BIL 717
Image Processing
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Variational Segmentation Models

Review – Nonlinear Diffusion

- use nonlinear PDEs to create a scale space representation
  - consists of gradually simplified images
  - some image features such as edges are maintained or even enhanced.
- Perona-Malik Type Nonlinear Diffusion (1990)
- Total Variation (TV) Regularization (1992)
- Weickert's Edge Enhancing Diffusion (1994)

Review - Perona-Malik Type Nonlinear Diffusion

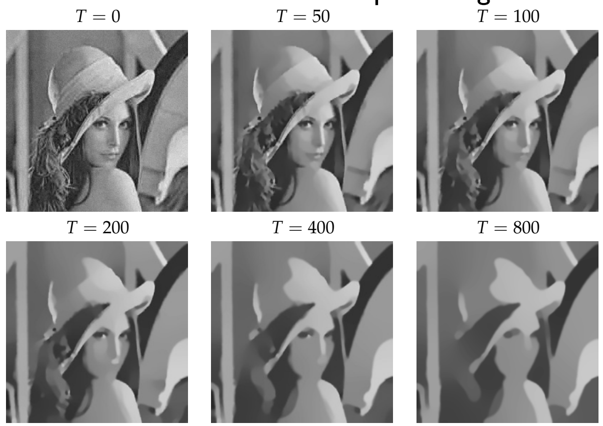
- The Perona-Malik equation is:

$$\frac{\partial u}{\partial t} = \nabla \cdot (g(|\nabla u|)\nabla u)$$

  with homogeneous Neumann boundary conditions and the initial condition $u0(x) = f(x)$, f denoting the input image.
- Constant diffusion coefficient of linear equation is replaced with a smooth non-increasing diffusivity function g satisfying
  - $g(0) = 1$,
  - $g(s) \geq 0$,
  - $\lim_{s\to\infty} g(s) = 0$
- <u>The diffusivities become variable in both space and time</u>.

Review - Perona-Malik Type Nonlinear Diffusion

- Smoothing process diminishes noise while retaining or enhancing edges since it considers a kind of a priori edge knowledge



$(\lambda = 1, \sigma = 1)$

Review - Total Variation (TV) Regularization

- Rudin et al. (1992) formulated image restoration as minimization of the total variation (TV) of a given image under certain assumptions on the noise.
- The Total Variation (TV) regularization model is generally defined as:

$$E_{TV}(u) = \int_{\Omega} \left(\frac{1}{2}(u-f)^2 + \alpha |\nabla u| \right) dx$$

  - $\Omega \subset \mathbf{R}^2$ is connected, bounded, open subset representing the image domain,
  - f is an image defined on Ω,
  - u is the smooth approximation of f,
  - $\alpha > 0$ is a scalar.

Review - TV Restoration results

$\alpha = 50$ $\alpha = 100$ $\alpha = 200$

- The value of α specifies the relative importance of the fidelity term and thus the level of smoothing.

Review - Total Variation (TV) Regularization

- The Total Variation (TV) regularization model:

$$E_{TV}(u) = \int_{\Omega} \left(\frac{1}{2}(u-f)^2 + \alpha |\nabla u| \right) dx$$

- The gradient descent equation for Equation (10) is defined by:

$$\frac{\partial u}{\partial t} = \nabla \cdot \left(\frac{\nabla u}{|\nabla u|} \right) - \frac{1}{\alpha}(u-f); \quad \left. \frac{\partial u}{\partial n} \right|_{\partial \Omega} = 0$$

- The value of α specifies the relative importance of the fidelity term.
- It can be interpreted as a scale parameter that determines the level of smoothing.

Review - TV Regularization and TV Flow

- TV regularization can be associated with a nonlinear diffusion filter, the so-called *TV flow*
- Ignoring the fidelity term in the TV regularization model leads to the PDE:

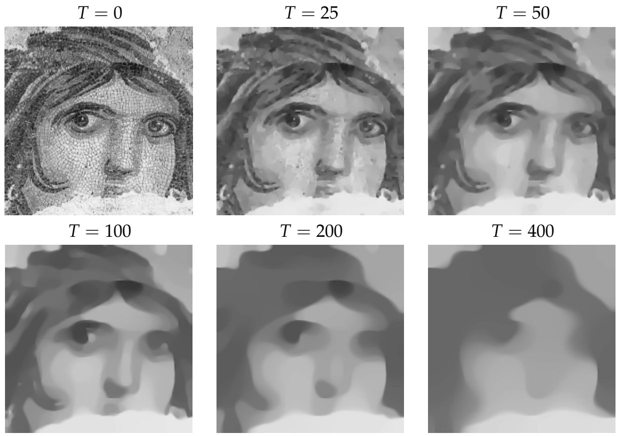
$$\frac{\partial u}{\partial t} = \nabla \cdot (g(|\nabla u|)\nabla u)$$

with $u^0 = f$ and the diffusivity function $g(|\nabla u|) = \frac{1}{|\nabla u|}$

- Notice that this diffusivity function has no additional contrast parameter as compared with the Perona-Malik diffusivities.

Review - Sample TV Flow results

- Corresponding smoothing process yields segmentation-like, piecewise constant images.



Review - Edge Enhancing Diffusion

- Proposed by Weickert (1994)
- an anisotropic nonlinear diffusion model with better edge enhancing capabilities than the Perona-Malik model
- can be described by the equation:

$$\frac{\partial u}{\partial t} = \nabla \cdot (D(\nabla u)\nabla u)$$

where

- u is the smoothed image,
- f is the input image ($u^0(x) = f(x)$),
- D represents a matrix-valued diffusion tensor that describes the smoothing directions and the corresponding diffusivities

Review - Edge Enhancing Diffusion

- Suggested eigenvalues are

$$\begin{aligned} \lambda_1(|\nabla u_\sigma|) &= \begin{cases} 1 & \text{if } |\nabla u_\sigma| = 0 \\ 1 - exp\left(-\frac{3.31488}{(|\nabla u_\sigma|/\lambda)^8}\right) & \text{otherwise,} \end{cases} \\ \lambda_2(|\nabla u_\sigma|) &= 1 \end{aligned}$$

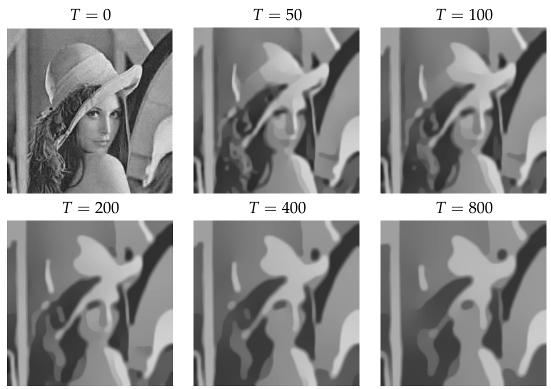
where λ denotes the contrast parameter.

- preserves and enhances image edges by reducing the diffusivity λ_1 perpendicular to edges for sufficiently large values of $|\nabla u_\sigma|$.
- Specifically, the diffusion tensor is given by the formula:

$$D = \begin{bmatrix} (u_\sigma)_x & -(u_\sigma)_y \\ (u_\sigma)_y & (u_\sigma)_x \end{bmatrix} \cdot \begin{bmatrix} \lambda_1(|\nabla u_\sigma|) & 0 \\ 0 & \lambda_2(|\nabla u_\sigma|) \end{bmatrix} \cdot \begin{bmatrix} (u_\sigma)_x & -(u_\sigma)_y \\ (u_\sigma)_y & (u_\sigma)_x \end{bmatrix}^{-1}$$

Review - Sample Results of Edge Enhancing Diffusion

- Smoothing process diminishes noise and fine image details while retaining and enhancing edges as in the Perona-Malik type nonlinear diffusion.



$(\lambda = 2, \sigma = 1)$

Variational Segmentation Models

- Segmentation is formalized as a functional minimization.

- Mumford-Shah Model (1989)
- Ambrosio-Tortorelli Model (1990)
- Shah's Model (1996)
- Context-guided Mumford-Shah Model (2009)
- Chan-Vese Model (2001)

Mumford-Shah (MS) Segmentation Model

- Mumford & Shah, Comm. Pure Appl. Math., 1989
- Segmentation is formalized as a functional minimization: Given an image *f*, compute a piecewise smooth image *u* and an edge set Γ

$$E_{MS}(u,\Gamma) = \beta \int_{\Omega} (u-f)^2 dx + \alpha \int_{\Omega \setminus \Gamma} |\nabla u|^2 dx + length(\Gamma)$$

  - $\Omega \subset \mathbf{R}^2$ is connected, bounded, open subset representing the image domain,
  - f is an image defined on Ω,
  - $\Gamma \subset \Omega$ is the edge set segmenting Ω,
  - u is the piecewise smooth approximation of f,
  - $\alpha,\ \beta > 0$ are the scale space parameters.

Mumford-Shah (MS) Segmentation Model

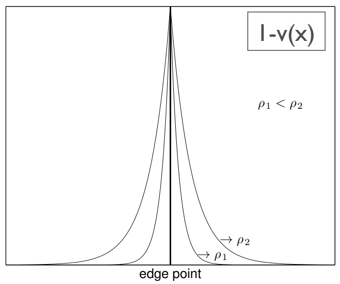
$$E_{MS}(u,\Gamma) = \underbrace{\beta \int_{\Omega} (u-f)^2 dx}_{\text{data fidelity term}} + \underbrace{\alpha \int_{\Omega \setminus \Gamma} |\nabla u|^2 dx + length(\Gamma)}_{\text{regularization or smoothness term}}$$

- Smoothing and edge detection processes work jointly to partition an image into segments.
- Unknown edge set Γ of a lower dimension makes the minimization of the MS model very difficult.
- In literature several approaches for approximating the MS model are suggested.

Ambrosio-Tortorelli (AT) Approximation

$$E_{AT}(u,v) = \int_{\Omega} \left(\beta(u-f)^2 + \alpha(v^2|\nabla u|^2) + \underbrace{\frac{1}{2}\left(\rho|\nabla v|^2 + \frac{(1-v)^2}{\rho}\right)}_{\text{length}(\Gamma)} \right) dx$$

- Unknown edge set Γ is replaced with a continuous function v(x)
  - $v \approx 0$ along image edges
  - v grows rapidly towards 1 away from edges
- The function *v* can be interpreted as a blurred version of the edge set.
- The parameter ρ specifies the level of blurring.



Ambrosio-Tortorelli (AT) Approximation: u and v processes

- Piecewise smooth image *u* and the edge strength function *v* are simultaneously computed via the solution of the following system of coupled PDEs:

$$\frac{\partial u}{\partial t} = \nabla \cdot (v^2 \nabla u) - \frac{\beta}{\alpha}(u - f); \quad \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0$$

$$\frac{\partial v}{\partial t} = \nabla^2 v - \frac{2\alpha|\nabla u|^2 v}{\rho} - \frac{(v-1)}{\rho^2}; \quad \left.\frac{\partial v}{\partial n}\right|_{\partial\Omega} = 0$$

Ambrosio-Tortorelli (AT) Approximation: u and v processes


f: raw image


u: smooth image


v: edge strength function

Ambrosio-Tortorelli (AT) Approximation: u and v processes

- Piecewise smooth image *u* and the edge strength function *v* are simultaneously computed via the solution of the following system of coupled PDEs:

$$\frac{\partial u}{\partial t} = \nabla \cdot (v^2 \nabla u) - \frac{\beta}{\alpha}(u - f); \quad \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0$$

$$\frac{\partial v}{\partial t} = \nabla^2 v - \frac{2\alpha|\nabla u|^2 v}{\rho} - \frac{(v-1)}{\rho^2}; \quad \left.\frac{\partial v}{\partial n}\right|_{\partial\Omega} = 0$$

- PDE for each variable can be interpreted as a biased diffusion equation that minimizes a convex quadratic functional in which the other variable is kept fixed.

Ambrosio-Tortorelli (AT) Approximation: u process

- Keeping *v* fixed, PDE for the process *u* minimizes the following convex quadratic functional:

$$\int_\Omega \left(\alpha v^2 |\nabla u|^2 + \beta (u - f)^2\right) dx$$

- The data fidelity term provides a bias that forces *u* to be close to the original image *f* .
- In the regularization term, the edge strength function *v* specifies the boundary points and guides the smoothing accordingly.
- Since $v \approx 0$ along the boundaries, no smoothing is carried out at the boundary points, thus the edges are preserved.

Ambrosio-Tortorelli (AT) Approximation: v process

- Keeping *u* fixed, PDE for the process *v* minimizes the following convex quadratic functional:

$$\frac{\rho}{2}\int_{\Omega}\left(|\nabla v|^2 + \frac{1+2\alpha\rho|\nabla u|^2}{\rho^2}\left(v - \frac{1}{1+2\alpha\rho|\nabla u|^2}\right)^2\right)dx$$

- The function *v* is nothing but a smoothing of $\frac{1}{1+2\alpha\rho|\nabla u|^2}$
- The smoothness term forces some spatial organization by requiring the edges to be smooth.
- Ignoring the smoothness term and letting ρ go to 0, we have

$$v \approx \frac{1}{1+2\alpha\rho|\nabla u|^2}$$

Relating with the Perona-Malik Diffusion

- Replacing *v* with $1/(1+2\alpha\rho|\nabla u|^2)$, PDE for the process *u* can be interpreted as a biased Perona-Malik type nonlinear diffusion:

$$\frac{\partial u}{\partial t} = \nabla\cdot(g(|\nabla u|)\nabla u) - \frac{\beta}{\alpha}(u-f)$$

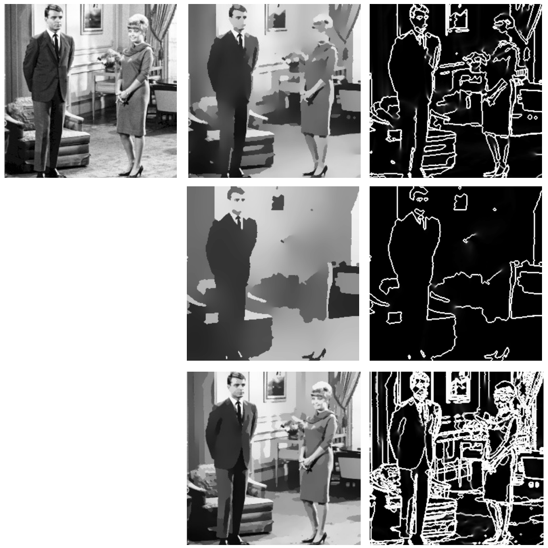
with

$$g(|\nabla u|) = \left(\frac{1}{1+|\nabla u|^2/\lambda^2}\right)^2$$

$$\lambda^2 = 1/\,(2\alpha\rho)$$

- $\sqrt{1/\,(2\alpha\rho)}$ as a contrast parameter
- Relative importance of the regularization term (scale) depends on the ratio between α and β.

Sample Results of the AT model



$\alpha = 1, \beta = 0.01, \rho = 0.01$

$\alpha = 1, \beta = 0.001, \rho = 0.01$

$\alpha = 4, \beta = 0.04, \rho = 0.01$

Numerical Implementation

- Original model:

$$\frac{\partial u}{\partial t} = \nabla\cdot(v^2\nabla u) - \frac{\beta}{\alpha}(u-f); \quad \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0$$

- Space discrete version:

$$\begin{aligned}\frac{du_{i,j}}{dt} &= v^2_{i+\frac{1}{2},j}\cdot(u_{i+1,j}-u_{i,j}) - v^2_{i-\frac{1}{2},j}\cdot(u_{i,j}-u_{i-1,j})\\ &+ v^2_{i,j+\frac{1}{2}}\cdot(u_{i,j+1}-u_{i,j}) - v^2_{i,j-\frac{1}{2}}\cdot(u_{i,j}-u_{i,j-1})\\ &- \frac{\beta}{\alpha}(u_{i,j}-f_{i,j})\ ,\end{aligned}$$

with $v_{i,j\pm\frac{1}{2}} = \frac{v_{i,j\pm 1}+v_{i,j}}{2}$ and $v_{i\pm\frac{1}{2},j} = \frac{v_{i\pm 1,j}+v_{i,j}}{2}$

Numerical Implementation

- Original model:

$$\frac{\partial v}{\partial t} = \nabla^2 v - \frac{2\alpha|\nabla u|^2 v}{\rho} - \frac{(v-1)}{\rho^2}; \quad \left.\frac{\partial v}{\partial n}\right|_{\partial\Omega} = 0$$

- Space discrete version:

$$\begin{aligned}\frac{dv_{i,j}}{dt} &= v_{i+1,j} + v_{i-1,j} + v_{i,j+1} + v_{i,j-1} - 4v_{i,j} \\ &- \frac{2\alpha|\nabla u_{i,j}|^2 v_{i,j}}{\rho} - \frac{(v_{i,j}-1)}{\rho^2}.\end{aligned}$$

Numerical Implementation

- Space-time discrete versions:

$$\begin{aligned}\frac{u_{i,j}^{k+1} - u_{i,j}^{k}}{\Delta t} &= \left(v_{i+\frac{1}{2},j}^{k}\right)^2 \cdot u_{i+1,j}^{k} + \left(v_{i-\frac{1}{2},j}^{k}\right)^2 \cdot u_{i-1,j}^{k} \\ &+ \left(v_{i,j+\frac{1}{2}}^{k}\right)^2 \cdot u_{i,j+1}^{k} + \left(v_{i,j-\frac{1}{2}}^{k}\right)^2 \cdot u_{i,j-1}^{k} \\ &- \left(\left(v_{i+\frac{1}{2},j}^{k}\right)^2 + \left(v_{i-\frac{1}{2},j}^{k}\right)^2 + \left(v_{i,j+\frac{1}{2}}^{k}\right)^2 + \left(v_{i,j-\frac{1}{2}}^{k}\right)^2\right) \cdot u_{i,j}^{k} \\ &- \frac{\beta}{\alpha}\left(u_{i,j}^{k+1} - f_{i,j}\right),\end{aligned}$$

$$\begin{aligned}\frac{v_{i,j}^{k+1} - v_{i,j}^{k}}{\Delta t} &= v_{i+1,j}^{k} + v_{i-1,j}^{k} + v_{i,j+1}^{k} + v_{i,j-1}^{k} - 4v_{i,j}^{k} \\ &- \frac{\alpha\left(\left(u_{i+1,j}^{k} - u_{i-1,j}^{k}\right)^2 + \left(u_{i,j+1}^{k} - u_{i,j-1}^{k}\right)^2\right) v_{i,j}^{k+1}}{2\rho} - \frac{\left(v_{i,j}^{k+1} - 1\right)}{\rho^2}\end{aligned}$$

A Common Framework for Curve Evolution, Segmentation and Anisotropic Diffusion

- Quadratic cost functions in the data fidelity and the smoothing terms are replaced with $L1$-functions (Shah, CVPR 1996):

$$E_S(u,v) = \int_\Omega \left(\beta\,|u-f| + \alpha\, v^2|\nabla u| + \frac{1}{2}\left(\rho|\nabla v|^2 + \frac{(1-v)^2}{\rho}\right)\right) dx$$

- As $\rho \to 0$, this energy functional converges to the following functional:

$$E_{S2}(u,\Gamma) = \frac{\beta}{\alpha}\int_\Omega |u-f|dx + \int_{\Omega\setminus\Gamma} |\nabla u|dx + \int_\Gamma \frac{J_u}{1+\alpha J_u} ds$$

with $J_u = |u^+ - u^-|$ indicating the jump in u across Γ, and $u+$ and $u-$ denote inten-sity values on two sides of Γ

A Common Framework for Curve Evolution, Segmentation and Anisotropic Diffusion

- Minimizing the energy functional results in the following system of coupled PDEs:

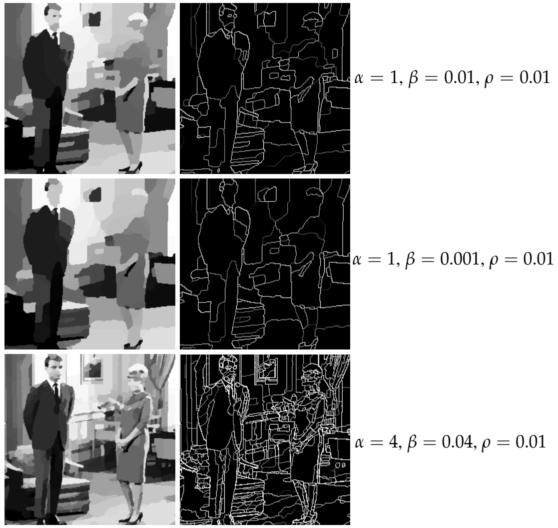
$$\frac{\partial u}{\partial t} = 2\nabla v \cdot \nabla u + v|\nabla u|\ curv(u) - \frac{\beta}{\alpha v}|\nabla u|\frac{(u-f)}{|u-f|}; \quad \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0$$

$$\frac{\partial v}{\partial t} = \nabla^2 v - \frac{2\alpha|\nabla u|v}{\rho} - \frac{(v-1)}{\rho^2}; \quad \left.\frac{\partial v}{\partial n}\right|_{\partial\Omega} = 0$$

with $curv(u) = \nabla \cdot \left(\frac{\nabla u}{|\nabla u|}\right)$

- Replacing $L2$-norms in both the data fidelity and the smoothness terms by their $L1$-norms generates shocks in u and thus object boundaries are recovered as actual discontinuities.

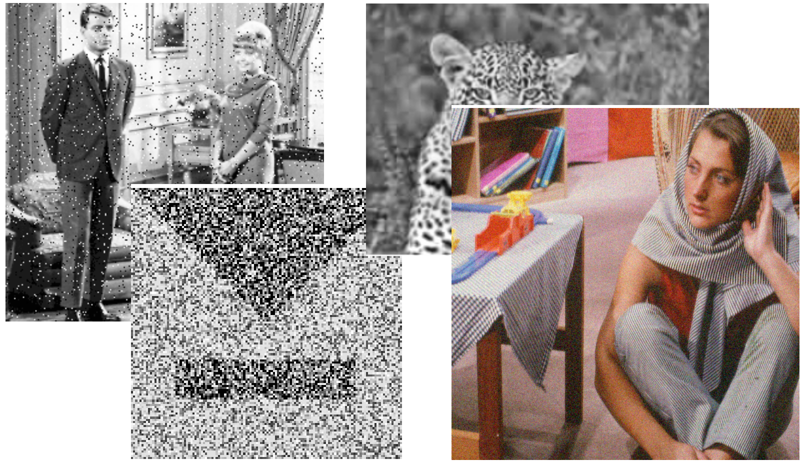
Sample Results of Shah (CVPR96)



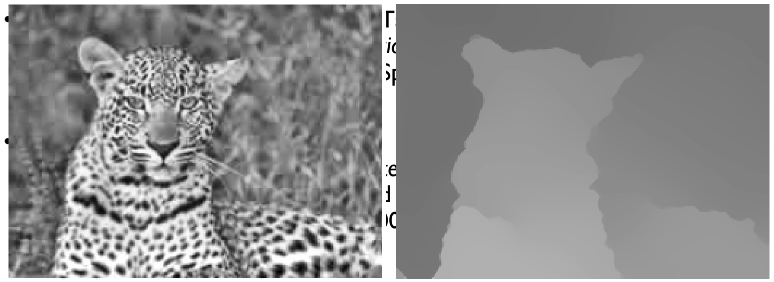
- Smoothing process of *u* gives rise to more cartoon-like, piecewise constant images

  but with some unintuitive regions

Challenging Cases

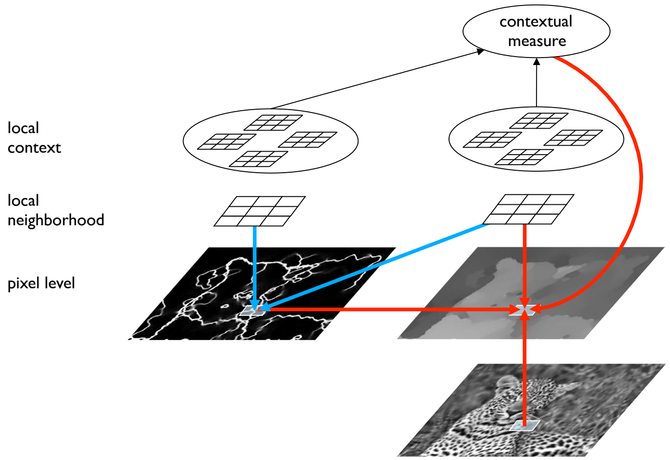


Context-Guided Image Smoothing



- Contextual knowledge extracted from local image regions guides the regularization process.

Context-Guided Image Smoothing



Context-Guided Image Smoothing

- 2 coupled processes (u and v modules)

$$\frac{\partial v}{\partial t} = \nabla^2 v - \frac{2\alpha|\nabla u|^2 v}{\rho} - \frac{(v-1)}{\rho^2}; \quad \left.\frac{\partial v}{\partial n}\right|_{\partial\Omega} = 0$$

$$\frac{\partial u}{\partial t} = \nabla \cdot ((cv)^2 \nabla u) - \frac{\beta}{\alpha}(u-f); \quad \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0$$

$$cv = \phi v + (1-\phi)V$$

$\phi \in [0,1]$ $\quad$ $V \in \{0,1\}$

The Roles of ϕ and V

1. Eliminating an accidentally occurring event
   - e.g., a high gradient due to noise
   - V=1, ϕ is low for accidental occurrences

$$(cv)_i^2 = (\phi_i v_i + (1-\phi_i)\,1)^2$$

2. Preventing an accidental elimination of a feature of interest
   - e.g., encourage edge formation
   - V=0, ϕ is low for meaningful occurrences

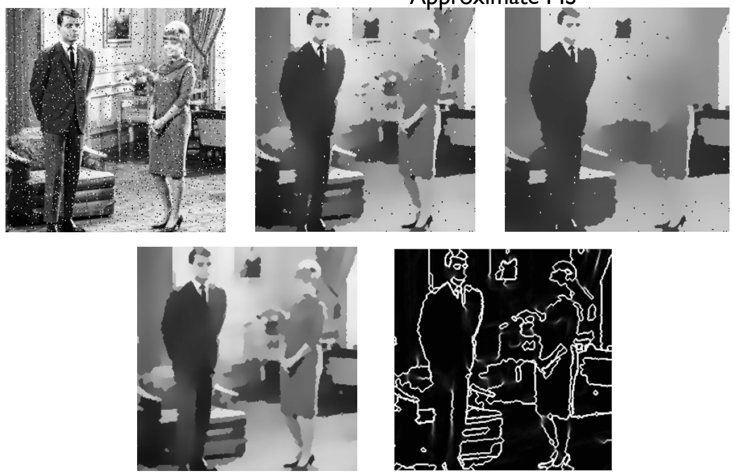
$$(cv)_i^2 = (\phi_i v_i + (1-\phi_i)\,0)^2$$

Experimental Results

- Suggested contextual measures:
  1. Directional consistency of edges
     - shapes have smooth boundaries
  2. Edge Continuity
     - gap filling
  3. Texture Edges
     - boundary between different textured regions
  4. Local Scale
     - Resolution varies throughout the image

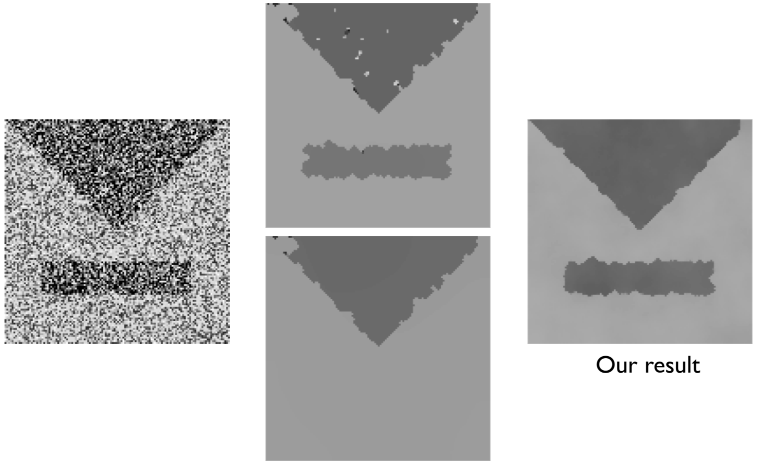
Directional Consistency

Approximate MS



Our result

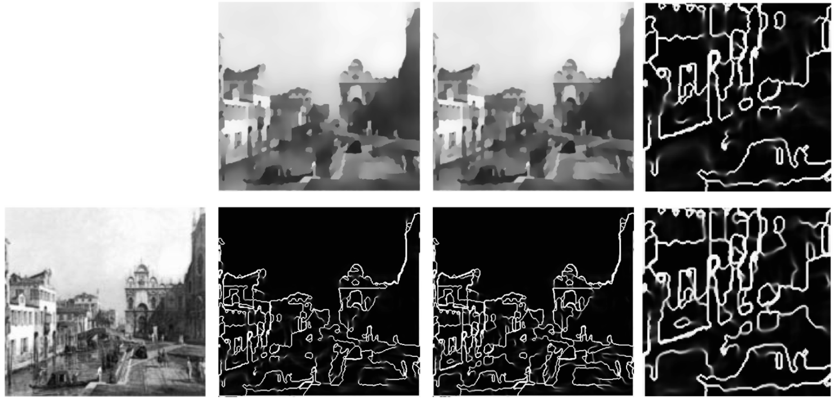
Directional Consistency



Our result

Approximate MS

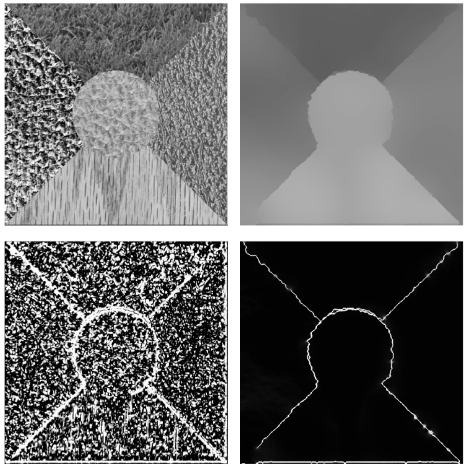
Edge Continuity



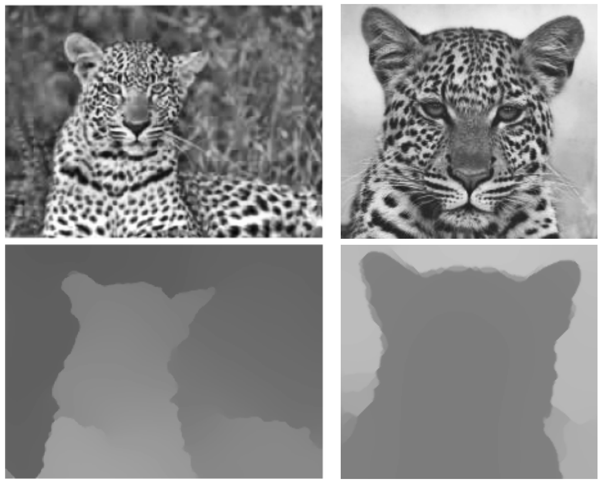
Approximate MS   Our result

Coalition of Directional Consistency and Texture Edges

Φ^{te}



Coalition of Directional Consistency, Edge Continuity and Texture Edges



Local Scale

Approximate MS

Our result

ϕ^{ls}

Active Contours Without Edges

- A level-set based approximation of the Mumford-Shah model proposed by Chan and Vese (2001).
- Level sets provide an implicit contour representation where an evolving curve is represented with the zero-level line of a level set function.

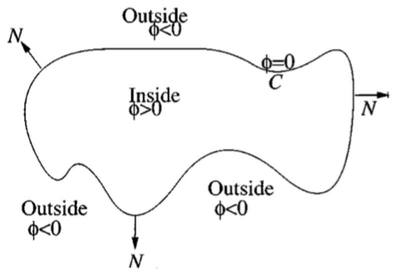


Image credit: Chan & Vese, 2001

Active Contours Without Edges

- **Basic idea:** Fitting term

$$\int_{inside(C)} |u_0 - c_1|^2 dxdy + \int_{outside(C)} |u_0 - c_2|^2 dxdy$$

$$\text{where} \begin{cases} c_1 = \text{average of } u_0 \text{ inside } C \\ c_2 = \text{average of } u_0 \text{ outside } C \end{cases}$$

Fit > 0 Fit > 0 Fit > 0 Fit ≈ 0

- **Minimize:** the Fitting term + Length(C)

Slide credit: L.Vese

Active Contours Without Edges

- A level-set based approximation of the Mumford-Shah model proposed by Chan and Vese (2001):

$$\begin{aligned} E_{CV}(c_1, c_2, \phi) &= \lambda_1 \int_\Omega (f - c_1)^2 H(\phi) dx + \lambda_2 \int_\Omega (f - c_2)^2 (1 - H(\phi)) dx \\ &+ \mu \int_\Omega |\nabla H(\phi)| dx \end{aligned}$$

where λ_1, $\lambda_2 > 0$ and $\mu \geq 0$ are fixed parameters.

- The length parameter μ can be interpreted as a scale parameter. It determines the relative importance of the length term.
- The possibility of detecting smaller objects/regions increases with decreasing μ.

Active Contours Without Edges

- A level-set based approximation of the Mumford-Shah model proposed by Chan and Vese (2001):

$$E_{CV}(c_1, c_2, \phi) = \lambda_1 \int_\Omega (f - c_1)^2 H(\phi) dx + \lambda_2 \int_\Omega (f - c_2)^2 (1 - H(\phi)) dx + \mu \int_\Omega |\nabla H(\phi)| dx$$

- The model represents the segmented image with the variables c_1, c_2 and $H(\phi)$, where $H(\phi)$ denotes the Heaviside function of the level set function ϕ:

$$H(z) = \begin{cases} 1 & \text{if } z \geq 0 \\ 0 & \text{if } z < 0 \end{cases}$$

Active Contours Without Edges

- A level-set based approximation of the Mumford-Shah model proposed by Chan and Vese (2001):

$$E_{CV}(c_1, c_2, \phi) = \lambda_1 \int_\Omega (f - c_1)^2 H(\phi) dx + \lambda_2 \int_\Omega (f - c_2)^2 (1 - H(\phi)) dx + \mu \int_\Omega |\nabla H(\phi)| dx$$

- c_1 and c_2 denote the average gray values of object and background regions indicated by $\phi \geq 0$ and $\phi < 0$, respectively.
- Chan-Vese model can be seen as a two-phase piecewise constant approximation of the MS model.

Active Contours Without Edges

- A level-set based approximation of the Mumford-Shah model proposed by Chan and Vese (2001):

$$E_{CV}(c_1, c_2, \phi) = \lambda_1 \int_\Omega (f - c_1)^2 H(\phi) dx + \lambda_2 \int_\Omega (f - c_2)^2 (1 - H(\phi)) dx + \mu \int_\Omega |\nabla H(\phi)| dx$$

where $\lambda_1, \lambda_2 > 0$ and $\mu \geq 0$ are fixed parameters.

Active Contours Without Edges

- Segmentation involves minimizing the energy functional with respect to c_1, c_2, and ϕ.
- Keeping ϕ fixed, the average gray values c_1 and c_2 can be estimated as follows:

$$c_1 = \frac{\int_\Omega f(x) H(\phi(x)) dx}{\int_\Omega H(\phi(x)) dx},$$

$$c_2 = \frac{\int_\Omega f(x)(1 - H(\phi(x))) dx}{\int_\Omega (1 - H(\phi(x))) dx}$$

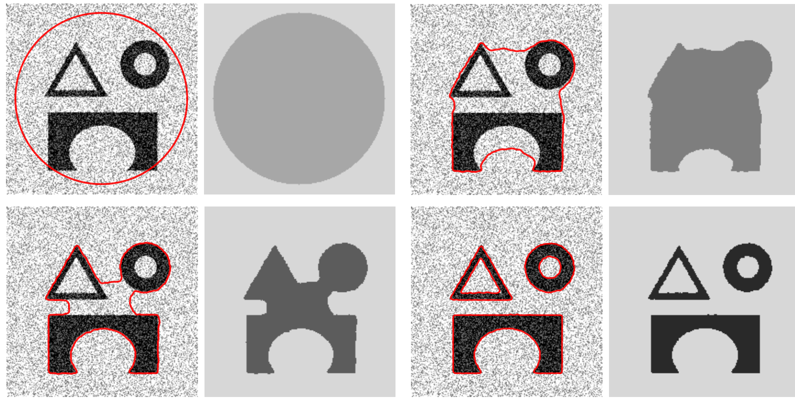
Active Contours Without Edges

- Segmentation involves minimizing the energy functional with respect to c_1, c_2, and ϕ.
- Keeping c_1 and c_2 fixed and using the calculus of variations for the given functional, the gradient descent equation for the evolution of ϕ is derived as:

$$\frac{\partial \phi}{\partial t} = \delta(\phi)\left[\mu\, \nabla \cdot \left(\frac{\nabla \phi}{|\nabla \phi|}\right) - \lambda_1 (f - c_1)^2 + \lambda_2 (f - c_2)^2\right]$$

Sample result of the Chan-Vese Model

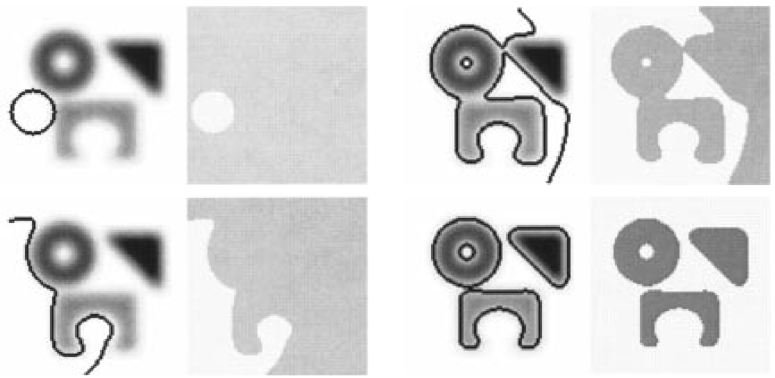
- As the zero-level line of the evolving level set function ϕ is attracted to object boundaries, a more accurate piecewise constant approximations of the original image f is recovered.



$\phi_0 = -\sqrt{(x-100)^2 + (y-100)^2} + 90$ $\quad$ $\lambda_1 = \lambda_2 = 1,\ \mu = 0.5 \cdot 255^2$

Sample result of the Chan-Vese Model

- As the zero-level line of the evolving level set function ϕ is attracted to object boundaries, a more accurate piecewise constant approximations of the original image f is recovered.



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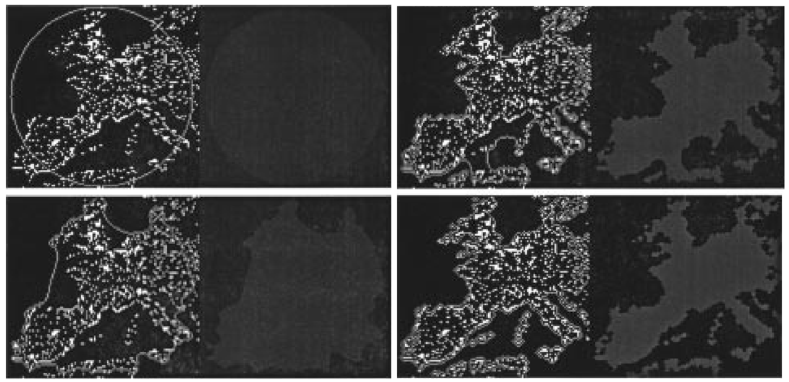

Numerical Implementation

- In the numerical approximation, regularized form of the Heaviside function is used:

$$
\begin{aligned}
H_\varepsilon(z) &= \frac{1}{2}\left(1+\frac{2}{\pi}arctan\left(\frac{z}{\varepsilon}\right)\right) \\
\delta_\varepsilon(z) &= \frac{dH_\varepsilon(z)}{dz} = \frac{1}{\pi}\frac{\varepsilon}{\varepsilon^2+z^2}
\end{aligned}
$$

Numerical Implementation

- Space-time discrete version:

$$
\begin{aligned}
\frac{\phi_{i,j}^{k+1}-\phi_{i,j}^{k}}{\Delta t} = \delta(\phi_{i,j}^{k}) \quad & \left[\mu\Delta_-^x\cdot\left(\frac{\Delta_+^x\phi_{i,j}^{k+1}}{\sqrt{(\Delta_+^x\phi_{i,j}^{k})^2+(\phi_{i,j+1}^{k}-\phi_{i,j-1}^{k})^2/4}}\right)\right. \\
& + \mu\Delta_-^y\cdot\left(\frac{\Delta_+^y\phi_{i,j}^{k+1}}{\sqrt{(\phi_{i+1,j}^{k}-\phi_{i-1,j}^{k})^2/4+(\Delta_+^y\phi_{i,j}^{k})^2}}\right) \\
& \left. - \lambda_1\left(f_{i,j}-c_1(\phi^k)\right)^2+\lambda_2\left(f_{i,j}-c_2(\phi^k)\right)^2\right]
\end{aligned}
$$

with

$$
\begin{aligned}
\Delta_-^x\phi_{i,j} = \phi_{i,j}-\phi_{i-1,j}, \quad \Delta_+^x\phi_{i,j} = \phi_{i+1,j}-\phi_{i,j}, \\
\Delta_-^y\phi_{i,j} = \phi_{i,j}-\phi_{i,j-1}, \quad \Delta_+^y\phi_{i,j} = \phi_{i,j+1}-\phi_{i,j}.
\end{aligned}
$$