Application of Backstepping Control Technique to Fractional Order Dynamic Systems

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Abstract This paper focuses on the application of backstepping control scheme for fractional order dynamic systems. As in the case of integer order version, the control scheme is applicable to a particular class of systems letting the designer obtain a closed loop control law in a nested structure. A Lyapunov function is defined at each stage and the negativity of an overall Lyapunov function is ensured by proper selection of the control law. Two exemplar cases are considered in the paper.

1 Introduction

Recently there has been a dramatic increase in the number of research outcomes regarding the theory and applications of fractional order systems and control, Oldham and Spanier (1974), Podlubny (1998), Das (2008). Despite the emergence of the theory dates back to a letter from Leibniz to L'Hôpital in 1695, asking the possible consequences of choosing a derivative of order ½, the theory has been stipulated and with the advances in the computational facilities, many important tools of classical control have been reformulated for (or adapted to) fractional order case, such as PID controllers Podlubny (1999), Zhao et al (2005), stability considerations, Matignon (1996), Matignon (1998), Chen et al (2006), Ahmed (2006), Kalman filtering Sierociuk and Dzielinski (2006), state space models and approaches Das (2008), Ortigueira (2000), Raynaud and Zerganoh (2000), root locus technique Merrikh-Bayat and Afshar (2008), applications involved with the partial differential equations Meerschaert and Tadjeran (2006), Podlubny et al (2009), discrete time issues Oldham and Spanier (1974), Podlubny (1998), Das (2008), Sierociuk and Dzielinski (2006) and so on. A system to be identified can well be approximated by an integer order model or it can be approximated by a much simpler model that is a fractional order one. Having the necessary techniques and tools for such cases becomes a critical issue and with this motivation in mind, this paper focuses on adapting the backstepping control technique for fractional order plant dynamics.

Backstepping technique has been a frequently used nonlinear control technique that is based on the definition of a set of intermediate variables and the procedure of ensuring the negativity of Lyapunov functions that add up to build a common control

Lyapunov function for the overall system. Due to this nature, the backstepping technique is applicable to a particular –yet wide– class of systems, which includes most mechanical systems, biochemical processes etc. The technique has successfully been implemented in the field of robotics to as one of the state variables is of type position and the other is of type velocity, Krstic (1995), Madani and Benallegue (2006), Adigbli et al (2007), Hua et al (2009).

Although the tools and approaches of fractional order mathematics and backstepping control are not new, implementation of backstepping control for fractional order system dynamics is. The reason is the definition of derivative that is generalized by Leibniz rule. The rule, which also generalizes the integer order cases, yields infinitely many terms for the product and it becomes difficult to figure out stability by choosing a square type Lyapunov function and obtaining its time derivative. This paper discusses a remedy to this within the context of backstepping control method. The contribution of the current study is to extend the backstepping technique to fractional order plants.

This paper is organized as follows: The second section briefly gives the definitions of widely used fractional differintegration formulas and basics of fractional calculus, the third section describes the backstepping technique for fractional order plant dynamics, the fourth section presents a set of simulation studies covering a second order linear system with known dynamics, and a third order nonlinear system having uncertainties and disturbances, and the concluding remarks are given at the end of the paper.

2 Fractional Order Differintegration Operators

Let $\mathbf{D}^{\beta}$ denote the differintegration operator of order β, where $\beta\in\Re$. For positive values of β, the operator is a differentiator whereas the negative values of β correspond to integrators. This representation lets $\mathbf{D}^{\beta}$ to be a differintegration operator whose functionality depends upon the numerical value of β. With n being an integer and $n-1 \leq \beta < n$, Riemann-Liouville definition of the β-fold fractional differintegration is defined by (1) where Caputo's definition for which is in (2).

$$\mathbf{D}^{\beta} f(t) = \frac{1}{\Gamma(n-\beta)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{n} \int_{0}^{t} \frac{f(\tau)}{(t-\tau)^{\beta-n+1}} \mathrm{d}\tau \tag{1}$$

$$\mathbf{D}^{\beta} f(t) = \frac{1}{\Gamma(n-\beta)} \int_{0}^{t} \frac{f^{(n)}(\tau)}{(t-\tau)^{\beta-n+1}} \mathrm{d}\tau \tag{2}$$

where $\Gamma(\beta)=\int_0^{\infty} e^{-t} t^{\beta-1}\mathrm{d}t$ is the well known Gamma function. In both definitions, we assumed the lower terminal zero and the integrals start from zero. Considering a_k, $b_k \in\Re$ and α_k, $\beta_k\in\Re^+$, one can define the following differential equation

$$(a_n \mathbf{D}^{\alpha_n} + a_{n-1}\mathbf{D}^{\alpha_{n-1}} + \cdots + a_0)y(t) = (b_m \mathbf{D}^{\beta_m} + b_{m-1}\mathbf{D}^{\beta_{m-1}} + \cdots + b_0)u(t) \tag{3}$$

and with the assumption that all initial conditions are zero, obtain the transfer function given by (4).

$$\frac{Y(s)}{U(s)} = \frac{b_m s^{\beta_m} + b_{m-1}s^{\beta_{m-1}} + \cdots + b_0}{a_n s^{\alpha_n} + a_{n-1}s^{\alpha_{n-1}} + \cdots + a_0} \tag{4}$$

Denoting frequency by ω and substituting $s = j\omega$ in (4), one can exploit the techniques of frequency domain. A significant difference in the Bode magnitude plot is to observe that the asymptotes can have any slope other that the integer multiples of 20 dB/decade and this is a substantially important flexibility for modeling and identification research. When it comes to consider state space models, one can define

$$\begin{aligned} \mathbf{D}^{\beta}\mathbf{x} &= \mathbf{A}\mathbf{x} + \mathbf{B}u \\ y &= \mathbf{C}\mathbf{x} + Du \end{aligned} \tag{5}$$

and obtain the transfer function via taking the Laplace transform in the usual sense, i.e.

$$H(s) = \mathbf{C}\left(s^{\beta}\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B} + D \tag{6}$$

For the state space representation in (5), if λ_i is an eigenvalue of the matrix **A**, the condition

$$|\arg(\lambda_i)| > \beta\frac{\pi}{2} \tag{7}$$

is required for stability. It is possible to apply the same condition for the transfer function representation in (4), where λ_is denote the roots of the expression in the denominator.

The implementation issues are tightly related to the numerical realization of the operators defined in (1) and (2). There are several approaches in the literature and Crone is the most frequently used scheme in approximating the fractional order differintegration operators, Das (2008). More explicitly, the algorithm determines a number of poles and zeros and approximates the magnitude plot over a predefined range of the frequency spectrum. In (8), the expression used in Crone approximation is given and the approximation accuracy is depicted for $N = 9$ in Figure 1 and for $N = 40$ in Figure 2. According to the shown approximates, it is clearly seen that the accuracy is improved as N gets larger, yet the price paid for this is the complexity.

$$s^{\beta} \approx K\frac{\prod_{k=1}^{N} 1 + s/w_{p_k}}{\prod_{k=1}^{N} 1 + s/w_{z_k}} \tag{8}$$

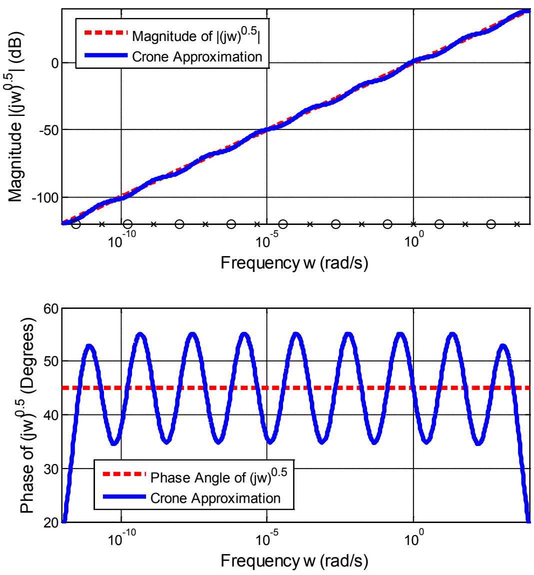

Fig. 1. Crone approximation to the operator $s^{0.5}$ with ω_{min}=1e–12 rad/s, ω_{max}=1e+4 rad/s, N= 9

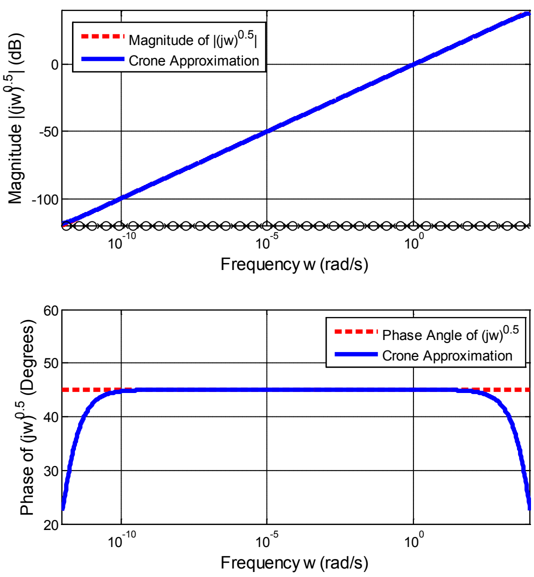

Fig. 2. Crone approximation to the operator $s^{0.5}$ with ω_{min}=1e–12 rad/s, ω_{max}=1e+4 rad/s, N= 35

3 Backstepping Control Technique for Fractional Order Plant Dynamics

Denote the β-fold differintegration operator $\mathbf{D}^{\beta}x$ by $x^{(\beta)}$ and consider the system

$$\begin{aligned} x_1^{(\beta_1)} &= x_2 \\ x_2^{(\beta_2)} &= f(x_1,x_2) + g(x_1,x_2)u \end{aligned} \tag{9}$$

where x_1 and x_2 are the state variables, $0<\beta_1, \beta_2<1$ are positive fractional differentiation orders, $f(x_1,x_2)$ and $g(x_1,x_2)$ are known and smooth functions of the state variables and $g(x_1,x_2)\neq 0$. Define the following intermediate variables of backstepping design.

$$\begin{aligned} z_1 &:= x_1 - r_1 - A_1 \\ z_2 &:= x_2 - r_2 - A_2 \end{aligned} \tag{10}$$

where $A_1=0$ and $r_1^{(\beta_1)} = r_2$.

Theorem: Let z be the variable of interest and choose the Lyapunov function given by (11).

$$V = \frac{1}{2}z^2 \tag{11}$$

If $zz^{(\beta)} < 0$ if $0<\beta<1$ is maintained then $z\dot{z} < 0$ is satisfied.

Proof: Consider the Riemann-Liouville definition, which is rewritten for the given conditions in (12).

$$zz^{(\beta)} = \frac{z}{\Gamma(1-\beta)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{z(\tau)}{(t-\tau)^{\beta}}\mathrm{d}\tau \tag{12}$$

If $zz^{(\beta)} < 0$ is satisfied, then the variable z and the integral

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{z(\tau)}{(t-\tau)^{\beta}}\mathrm{d}\tau \tag{13}$$

are opposite signed, i.e. $\int_0^t \frac{z(\tau)}{(t-\tau)^{\beta}}\mathrm{d}\tau$ is monotonically decreasing for positive z, and monotonically increasing for negative z. Since the denominator of the integrand is always positive, this can only arise if $z\dot{z} < 0$ is satisfied.

Considering the Caputo's definition in (14), having $zz^{(\beta)} < 0$ can arise when $z\dot{z} < 0$.

$$zz^{(\beta)} = \frac{z}{\Gamma(1-\beta)}\int_0^t \frac{\dot{z}(\tau)}{(t-\tau)^{\beta}}\mathrm{d}\tau \tag{14}$$

This proves that forcing $zz^{(\beta)} < 0$ implies $z\dot{z} < 0$. □

Now we will formulate the backstepping control technique for the plant described by (9) by repetitively checking the quantities $z_1 z_1^{(\beta_1)}$ and $z_1 z_1^{(\beta_1)} + z_2 z_2^{(\beta_2)}$ as explained below.

Step 1: Check $z_1 z_1^{(\beta_1)}$

$$\begin{aligned} z_1 z_1^{(\beta_1)} &= z_1\left(x_1^{(\beta_1)} - r_1^{(\beta_1)}\right) \\ &= z_1\left(x_2 - r_2\right) \\ &= z_1\left(z_2 + r_2 + A_2 - r_2\right) \\ &= z_1\left(z_2 + A_2\right) \end{aligned} \tag{15}$$

Step 2: With k_1>0, choose $A_2 = -k_1 z_1$, this would let us have

$$z_1 z_1^{(\beta_1)} = -k_1 z_1^2 + z_1 z_2 \tag{16}$$

Step 3: Check $z_1 z_1^{(\beta_1)} + z_2 z_2^{(\beta_2)}$

$$\begin{aligned} z_1 z_1^{(\beta_1)} + z_2 z_2^{(\beta_2)} &= -k_1 z_1^2 + z_1 z_2 + z_2\left(x_2^{(\beta_2)} - r_2^{(\beta_2)} - A_2^{(\beta_2)}\right) \\ &= -k_1 z_1^2 + z_2\left(x_2^{(\beta_2)} - r_2^{(\beta_2)} - A_2^{(\beta_2)} + z_1\right) \\ &= -k_1 z_1^2 + z_2\left(f + gu - r_2^{(\beta_2)} - A_2^{(\beta_2)} + z_1\right) \end{aligned} \tag{17}$$

Step 4: Force $z_1 z_1^{(\beta_1)} + z_2 z_2^{(\beta_2)} = -k_1 z_1^2 - k_2 z_2^2$, k_2>0, this requires

$$f + gu - r_2^{(\beta_2)} - A_2^{(\beta_2)} + z_1 := -k_2 z_2 \tag{18}$$

Step 5: Solve for u

$$u = -\frac{1}{g(x_1,x_2)}\left(f(x_1,x_2) - r_2^{(\beta_2)} + k_1 z_1^{(\beta_2)} + z_1 + k_2 z_2\right) \tag{19}$$

It is possible to generalize the above procedure for higher order systems of the form

$$\begin{aligned} x_i^{(\beta_i)} &= x_{i+1},\ i = 1,2,\cdots,q-1 \\ x_q^{(\beta_q)} &= f(x_1,x_2,\cdots,x_q) + g(x_1,x_2,\cdots,x_q)u \end{aligned} \tag{20}$$

and the control law

$$u = -\frac{1}{g}\left(f - r_q^{(\beta_q)} - A_q^{(\beta_q)} + z_{q-1} + k_q z_q\right) \tag{21}$$

where k_q>0 and

$$A_1 = 0, z_0 = 0 \tag{22}$$

$$A_{i+1} = -k_i z_i + A_i^{(\beta_i)} - z_{i-1}, i = 1,2,q-1 \tag{23}$$

and the result of applying the control law in (21) is as below.

$$\sum_{i=1}^{q} z_i z_i^{(\beta_i)} = -\sum_{i=1}^{q} k_i z_i^2 \tag{24}$$

According to the aforementioned theorem, ensuring the negativeness of the right hand side of (24) equivalent to ensuring the negativity of $\sum_{i=1}^{q} z_i \dot{z}_i$, and the trajectories in the coordinate system spanned by $z_1,\ldots,z_q$ converge the origin.

4 Simulation Studies

In this section, we consider two sets of simulations so justify the claims. The first system is linear and a second order one with all necessary parameters are known perfectly. The system is given by (25).

$$\begin{pmatrix} x_1^{(0.7)} \\ x_2^{(0.6)} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \tag{25}$$

The system is desired to track a sinusoidal profile for a period of 50 seconds, and then the following of a pulse like command is claimed. The results are illustrated in Figure 3-4.

According to the presented results, precise tracking of the command signals is achieved with N=35 term approximation for the fractional order differentiation operators. The numerical realization has been performed in Matlab environment with Ninteger toolbox, Valerio (2005). The results seen in Figure 3 have been obtained with k_1=k_2=10, and those in Figure 4 are obtained with k_1=k_2=0.1. The former case reveals better tracking performance while the latter produces smother control signals and the comparison guides the designer for setting the best parameter values for the design expectations.

In the second set of simulations, a third order system dynamics with several uncertainty terms is considered. The system dynamics is given by (26).

$$\begin{aligned} x_1^{(0.7)} &= x_2 \\ x_1^{(0.6)} &= x_3 \\ x_3^{(0.5)} &= f(x_1,x_2,x_3) + \Delta(x_1,x_2,x_3,t) + g(t)u + \xi(t) \end{aligned} \tag{26}$$

where $\Delta(x_1,x_2,x_3)$ and $\xi(t)$ are uncertainties and disturbance terms that are not available to the designer. In above, we have

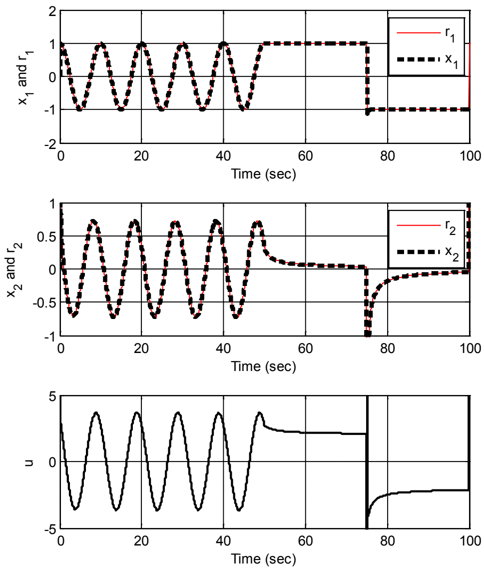

Fig. 3. Simulation results for the system described by (24). $k_1=k_2=10$, $N=35$

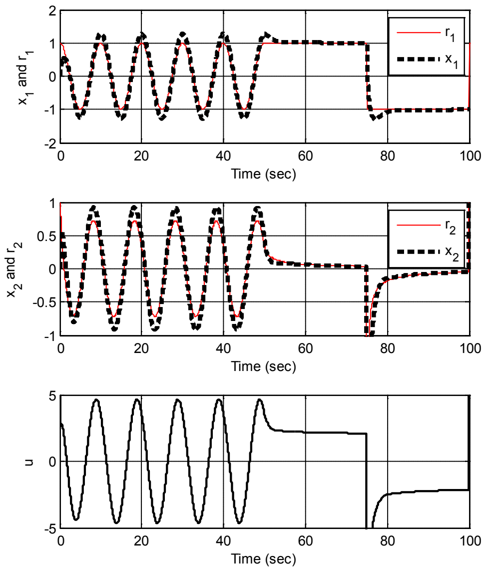

Fig. 4. Simulation results for the system described by (24). $k_1=k_2=0.1$, $N=35$

$$f(x_1,x_2,x_3) = -0.5x_1 - 0.5x_2^3 - 0.5x_3|x_3| \tag{27}$$

$$g(t) = 1 + 0.1\sin\left(\frac{\pi t}{3}\right) \tag{28}$$

$$\begin{aligned}\Delta(x_1,x_2,x_3,t) &= \left(-0.05 + 0.25\sin(5\pi t)\right)x_1 + \\ &\left(-0.03 + 0.3\cos(5\pi t)\right)x_2^3 + \left(-0.05 + 0.25\sin(7\pi t)\right)x_3|x_3|\end{aligned} \tag{29}$$

$$\xi(t) = 0.2\sin(4\pi t) \tag{30}$$

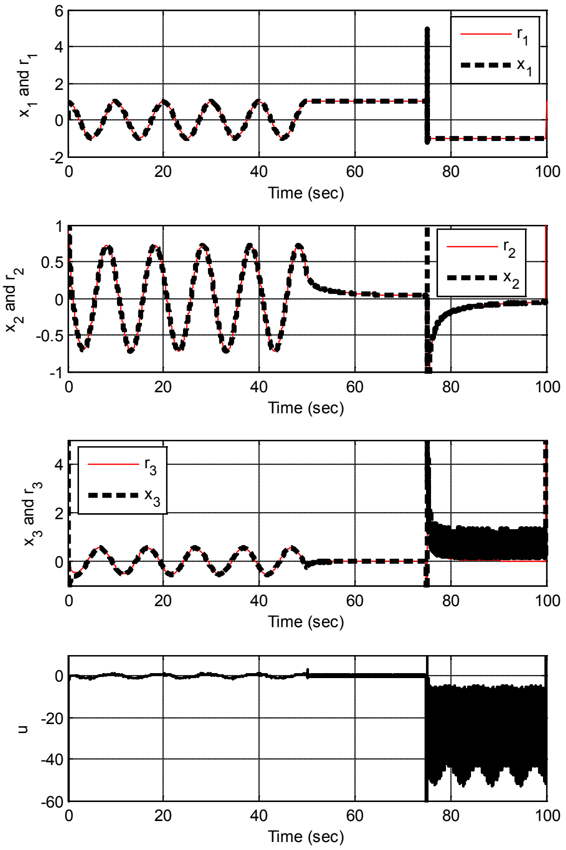

Fig. 5. Simulation results for the system described by (26)-(30), k_1=k_2=k_3=10 and N=35

The results of the simulations are shown in Figure 5, where it is seen that the reference signal for the first state variable is followed very precisely when k_1=k_2=10 and N=35. Regarding the second state variable, due to the sharp changes in the reference signal, several instantaneous peaks are visible. The effect of the disturbances and approximation errors are seen as a slight degradation in the tracking performance of the third state variable. The last row of Figure 5 shows the control signal that yields the shown tracking performances. Clearly the control signal has very sharp responses when there are sudden changes in the command signal. In Figure 6, the approximation parameter is reduced to N=9 and the simulations were repeated. Apparently in this case the state tracking performance even for the second state is visibly degraded and we conclude that the numerical issues in implementing the fractional order differintegration operators influence the performance significantly.

Since the reference signal contains instantaneous changes, the responses are affected at these instants. In order to clarify this situation, we study the second example once again but in this time, we choose the reference signal as a filtered version of the reference signal considered in the previous cases. More explicitly, we choose

$$R_1(s) = \frac{1}{(s+1)^6} C(s) \tag{31}$$

where $C(s)$ is the command signal used so far and $R_1(s)$ is the Laplace transform of $r_1(t)$. The results are shown in Figure 7, where it is seen that both the trajectory tracking performance and the control signal smoothness are very good provided that the smoothness of the command signal is assured.

The presented results demonstrate that the backstepping design can be adapted for fractional order plant dynamics and the use of better approximations for fractional order operators can lead to improved performance indications.

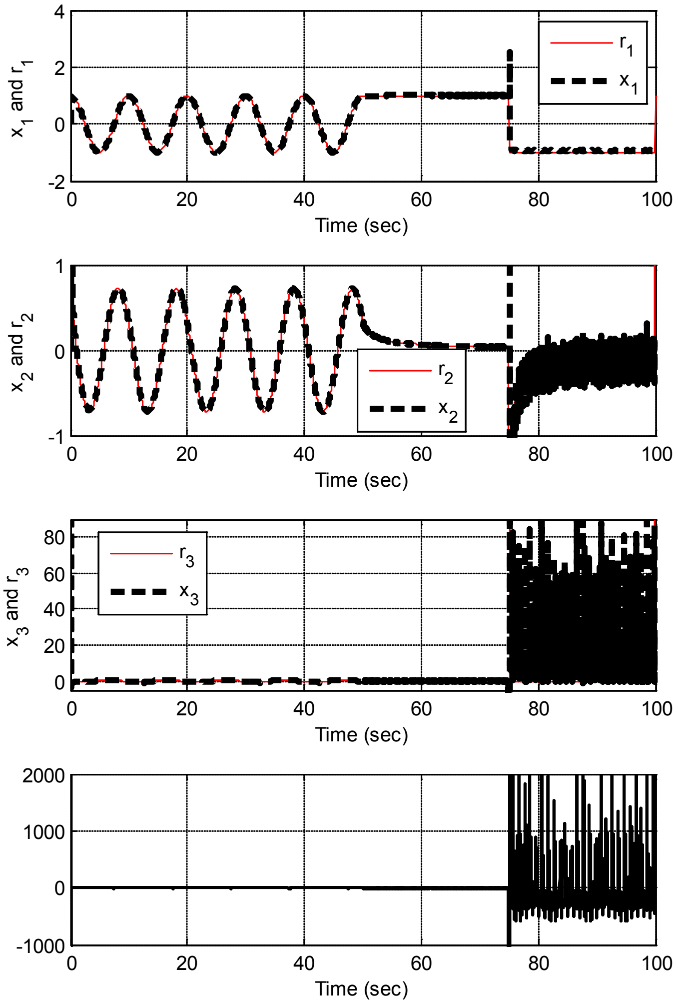

Fig. 6. Simulation results for the system described by (26)-(30), k_1=k_2=k_3=10 and N=9

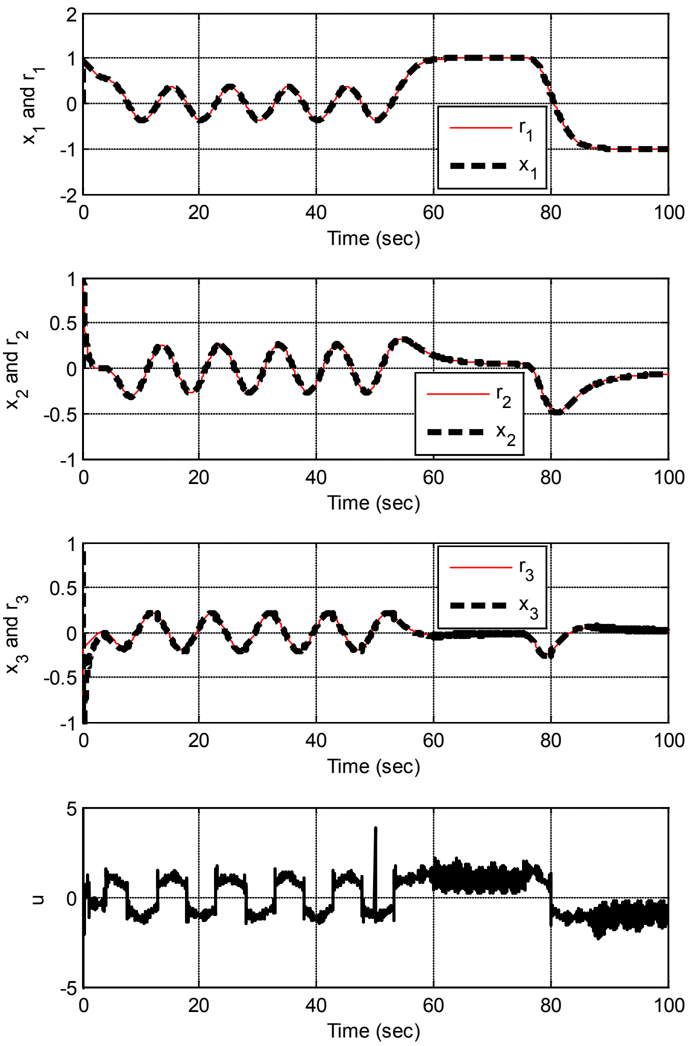

Fig. 7. Simulation results for the system described by (26)-(30), k_1=k_2=k_3=10, N=35 and the reference signal is a filtered one as described by (31).

5 Conclusions

This paper focuses on the adaptation of backstepping control technique for fractional order plant dynamics. The derivation of the control law for a second order plant is given, the result is generalized for q-th order case and it is shown that ensuring $zz^{(\beta)}<0$ implies $z\dot{z}<0$ and stability conclusions for the control laws maintaining $zz^{(\beta)}<0$ are tied to the integer order case. Two application examples are scrutinized. The first is a linear second order system, the analytical details embodying which is known thoroughly. The second example is a nonlinear system that possesses some uncertainty terms as well as disturbances, which are all bounded. The adapted backstepping scheme is applied to both systems and it is seen that the analytical claims are met perfectly for the first case and some degradation in the performance due to the uncertainties is seen in the second case. If the smoothness of the command signal is assured, then a significant improvement in the trajectory tracking performance and the command signal smoothness is observed.

Briefly, the paper demonstrates the use of backstepping control technique for fractional order plant dynamics and several illustrative examples are discussed. The results show that the design parameters N and k_is have a strong influence on the overall performance of the control system as well as the smoothness of the command signal is seen to be an important parameter influencing the closed loop performance.

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