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—Brief Paper—

A SUFFICIENT CONDITION FOR CHECKING THE ATTRACTIVENESS OF A SLIDING MANIFOLD IN FRACTIONAL ORDER SLIDING MODE CONTROL

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ABSTRACT

Stability issues of fractional order sliding mode control laws are analyzed in this paper. For differentiation orders less than unity, it is shown that a stable reaching law in the fractional order case corresponds to a stable reaching law in the integer order case. The contribution of the current study is to explain the stability of the closed loop by the use of the Caputo and Riemann-Liouville definitions of fractional order differentiation.

Key Words: Sliding mode control, fractional order systems.

I. INTRODUCTION

Sliding mode control (SMC) is a well-established robust control scheme that produces a switching type control signal that alleviates uncertainties that are matched and bounded. In the past, a number of variants of the sliding mode control scheme have been developed and successful results have been obtained. Most studies reported so far have a common property; for the continuous time case, the derivatives and integrators are of integer order and the differences for the discrete-time cases involve a finite number of terms. Motivated by this and the fact that some physical processes are described by fractional order operators, e.g. heat conduction, lossy transmission lines, etc., this paper aims to explain the stability properties of fractional order sliding mode control systems.

Fractional order control offers more degrees of freedom to designers to meet a predefined set of performance criteria. Order selection for differentiation and integration in a proportional, integral plus derivative (PID) controller is an example of this. Many successful outcomes have appeared in the literature on linear control applications. Recently, there has been a dramatic increase in the number of research outcomes regarding the theory and applications of fractional order systems and control [1–3]. Since the emergence of the theory dating back to a letter from Leibniz to L'Hôpital in 1695, asking about the possible consequences of choosing a derivative of order $1/2$, the theoretical foundations have been stipulated and, with advances in computational facilities, many important tools of classical control have been reformulated for (or adapted to) the fractional order case, such as PID controllers [4, 5], stability considerations [6–9], Kalman filtering [10], state space models and approaches [3, 11, 12], root locus technique [13], applications involving partial differential equations [14, 15], discrete time issues [1–3, 10], and so on. A system to be identified can be well approximated by an integer order model, or it can be approximated by a much simpler fractional order model. Having the necessary techniques and tools for such cases becomes a critical issue. With this motivation in mind, this paper focuses on the sliding mode control technique, although applications to nonlinear systems are highly limited due to the lack of a tool such as Lyapunov analysis for explaining the stability. This paper aims to fill this gap to some extent. A sufficient condition is derived and an application example is discussed. SMC is considered, as its robustness and

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invariance properties are exploited strongly to observe a desired response. Introductory work considering SMC of a double integrator is discussed in [16]. This paper gives a practical condition for checking whether the switching manifold is an attractor. The paper is organized as follows. Section II briefly describes fractional order differentiation and the main result of the letter. Section III presents the SMC design. A numerical example is given in Section IV, and concluding remarks constitute Section V.

II. FRACTIONAL ORDER DIFFERENTIATION AND INTEGRATION

Let σ be a function of time and $\beta \in (0, 1)$ be a positive real number. With the lower terminal equal to zero, Caputo's definition of derivative at order β is given by (1) and the Riemann-Liouville definition is given in (2):

$$\frac{\mathrm{d}^\beta}{\mathrm{d}t^\beta}\sigma(t) = \frac{1}{\Gamma(1-\beta)}\int_0^t \frac{\dot\sigma(\tau)}{(t-\tau)^\beta}\mathrm{d}\tau := \mathbf{C}^\beta\sigma(t) \qquad (1)$$

$$\frac{\mathrm{d}^\beta}{\mathrm{d}t^\beta}\sigma(t) = \frac{1}{\Gamma(1-\beta)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{\sigma(\tau)}{(t-\tau)^\beta}\mathrm{d}\tau := \mathbf{R}^\beta\sigma(t) \qquad (2)$$

where $\Gamma(\beta) = \int_0^t \mathrm{e}^{-t}t^{\beta-1}\mathrm{d}t$ is the Gamma function. It should be noted that fractional integration in the sense of Riemann-Liouville and Caputo corresponds to the same expression given as $\frac{\mathrm{d}^{-\beta}}{\mathrm{d}t^{-\beta}}\sigma(t) := \mathbf{D}^{-\beta}\sigma(t) = \frac{1}{\Gamma(\beta)}\int_0^t \frac{\sigma(\tau)}{(t-\tau)^{1-\beta}}\mathrm{d}\tau$. This is due to the selection of $\beta \in (0, 1)$.

Theorem 1. Let $\mathbf{D} \in \{\mathbf{C}, \mathbf{R}\}$ be one of the fractional differentiation operators. Let $\sigma(t)$ be a function of time, and let $\sigma(t)$ be a signal satisfying $\mathbf{D}^\beta\sigma = -k\,\mathrm{sgn}(\sigma)$, where $k>0$ and $\beta \in (0, 1)$. Such a nonzero $\sigma(t)$ also satisfies $\sigma\dot\sigma<0$, indicating a convergence toward $\sigma=0$.

Proof. As $0<\beta<1$, equality $\mathbf{D}^\beta\sigma = -k\,\mathrm{sgn}(\sigma)$ can be rearranged as $\dot\sigma = -k\mathbf{D}^{1-\beta}\mathrm{sgn}(\sigma)$. Since $\mathrm{sgn}(\mathbf{D}^{1-\beta}\mathrm{sgn}(\sigma)) = \mathrm{sgn}(\sigma)$ the inequality $\sigma\mathbf{D}^\beta\sigma<0$ is satisfied for $\forall t>0$. Let us consider the implications of this for (1) and (2) separately. According to (1), Caputo's definition yields $\sigma\mathbf{C}^\beta\sigma := \frac{\sigma}{\Gamma(1-\beta)}\int_0^t \frac{\dot\sigma(\tau)}{(t-\tau)^\beta} = \mathrm{d}\tau$. If $\sigma\mathbf{C}^\beta\sigma<0$, for $\forall t>0$, then σ and $\dot\sigma$ must have opposite signs, *i.e.* $\sigma\dot\sigma<0$ is obtained. According to (2), we have $\sigma\mathbf{R}^\beta\sigma := \frac{\sigma}{\Gamma(1-\beta)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{\sigma(\tau)}{(t-\tau)^\beta}\mathrm{d}\tau$ in terms of the Riemann-Liouville definition of the fractional order derivative. We can obtain $\sigma\mathbf{R}^\beta\sigma<0$ in the following cases. In the first case, $\sigma(t)<0$ and the integral $\int_0^t \sigma(\tau)(t-\tau)^{-\beta}\mathrm{d}\tau$ need to monotonically decrease. In the second case, $\sigma(t)<0$ and the integral $\int_0^t \sigma(\tau)(t-\tau)^{-\beta}\mathrm{d}\tau$ must monotonically increase. In both cases, the signal $\sigma(t)$ is forced to approach the origin faster than $t^{-\beta}$. According to both definitions, $\mathbf{D}^\beta\sigma = -k\,\mathrm{sgn}(\sigma)$ ensures that $\sigma=0$ is a global attractor for $\sigma \in \Re$. □

A numerical example justifying this result is shown in Fig. 1, where $\beta=0.5$ and $k=1$. The solutions of $\mathbf{D}^\beta\sigma = -k\,\mathrm{sgn}(\sigma)$ for $\sigma(0)=1$ and $\sigma(0)=-1$ are plotted on integer order axes, and it is clear from Fig. 1 that $\sigma\dot\sigma<0$ is satisfied. According to the aforementioned rule of fractional integration, if we apply the integration operator to both sides of $\mathbf{D}^\beta\sigma = -k\,\mathrm{sgn}(\sigma)$ we obtain

$$\begin{aligned}\sigma(t)-\sigma(0) &= \mathbf{D}^{-\beta}(-k\,\mathrm{sgn}\sigma(t))\\ &= \mathbf{D}^{-\beta}(-k\,\mathrm{sgn}\sigma(0))\\ &= -k\,\mathrm{sgn}\sigma(0)\frac{1}{\Gamma(\beta)}\int_0^t \frac{\mathrm{d}\tau}{(t-\tau)^{1-\beta}}\end{aligned} \qquad (3)$$

and taking the integral leads to the following solution:

$$\sigma(t) = \begin{cases}\left(|\sigma(0)| - \dfrac{k}{\Gamma(\beta+1)}t^\beta\right)\mathrm{sgn}(\sigma(0)) & t<t_h\\ 0 & t \ge t_h\end{cases} \qquad (4)$$

where $t_h = (k^{-1}|\sigma(0)|\Gamma(\beta+1))^{1/\beta}$. It is important to note that the solution does not change sign until it reaches zero, which allows us to write the second line of (3).

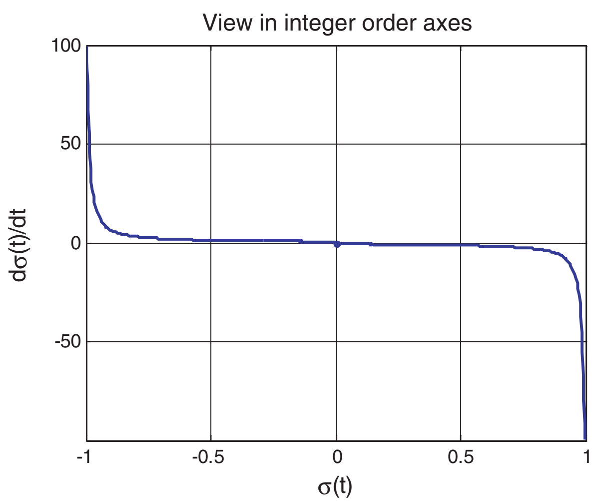

Fig. 1. The solution of $\mathbf{D}^\beta\sigma = -k\,\mathrm{sgn}(\sigma)$ for $\sigma(0)=1$ and $\sigma(0)=-1$ are obtained and $\sigma(t)$ is plotted versus its first derivative to show that $\sigma(0)=0$ is a global attractor.

III. SLIDING MODE CONTROLLER DESIGN

Let $\mathbf{x}\in\Re^n$ and $u\in\Re$ be the state vector and the input signal, respectively. Let $\beta\in(0,1)$ and $\mathbf{D}^\beta\mathbf{x}=(\mathbf{A}+\Delta\mathbf{A})\mathbf{x}+(\mathbf{B}+\Delta\mathbf{B})u$ be the system to be controlled. Let $\mathbf{r}\in\Re^n$ be the vector of reference signals, and let σ be the switching function defined as

$$\sigma=\mathbf{\Lambda}(\mathbf{x}-\mathbf{r}) \tag{5}$$

where $\mathbf{\Lambda}$ is chosen such that $\sigma=0$ is a subspace in $\Re^{n-1}$ and its global equilibrium is the origin. With $k>0$, the control law in (6) forces the dynamics in (7).

$$u=\frac{-\mathbf{\Lambda A x}+\mathbf{\Lambda D}^\beta\mathbf{r}-k\,\mathrm{sgn}(\sigma)}{\mathbf{\Lambda B}} \tag{6}$$

$$\mathbf{D}^\beta\sigma=-\left(1+\frac{\mathbf{\Lambda}\Delta\mathbf{B}}{\mathbf{\Lambda B}}\right)k\,\mathrm{sgn}(\sigma)+\frac{\mathbf{\Lambda}\Delta\mathbf{B}}{\mathbf{\Lambda B}}\mathbf{\Lambda}(\mathbf{D}^\beta\mathbf{r}-\mathbf{Ax})+\mathbf{\Lambda}\Delta\mathbf{A}\mathbf{x} \tag{7}$$

1. If there are no uncertainties, *i.e.* $\Delta\mathbf{A}=\mathbf{0}$ and $\Delta\mathbf{B}=\mathbf{0}$, then we have $\mathbf{D}^\beta\sigma=-k\,\mathrm{sgn}(\sigma)$. This ensures $\sigma\mathbf{D}^\beta\sigma<0$ and the sliding regime emerges after hitting the sliding hypersurface $\sigma=0$.
2. If $\Delta\mathbf{B}=\mathbf{0}$ and the columns of $\Delta\mathbf{A}$ are in the range space of $\mathbf{B}$, then $\mathbf{D}^\beta\sigma=-k\,\mathrm{sgn}(\sigma)+\mathbf{\Lambda}\Delta\mathbf{A}\mathbf{x}$. This case further requires that the condition in (8) holds to maintain $\sigma\mathbf{D}^\beta\sigma<0$.

$$k>|\mathbf{\Lambda}\Delta\mathbf{A}\mathbf{x}| \tag{8}$$

3. If there are nonzero uncertainty terms, then (7) is valid and the designer needs to set k carefully to maintain the attractiveness of the subspace defined by $\sigma=0$. The conditions in (9)–(10) are needed to maintain $\sigma\mathbf{D}^\beta\sigma<0$.

$$\left|\frac{\mathbf{\Lambda}\Delta\mathbf{B}}{\mathbf{\Lambda B}}\right|<1 \tag{9}$$

$$k>\left(1+\frac{\mathbf{\Lambda}\Delta\mathbf{B}}{\mathbf{\Lambda B}}\right)^{-1}\left|\frac{\mathbf{\Lambda}\Delta\mathbf{B}}{\mathbf{\Lambda B}}\mathbf{\Lambda}(\mathbf{D}^\beta\mathbf{r}-\mathbf{Ax})+\mathbf{\Lambda}\Delta\mathbf{A}\mathbf{x}\right| := K(t) \tag{10}$$

IV. A NUMERICAL EXAMPLE

Consider

$$\mathbf{A}=\begin{bmatrix}0&1&0\\0&0&1\\-1&-2&-1\end{bmatrix},\quad \mathbf{B}=\begin{bmatrix}0\\0\\1\end{bmatrix},\quad \beta=0.5$$

and

$$\Delta\mathbf{A}=\begin{bmatrix}0&0&0\\0&0&0\\0.1x_2&0.12\sin 6\pi t+0.15x_3&0.15\sin 10\pi t-0.11x_1\end{bmatrix}$$

$$\Delta\mathbf{B}=\begin{bmatrix}0\\0\\0.02\sin 7\pi t+0.1x_1\end{bmatrix}$$

Due to the uncertainty terms, the plant dynamics is nonlinear. With $\mathbf{\Lambda}=[1\ 2\ 1]$, the switching function becomes $\sigma=(\mathbf{D}^\beta+1)^2e_1$, where $e_i=x_i-r_i$, $i=1,2,3$, and $\mathbf{x}=[x_1\ x_2\ x_3]^T$. Due to the above choice, $\sigma=0$ is a plane and has a unique global equilibrium at the origin. In the simulations, we need to realize the fractional order operators via numerical approximations exploiting integer order terms. Crone approximation in (11) is selected to realize the operator $\mathbf{D}$, and sinusoidal reference profiles are chosen as described by (12)–(13).

$$\mathbf{D}^\beta := s^\beta\approx G\frac{\prod_{k=1}^N 1+s/w_{p_k}}{\prod_{k=1}^N 1+s/w_{z_k}} \tag{11}$$

$$r_1 = \sin(10t) \tag{12}$$

$$r_2 = \mathbf{D}^\beta r_1=\sqrt{10}\sin\left(10t+\frac{\pi}{4}\right) \tag{13}$$

$$r_3 = \mathbf{D}^\beta r_2=10\sin\left(10t+\frac{\pi}{2}\right) \tag{14}$$

where G in (11) is set so that the expression passes through 0 dB level when $w=1$ rad/s. The variables w_{pk} and w_{zk} are scheduled by the Crone algorithm with a given realization order N and a frequency region. We choose $N=38$ for the frequency range $[1e-12\ \ 1e+4]$ rad/s. In setting these values, we considered the highest possible N value not causing numerical problems and an adequately wide band of the frequency spectrum to approximate the fractional order operator. With these selections, we simulate the control loop and, in Fig. 2, the reference signals and the process responses are shown together. In spite of the initial errors, the process states very quickly reach their desired values. Figure 2 shows that the sliding mode starts very quickly as the errors very quickly reach zero. Figure 3 illustrates the control

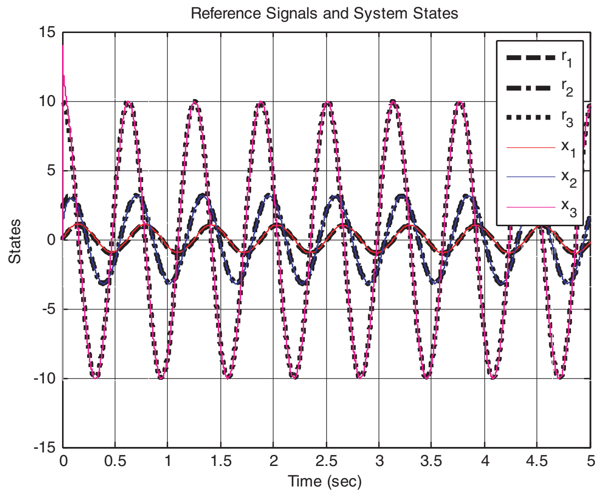

Fig. 2. Reference signals and the plant responses.

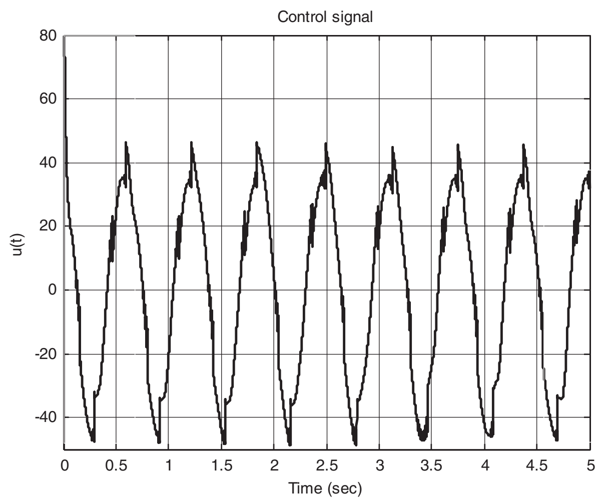

Fig. 3. Applied control signal.

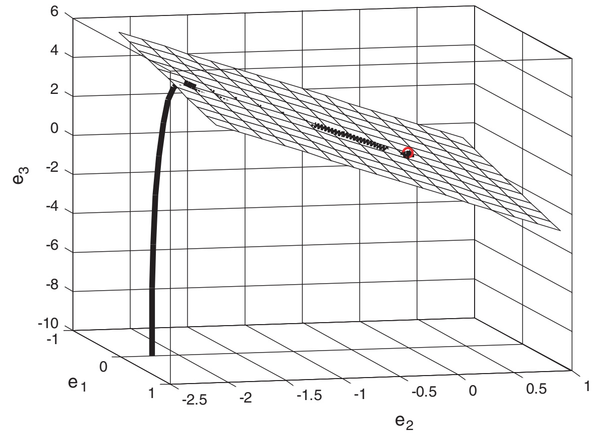

Fig. 4. Phase space behavior.

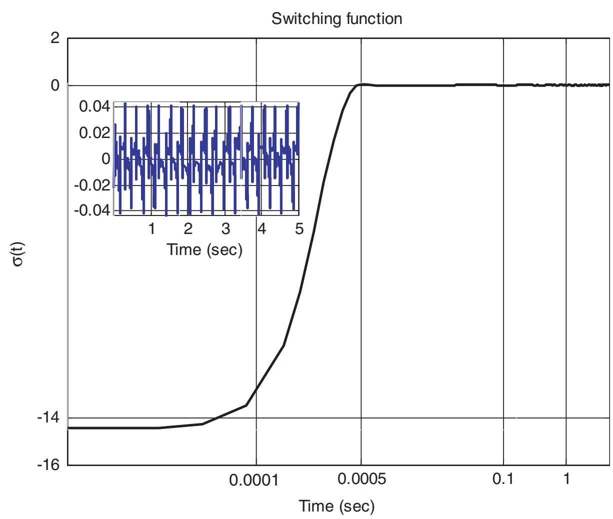

Fig. 5. The evolution of the switching variable $\sigma(t)$.

signal produced by the proposed scheme. The smoothness of the control signal needs emphasis as it is relevant to the actuation periphery in real control systems. Since the control signal depends on the sign of a quantity that is very close to zero, noise in the observations can introduce undesired high frequency switching in the control signal. This is known in the literature as chattering. A remedy for this is to utilize the approximation $\mathrm{sgn}(\sigma) \approx \sigma/(|\sigma|+\delta)$ with a small δ, e.g. $\delta = 0.01$, to obtain a smoother control signal and to reduce chattering to some extent. In Fig. 4, the behavior in the phase space is shown with a circle marking the origin. The error vector hits the sliding surface at approximately $t = 0.5$ ms and remains in the vicinity. Since the prescribed dynamics require convergence to zero, the trajectory tends toward the origin. To see this explicitly, in Fig. 5, the time evolution of the switching variable is shown on a logarithmic time axis. Clearly, the switching function is maintained around zero for approximately $t > 0.5$ ms. The regime around zero is depicted separately as a window plot in Fig. 5, where the results support the theoretical claims. Figure 6 depicts $K(t)$ defined in (10), which is always less than $k = 10$. The condition in (9) is satisfied as $\mathbf{\Lambda B} = 1$ and $|\mathbf{\Lambda \Delta B}| \leq 0.12$.

V. CONCLUSIONS

In the SMC approach for fractional order nonlinear systems with nonlinearities entering as the uncertainty terms, it is sufficient to check whether $\sigma \mathbf{D}^{\beta} \sigma < 0$ is satisfied if $\sigma = 0$ is desired as a global attractor. Due to the design of the switching function σ, the trajectories lying

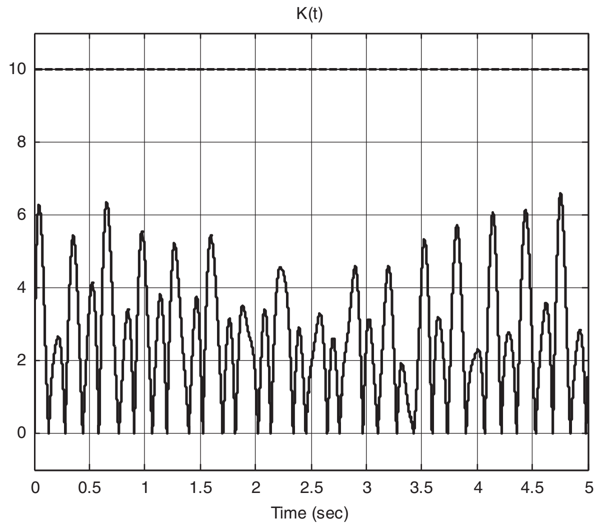

Fig. 6. The evolution of $K(t)$ of (10).

on the sliding hypersurface defined by $\sigma = 0$ converge to the origin of the phase space.

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