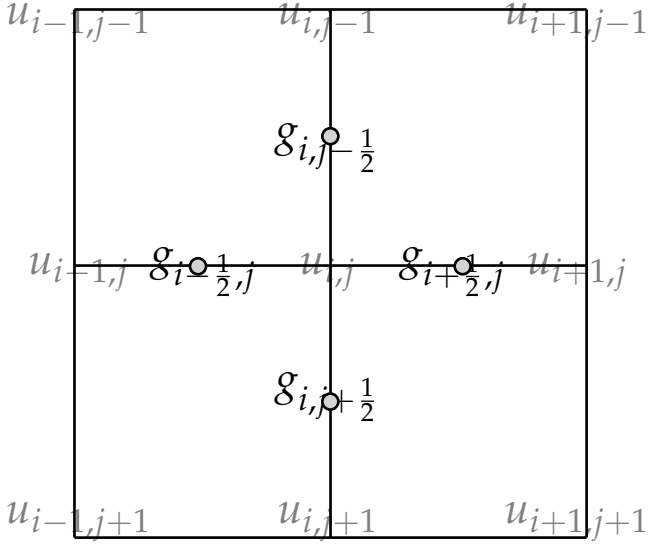
- Space discrete version:

$$\begin{aligned}\frac{du_{i,j}}{dt} &= g_{i+\frac{1}{2},j}\cdot\left(u_{i+1,j}-u_{i,j}\right)-g_{i-\frac{1}{2},j}\cdot\left(u_{i,j}-u_{i-1,j}\right)\\ &+ g_{i,j+\frac{1}{2}}\cdot\left(u_{i,j+1}-u_{i,j}\right)-g_{i,j-\frac{1}{2}}\cdot\left(u_{i,j}-u_{i,j-1}\right)\end{aligned}$$

- This discretization scheme requires the diffusivities to be estimated at mid-pixel points.
- They are computed by taking averages of the diffusivities over neighboring pixels:

$$g_{i\pm\frac{1}{2},j}=\frac{g_{i\pm1,j}+g_{i,j}}{2}$$

$$g_{i,j\pm\frac{1}{2}}=\frac{g_{i,j\pm1}+g_{i,j}}{2}$$



Numerical Implementation

- Space discrete version:

$$\begin{aligned}\frac{du_{i,j}}{dt} &= g_{i+\frac{1}{2},j}\cdot\left(u_{i+1,j}-u_{i,j}\right)-g_{i-\frac{1}{2},j}\cdot\left(u_{i,j}-u_{i-1,j}\right)\\ &+ g_{i,j+\frac{1}{2}}\cdot\left(u_{i,j+1}-u_{i,j}\right)-g_{i,j-\frac{1}{2}}\cdot\left(u_{i,j}-u_{i,j-1}\right)\end{aligned}$$

- Space-time discrete version:

$$\begin{aligned}\frac{u_{i,j}^{k+1}-u_{i,j}^{k}}{\Delta t} &= g_{i+\frac{1}{2},j}^{k}\cdot u_{i+1,j}^{k}+g_{i-\frac{1}{2},j}^{k}\cdot u_{i-1,j}^{k}+g_{i,j+\frac{1}{2}}^{k}\cdot u_{i,j+1}^{k}+g_{i,j-\frac{1}{2}}^{k}\cdot u_{i,j-1}^{k}\\ &- \left(g_{i+\frac{1}{2},j}^{k}+g_{i-\frac{1}{2},j}^{k}+g_{i,j+\frac{1}{2}}^{k}+g_{i,j-\frac{1}{2}}^{k}\right)\cdot u_{i,j}^{k}\end{aligned}$$

homogeneous Neumann boundary condition along the image boundary

$\Delta t \leq 0.25$ is required for numerical stability

Total Variation (TV) Regularization

- Rudin et al. (1992) formulated image restoration as minimization of the total variation (TV) of a given image under certain assumptions on the noise.
- The Total Variation (TV) regularization model is generally defined as:

$$E_{TV}(u) = \int_{\Omega} \left(\frac{1}{2}(u - f)^2 + \alpha |\nabla u| \right) dx$$

  - $\Omega \subset \mathbf{R}^2$ is connected, bounded, open subset representing the image domain,
  - f is an image defined on Ω,
  - u is the smooth approximation of f,
  - $\alpha > 0$ is a scalar.

Total Variation (TV) Regularization

- The Total Variation (TV) regularization model:

$$E_{TV}(u) = \int_{\Omega} \left(\frac{1}{2}(u - f)^2 + \alpha |\nabla u| \right) dx$$

- The gradient descent equation for Equation (10) is defined by:

$$\frac{\partial u}{\partial t} = \nabla \cdot \left(\frac{\nabla u}{|\nabla u|} \right) - \frac{1}{\alpha}(u - f); \quad \left. \frac{\partial u}{\partial n} \right|_{\partial \Omega} = 0$$

- The value of α specifies the relative importance of the fidelity term.
- It can be interpreted as a scale parameter that determines the level of smoothing.

Sample TV Restoration results

$\alpha = 50$

$\alpha = 100$

$\alpha = 200$

- The value of α specifies the relative importance of the fidelity term and thus the level of smoothing.

TV Regularization

- In the original formulation, the observed image *f* was assumed to be degraded by additive Gaussian noise with zero mean and known variance σ^2.
- In order to restore a given image, Rudin et al. proposed to solve the following constrained optimization problem:

$$\min_u \int_\Omega |\nabla u| dx$$

subject to

$$\int_\Omega (u - f)^2 dx = \sigma^2$$

- $\frac{1}{\alpha}$ can be considered as a Lagrange multiplier.

TV Regularization and TV Flow

- TV regularization can be associated with a nonlinear diffusion filter, the so-called *TV flow*
- Ignoring the fidelity term in the TV regularization model leads to the PDE:

$$\frac{\partial u}{\partial t} = \nabla \cdot (g(|\nabla u|)\nabla u)$$

with $u^0 = f$ and the diffusivity function $g(|\nabla u|) = \frac{1}{|\nabla u|}$

- Notice that this diffusivity function has no additional contrast parameter as compared with the Perona-Malik diffusivities.

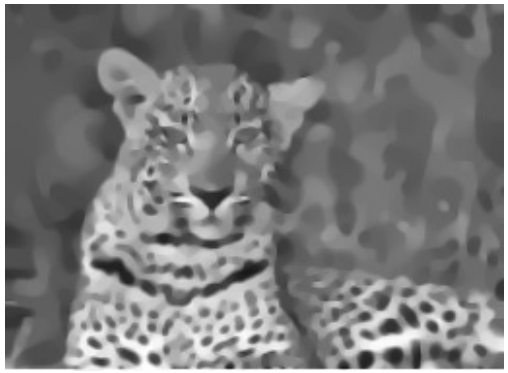
Sample TV Flow results

- Corresponding smoothing process yields segmentation-like, piecewise constant images.

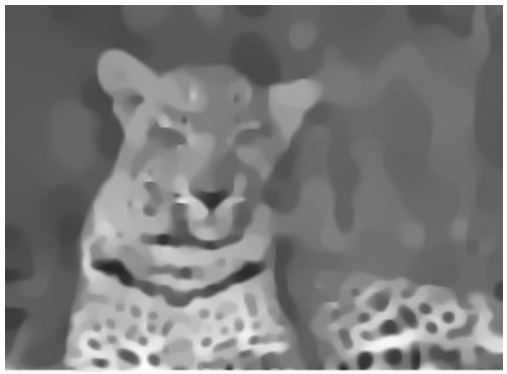
$T = 0$

$T = 25$

$T = 50$

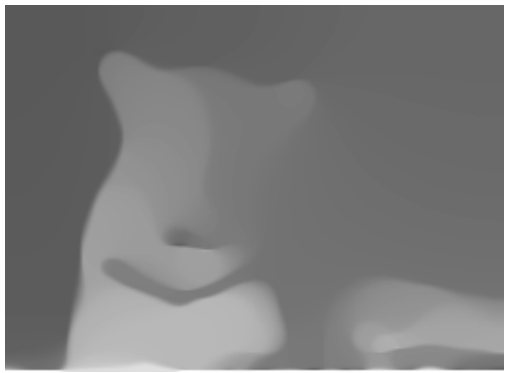


$T = 100$



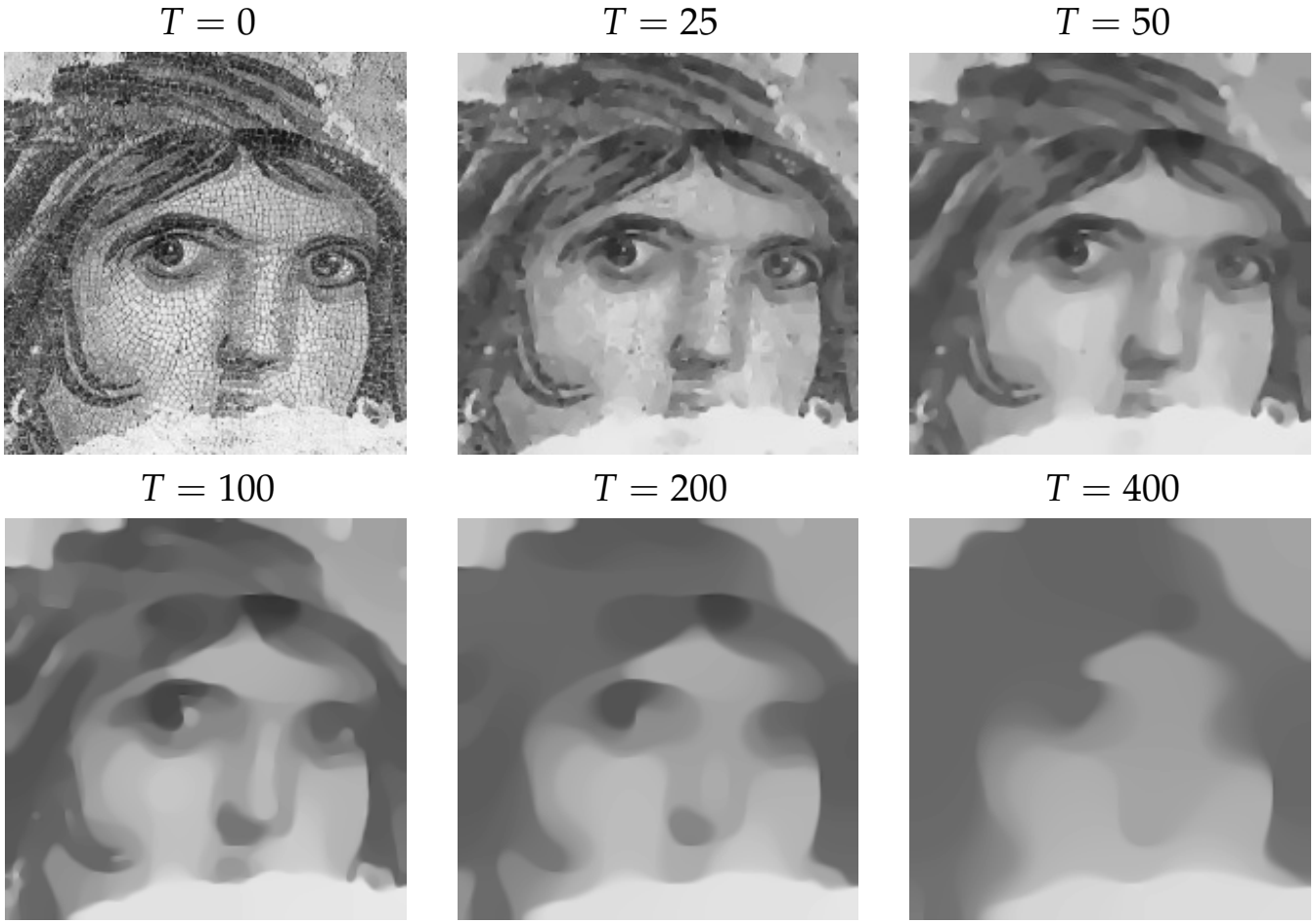
$T = 200$

$T = 400$



Sample TV Flow results

- Corresponding smoothing process yields segmentation-like, piecewise constant images.



Numerical Implementation

- The evolution equation can be discretized by using standard finite differences.
- The solution of TV regularization or equivalently TV flow leads to singular diffusivities.
- In numerical implementations based on standard discretization, this leads to stability problems as the image gradient tends to zero.
- A common solution to this problem is to add a small positive constant ε to image gradients/
- More accurate numerical implementations are suggested.

Numerical Implementation

- Space discrete version:

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial}{\partial x}\left(\frac{u_x}{\sqrt{|\nabla u|^2+\epsilon^2}}\right)+\frac{\partial}{\partial y}\left(\frac{u_y}{\sqrt{|\nabla u|^2+\epsilon^2}}\right)-\frac{1}{\alpha}(u-f) \\
&= \frac{u_{xx}\left(u_y^2+\epsilon^2\right)-2u_x u_y u_{xy}+u_{yy}\left(u_x^2+\epsilon^2\right)}{\left(u_x^2+u_y^2+\epsilon^2\right)^{\frac{3}{2}}}-\frac{1}{\alpha}(u-f)\,,
\end{aligned}$$

Numerical Implementation

- Space discrete version:

$$\frac{du_{i,j}}{dt} = \frac{\frac{d^2u_{i,j}}{dx^2}\left(\left(\frac{du_{i,j}}{dy}\right)^2+\epsilon^2\right) - 2\left(\frac{du_{i,j}}{dx}\right)\left(\frac{du_{i,j}}{dy}\right)\left(\frac{d^2u_{i,j}}{dxdy}\right) + \frac{d^2u_{i,j}}{dy^2}\left(\left(\frac{du_{i,j}}{dx}\right)^2+\epsilon^2\right)}{\left(\left(\frac{du_{i,j}}{dx}\right)^2+\left(\frac{du_{i,j}}{dy}\right)^2+\epsilon^2\right)^{\frac{3}{2}}}$$

$$- \frac{1}{\alpha}\left(u_{i,j} - f_{i,j}\right)$$

with $$\frac{d^2u_{i,j}}{dxdy} \approx \frac{u_{i+1,j+1} - u_{i+1,j-1} - u_{i-1,j+1} + u_{i-1,j-1}}{4}$$

Numerical Implementation

- Space-time discrete version:

$$
\begin{aligned}
\frac{u_{i,j}^{k+1}-u_{i,j}^{k}}{\Delta t} &= \left(\left(\frac{u_{i+1,j}^{k}-u_{i-1,j}^{k}}{2}\right)^2+\left(\frac{u_{i,j+1}^{k}-u_{i,j-1}^{k}}{2}\right)^2+\epsilon^2\right)^{-\frac{3}{2}} \\
&\cdot\Bigg[\left(u_{i+1,j}^{k}-2u_{i,j}^{k}+u_{i-1,j}^{k}\right)\left(\left(\frac{u_{i,j+1}^{k}-u_{i,j-1}^{k}}{2}\right)^2+\epsilon^2\right) \\
&- \frac{1}{8}\left(u_{i+1,j}^{k}-u_{i-1,j}^{k}\right)\left(u_{i,j+1}^{k}-u_{i,j-1}^{k}\right) \\
&\quad \left(u_{i+1,j+1}^{k}-u_{i+1,j-1}^{k}-u_{i-1,j+1}^{k}+u_{i-1,j-1}^{k}\right) \\
&+ \left(u_{i,j+1}^{k}-2u_{i,j}^{k}+u_{i,j-1}^{k}\right)\left(\left(\frac{u_{i+1,j}^{k}-u_{i-1,j}^{k}}{2}\right)^2+\epsilon^2\right)\Bigg] \\
&- \frac{1}{\alpha}\left(u_{i,j}^{k}-f_{i,j}\right)
\end{aligned}
$$

homogeneous Neumann boundary condition along the image boundary

$\Delta t \leq 0.25\ \varepsilon$ is required for numerical stability

Edge Enhancing Diffusion

- Proposed by Weickert (1994)
- an anisotropic nonlinear diffusion model with better edge enhancing capabilities than the Perona-Malik model
- can be described by the equation:

$$\frac{\partial u}{\partial t} = \nabla \cdot (D(\nabla u)\nabla u)$$

where

- u is the smoothed image,
- f is the input image ($u^0(x) = f(x)$),
- D represents a matrix-valued diffusion tensor that describes the smoothing directions and the corresponding diffusivities

Edge Enhancing Diffusion

$$\frac{\partial u}{\partial t} = \nabla \cdot (D(\nabla u)\nabla u)$$

- For linear diffusion the diffusion tensor can be defined as $D(\nabla u) = I$ with I denoting the identity matrix.
  - This results in a constant diffusion coefficient for all image points in all directions.
- For Perona-Malik type nonlinear diffusion, $D(\nabla u) = g(|\nabla u_\sigma|)I$.
  - Such a choice reduces the amount of smoothing at image edges, but in an equal amount in all directions.
- In actual anisotropic setting, the diffusion tensor D is defined as a function of the structure tensor $J(\nabla u)$.

Structure Tensor

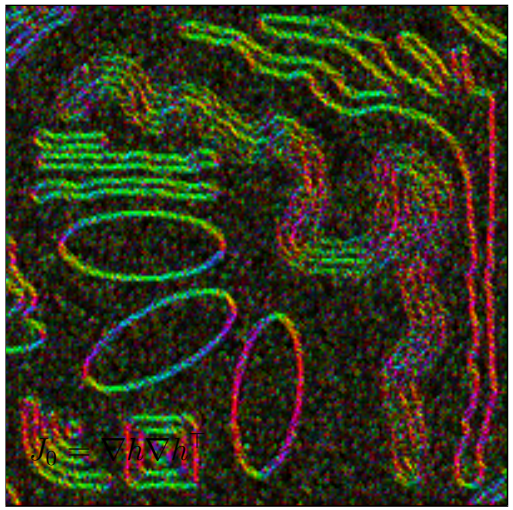
- The structure tensor $J(\nabla u)$ is described by:

$$J(\nabla u) = \nabla u \nabla u^T = \begin{bmatrix} u_x^2 & u_x u_y \\ u_x u_y & u_y^2 \end{bmatrix}$$

- The structure tensor $J(\nabla u)$ can be interpreted as an image feature describing the local orientation information.
- It has
  - an orthonormal basis of eigenvectors v_1 and v_2 with $v_1 \parallel \nabla u$ and $v_2 \perp \nabla u$, and
  - the corresponding eigenvalues $\lambda_1 = |\nabla u|^2$ and $\lambda_2 = 0$.

Structure Tensor

$$J(\nabla u) = \nabla u \nabla u^T = \begin{bmatrix} u_x^2 & u_x u_y \\ u_x u_y & u_y^2 \end{bmatrix}$$

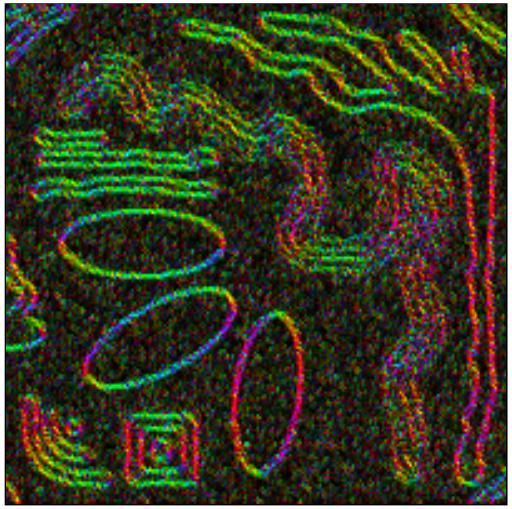


$$J(\nabla u) = \nabla u \nabla u^T$$

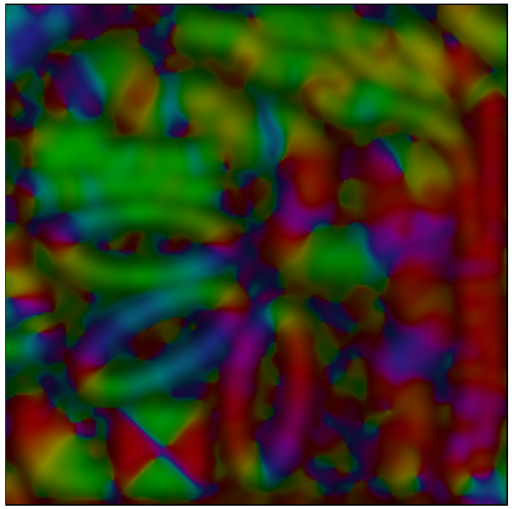
Images are taken from Brox et al., 2004

Structure Tensor

- Noise significantly affects the tensor estimation.
- The given image u is usually convolved with a Gaussian kernel G_σ with a relatively small standard deviation σ
- The (linear) structure tensor is computed accordingly by using $\nabla u_\sigma = \nabla(G_\sigma * u)$ instead of ∇u.



$J(\nabla u) = \nabla u \nabla u^T$



J_ρ with $\rho = 3$

Images are taken from Brox et al., 2004

Edge Enhancing Diffusion

- use the structure tensor as an image/edge descriptor to construct a diffusion tensor that
  - reduces the amount of smoothing across the edges
  - while smoothing is still carried out along the edges
- Weickert proposed to utilize same orthonormal basis of eigenvectors $v_1 \parallel \nabla u_\sigma$ and $v_2 \perp \nabla u_\sigma$ estimated from the structure tensor $J(\nabla u_\sigma)$ with the following choice of eigenvalues satisfying
  - $\frac{\lambda_1(|\nabla u_\sigma|)}{\lambda_2(|\nabla u_\sigma|)} \to 0$ for $|\nabla u_\sigma| \to \infty$

Edge Enhancing Diffusion

- Suggested eigenvalues are

$$\lambda_1(|\nabla u_\sigma|) = \begin{cases} 1 & \text{if } |\nabla u_\sigma| = 0 \\ 1 - exp\left(-\frac{3.31488}{(|\nabla u_\sigma|/\lambda)^8}\right) & \text{otherwise,} \end{cases}$$

$$\lambda_2(|\nabla u_\sigma|) = 1$$

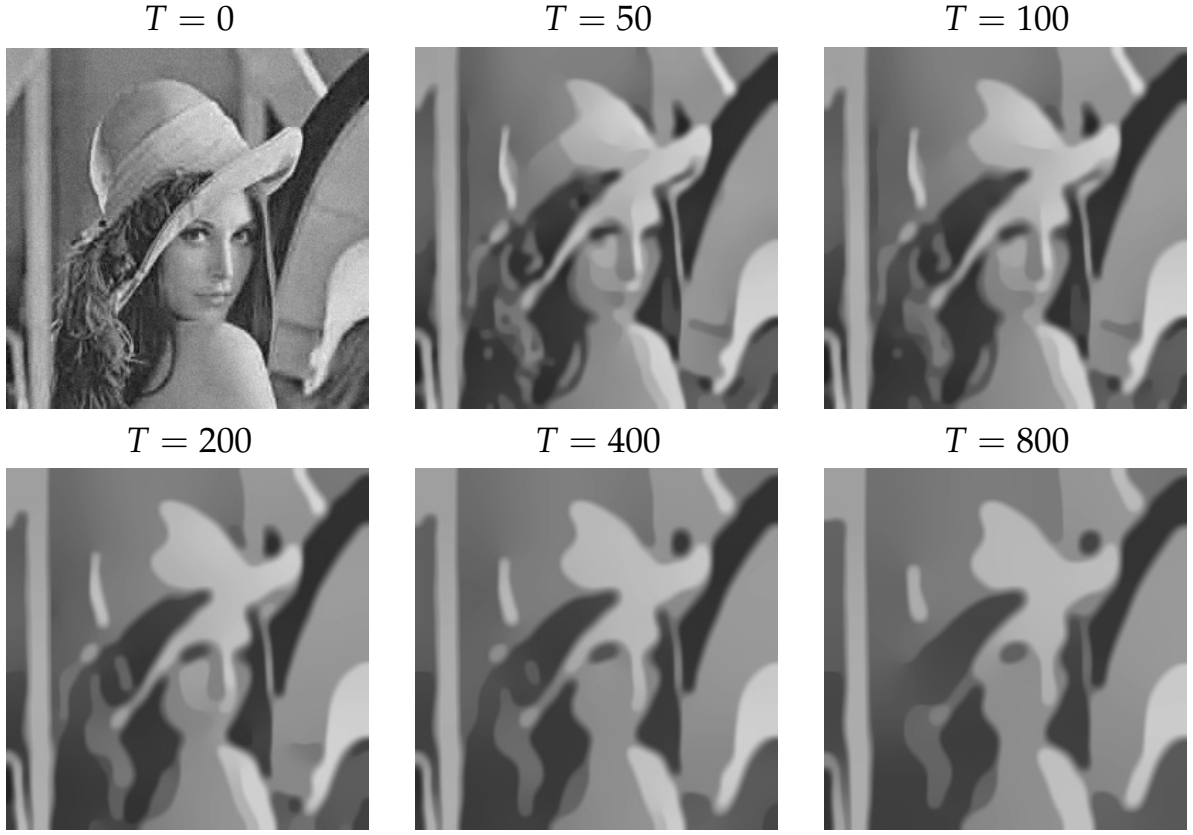
  where λ denotes the contrast parameter.

- preserves and enhances image edges by reducing the diffusivity λ_1 perpendicular to edges for sufficiently large values of $|\nabla u_\sigma|$.
- Specifically, the diffusion tensor is given by the formula:

$$D = \begin{bmatrix} (u_\sigma)_x & -(u_\sigma)_y \\ (u_\sigma)_y & (u_\sigma)_x \end{bmatrix} \cdot \begin{bmatrix} \lambda_1(|\nabla u_\sigma|) & 0 \\ 0 & \lambda_2(|\nabla u_\sigma|) \end{bmatrix} \cdot \begin{bmatrix} (u_\sigma)_x & -(u_\sigma)_y \\ (u_\sigma)_y & (u_\sigma)_x \end{bmatrix}^{-1}$$

Sample Results of Edge Enhancing Diffusion

- Smoothing process diminishes noise and fine image details while retaining and enhancing edges as in the Perona-Malik type nonlinear diffusion.

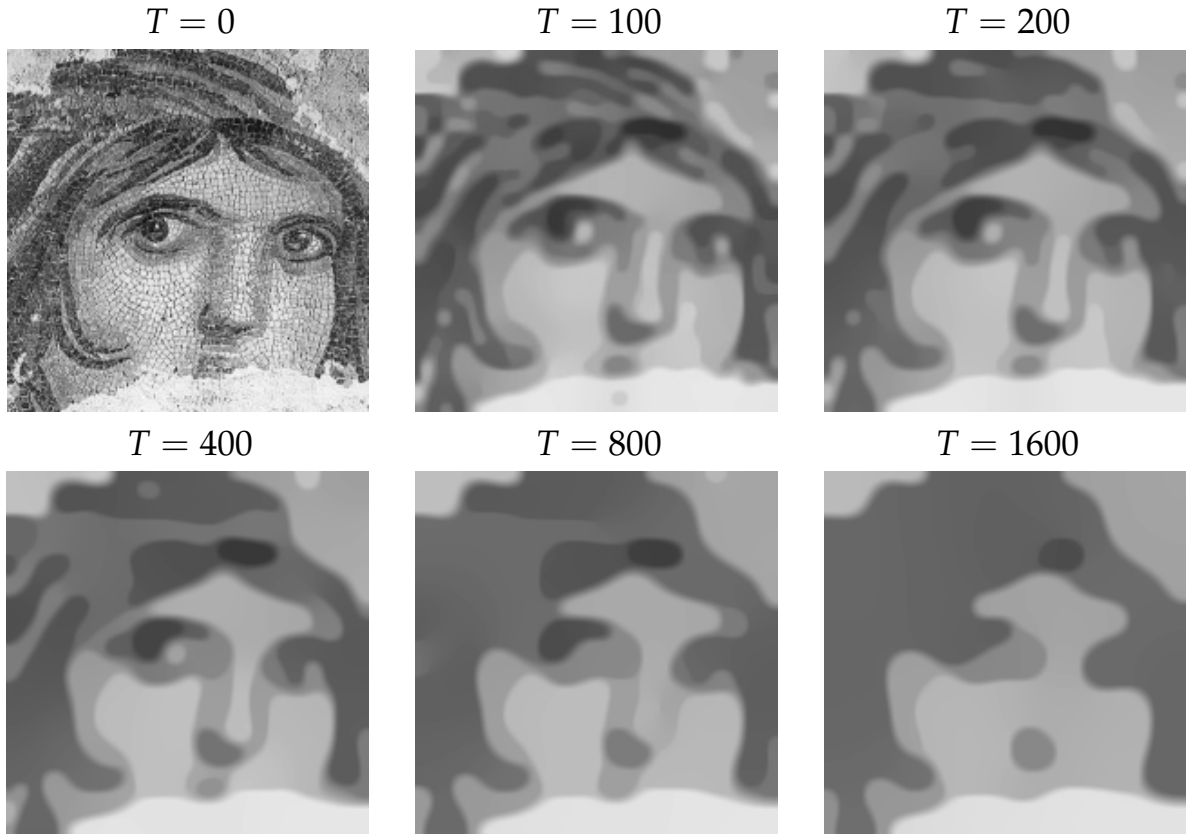


$(\lambda = 2, \sigma = 1)$

Sample Results of Edge Enhancing Diffusion

- Corners become more rounded in the anisotropic model compared to the Perona-Malik filter.

- Smoothing along edges and not across them causes a slight shrinking effect in the image structures, eliminating fine or thin structures



$(\lambda = 1.8,\ \sigma = 1)$