**Review - Perona-Malik Type Nonlinear Diffusion**

- The Perona-Malik equation is:

\[
\frac{\partial u}{\partial t} = \nabla \cdot (g(|\nabla u|) \nabla u)
\]

with homogeneous Neumann boundary conditions and the initial condition \(u(0, x) = f(x)\), \(f\) denoting the input image.

- Constant diffusion coefficient of linear equation is replaced with a smooth non-increasing diffusivity function \(g\) satisfying
  - \(g(0) = 1\),
  - \(g(s) \geq 0\),
  - \(\lim_{s \to \infty} g(s) = 0\)

- The diffusivities become variable in both space and time.

**Review – Nonlinear Diffusion**

- use nonlinear PDEs to create a scale space representation
  - consists of gradually simplified images
  - some image features such as edges are maintained or even enhanced.

- Perona-Malik Type Nonlinear Diffusion (1990)
- Total Variation (TV) Regularization (1992)
- Weickert’s Edge Enhancing Diffusion (1994)
**Review - Total Variation (TV) Regularization**

- Rudin et al. (1992) formulated image restoration as minimization of the total variation (TV) of a given image under certain assumptions on the noise.
- The Total Variation (TV) regularization model is generally defined as:

\[
E_{TV}(u) = \int_{\Omega} \left( \frac{1}{2} (u - f)^2 + \alpha |\nabla u| \right) \, dx
\]

- \( \Omega \subset \mathbb{R}^2 \) is connected, bounded, open subset representing the image domain,
- \( f \) is an image defined on \( \Omega \),
- \( u \) is the smooth approximation of \( f \),
- \( \alpha > 0 \) is a scalar.

**Review - TV Restoration results**

- The value of \( \alpha \) specifies the relative importance of the fidelity term and thus the level of smoothing.

- Figure 6: Example TV restoration results. (a) Source image. (b)-(d) Corresponding segmentations obtained with (b) \( T = 0 \), (c) \( T = 50 \), (d) \( T = 200 \).

**Review - Total Variation (TV) Regularization**

- The Total Variation (TV) regularization model:

\[
E_{TV}(u) = \int_{\Omega} \left( \frac{1}{2} (u - f)^2 + \alpha |\nabla u| \right) \, dx
\]

- The gradient descent equation for Equation (10) is defined by:

\[
\frac{\partial u}{\partial t} = \nabla \cdot \left( \frac{\nabla u}{|\nabla u|} \right) - \frac{1}{\alpha} (u - f); \quad \frac{\partial u}{\partial t} \big|_{\partial \Omega} = 0
\]

- The value of \( \alpha \) specifies the relative importance of the fidelity term.
- It can be interpreted as a scale parameter that determines the level of smoothing.

**Review - TV Regularization and TV Flow**

- TV regularization can be associated with a nonlinear diffusion filter, the so-called TV flow
- Ignoring the fidelity term in the TV regularization model leads to the PDE:

\[
\frac{\partial u}{\partial t} = \nabla \cdot (g(|\nabla u|) \nabla u)
\]

with \( u^0 = f \) and the diffusivity function \( g(|\nabla u|) = \frac{1}{|\nabla u|} \)

- Notice that this diffusivity function has no additional contrast parameter as compared with the Perona-Malik diffusivities.
Review - Sample TV Flow results

- Corresponding smoothing process yields segmentation-like, piecewise constant images.

![Image](image1.png)

Review - Edge Enhancing Diffusion

- Proposed by Weickert (1994)
- An anisotropic nonlinear diffusion model with better edge enhancing capabilities than the Perona-Malik model
- Can be described by the equation:
  \[
  \frac{\partial u}{\partial t} = \nabla \cdot (D(\nabla u) \nabla u)
  \]
  where
  - \( u \) is the smoothed image,
  - \( f \) is the input image (\( u^0(x) = f(x) \)),
  - \( D \) represents a matrix-valued diffusion tensor that describes the smoothing directions and the corresponding diffusivities

Review - Sample Results of Edge Enhancing Diffusion

- Smoothing process diminishes noise and fine image details while retaining and enhancing edges as in the Perona-Malik type nonlinear diffusion.

![Image](image2.png)
Variational Segmentation Models

- Segmentation is formalized as a functional minimization.
- Ambrosio-Tortorelli Model (1990)
- Shah’s Model (1996)
- Chan-Vese Model (2001)

Mumford-Shah (MS) Segmentation Model

\[ E_{MS}(u, \Gamma) = \beta \int_{\Omega} (u - f)^2 dx + \alpha \int_{\Omega \setminus \Gamma} |\nabla u|^2 dx + \text{length}(\Gamma) \]

- Smoothing and edge detection processes work jointly to partition an image into segments.
- Unknown edge set \( \Gamma \) of a lower dimension makes the minimization of the MS model very difficult.
- In literature several approaches for approximating the MS model are suggested.

Mumford-Shah (MS) Functional

\[ E_{MS}(u, \Gamma) = \beta \int_{\Omega} (u - f)^2 dx + \alpha \int_{\Omega \setminus \Gamma} |\nabla u|^2 dx + \text{length}(\Gamma) \]

- \( \Omega \subset \mathbb{R}^2 \) is connected, bounded, open subset representing the image domain.
- \( f \) is an image defined on \( \Omega \).
- \( \Gamma \subset \Omega \) is the edge segmenting \( \Omega \).
- \( u \) is the piecewise smooth approximation of \( f \).
- \( \alpha, \beta > 0 \) are the scale space parameters.

Ambrosio-Tortorelli (AT) Approximation

\[ E_{AT}(u, v) = \int_{\Omega} \left( \beta (u - f)^2 + \alpha (v^2 |\nabla u|^2) + \frac{1}{2} \left( \rho |\nabla v|^2 + \frac{(1 - v)^2}{\rho^2} \right) \right) dx \]

- Unknown edge set \( \Gamma \) is replaced with a continuous function \( v(x) \)
  - \( v \approx 0 \) along image edges
  - \( v \) grows rapidly towards 1 away from edges
- The function \( v \) can be interpreted as a blurred version of the edge set.
- The parameter \( \rho \) specifies the level of blurring.
### Ambrosio-Tortorelli (AT) Approximation: $u$ and $v$ processes

- Piecewise smooth image $u$ and the edge strength function $v$ are simultaneously computed via the solution of the following system of coupled PDEs:

\[
\frac{\partial u}{\partial t} = \nabla \cdot (v^2 \nabla u) - \frac{\beta}{\alpha} (u - f); \quad \frac{\partial u}{\partial n}\bigg|_{\partial \Omega} = 0
\]

\[
\frac{\partial v}{\partial t} = \Delta v - 2 \alpha |\nabla u|^2 v - \frac{(v - 1)}{\rho} - \frac{(v - 1)}{\rho^2}; \quad \frac{\partial v}{\partial n}\bigg|_{\partial \Omega} = 0
\]

- PDE for each variable can be interpreted as a biased diffusion equation that minimizes a convex quadratic functional in which the other variable is kept fixed.

### Ambrosio-Tortorelli (AT) Approximation: $u$ process

- Keeping $v$ fixed, PDE for the process $u$ minimizes the following convex quadratic functional:

\[
\int_{\Omega} \left( \alpha v^2 |\nabla u|^2 + \beta (u - f)^2 \right) dx
\]

- The data fidelity term provides a bias that forces $u$ to be close to the original image $f$.
- In the regularization term, the edge strength function $v$ specifies the boundary points and guides the smoothing accordingly.
- Since $v \approx 0$ along the boundaries, no smoothing is carried out at the boundary points, thus the edges are preserved.
**Ambrosio-Tortorelli (AT) Approximation: v process**

- Keeping \( u \) fixed, PDE for the process \( v \) minimizes the following convex quadratic functional:
  \[
  \frac{\rho}{2} \int_{\Omega} \left( |\nabla v|^2 + \frac{1 + 2\alpha \rho |\nabla u|^2}{\rho^2} \left( v - \frac{1}{1 + 2\alpha \rho |\nabla u|^2} \right)^2 \right) \, dx 
  \]
  - The function \( v \) is nothing but a smoothing of \( \frac{1}{1 + 2\alpha \rho |\nabla u|^2} \)
  - The smoothness term forces some spatial organization by requiring the edges to be smooth.
  - Ignoring the smoothness term and letting \( \rho \) go to 0, we have \( v \approx \frac{1}{1 + 2\alpha \rho |\nabla u|^2} \)

**Sample Results of the AT model**

\( a = 1, \beta = 0.01, \rho = 0.01 \)

\( a = 1, \beta = 0.001, \rho = 0.01 \)

\( a = 4, \beta = 0.04, \rho = 0.01 \)

**Relating with the Perona-Malik Diffusion**

- Replacing \( v \) with \( 1/(1 + 2\alpha \rho |\nabla u|^2) \), PDE for the process \( u \) can be interpreted as a biased Perona-Malik type nonlinear diffusion:
  \[
  \frac{\partial u}{\partial t} = \nabla \cdot \left( g(|\nabla u|) \nabla u \right) - \beta(u - f)
  \]
  with
  \[
  g(|\nabla u|) = \left( \frac{1}{1 + \lambda |\nabla u|^2} \right)^2 
  \]
  \[ \lambda^2 = 1/(2\alpha \rho) \]
  - \( \sqrt{1/(2\alpha \rho)} \) as a contrast parameter
  - Relative importance of the regularization term (scale) depends on the ratio between \( \alpha \) and \( \beta \).

**Numerical Implementation**

- Original model:
  \[
  \frac{\partial u}{\partial t} = \nabla \cdot (v^2 \nabla u) - \frac{\beta}{\alpha} (u - f); \quad \frac{\partial u}{\partial n} \bigg|_{\partial \Omega} = 0
  \]
  - Space discrete version:
    \[
    \frac{du_{i,j}}{dt} = v_{i+1/2,j}^2 \cdot (u_{i+1,j} - u_{i,j}) - v_{i-1/2,j}^2 \cdot (u_{i,j} - u_{i-1,j}) 
    + v_{i,j+1/2}^2 \cdot (u_{i,j+1} - u_{i,j}) - v_{i,j-1/2}^2 \cdot (u_{i,j} - u_{i,j-1}) 
    - \frac{\beta}{\alpha} (u_{i,j} - f_{i,j}) ,
    \]
    with \( v_{i+1/2,j} = \frac{v_{i+1,j} + v_{i,j}}{2} \) and \( v_{i-1/2,j} = \frac{v_{i+1,j} + v_{i,j}}{2} \)
**Numerical Implementation**

- Original model:

\[
\frac{\partial v}{\partial t} = \nabla^2 v - \frac{2\alpha |\nabla u|^2 v}{\rho} - (v - 1) \quad \frac{\partial v}{\partial n} \bigg|_{\partial \Omega} = 0
\]

- Space discrete version:

\[
\frac{d v_{i,j}}{d t} = v_{i+1,j} + v_{i-1,j} + v_{i,j+1} + v_{i,j-1} - 4 v_{i,j} - 2\alpha |\nabla u_{i,j}|^2 v_{i,j} - \frac{(v_{i,j} - 1)}{\rho^2}
\]

**A Common Framework for Curve Evolution, Segmentation and Anisotropic Diffusion**

- Quadratic cost functions in the data fidelity and the smoothing terms are replaced with L1-functions (Shah, CVPR 1996):

\[
E_S(u, v) = \int_{\Omega} \left( \beta |u - f| + \alpha v^2 |\nabla u| + \frac{1}{2} \left( \rho |\nabla v|^2 + \frac{(1 - v)^2}{\rho} \right) \right) dx
\]

- As \(\rho \to 0\), this energy functional converges to the following functional:

\[
E_{S2}(u, \Gamma) = \frac{\beta}{\alpha} \int_{\Omega} |u - f| dx + \int_{\Omega \setminus \Gamma} |\nabla u| dx + \int_{\Gamma} \frac{J_u}{1 + \alpha |u|} ds
\]

with \(J_u = |u^+ - u^-|\) indicating the jump in \(u\) across \(\Gamma\), and \(u^+\) and \(u^-\) denote intensity values on two sides of \(\Gamma\)

**Numerical Implementation**

- Space-time discrete versions:

\[
\frac{u_{i,j}^{k+1} - u_{i,j}^k}{\Delta t} = \left( \frac{u_{i,j}^k}{2} \right)^2 \cdot u_{i+1,j}^k + \left( \frac{u_{i,j}^k}{2} \right)^2 \cdot u_{i-1,j}^k + \left( \frac{u_{i,j}^k}{2} \right)^2 \cdot u_{i,j+1}^k + \left( \frac{u_{i,j}^k}{2} \right)^2 \cdot u_{i,j-1}^k - \frac{\beta}{\alpha} \left( \frac{u_{i,j}^{k+1} - f_{i,j}}{u_{i,j}^k} \right)
\]

**A Common Framework for Curve Evolution, Segmentation and Anisotropic Diffusion**

- Minimizing the energy functional results in the following system of coupled PDEs:

\[
\frac{\partial u}{\partial t} = 2 \nabla \cdot \nabla u + \nabla |\nabla u| \text{curv}(u) - \frac{\beta}{\alpha v} |\nabla u| \frac{(u - f)}{|u - f|} \quad \frac{\partial u}{\partial n} \bigg|_{\partial \Omega} = 0
\]

\[
\frac{\partial v}{\partial t} = \nabla^2 v - \frac{2\alpha |\nabla u|^2 v}{\rho} - (v - 1) \quad \frac{\partial v}{\partial n} \bigg|_{\partial \Omega} = 0
\]

with \text{curv}(u) = \nabla \cdot \left( \frac{\nabla u}{|\nabla u|} \right)

- Replacing L2-norms in both the data fidelity and the smoothness terms by their L1-norms generates shocks in \(u\) and thus object boundaries are recovered as actual discontinuities.
Sample Results of Shah (CVPR96)

- Smoothing process of \( u \) gives rise to more cartoon-like, piecewise constant images but with some unintuitive regions.

\[ a = 1, \beta = 0.001, \rho = 0.01 \]

\[ a = 1, \beta = 0.001, \rho = 0.01 \]

\[ a = 4, \beta = 0.04, \rho = 0.01 \]

Context-Guided Image Smoothing

- Contextual knowledge extracted from local image regions guides the regularization process.

Challenging Cases

Context-Guided Image Smoothing

- Local context
- Local neighborhood
- Pixel level

contextual measure
Context-Guided Image Smoothing

- 2 coupled processes (u and v modules)

\[
\frac{\partial v}{\partial t} = \nabla^2 v - \frac{2\alpha}{\rho} |\nabla u|^2 v - \frac{(v - 1)}{\rho^2}; \quad \frac{\partial v}{\partial n}\bigg|_{\partial \Omega} = 0 \\
\frac{\partial u}{\partial t} = \nabla \cdot ((cv)^2 \nabla u) - \frac{\beta}{\alpha} (u - f); \quad \frac{\partial u}{\partial n}\bigg|_{\partial \Omega} = 0 \\
cv = \phi v + (1 - \phi)V \\
\phi \in [0,1] \quad V \in \{0,1\}
\]

The Roles of $\phi$ and $V$

1. Eliminating an accidentally occurring event
   - e.g., a high gradient due to noise
   - $V=1$, $\phi$ is low for accidental occurrences
   \[(cv)^2_i = (\phi_i v_i + (1 - \phi_i) 1)^2\]

2. Preventing an accidental elimination of a feature of interest
   - e.g., encourage edge formation
   - $V=0$, $\phi$ is low for meaningful occurrences
   \[(cv)^2_i = (\phi_i v_i + (1 - \phi_i) 0)^2\]

Experimental Results

- Suggested contextual measures:
  1. Directional consistency of edges
     - shapes have smooth boundaries
  2. Edge Continuity
     - gap filling
  3. Texture Edges
     - boundary between different textured regions
  4. Local Scale
     - Resolution varies throughout the image

Directional Consistency

Approximate MS

Our result
**Directional Consistency**

Approximate MS

Our result

**Edge Continuity**

Approximate MS

Our result

**Coalition of Directional Consistency and Texture Edges**

\( \phi_{10} \)

**Coalition of Directional Consistency, Edge Continuity and Texture Edges**
**Local Scale**

Active Contours Without Edges

- Level sets provide an implicit contour representation where an evolving curve is represented with the zero-level line of a level set function.

**Active Contours Without Edges**

- **Basic idea:** Fitting term
  \[
  \int_{\text{inside}(C)} |u_0 - c_1|^2 \, dx \, dy + \int_{\text{outside}(C)} |u_0 - c_2|^2 \, dx \, dy
  \]
  where
  \[
  \begin{align*}
  c_1 &= \text{average of } u_0 \text{ inside } C \\
  c_2 &= \text{average of } u_0 \text{ outside } C
  \end{align*}
  \]

  - **Fit** > 0
  - **Fit** > 0
  - **Fit** > 0
  - **Fit** ≈ 0

  - **Minimize:** the Fitting term + Length(C)

**Active Contours Without Edges (continued)**

- The length parameter \( \mu \) can be interpreted as a scale parameter. It determines the relative importance of the length term.
- The possibility of detecting smaller objects/regions increases with decreasing \( \mu \).
Active Contours Without Edges


\[
E_{CV}(c_1, c_2, \phi) = \lambda_1 \int_\Omega (f - c_1)^2 H(\phi) dx + \lambda_2 \int_\Omega (f - c_2)^2 (1 - H(\phi)) dx \\
+ \mu \int_\Omega |\nabla H(\phi)| dx
\]

- The model represents the segmented image with the variables \(c_1\), \(c_2\) and \(H(\phi)\), where \(H(\phi)\) denotes the Heaviside function of the level set function \(\phi\):

\[
H(z) = \begin{cases} 
1 & \text{if } z \geq 0 \\
0 & \text{if } z < 0 
\end{cases}
\]

Active Contours Without Edges


\[
E_{CV}(c_1, c_2, \phi) = \lambda_1 \int_\Omega (f - c_1)^2 H(\phi) dx + \lambda_2 \int_\Omega (f - c_2)^2 (1 - H(\phi)) dx \\
+ \mu \int_\Omega |\nabla H(\phi)| dx
\]

- \(c_1\) and \(c_2\) denote the average gray values of object and background regions indicated by \(\phi \geq 0\) and \(\phi < 0\), respectively.

- Chan-Vese model can be seen as a two-phase piecewise constant approximation of the MS model.

Active Contours Without Edges

- Segmentation involves minimizing the energy functional with respect to \(c_1\), \(c_2\), and \(\phi\).

- Keeping \(\phi\) fixed, the average gray values \(c_1\) and \(c_2\) can be estimated as follows:

\[
c_1 = \frac{\int_\Omega f(x) H(\phi(x)) dx}{\int_\Omega H(\phi(x)) dx},
\]

\[
c_2 = \frac{\int_\Omega f(x)(1 - H(\phi(x))) dx}{\int_\Omega (1 - H(\phi(x))) dx}
\]
Active Contours Without Edges

- Segmentation involves minimizing the energy functional with respect to $c_1$, $c_2$, and $\phi$.
- Keeping $c_1$ and $c_2$ fixed and using the calculus of variations for the given functional, the gradient descent equation for the evolution of $\phi$ is derived as:

$$\frac{\partial \phi}{\partial t} = \delta(\phi) \left[ \mu \nabla \cdot \left( \frac{\nabla \phi}{|\nabla \phi|} \right) - \lambda_1 (f - c_1)^2 + \lambda_2 (f - c_2)^2 \right]$$

Sample result of the Chan-Vese Model

- As the zero-level line of the evolving level set function $\phi$ is attracted to object boundaries, a more accurate piecewise constant approximations of the original image $f$ is recovered.

Sample result of the Chan-Vese Model

- As the zero-level line of the evolving level set function $\phi$ is attracted to object boundaries, a more accurate piecewise constant approximations of the original image $f$ is recovered.
**Numerical Implementation**

- In the numerical approximation, regularized form of the Heaviside function is used:

\[
H_\varepsilon(z) = \frac{1}{2} \left( 1 + \frac{2}{\pi} \arctan \left( \frac{z}{\varepsilon} \right) \right)
\]

\[
\delta_\varepsilon(z) = \frac{dH_\varepsilon(z)}{dz} = \frac{1}{\pi} \frac{\varepsilon}{\varepsilon^2 + z^2}
\]

**Numerical Implementation**

- Space-time discrete version:

\[
\frac{\phi_{ij}^{k+1} - \phi_{ij}^k}{\Delta t} = \delta(\phi_{ij}^k) \left[ \mu \Delta_y \alpha \left( \frac{\Delta_x \phi_{ij}^{k+1}}{\sqrt{(\Delta_x \phi_{ij}^k)^2 + (\phi_{ij+1} - \phi_{ij-1})^2 / 4}} \right) \right.
\]

\[
+ \mu \Delta_y \alpha \left( \frac{\Delta_y \phi_{ij}^{k+1}}{\sqrt{(\phi_{i+1,ij} - \phi_{i-1,ij})^2 / 4 + (\Delta_y \phi_{ij}^k)^2}} \right)
\]

\[
- \lambda_1 (f_{ij} - c_1(\phi^k))^2 + \lambda_2 (f_{ij} - c_2(\phi^k))^2
\]

with

\[
\Delta_x \phi_{ij} = \phi_{i,j} - \phi_{i-1,j}, \quad \Delta_x \phi_{ij} = \phi_{i+1,j} - \phi_{ij},
\]

\[
\Delta_y \phi_{ij} = \phi_{i,j} - \phi_{i,j-1}, \quad \Delta_y \phi_{ij} = \phi_{i,j+1} - \phi_{ij}.
\]