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# Modeling of PDE processes with finite dimensional non-autonomous ODE systems*

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Processes governed by Partial Differential Equations (PDE) display very rich dynamical behavior, which is continuous spatially. Influencing the behavior of PDE systems through boundaries is an interesting research as it is involves the handling of infinite dimensionality, due to which the traditional tools of control theory do not apply directly. This study demonstrates how a nonlinear PDE is converted into a reasonably descriptive Ordinary Differential Equation (ODE) model. The approach is based on Proper Orthogonal Decomposition (POD), which separates the temporal and spatial components of the dynamics. The finite term expansion of the solution results in an autonomous ODE and this paper demonstrates how the external excitations are made explicit in the dynamical model. 2D Burgers equation is used to illustrate the effectiveness of the approach and a finite dimensional dynamical model is shown to be capable of capturing the essential response.

## 1 Introduction

Systems and control theory is a well-founded framework and the research on the discovery and understanding of system dynamics is an every growing subset of the paradigm. Defining $\alpha \in \Re^n$ and $\gamma \in \Re^m$ as the state vector and the control input vector, respectively, one branch of the control research focuses on the affine nonlinear models having the representation $\dot{\alpha} = f(\alpha) + g(\alpha)\gamma$, [Kha02]. Upon suitably defining the functions $f(\cdot)$ and $g(\cdot)$, the linear state space systems of the form $\dot{\alpha} = A\alpha + B\gamma$ are obtained and these systems constitute a subset for the aforementioned nonlinear models, [Oga97]. Unsurprisingly, in both representations above, we have the control input explicitly. The problem studied in this paper is to obtain dynamic models for processes governed by PDEs. We particularly focus on the 2D Burgers equation

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$w_t = cw_{xx} - \mu w(w_x + w_y)$, [Ef06]. The problem is interesting not only because of its spatial continuity but also because of its nonlinearity. POD method is utilized to perform the modal decomposition and unfortunately an autonomous dynamical model lacking the control input(s) is obtained, [LT01]. The paper demonstrates how the boundary conditions are made explicit in the resulting finite dimensional ODE model having the structure $\dot{\alpha} = f(\alpha) + g(\alpha)\gamma$.

Various kinds of Burgers equation have been studied in the past. In [DH03, McDH04, Blen91, BE03, Hata98, NMT01], a simplified version of Navier-Stokes equations given by the Partial Differential Equation (PDE) set $\underline{w}_t + \epsilon(\underline{w} \cdot \nabla)\underline{w} = \mu\nabla^2\underline{w}$ with $\underline{w}$ being 2-by-1 vector function is described as the 2D Burgers equation. The 2D Burgers equation is therefore considered as a turbulence free cartoon for Navier-Stokes equations and has been studied in the past for modelling traffic flows, shock waves and acoustic transmission. Blender postulates a method to obtain the solution of the above mentioned PDE set iteratively, [Blen91]. In [BE03], Boules and Eick perform the model reduction with Fourier expansions. In [Sire99, Hie00, Zhu96], some other variants of 2D Burgers equation have been considered with the goal of finding exact solutions under certain circumstances. These types are $(w_t + ww_x - w_{xx})_x + w_{yy} = 0$ in [Sire99, Hie00], $w_t + uu_x + w_{xx} + w_{xxx} = 0$ in [Zhu96]. In [NMT01], Nishinari et al. focus on cellular automaton, which is extensively studied for developing models of traffic flow, fluids and immune systems, and therefore a good model to work on is a variant of Burgers equation. In [Hata98], the dynamics that arises upon discretization of 2D Burgers equation is analyzed. The effects of chosen time step ($\Delta t$) for getting physically reasonable numerical solutions are elaborated. Wescott et al. present a computational technique to obtain the numerical solutions of PDEs having nonlinear convection terms like 2D Burgers equation and Navier-Stokes equations, [Wes01], the goal in which is to reduce the computation time without giving concessions from the accuracy. Boules and Eick obtain the solution of Burgers equation for a specific boundary regime and initial conditions, [BE03]. Using a truncated Fourier series expansion yields an autonomous ODE set, the solution of which approximates the numerical solution, and the derived model rebuilds the situation implied by the chosen initial and boundary conditions. When the 1D version given by $w_t = -ww_x + w_{xx}$ is taken into consideration, it is seen that a significant amount of research outcome has been reported on modelling and control system design. A majority of the works on 1D Burgers equation emphasize the similar difficulties as the motivating factors and focus on the solutions and solvability issues. The current paper, on the other hand, presents an approach to low dimensional (LD) modeling of PDE processes which are to be controlled through boundaries. The contribution of this paper is to demonstrate that a LD nonlinear model can easily be obtained to represent the essential dynamics of a 2D Burgers equation excited continuously through the boundaries of a square domain. The POD algorithm is presented in the second section. The third section demonstrates how the autonomous ODE set is made non-autonomous. The fourth section is devoted to the justification of the devised LD model and the concluding remarks are given at the end of the paper.

## 2 Proper orthogonal decomposition

Consider the ensemble $W_i(x, y)$, $i = 1, 2, \ldots, N_s$, where $N_s$ is the number of elements. Every element of this set corresponds to a snapshot observed from a process, say for example the flow governed by 2D Burgers equation with initial and boundary conditions,

$$\begin{aligned} &w_t(x,y,t) = c\,(w_{xx}(x,y,t) + w_{yy}(x,y,t)) - \mu w(x,y,t)\,(w_x(x,y,t) + w_y(x,y,t)) \\ &w(x,0,t) = f_1(x)\gamma_1(t), w(1,y,t) = f_2(y)\gamma_2(t) \\ &w(x,1,t) = f_3(x)\gamma_3(t), w(0,y,t) = f_4(y)\gamma_4(t), \\ &w(x,y,0) = 0 \quad \forall(x,y), \end{aligned} \tag{1}$$

where, $c$ and $\mu$ are the known constant parameters, and the subscripts $_x$, $_y$ and $_t$ refer to the partial differentiation with respect to $x$, $y$ and time, respectively. The continuous time process takes place over the physical domain $\Omega := \{(x,y)|(x,y) \in [0,1] \times [0,1]\}$ and the solution is obtained on a spatial grid denoted by $\Omega_d$, which describes the coordinates of the pixels of every snapshot in the ensemble. The entities described over $\Omega_d$ are matrices in $\mathbb{R}^{N_y \times N_x}$. Note that in (1), $f_i(\cdot)$ for each $i$ is a function that describes how $\gamma_i(t)$ influences the behavior along the corresponding edge of $\Omega$. $f_i(\cdot)$s can be selected arbitrarily yet for every $i$, $f_i(0) = f_i(1) = 0$ so that the problem description is consistent at the corners of $\Omega$, and $\gamma_i(t)$ becomes independent from $\gamma_j(t)$ for $i \neq j$ and the external excitations can be selected arbitrarily.

With this problem description, the goal of applying POD is to find an orthonormal basis set letting us to write the solution as

$$w(x,y,t) = \sum_{i=1}^{R_L} \alpha_i(t)\Phi_i(x,y), \tag{2}$$

where $\alpha_i(t)$ is the $i^{\text{th}}$ temporal mode, $\Phi_i(x,y)$ is the $i^{\text{th}}$ spatial function (basis function or the eigenfunction), $R_L$ is the number of independent basis functions that can be synthesized from the given ensemble, or equivalently that spans the space described by the ensemble. It will later be clear that if the basis set $\{\Phi_i(x,y)\}_{i=1}^{R_L}$ is an orthonormal set, Galerkin projection yields the autonomous set of ODEs directly. Let us summarize the POD procedure.

*Step 1.* Calculate the $N_s \times N_s$ dimensional correlation matrix $L$, the $(ij)^{\text{th}}$ entry of which is $L_{ij} := \langle W_i, W_j \rangle_{\Omega_d}$, where $\langle \cdot, \cdot \rangle_{\Omega_d}$ is the inner product operator defined over $\mathbb{R}^{N_y \times N_x}$.

*Step 2.* Find the eigenvectors (denoted by $v_i$) and the associated eigenvalues ($\lambda_i$) of the matrix $L$. Sort them in a descending order in terms of the magnitudes of $\lambda_i$. Note that every $v_i$ is an $N_s \times 1$ dimensional vector satisfying $v_i^{\mathrm{T}} v_i = \frac{1}{\lambda_i}$, here, for simplicity of the exposition, we assume that the eigenvalues are distinct.

*Step 3.* Construct the basis set by using

$$\Phi_i(x,y) = \sum_{j=1}^{N_s} v_{ij} W_j(x,y), \tag{3}$$

where $v_{ij}$ is the $j^{\text{th}}$ entry of the eigenvector $v_i = (v_{i1}\ v_{i2}\ \ldots\ v_{iN_s})^{\mathrm{T}}$, and $i = 1, 2, ..., R_L$, with $R_L = \mathrm{rank}(L)$. It can be shown that $\langle \Phi_i(x,y), \Phi_j(x,y)\rangle_\Omega = \delta_{ij}$ with $\delta_{ij}$ being the Kronecker delta function. Notice that the basis functions are admixtures of the snapshots, [LT01].

*Step 4.* Calculate the temporal coefficients. Taking the inner product of both sides of (2) with $\Phi_i(x,y)$, the orthonormality property leads to

$$\begin{aligned} \alpha_i(t_0) &= \langle \Phi_i(x,y), w(x,y,t_0)\rangle_\Omega \\ &= \langle \phi_i, W_{t_0}\rangle_{\Omega_d} \\ &:= \frac{1}{N_s}\sum_{l=1}^{N_x}\sum_{j=1}^{N_y} \phi_i(x_l,y_j) W_{t_0}(x_l,y_j) \\ &:= \phi_i(x,y) \odot W_{t_0}(x,y), \end{aligned} \tag{4}$$

where $\phi_i \in \mathbb{R}^{N_y \times N_x}$ is a sampled form of the basis function $\Phi_i$ defined over $\Omega$. The operator denoted by $\odot$ computes a real number that is the sum of all elements of a matrix obtained through the elementwise multiplication of the two matrices that $\odot$ lies in between. Without loss of generality, an element of the ensemble $\{W_i(x,y)\}_{i=1}^{N_s}$ may be $W(x,y,t_0)$. Therefore, in order to generate the temporal gain, $\alpha_k(t)$, of the spatial eigenfunction $\phi_k$, one would take the inner product of $\phi_k$ with the elements of the ensemble as given below,

$$\langle W_i, \phi_k\rangle_{\Omega_d} \approx \alpha_k(t_i) \qquad i = 1, 2, \ldots, N_s \tag{5}$$

The above computation is important for making a comparison between the quantities obtained from the decomposition (See (5)) and the quantities obtained from the model. Note that the temporal coefficients satisfy orthogonality properties over the discrete set $t \in \{t_1, t_2, \ldots, t_{N_s}\}$ (See (6)).

$$\sum_{i=1}^{N_s} \langle W_i(x,y), \Phi_k(x,y)\rangle_{\Omega_d}^2 \approx \sum_{i=1}^{N_s} \alpha_i^2(t_i) = \lambda_k. \tag{6}$$

For a more detailed discussion on the POD method, the reader is referred to [LT01, Lum67, RCM04] and the references therein.

*Fundamental assumption:* The majority of works dealing with POD and model reduction applications presume that the flow is dominated by coherent modes, which means that the flow can be decomposed into distinguishable components in the order of dominance. Because of the dominance of coherent modes, the typical spread of the eigenvalues of the correlation matrix turns out to be logarithmic and the terms decay very rapidly in magnitude. This fact enables us to assume that a reduced order representation, say with $M$ modes ($M < R_L$) can also be written as an equality

$$W(x, y, t) = \sum_{i=1}^{M} \alpha_i(t)\Phi_i(x, y), \tag{7}$$

and the reduced order model is derived under the assumption that (7) satisfies the governing PDE in (1), [LT01, RCM04, Ravi00]. Unsurprisingly, such an assumption results in a model having uncertainties, however, one should keep in mind that the goal is to find a model, which matches the infinite dimensional system in some sense of approximation with typically $M \ll R_L \leq N_s$. To represent how good such an expansion is, a percent energy measure is defined as follows

$$E = \frac{\sum_{i=1}^{M} \lambda_i}{\sum_{i=1}^{R_L} \lambda_i} \times 100\,\%, \tag{8}$$

where the tendency of $E \to 100\%$ means that the model captures the dynamical information contained in the snapshots well. Conversely, an insufficient model will be obtained if $E$ is far below 100%. Clearly, POD lets us reduce the dimensionality of the problem from infinity to $R_L$, and the fundamental assumption further enables us to reduce the LD model order to $M$. In the next section, we demonstrate how the boundary conditions are transformed into explicit control terms in the corresponding set of ODEs.

## 3 Low dimensional modeling

In the order reduction phase, we need to obtain the autonomous ODE model first. Towards this goal, if (7) is a solution to the PDE in (1), then it has to satisfy the PDE. Substituting (7) into (1) with the fundamental assumption yields

$$\sum_{i=1}^{M} \dot{\alpha}_i(t)\Phi_i(x, y) = c\sum_{i=1}^{M} \alpha_i(t)\Psi_i(x, y) - \mu\sum_{i=1}^{M}\sum_{j=1}^{M} \alpha_i(t)\alpha_j(t)\Phi_i(x, y)\Upsilon_j(x, y), \tag{9}$$

where $\Psi_i(x, y) = \frac{\partial^2\Phi_i(x,y)}{\partial x^2} + \frac{\partial^2\Phi_i(x,y)}{\partial y^2}$ and $\Upsilon_j(x, y) = \frac{\partial\Phi_j(x,y)}{\partial x} + \frac{\partial\Phi_j(x,y)}{\partial y}$. Taking the inner product of both sides with $\Phi_k(x, y)$ and remembering $\langle\Phi_i(x, y), \Phi_k(x, y)\rangle_\Omega = \delta_{ik}$ with $\delta_{ik}$ being Kronecker delta results in

$$\dot{\alpha}_k = c\sum_{i=1}^{M}\alpha_i\langle\Phi_k,\Psi_i\rangle_\Omega - \mu\sum_{i=1}^{M}\sum_{j=1}^{M}\alpha_i\alpha_j\langle\Phi_k,\Phi_i\Upsilon_j\rangle_\Omega, \tag{10}$$

where we have dropped the arguments $x, y$ and $t$ for simplicity. Defining $\zeta_k$ and $\beta_j$ as the entities in $\Omega_d$ corresponding to the entities $\Psi_k$ and $\Upsilon_j$ in $\Omega$, respectively, one could rewrite (10) as

$$\dot{\alpha}_k(t) = c\sum_{i=1}^{M}\alpha_i(t)\left(\phi_k(x,y)\odot\zeta_i(x,y)\right) - \mu\sum_{i=1}^{M}\sum_{j=1}^{M}\alpha_i\alpha_j\left(\phi_k\odot(\phi_i\otimes\beta_j)\right)_{\Omega_d} \tag{11}$$

where $\otimes$ stands for the elementwise multiplication of the two matrices that it lies in between. For the first term above, notice that $\odot$ operator can be applied individually over $\Omega_d^1,\ldots,\Omega_d^n$ which are $n$ nonoverlapping subdomains of $\Omega_d$ such that $\Omega_d^1\bigcup\ldots\bigcup\Omega_d^n = \Omega_d$. This lets us separate the entries corresponding to boundaries without modifying the value of $\langle\phi_k,\zeta_i\rangle_{\Omega_d}$, i.e. $\phi_k(x,y)\odot\zeta_i(x,y)$ as seen in (12),

$$\begin{aligned}
\dot{\alpha}_k(t) = {} & c\sum_{i=1}^{M}\alpha_i(\phi_k(x,0)\odot\zeta_i(x,0)) + c\sum_{i=1}^{M}\alpha_i(\phi_k(1,y)\odot\zeta_i(1,y)) + \\
& c\sum_{i=1}^{M}\alpha_i(\phi_k(x,1)\odot\zeta_i(x,1)) + c\sum_{i=1}^{M}\alpha_i(\phi_k(0,y)\odot\zeta_i(0,y)) + \\
& c\sum_{i=1}^{M}\alpha_i\left(\phi_k^\circ(x,y)\odot\zeta_i^\circ(x,y)\right) - \mu\sum_{i=1}^{M}\sum_{j=1}^{M}\alpha_i\alpha_j\left(\phi_k^\circ\odot(\phi_i^\circ\otimes\beta_j^\circ)\right) \\
& -\mu\sum_{i=1}^{M}\alpha_i\phi_i(x,0)\odot\sum_{j=1}^{M}\alpha_j\left(\phi_k(x,0)\otimes\beta_j(x,0)\right) \\
& -\mu\sum_{i=1}^{M}\alpha_i\phi_i(1,y)\odot\sum_{j=1}^{M}\alpha_j\left(\phi_k(1,y)\otimes\beta_j(1,y)\right) \\
& -\mu\sum_{i=1}^{M}\alpha_i\phi_i(x,1)\odot\sum_{j=1}^{M}\alpha_j\left(\phi_k(x,1)\otimes\beta_j(x,1)\right) \\
& -\mu\sum_{i=1}^{M}\alpha_i\phi_i(0,y)\odot\sum_{j=1}^{M}\alpha_j\left(\phi_k(0,y)\otimes\beta_j(0,y)\right)
\end{aligned} \tag{12}$$

In above, $\phi_k^\circ(x,y)$ denotes a matrix which is obtained when the boundary elements of $\phi_k(x,y)$ are removed, i.e. the first and the last rows, and columns. Similarly, in the computation of terms like $\phi_k(x,0)\odot\zeta_i(x,0)$, the terms $\phi_k(x,0)$ and $\zeta_i(x,0)$ correspond to the first rows of the matrices $\phi_k(x,y)$ and $\zeta_i(x,y)$, respectively.

Due to the lengthy nature of the expression above, we will demonstrate how the terms $T_1 := \sum_{i=1}^{M}\alpha_i(\phi_k(x,0)\odot\zeta_i(x,0))$, which is responsible for the linear diffusion type term, and $T_2 := \sum_{i=1}^{M} \alpha_i \phi_i(x,0) \odot \sum_{j=1}^{M} \alpha_j \left(\phi_k(x,0) \otimes \beta_j(x,0)\right)$, which is responsible for the nonlinear term, are manipulated to postulate the model. Notice that the boundary condition along $y = 0$ edge of $\Omega$ is given by

$$\sum_{i=1}^{M} \alpha_i(t)\phi_i(x,0) = f_1(x)\gamma_1(t), \tag{13}$$

which states that if (7) is a solution, then is must be satisfied at the boundaries as well. Considering this fact constitutes the crux of the LD modeling effort. The boundary condition in (13) can be paraphrased as

$$\alpha_k(t)\phi_k(x,0) = f_1(x)\gamma_1(t) - \sum_{i=1}^{M} (1 - \delta_{ik})\,\alpha_i(t)\phi_i(x,0). \tag{14}$$

Separating the $k^{\text{th}}$ component of the term $T_1$, which is obtained when $i = k$, lets us embed the boundary conditions in (14) into the expression of $T_1$ as given below,

$$T_1 := (f_1(x) \odot \zeta_k(x,0))\gamma_1(t) + \sum_{i=1}^{M} \alpha_i \left(\phi_k(x,0) \odot \zeta_i(x,0) - \phi_i(x,0) \odot \zeta_k(x,0)\right) \tag{15}$$

Similarly, for the term $T_2$, we have rather simple arrangements to see the excitation terms explicitly,

$$T_2 = \gamma_1(t) \sum_{j=1}^{M} \alpha_j \left(f_1(x) \odot (\phi_k(x,0) \otimes \beta_j(x,0))\right). \tag{16}$$

The representations in (15) and (16) indicate that the terms seen in (12) can be concatenated and the following low dimensional dynamical model is obtained,

$$\dot{\alpha}(t) = A\alpha(t) - C(\alpha(t)) + \sum_{i=1}^{4} (B_i - D_i\alpha)\,\gamma_i(t), \tag{17}$$

where $\alpha(t) = (\alpha_1(t)\ \alpha_2(t)\ \ldots\ \alpha_M(t))^{\mathsf{T}}$ is the state vector and

$$C(\alpha) = (\alpha^{\mathsf{T}} C_1 \alpha \quad \alpha^{\mathsf{T}} C_2 \alpha \quad \ldots \quad \alpha^{\mathsf{T}} C_M \alpha)^{\mathsf{T}}.$$

The $(ki)^{\text{th}}$ entry of matrix $A$ is computed as

$$\begin{aligned}(A)_{ki} = c(&\phi_k(x,y) \odot \zeta_i(x,y) - \phi_i(x,0) \odot \zeta_k(x,0) - \phi_i(1,y) \odot \zeta_k(1,y) \\ &- \phi_i(x,1) \odot \zeta_k(x,1) - \phi_i(0,y) \odot \zeta_k(0,y)),\end{aligned} \tag{18}$$

and the $k^{\text{th}}$ entries of the column vectors $B_1, \ldots, B_4$ are

$$(B_1)_k = cf_1(x) \odot \zeta_k(x,0)\,, \quad (B_2)_k = cf_2(y) \odot \zeta_k(1,y),$$
$$(B_3)_k = cf_3(x) \odot \zeta_k(x,1)\,, \quad (B_4)_k = cf_4(y) \odot \zeta_k(0,y). \tag{19}$$

Likewise, we have the components $C(\alpha)$ and matrices $D_1, \ldots, D_4$ contributed by the nonlinear term $-\mu w(w_x + w_y)$ of the PDE in (1);

$$(C_k)_{ij} = \mu(\phi_k^\circ \odot (\phi_i^\circ \otimes \beta_j^\circ)),$$

where $C_k$ is an $M \times M$ matrix, and

$$(D_1)_{kj} = \mu f_1(x) \odot (\phi_k(x,0) \otimes \beta_j(x,0)), (D_2)_{kj} = \mu f_2(y) \odot (\phi_k(1,y) \otimes \beta_j(1,y)),$$
$$(D_3)_{kj} = \mu f_3(x) \odot (\phi_k(x,1) \otimes \beta_j(x,1)), (D_4)_{kj} = \mu f_4(y) \odot (\phi_k(0,y) \otimes \beta_j(0,y)) \tag{20}$$

This result practically lets us have the representative nonlinear dynamical model in (17) for the infinite dimensional process in (1), which needs to be validated. The next section presents to what extent the modelling strategy discussed here could be successful.

## 4 Validation of the nonlinear dynamical model

According to the described procedure, several tests have been done. Due to the numerical advantages, the PDE has been solved by using Crank-Nicholson method (See [Far93] for details), with a step size of 1 msec. The initial distribution is taken zero everywhere and we have chosen $c = 2$ and $\mu = 1$. In order to form the solution, a linear grid having $N_x = N_y = 40$ points in $x$-direction and $y$-direction respectively. According to the above parameter values, a set of 501 snapshots embodies the entire numerical solution, among which a linearly sampled $N = 251$ snapshots have been used for the POD scheme. Although one may use the entire set of snapshots, it has been shown by Sirovich, [Siro87], that a reasonably descriptive subset of them can be used for the same purpose. In the literature, this approach is called *method of snapshots*, which significantly reduces the computational intensity of the overall scheme, (See also [Ravi00, LT01]). Once the modes have been obtained, we truncate the solution at $M = 12$, which represents %99.9832 of the total energy described in the denominator of the expression in (8).

In order to demonstrate the performance of the dynamic model, we choose the functions that are effective along the boundaries as $f_1(x) = \sin(2\pi x)$, $f_2(y) = \sin(2\pi y)$, $f_3(x) = -\sin(2\pi x)$ and $f_4(y) = -\sin(2\pi y)$. As the temporal excitations we chose the following input signals, $\gamma_1(t) = \sin(2\pi 50t(T-t))$, $\gamma_2(t) =$

$\sin(2\pi 8t(T-\sin(4\pi t))$, $\gamma_3(t) = \sin(2\pi 65t(T-t))$ and $\gamma_4(t) = \sin(2\pi 8t(T-\cos(4\pi t))$, where $T = 0.5$ seconds. The choice of the above set of excitations signals is deliberate as they are spectrally rich enough, i.e, $\alpha_k(t)$'s will undergo regimes that change sometimes slowly and sometimes fast depending on the spectral composition of the external inputs. Under these conditions, the numerical content of the dynamical model is computed and a dynamic model is obtained. It is observed that the temporal variables obtained form the POD algorithm are very close to those obtained from the LD model and this observation indicates that the LD model is a good representative for the chosen conditions. Undoubtedly, one would expect a good match between the state variables obtained from the POD algorithm and the state variables obtained through the numerical solution of the ODE set in (17). One might question whether the model is specific to the boundary conditions above. Remedying this is accomplished by choosing another set of boundary conditions and obtaining the response of the model without modifying the model parameters. For this purpose, we set $\gamma_1(t) = \sin(2\pi 55t(T-t))$, $\gamma_2(t) = \sin(2\pi 9t(T-\sin(2\pi t))$, $\gamma_3(t) = \sin(2\pi 75t(T-t))$ and $\gamma_4(t) = \sin(2\pi 7t(T-\cos(5\pi t))$. The choice of this set of excitation signals is due to the spectral richness and aperiodicity within the selected time. With these excitation signals, without modifying the basis and the model contents, we have obtained the results illustrated in Figure 1, where every subplot contains two curves. It is seen that the state variables are obtained precisely when the relevant signal changes slowly. During the regions where the signals change quickly, there is some small discrepancy due to the neglected modes, chosen excitation signals, effects of numerical differentiation and so on. This result practically tells us that POD is a powerful technique for developing ODE models for PDE systems.

## 5 Conclusions

This paper considers POD based LD modeling of the flow governed by 2D Burgers equation. The studied problem is interesting due to its nonlinearity and the manner in which the boundary signals excite the process. The 2D nature of the problem makes it further appealing to dwell on. The paper validates the model and emphasizes that the model is useful over a set of operating conditions. The simulation results have shown that the model produces the temporal content of the dynamics precisely, indicating that the POD algorithm associated with the presented separation scheme are successful in deriving a representative LD model. Altholection of $M$ is absolutely a matter of the problem in hand and the auxiliary conditions it is subjectugh not discussed here, it is clear from (8) that the model performance is strictly dependent upon the number of modes chosen, i.e. increasing $M$ yields better yet mode complicated models, however, as $M$ decreases, the similarity of the modes from the LD model to those from the POD algorithm disappears according to the energy expression in (8). In short, the se to.

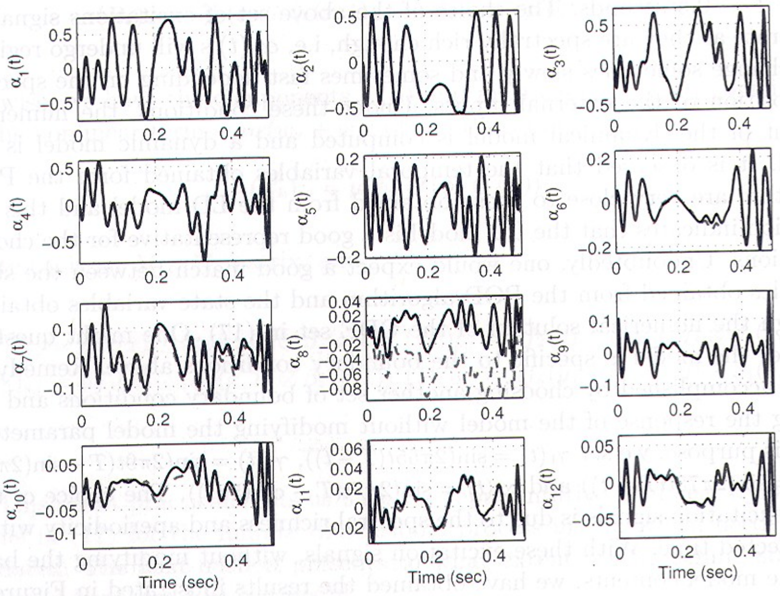

**Fig. 1.** Temporal variables; solid curves are form the dynamic model in (17) whereas dashed curves are the desired ones obtained from the POD algorithm (See (5))

As a result, POD is a powerful technique but its usefulness depends upon the PDE in hand, problem settings and the associated operating conditions.

## References


[Kha02] Khalil, H.K.: Nonlinear systems. Prentice-Hall, 3rd Ed., New Jersey (2002)

[Oga97] Ogata, K.: Modern control engineering. Prentice-Hall, New Jersey (1997)

[Ef06] Efe, M.Ö.: Observer based boundary control for 2D Burgers equation. Trans. of the Institute of Measurement and Control, **28** 177–185 (2006)

[LT01] Ly, H.V., Tran, H.T.: Modeling and control of physical processes using proper orthogonal decomposition. Mathematical and Computer Modelling of Dynamical Systems, **33**, 223-236 (2001)

[DH03] Donea, J., Huerta, A.: Finite element methods for flow problems. John Wiley & Sons, West Sussex (2003)

[McDH04] McDonough, J.M., Huang, M.T.: A 'poor man's NavierStokes equation': derivation and numerical experiments-the 2-D case. Int. J. for Numerical Methods in Fluids, **44**, 545-578 (2004)

[Blen91] Blender, R.: Iterative solution of nonlinear partial-differential equations. Journal of Physics A: Mathematical and General, **24**, L509-L512 (1991)

[BE03] Boules, A.N., Eick, I.J.: A spectral approximation of the two-dimensional Burgers equation. Indian Journal of Pure & Applied Mathematics, **34**, 299-309 (2003)

[Hata98] Hataue, I.: Mathematical and numerical analyses of dynamical structure of numerical solutions of two-dimensional fluid equations. Journal of the Physical Society of Japan, **67**, 1895-1911 (1998)

[NMT01] Nishinari, K., Matsukidaira, J., Takahashi, D.: Two-dimensional Burgers cellular automaton. Journal of the Physical Society of Japan, **70**, 2267-2272 (2001)

[Sire99] Sirendaoreji, S.J.: Exact solutions of the two-dimensional Burgers equation. Journal of Physics A: Mathematical and General, **32**, 6897-6900 (1999)

[Hie00] Hietarinta, L.: Comments on 'Exact solutions of the two-dimensional Burgers equation'. Journal of Physics A: Mathematical and General, **33**, 5157-5158 (2000)

[Zhu96] Zhu, Z.: Exact solutions for a two-dimensional KdV-Burgers-type equation. Chinese Journal of Physics, **34**, 1101-1105 (1996)

[Wes01] Wescott, B.L.: An efficient formulation of the modified nodal integral method and application to the two-dimensional Burgers equation. Nuclear Science and Eng., **139**, 293-305 (2001)

[Lum67] Lumley, J.: The structure of inhomogeneous turbulent flows. In: Yaglom, A., Tatarsky, V. (ed) Atmospheric Turbulence and Wave Propagation. Vol.3, Nauca, Moscow (1967)

[RCM04] Rowley, C.W., Colonius, T., Murray, R.M.: Model reduction for compressible flows using POD and Galerkin projection. Physica D-Nonlinear Phenomena, **189**, 115-129 (2004)

[Ravi00] Ravindran, S.S.: A reduced order approach for optimal control of fluids using proper orthogonal decomposition. Int. Journal for Numerical Methods in Fluids, **34**, 425-488 (2000)

[Far93] Farlow, S.J.: Partial differential equations for scientists and engineers. Dover Publications Inc., New York (1993)

[Siro87] Sirovich, L.: Turbulence and the dynamics of coherent structures. Quarterly of Applied Mathematics, **XLV**, 561-590 (1987)

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