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A non-fragile robust observer design for uncertain time-delay fractional Itô stochastic systems with input nonlinearity: An SMC approach

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Abstract

This article considers the problem of non-fragile observer design for uncertain fractional Itô stochastic systems. The design is based on a sliding surface whose reachability in finite time is guaranteed by introducing a novel sliding mode control law. Employing the fractional infinitesimal operator and the related lemmas, the stochastic stability of the overall closed-loop system is transformed to the problem of solving a set of linear matrix inequalities. Addressing the fragility issue, a norm-bounded term is added to the observer gain, which prevents failure of the estimation error system. The adverse effects of the input nonlinearity and time-varying delay are alleviated by the proposed approach. Furthermore, the present method is investigated for the fractional Itô stochastic systems with known states. A numerical example is presented to illustrate the effectiveness of the proposed method.

Keywords

Non-fragile observer, fractional Itô stochastic systems, sliding mode control, linear matrix inequalities, input nonlinearity

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Introduction

In many practical systems, some or all of the states of the system are not measurable. Hence, it very important to develop state estimation methods and observer design strategies. Among various state estimation and filtering methods, sliding mode observers have been very popular due to their fast convergence, robustness against disturbances and the ability to cope with uncertainties.[1,2] A class of sliding mode observers, called non-fragile observers, have been introduced which are resilient against perturbations of the observer coefficients.[3–9] Stochastic systems have found applications in various fields of science and numerous stochastic models have been proposed to describe real-world randomly varying processes.[10–12] For such systems, several Sliding Mode Control (SMC) based observation strategies have been introduced in the literature. For instance, in Ma et al.[13] an adaptive sliding mode observer has been proposed to estimate the states of Itô stochastic jump systems. A non-fragile observer-based adaptive SMC method has been developed for fractional-order time-delay Markovian jump systems with time delay in Parvizian et al.[14] In Niu and Ho,[15] SMC has been utilized for obtaining an observer for Itô stochastic systems with time delay and unmatched uncertainties. In Basin et al.,[16] an integral SMC filtering approach has been proposed for linear stochastic systems, and employing the approach, the disturbances in the observation equation are successfully suppressed. For a class of nonlinear stochastic systems, an observer-based adaptive SMC design method has been proposed in Jiang et al.[17] and the approach has been applied on a single-link robot arm model. In Li et al.,[18] an augmented sliding mode observer has been proposed for a class of stochastic systems for the sake of eliminating the effects of sensor faults and disturbances. In Kao et al.,[19] a non-fragile sliding mode observer for uncertain Markovian neutral-type stochastic systems has

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been introduced, where the estimation error system is stochastically asymptotically stabilized with a certain disturbance attenuation level. In the aforementioned works, the stochastic models are driven by Brownian motion. Such systems can be modeled in the stochastic Itô form and also can be analyzed utilizing classical stochastic calculus and Itô's formula. A generalization of such models, which are called fractional Itô stochastic systems, have gained increasing attention in the recent years. Such stochastic systems are driven by fractional Brownian motion (fBm), which is characterized by a special parameter called Hurst index (H). For $0.5 < H < 1$, fBm represents a self-similar, non-semi-martingale process with long range dependence.[20] Finding solutions for fractional Itô stochastic differential equations has been investigated in a few articles such as Zeng et al.,[21] Nguyen,[22] Arthi et al.[23] and Khandani et al.[24] For stability analysis of such systems, a mathematical tool has been proposed in Khandani et al.,[25] where the fractional infinitesimal operator has been introduced and a Lyapunov stability theorem has been proposed for linear Itô-type fractional stochastic models. In Tamilalagan and Balasubramaniam,[26] the approximate controllability of a class of fractional stochastic differential equations has been proposed. Very few control methods for fractional Itô stochastic systems have been developed. For example, an SMC strategy for linear fractional Itô stochastic systems driven by fBm has been developed in Khandani et al.[25,27] where it has been assumed that the states are available for the controller design. In addition, Shi and Zhang[28] addressed H_∞ filtering and control problem of linear fractional stochastic systems. In fractional stochastic systems driven by fBm, estimating the states of systems is even more critical since the system inherently features fractional stochastic perturbations, which complicate the observer design methods. There are numerous systems with fractional stochastic perturbations in which obtaining the estimates of states is of great importance, to name a few, wind turbine models,[29] network traffic,[30] fluids models,[31] and super-diffusion and sub-diffusion.[32] Estimation of the states of fractional stochastic systems is vital for developing a successful control approach. This is the motivation of the current study.

In this article, we consider uncertain fractional Itô stochastic systems with time-varying delay. We also consider the effects of input nonlinearity in the system, which is a practical issue. An SMC observer is developed to obtain the states of the system. Since the stochastic system is driven by fractional Brownian motion and the dynamics of the closed-loop system cannot be analyzed by traditional stochastic calculus, we utilized fractional infinitesimal operator to obtain the error stability conditions in terms of LMIs, which can be straightforwardly checked for feasibility. It has also been shown that the proposed SMC approach can be applied to control a fractional Itô system with known states. Briefly, the novelties of this article lie in the following directions:

- A novel non-fragile observer system, which is robust against parameter variations, is proposed. The system can handle the mentioned issues in estimating the states of Itô-type fractional stochastic systems driven by fractional Brownian motion.
- The overall stability of the closed-loop system is proven via generalized infinitesimal generator and novel Lyapunov functionals.

The rest of the article is organized as follows: In Section 2, the problem is defined and the preliminaries are given. In Section 3, the main results of the article are proposed and proven. In Section 4, a numerical example is given. In Section 5, we present the conclusion.

The problem statement and preliminaries

Consider the following fractional Itô stochastic system with input nonlinearity and time-varying delay

$$\begin{cases} dx(t) = ((A + \Delta A(t))x(t) + (A_1 + \Delta A_1(t))x(t-h(t)) \\ \qquad + B\emptyset(u) + f(x,t))dt + Gx(t)dB^H(t) \\ y(t) = Cx(t) \\ x(t) = \varphi(t), t \in [-h, 0] \end{cases} \tag{1}$$

where $x(t)$, $u(t)$ and $y(t)$ are the state variables, input and measured output vectors, respectively; $\varphi(t)$ is a continuous vector-valued initial function; A, A_1, B, G, and C are real constant matrices of appropriate dimensions and B is a full-column-rank matrix.$B^H(t)$ is the fractional Brownian motion with Hurst parameter (H) $0.5 < H < 1$; $\emptyset(u)$ is a continuous function vector; $f(x,t)$ is the mismatched function satisfying $f(0,t) = 0$ and

$$\|f(x_1,t) - f(x_2,t)\| \leq \rho\|x_1 - x_2\| \tag{2}$$

where ρ is a known positive constant. $h(t)$ denotes the time-varying delay and satisfies

$$0 \leq h(t) \leq h, \dot{h}(t) \leq \bar{h} < 1 \tag{3}$$

$\Delta A(t)$ and $\Delta A_1(t)$ are the uncertainties in the form of

$$[\Delta A(t)\Delta A_1(t)] = DF(t)[EE_1] \tag{4}$$

where D, E and E_1 are real constant matrices and $F(.)$ is an unknown time-varying matrix function satisfying the following inequality

$$F^T(t)F(t) \leqslant I \tag{5}$$

Assumption 1. The matrix pairs (A, B) and (A, C) are stabilizable and detectable, respectively.

Assumption 2. The mismatched function $f(x, t)$ satisfies

$$f(x, t) = \Xi(t)x(t) + \Xi_1(t)x(t - h(t)) \tag{6}$$

where $\Xi(t)$ and $\Xi_1(t)$ are matrix functions of appropriate dimensions.

Assumption 3. The nonlinear input $\emptyset(u)$ applied to the system satisfies

$$u^T\emptyset(u) \geqslant \chi u^T u \tag{7}$$

where χ is a positive constant and $\emptyset(0) = 0$.

Lemma 1. *Schur complement.* Given a symmetric matrix $\Omega = \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{bmatrix}$, the following three statements are equivalent

$$\begin{aligned} &\Omega < 0, \\ &\Omega_{11} < 0, \Omega_{22} - \Omega_{12}{}^T\Omega_{11}{}^{-1}\Omega_{12} < 0, \\ &\Omega_{22} < 0, \Omega_{11} - \Omega_{12}\Omega_{22}{}^{-1}\Omega_{12}{}^T < 0 \end{aligned} \tag{8}$$

Lemma 2. Let R, J, and $\Pi(t)$ be real matrices of appropriate dimensions with $\Pi(t)$ satisfying equation (5) and scalar $\varepsilon > 0$, then the following inequality holds

$$R\Pi(t)J + J^T\Pi^T(t)R^T \leqslant \varepsilon RR^T + \frac{1}{\varepsilon}J^TJ \tag{9}$$

Lemma 3. Let R, J, Z and $\Pi(t)$ be real matrices of appropriate dimensions with $\Pi(t)$ satisfying equation (5). Then we can write

$$R + J\Pi(t)Z + Z^T\Pi^T(t)J^T < 0 \tag{10}$$

if and only if there exists some scalar $\varepsilon > 0$ such that

$$R + \varepsilon JJ^T + \varepsilon^{-1}Z^TZ < 0 \tag{11}$$

Definition 1. Consider the stochastic differential equation[25]

$$dx(t) = a(x, t)dt + b(x, t)dB^H(t) \tag{12}$$

where $a(x, t)$ and $b(x, t)$ are real-valued linear functions and the Hurst parameter satisfies $0.5 < H < 1$. From fractional Itô's formula, the fractional infinitesimal operator $\mathcal{L}^H$ is defined as

$$\mathcal{L}^H : = \frac{\partial}{\partial t} + \frac{\partial}{\partial x}a(x, t) + b(x, t)\frac{\partial^2}{\partial x^2}D_t^{\varphi}x(t) \tag{13}$$

where $D_t^{\varphi}x(t)$ is the Malliavin derivative of $x(t)$ defined as

$$D_t^{\varphi}x(t) = x(t)\int_0^t \varphi(t, s)G(s)ds \forall t \in [0, T] \tag{14}$$

and the function $\varphi(s, t)$ is as given below:

$$\varphi(s, t) = H(2H - 1)|s - t|^{2H-2} \tag{15}$$

Theorem 1. Consider a stochastic system driven by fBm given in equation (12).[25] Provided that there exists a function $V(x, t) \in C^{2,1}$ such that

$$V(0, t) = 0, \alpha_1\|x\| \leqslant V(x, t), V_x(x, t)b(x, t) \in \mathcal{L}(0, T) \tag{16}$$

for all $(x, t) \in S_h \times R_+$ and $\mathcal{L}(0, T)$ defined in Definition 1 in Khandani et al.,[25] and $V_x(x, t)$ the derivative of $V(x, t)$ with respect to x, then the trivial solution of equation (12) is stochastically stable if:

$$\mathcal{L}^H V(x, t) \leqslant 0 \tag{17}$$

where $\mathcal{L}^H$ is defined in equation (13).

Remark 1. Note that the order of the dynamics of the fractional stochastic system introduced in equation (1) is integer, which is driven by fractional Brownian motion. This should not be confused with fractional-order systems with non-integer derivatives.

Remark 2. In Parvizian et al.,[9] the problem of SMC design for fractional-order Markovian jump systems has been addressed, where the order of the systems is fractional and the systems are subject to random parameter changes. For such systems, Itô's formula and the classical infinitesimal generator can be utilized to obtain the results. However, in the present article, the order of the systems is integer but the systems are driven by fractional Brownian motion. Such complicated systems presented in equation (1) cannot be investigated via classical tools in stochastic mathematics and the diffusive representation approach used in Parvizian et al.[9] cannot be applied here. We have used the generalized infinitesimal generator equation (13) to prove the stability theorems. Challenging limitations are caused by the appearance of the Malliavin derivative in the fractional infinitesimal operator, which are dealt with by novel Lyapunov functionals.

Main results

Non-fragile observer design for the fractional Itô stochastic system

We propose the following non-fragile state observer to estimate the states of the uncertain fractional stochastic system equation (1) as

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + A_1\hat{x}(t-h(t)) + B\emptyset(u) + f(\hat{x},t) \\ \qquad + (L+\Delta L(t))(y(t) - C\hat{x}(t)) \\ \hat{y}(t) = C\hat{x}(t) \end{cases} \tag{18}$$

where L is the observer gain to be designed later and $\Delta L(t)$ is a nonlinear function matrix satisfying

$$\|\Delta L(t)\| \leqslant \delta \tag{19}$$

where δ is a positive constant. Defining the estimation error as $e := x - \hat{x}$, the error system is obtained from equations (1) and (18) as

$$\begin{cases} \dot{e}(t) = ((A + \Delta A - LC - \Delta LC)e(t) + (A_1 + \Delta A_1)e(t-h(t)) + \Delta A_1\hat{x}(t-h(t)) \\ \qquad + \Delta A\hat{x}(t) + f(x,t) - f(\hat{x},t))dt + (Ge(t) + G\hat{x}(t))dB^H(t) \\ y_e(t) = Ce(t) \end{cases} \tag{20}$$

A switching function is chosen as follows

$$s(t) = \sigma(t) + B^T\hat{x}(t) \tag{21}$$

with

$$\dot{\sigma}(t) = B^TBK\hat{x}(t) - B^TA\hat{x}(t) - B^TA_1\hat{x}(t-h(t)) - B^Tf(\hat{x},t)$$

where the matrix K is to be chosen later and B^TB is nonsingular.

The control input $u(t)$ in equation (1) should be appropriately designed such that the estimated states in system equation (18) are driven toward the sliding surface even when the input nonlinearity is present. The SMC law is derived as follows

$$u(t) = -\frac{s(t)}{\|s(t)\|}\psi(\hat{x}) \tag{22}$$

where

$$\psi(\hat{x}) = \frac{1}{\chi}\Big(\|K\hat{x}(t)\| + \delta\left\|\left(B^TB\right)^{-1}B^T\right\|\|(y(t) - C\hat{x}(t))\| + \left\|\left(B^TB\right)^{-1}B^TL(\mathrm{y(t)} - C\hat{x}(t))\right\| + \xi\Big)$$

where ξ is an arbitrarily-chosen positive scalar. The following theorem establishes the reachability of the sliding surface.

Theorem 2. If the control input $u(t)$ is designed as in equation (22), then the trajectories of the observer system equation (18) will converge to the sliding surface $s(t) = 0$ in finite time.

Proof. Let

$$V_1(t) = \mathrm{s(t)}^T\left(B^TB\right)^{-1}\mathrm{s(t)} \tag{23}$$

From equations (18) and (21), we have

$$\dot{\mathrm{s}}(\mathrm{t}) = B^TBK\hat{x}(t) + B^TB\emptyset(u) + B^T(L+\Delta L(t))(y(t) - C\hat{x}(t)) \tag{24}$$

Then it follows from equations (23) and (24) that

$$\begin{aligned} \dot{V}_1(t) &= 2\mathrm{s(t)}^TK\hat{x}(t) + 2\mathrm{s(t)}^T\emptyset(\mathrm{u}) + 2s(t)^T\left(B^TB\right)^{-1} B^T(L+\Delta L(t))(y(t) - C\hat{x}(t)) \leqslant \\ & 2s(t)^T\emptyset(u) + 2\|s(t)\|\Big(\|K\hat{x}(t)\| + \delta\left\|\left(B^TB\right)^{-1}B^T\right\| \\ & \times \|(y(t) - C\hat{x}(t))\| + \\ & \left\|\left(B^TB\right)^{-1}B^TL(y(t) - C\hat{x}(t))\right\|\Big) \end{aligned} \tag{25}$$

From equation (22) and Assumption 3 we obtain

$$u^T\emptyset(u) = -\frac{s(t)^T}{\|s(t)\|}\psi(\hat{x})\emptyset(u) \geqslant \chi(\psi(\hat{x}))^2 \tag{26}$$

which results in

$$s(t)^T\emptyset(u) \leqslant -\chi\psi(\hat{x})\|s(t)\| \tag{27}$$

Substituting equation (27) into equation (25) we obtain

$$\dot{V}_1(t) \leqslant -\xi\|s(t)\| < 0 \; for \; \|s(t)\| \neq 0 \tag{28}$$

which proves that the error trajectories reach the sliding surface in finite time. ■

Stability analysis

From $\dot{s}(t) = 0$, the following equivalent control law can be obtained

$$\emptyset_{eq}(u) = -\left(K\hat{x}(t) + \left(B^TB\right)^{-1}B^T(L + \Delta L(t))(y(t) - C\hat{x}(t))\right) \tag{29}$$

In order to obtain the sliding mode dynamics in the state estimation space, we substitute equation (29) into the non-fragile state observer dynamics equation (18), which results in

$$\dot{\hat{x}}(t) = (A - BK)\hat{x}(t) + A_1\hat{x}(t - h(t)) + f(\hat{x}, t) + \left(I - B\left(B^TB\right)^{-1}B^T\right)(L + \Delta L(t))(y(t) - C\hat{x}(t)) \tag{30}$$

In the following theorem, the sufficient conditions for the stability of the overall closed-loop system are presented. We will derive the results in terms of LMIs which are straightforward to check the stability.

Theorem 3. Consider the error system equation (20), the switching function equation (21), and the sliding mode dynamics equation (30); The SMC law is chosen as in equation (22). Then the overall closed-loop system described by equations (20) and (30) is stochastically stable (stable in probability) provided that there exist matrices $X > 0$, $Y > 0$, $Q_1 > 0$, $Q_2 > 0$, and scalars $\varepsilon_i > 0$ $(i = 1 - 6)$ satisfying the following LMI

$$\Psi = \begin{bmatrix} \Psi_{11} & XA_1 + \varepsilon_1 E^T E_1 & 0 & 0 & N_1 \\ * & \varepsilon_1 E_1{}^T E_1 - (1-\bar{h})Q_2 & 0 & 0 & 0 \\ * & * & \Psi_{33} & XA_1 + \varepsilon_2 E^T E_1 & N_2 \\ * & * & * & \varepsilon_2 E_1{}^T E_1 - (1-\bar{h})Q_1 & 0 \\ * & * & * & * & N_3 \end{bmatrix} < 0 \tag{31}$$

with

$$\Upsilon = \left(I - B\left(B^TB\right)^{-1}B^T\right)$$

$$\Psi_{11} = \left(X(A - BK) + (A - BK)^TX\right) + \varepsilon_1 E^TE + Q_2 + \varepsilon_6\rho^2 I$$

$$\Psi_{33} = A^TX + XA - YC - C^TY^T + \varepsilon_2 E^TE + \varepsilon_3\delta^2C^TC + \varepsilon_4\delta^2C^TC + Q_1 + \varepsilon_5\rho^2 I$$

$$N_1 = [\,X\Upsilon \quad X\Upsilon \quad X \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\,]$$

$$N_2 = \left[0 \quad 0 \quad 0 \quad C^TY^T \quad XD \quad XD \quad X \quad X\right]$$

$$N_3 = diag\{-X \quad -\varepsilon_3 I \quad -\varepsilon_6 I \quad -X \quad -\varepsilon_1 I \quad -\varepsilon_2 I \quad -\varepsilon_4 I \quad -\varepsilon_5 I\}$$

and $\bar{h}$ defined in equation (3). The state observer gain is also obtained as:

$$L = X^{-1}Y$$

Proof. Choose the following Lyapunov functional candidate

$$V_2(t) = \left\{ e(t)^TXe(t) + \int_{t-h(t)}^{t} e(s)^TQ_1e(s)ds + \hat{x}(t)^TX\hat{x}(t) + \int_{t-h(t)}^{t} \hat{x}(s)^TQ_2\hat{x}(s)ds \right) \times e^{-\lambda\int_0^t\int_0^s \varphi(s,\tau)d\tau ds} \tag{32}$$

Then, by utilizing equations (13), (20) and (30), it follows that

$$\begin{aligned}\mathcal{L}^HV_2(t) = &\Big\{2e(t)^TX(A - LC)e(t) - 2e(t)^TX(\Delta LC)e(t) + 2e(t)^TX(\Delta A)e(t) + \\ &2e(t)^TXA_1e(t - h(t)) + 2e(t)^TX(\Delta A_1)e(t - h(t)) \\ &+ 2e(t)^TX(f(x, t) - f(\hat{x}, t)) \\ &+ 2e(t)^TX\Delta A\hat{x}(t) + 2e(t)^TX\Delta A_1\hat{x}(t - h(t)) \\ &+ e(t)^TQ_1e(t) - \left(1 - \dot{h}(t)\right)e(t - h(t))^TQ_1e(t - h(t)) \\ &+ 2\hat{x}(t)^TX(A - BK)\hat{x}(t) + 2\hat{x}(t)^TXA_1\hat{x}(t - h(t)) \\ &+ 2\hat{x}(t)^TX\left(\left(I - B\left(B^TB\right)^{-1}B^T\right)LC\right)e(t) \\ &+ 2\hat{x}(t)^TX\left(\left(I - B\left(B^TB\right)^{-1}B^T\right)\Delta LC\right)e(t) \\ &+ 2\hat{x}(t)^TXf(\hat{x}, t) + \hat{x}(t)^TQ_2\hat{x}(t) \\ &-\left(1 - \dot{h}(t)\right)\hat{x}(t - h(t))^TQ_2\hat{x}(t - h(t))\Big\} \times \\ &e^{-\lambda\int_0^t\int_0^s\varphi(s,\tau)d\tau ds} \\ &+ \Big\{e(t)^TG^TXGe(t) + \hat{x}(t)^TG^TXG\hat{x}(t) \\ &+ 2e(t)^TG^TXG\hat{x}(t). - \lambda e(t)^TXe(t) - \\ &\lambda\int_{t-h(t)}^{t} e(s)^TQ_1e(s)ds - \lambda\hat{x}^T(t)X\hat{x}(t) - \lambda \\ &\int_{t-h(t)}^{t}\hat{x}^T(s)Q\hat{x}(s)ds\Big\} \times \\ &\int_0^t\varphi(t, s)dse^{-\lambda\int_0^t\int_0^s\varphi(s,\tau)d\tau ds}\end{aligned} \tag{33}$$

Provided that a sufficiently large value is chosen for λ, the last term in equation (33) will vanish, then by employing Lemma 2, we obtain from equation (33)

$$\mathcal{L}^H V_2(t) \leqslant w_1(t)^T \Phi w_1(t) \times e^{-\lambda \int_0^t \int_0^s \varphi(s,\tau) d\tau ds} \tag{34}$$

where
$w_1(t) = \begin{bmatrix} \hat{x}^T(t) & \hat{x}^T(t-h(t)) & e^T(t) & e^T(t-h(t)) \end{bmatrix}^T$,
and

$$\Phi = \begin{bmatrix} \Phi_{11} & XA_1 + \varepsilon_1 E^T E_1 & 0 & 0 \\ * & \varepsilon_1 E_1{}^T E_1 - (1-\bar{h})Q_2 & 0 & 0 \\ * & * & \Phi_{33} & XA_1 + \varepsilon_2 E^T E_1 \\ * & * & * & \varepsilon_2 E_1{}^T E_1 - (1-\bar{h})Q_1 \end{bmatrix} \tag{35}$$

with

$$\Phi_{11} = \left(X(A-BK) + (A-BK)^T X \right) + XYX^{-1}Y^T X + \frac{1}{\varepsilon_3} XYY^T X + \varepsilon_1 E^T E + Q_2 + \frac{1}{\varepsilon_6} XX + \varepsilon_6 \rho^2 I$$

$$\Phi_{33} = A^T X + XA + \varepsilon_1{}^{-1} XDD^T X + C^T Y^T X^{-1} YC + \varepsilon_2{}^{-1} XDD^T X + \varepsilon_2 E^T E + \varepsilon_3 \delta^2 C^T C - YC - C^T Y^T + \varepsilon_4{}^{-1} XX + \varepsilon_4 \delta^2 C^T C + Q_1 + \frac{1}{\varepsilon_5} XX + \varepsilon_5 \rho^2 I$$

Therefore from Lemma 1, $\Phi < 0$ is resulted if LMIs equation (31) hold, this means that $\mathcal{L}^H V_2(t) < 0$, which shows the closed-loop system is stochastically stable. Then the proof is completed. ■

Full state feedback for SMC

In this section, we investigate the state-feedback SMC problem for the fractional stochastic system equation (1) with known states. The sliding mode switching function is chosen as

$$s_1(t) = \sigma_1(t) + B^T x(t) \tag{36}$$

with

$$d\sigma_1(t) = \left(B^T BKx(t) - B^T Ax(t) - B^T A_1 x(t-h(t)) - B^T f(x,t) \right) dt - B^T Gx(t) dB^H(t)$$

From equations (1) and (36), we have

$$ds_1(\mathrm{t}) = B^T BKx(t) + B^T \Delta Ax(t) + B^T \Delta A_1 x(t-h(t)) + B^T B \emptyset(u) \tag{37}$$

and the SMC law is derived as follows

$$u(t) = -\frac{s_1(t)}{\|s_1(t)\|} \psi(x) \tag{38}$$

where

$$\psi(x) = \frac{1}{\chi} \left(\|Kx(t)\| + \left\| \left(B^T B\right)^{-1} B^T U \right\| . (\|Ex(t)\| + \|E_1 x(t-h(t))\|) + \xi \right)$$

Remark 3. In order to avoid singularity we replace $\frac{s(t)}{\|s(t)\|}$ in equation (22) with $\mathrm{sgn}(s(t))$, $\mathrm{sgn}(s(t)) \triangleq \frac{s(t)}{\|s(t)\|}$ *for* $s(t) \neq 0$ *and* $sgn(s(t)) \triangleq 0$ *for* $s(t) = 0$, $s(t) \in R^q$ &$0 \in R^q$. With this replacement, the stability proof will not be different from the presented proof. Also $s(t) \in R^q$ is a vector with the same dimension as $u(t) \in R^q$.

Remark 4. To avoid singularity in simulations, we replace $\frac{s(t)}{\|s(t)\|}$ $\left(\frac{s_1(t)}{\|s_1(t)\|}\right)$ with $\frac{s(t)}{\|s(t)\| + \mu}$ $\left(\frac{s_1(t)}{\|s_1(t) + \mu\|}\right)$ in the SMC law equations (22) and (38) (μ is a small constant). It is also worth mentioning that since the states slide on the sliding surface $s(t) = 0$, chattering is inevitable. However, by taking ξ in equations (22) and (38) a small constant, we can reduce this effect.

Theorem 4. If the control input $u(t)$ is designed as equation (38), then the trajectories of the system equation (1) will converge to the sliding surface $s_1(t) = 0$ in finite time.

Proof. By choosing the Lyapunov function $V_3(t) = 1/2\left(s_1^T(t)(B^T B)^{-1} s_1(t)\right)$, we can prove that the sliding surface equation (36) is reachable. Due to space limit, the details are omitted. ■

The equivalent control law can be obtained as follows

$$\emptyset_{eq}(u) = -\left(K\hat{x}(t) + \left(B^T B\right)^{-1} B^T (\Delta Ax(t) + \Delta A_1 x(t-h(t))) \right) \tag{39}$$

From equations (1) and (39), we obtain the sliding mode dynamics as follows

$$dx(t) = ((A - BK)x(t) + A_1x(t - h(t)) + \Upsilon(\Delta Ax(t) + \Delta A_1x(t - h(t))) + f(x,t))dt + Gx(t)dB^H(t) \tag{40}$$

$$\mathcal{L}^H V_4(x,t) = \{2x^T(t)X(A - BK)x(t) + 2x^T(t)X\Upsilon\Delta A(t)x(t) + 2x^T(t)XA_1x(t - h(t)) + 2x^T(t)X\Upsilon\Delta A_1(t)x(t - h(t)) + 2x^T(t)Xf(x,t) + x^T(t)Qx(t) - \left(1 - \dot{h}(t)\right)x^T(t - h(t))Qx(t - h(t))\Big\} \times e^{-\lambda\int_0^t\int_0^s \varphi(s,\tau)d\tau ds} + 2x^T(t)G^TXGx(t) - \lambda x^T(t)Xx(t) - \lambda\int_{t-h(t)}^t x^T(s)Qx(s)ds \int_0^t \varphi(t,s)ds \times e^{-\lambda\int_0^t\int_0^s \varphi(s,\tau)d\tau ds} \tag{44}$$

In the following, the stability conditions of the dynamics in equation (40) are obtained in terms of LMIs.

Theorem 5. Consider the fractional stochastic system equation (1) and the switching function equation (36). The SMC law is chosen as in equation (38). This system is robustly stable in probability if there exist matrices $X > 0$, $Q > 0$, K and scalars $\varepsilon_i > 0$ $(i = 1 - 5)$ that satisfy the following:

$$\Theta = \begin{bmatrix} \Theta_1 & A_1 + \varepsilon_3E^TE_1 & XB & XB & XU & X & 0 & 0 & 0 & 0 \\ * & \Theta_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & -\varepsilon_1B^TB & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\varepsilon_2B^TB & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -\varepsilon_3 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -\varepsilon_4 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & -\varepsilon_1 & 0 & \varepsilon_1U & 0 \\ * & * & * & * & * & * & * & -\varepsilon_2 & 0 & \varepsilon_1U \\ * & * & * & * & * & * & * & * & -\varepsilon_5 & 0 \\ * & * & * & * & * & * & * & * & * & -\varepsilon_5 \end{bmatrix} < 0 \tag{41}$$

with

$$\Theta_1 = \left(X(A - BK) + (A - BK)^TX\right) + \varepsilon_3E^TE + Q + \varepsilon_4\rho^2I + \varepsilon_5E^TE$$

$$\Theta_2 = \varepsilon_3E_1^TE_1 - \left(1 - \bar{h}\right)Q + \varepsilon_5E_1^TE_1$$

Proof. Let the Lyapunov function candidate be chosen as follows

$$V_4(x,t) = \left(x^T(t)Xx(t) + \int_{t-h(t)}^t x^T(s)Qx(s)ds\right) \times e^{-\lambda\int_0^t\int_0^s \varphi(s,\tau)d\tau ds} \tag{42}$$

By using fractional It $\hat{o}$'s formula, we have

$$dV_4(x,t) = \mathcal{L}^H V_4(x,t) + 2x^T(t)XGx(t) \times e^{-\lambda\int_0^t\int_0^s \varphi(s,\tau)d\tau ds}dB^H(t) \tag{43}$$

with

Similar to the proof of Theorem 3, by choosing λ a sufficiently large number such that the last term in equation (44) vanishes, and also using Lemma 2, it is resulted that

$$\mathcal{L}^H V_4(x,t) < w_2^T(t)\Xi w_2(t) \times e^{-\lambda\int_0^t\int_0^s \varphi(s,\tau)d\tau ds} \tag{45}$$

where $w_2(t) = \left[\,x^T(t) \quad x^T(t - h(t))\,\right]^T$, and

$$\Xi = \begin{bmatrix} \Xi_1 & XA_1 + \varepsilon_3E^TE_1 \\ * & \varepsilon_3E_1^TE_1 - \left(1 - \bar{h}\right)Q + \varepsilon_2\Delta A_1^T(t)\Delta A_1(t) \end{bmatrix}$$

with

$$\Xi_1 = \left(X(A - BK) + (A - BK)^TX\right) + \varepsilon_1^{-1}XB\left(B^TB\right)^{-1}B^TX + \varepsilon_2^{-1}XB\left(B^TB\right)^{-1}B^TX + \varepsilon_3{}^{-1}XUU^TX + \varepsilon_3E^TE + \varepsilon_1\Delta A^T(t)\Delta A(t) + Q + \varepsilon_4^{-1}XX + \varepsilon_4\rho^2I$$

Using Lemma 1 several times, the condition $\Xi < 0$ is equivalent to

$$\Pi + \Upsilon_1F(t)\Upsilon_2 + (\Upsilon_1F(t)\Upsilon_2)^T < 0 \tag{46}$$

where

$$\Pi = \begin{bmatrix} \Pi_1 & XA_1 + \varepsilon_3 E^T E_1 & XB & XB & XU & X & 0 & 0 \\ * & \varepsilon_3 E_1^T E_1 - (1-\bar{h})Q & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & -\varepsilon_1 B^T B & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\varepsilon_2 B^T B & 0 & 0 & 0 & 0 \\ * & * & * & * & -\varepsilon_3 & 0 & 0 & 0 \\ * & * & * & * & * & -\varepsilon_4 & 0 & 0 \\ * & * & * & * & * & * & -\varepsilon_1 & 0 \\ * & * & * & * & * & * & * & -\varepsilon_2 \end{bmatrix}$$

with

$$\Pi_1 = \left(X(A - BK) + (A - BK)^T X\right) + \varepsilon_3 E^T E + Q + \varepsilon_4 \rho^2 I$$

and

$$\Upsilon_1 = \begin{bmatrix} E^T & 0 \\ 0 & E_1^T \\ 0_{6n\times n} & 0_{6n\times n} \end{bmatrix}, F(t) = \begin{bmatrix} F(t) & 0 \\ 0 & F(t) \end{bmatrix},$$
$$\Upsilon_2 = \begin{bmatrix} 0_{n\times 6n} & U^T\varepsilon_1 & 0 \\ 0_{n\times 6n} & 0 & U^T\varepsilon_2 \end{bmatrix}$$

Using Lemma 3, inequality equation (46) holds if and only if the following is satisfied

$$\Pi + \varepsilon_5 \Upsilon_1 \Upsilon_1^T + \varepsilon_5^{-1} \Upsilon_2^T \Upsilon_2 < 0 \tag{47}$$

Using Lemma 1, inequality equation (47) is equivalent to equation (41). This completes the proof.

Remark 5. The number of decision variables and the number of LMIs are the two important factors in solving LMIs. In our problem, the number of LMIs is fixed and the number of decision variables is a function of the system order. As we know, any LMI toolbox such as YALMIP in MATLAB can solve any arbitrary-dimensional LMIs (increasing the number of decision variables cause more processing). Therefore, the stability conditions equations (31) and (41) can be solved easily by YALMIP toolbox and verifying the feasibility of the proposed LMIs proves the stability. Note that since sufficient stability conditions have been proposed, infeasibility of the proposed LMIs does not imply instability. However, regarding the feasibility of the proposed LMIs in comparison with similar approaches such as integral SMC proposed in Khandani et al.,[25] it is worth mentioning that since the restricting condition on the design matrix (equation (32) in Khandani et al.[25]) does not exist in this work, then the proposed LMIs are less conservative.

Simulation results

We consider the system equations (1) and (18) with the following numerical values

$$A = \begin{bmatrix} -3 & 1 & 0 \\ 0.3 & -2.5 & 1 \\ -0.1 & 0.3 & -3.8 \end{bmatrix},$$
$$A_1 = \begin{bmatrix} -0.1 & 0 & 0.2 \\ 0.1 & 0 & -0.1 \\ 0 & 0.1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 2 & 1 \end{bmatrix},$$
$$C = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 1 \end{bmatrix},$$
$$D = \begin{bmatrix} 0.2 & 0.2 & 0.2 \\ 0 & 0.2 & 0.1 \\ 0 & 0 & 0.3 \end{bmatrix}, E = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0.2 & 0.2 & 1 \end{bmatrix},$$
$$E_1 = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0.1 & 0 & 0.1 \end{bmatrix},$$
$$G = \begin{bmatrix} 0.012 & 0.012 & 0.012 \\ 0.012 & 0.012 & 0.012 \\ 0.012 & 0.012 & 0.012 \end{bmatrix}$$

The perturbation observer gain, the nonlinear input and the other parameters are given as follows

$$\Delta L = \begin{bmatrix} 0.05\sin(t) & 0.2\sin(t) \\ 0.1\sin(t) & 0.3\sin(t) \\ 0.1\sin(t) & 0.3\sin(t) \end{bmatrix},$$
$$\emptyset(u) = (0.8 + 0.3sin(u))u,\ \chi = 0.5,\ \delta = 0.5,$$
$$\bar{h} = 0.5,\ \xi = 0.001,\ \rho = 0.5,\ H = 0.7$$

The function $F(t)$, the time-varying delay $h(t)$ and the function $f(x, t)$ are chosen as

$$F(t) = 0.5sin(4t),\ h(t) = 0.5\cos(t),$$
$$f(x, t) = 0.1\sin(t)[x_3 \quad x_1 \quad x_2]$$

We first choose K and then we solve LMI equation (31), and determine other parameters. So by choosing

$$K = \begin{bmatrix} 0.1450 & 5.9899 & 10.6121 \\ 12.1506 & 7.8498 & -14.7590 \end{bmatrix}$$

and then solving the proposed LMIs in Theorem 3, the following results are obtained

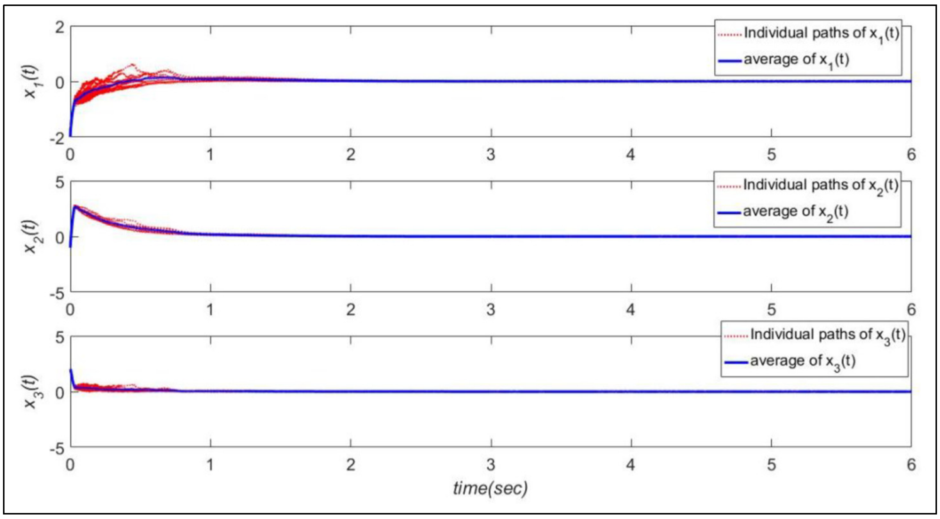

Figure 1. Trajectories of the states $x(t)$ for $H = 0.7$.

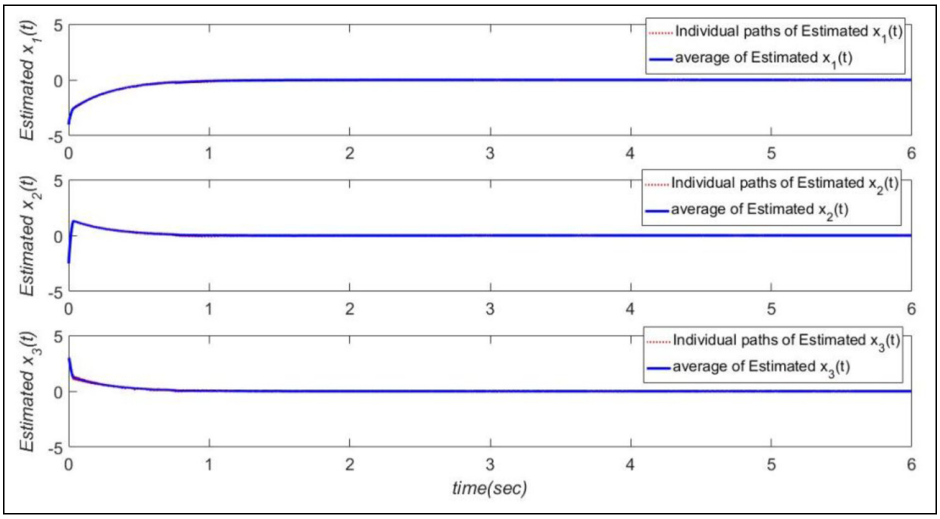

Figure 2. Trajectories of the state estimates $\hat{x}(t)$ for $H = 0.7$.

$$L = \begin{bmatrix} 0.5618 & 0.4655 \\ 0.2829 & 0.6249 \\ 0.3137 & 0.4361 \end{bmatrix},$$

$$X = \begin{bmatrix} 22.2283 & -7.6382 & -8.3119 \\ -7.6382 & 13.6318 & 3.7571 \\ -8.3119 & 3.7571 & 16.3889 \end{bmatrix},$$

$$Q_1 = \begin{bmatrix} 19.9702 & -6.5747 & -11.1064 \\ -6.5747 & 21.6024 & -5.1074 \\ -11.1064 & -5.1074 & 35.0890 \end{bmatrix},$$

$$Q_2 = \begin{bmatrix} 78.6196 & 45.7292 & 18.7516 \\ 45.7292 & 72.1607 & 15.5245 \\ 18.7516 & 15.5245 & 88.0699 \end{bmatrix}$$

The initial states are set as $\hat{x}(0) = [-4 \quad -2.5 \quad 3]^T$ and $x(0) = [-2 \quad -1 \quad 2]^T$. For two different values of the Hurst parameter (H), the simulation results are given in Figures 1–6. For $H = 0.7$, the state trajectories, the estimated states and the corresponding errors are shown in Figures 1–3, which illustrate that the estimation error converges to zero by employing the SMC law in equation (22). The sliding mode variables are shown in Figure 4. The simulations have been carried out for 10 individual fBm paths that are shown in dotted lines with the corresponding averages in solid lines. It can be observed that the trajectories of the states converge to zero and the overall closed-loop system is stochastically stabilized under the proposed SMC law. Furthermore, for $H = 0.85$, the simulation results are given in Figures 5 and 6 which show that the observer and controller can deal with the impact of the fractional Brownian motion and the results are satisfactory. By increasing the Hurst parameter, the effect of fractional Brownian motion is more destructive though, and despite the fact that the stability is ensured, the simulation results

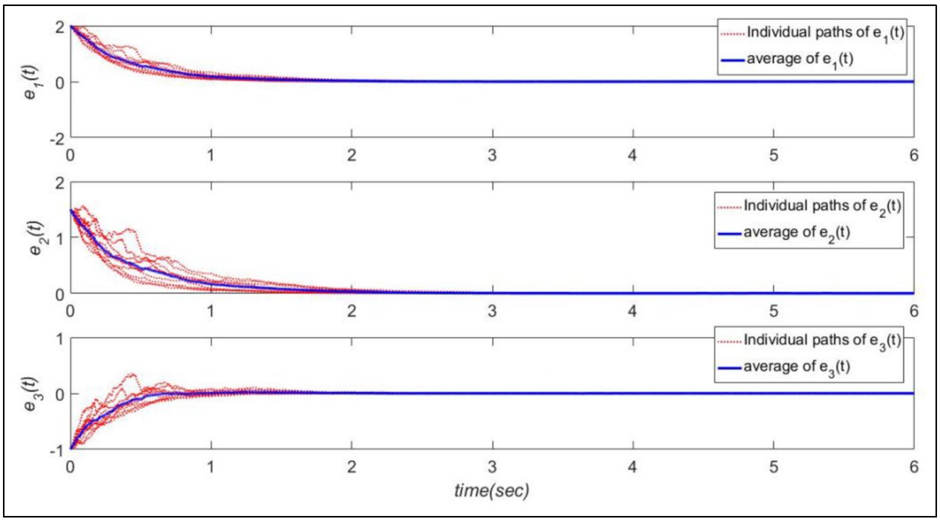

Figure 3. Trajectories of the estimation errors $e(t)$ for $H = 0.7$.

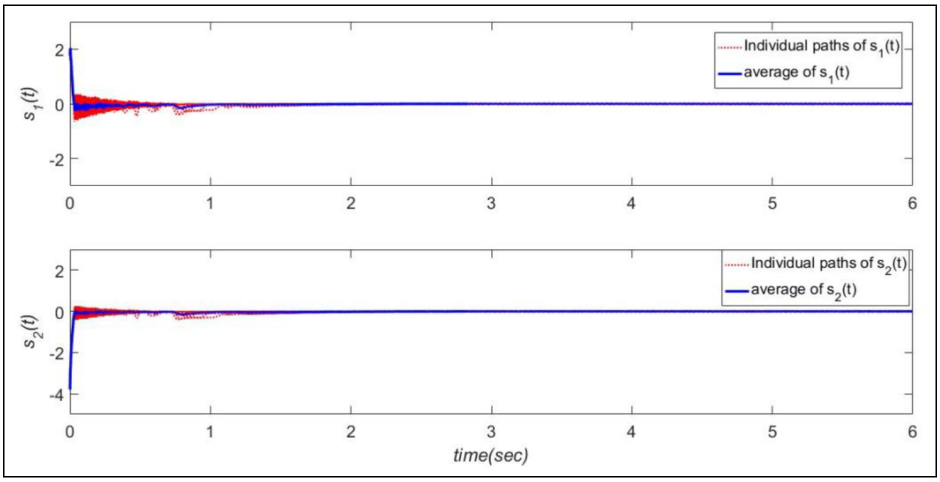

Figure 4. Trajectories of the sliding mode variable $s(t)$ for $H = 0.7$.

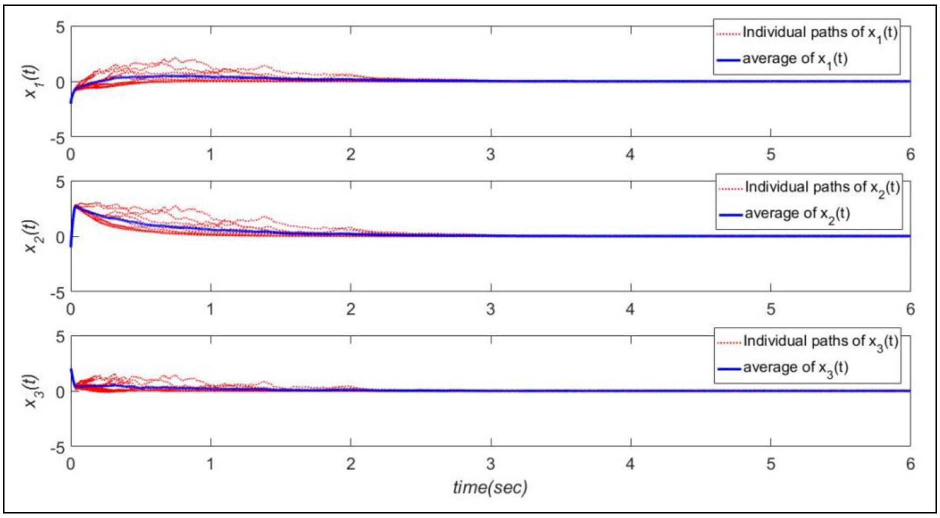

Figure 5. Trajectories of the states $x(t)$ for $H = 0.85$.

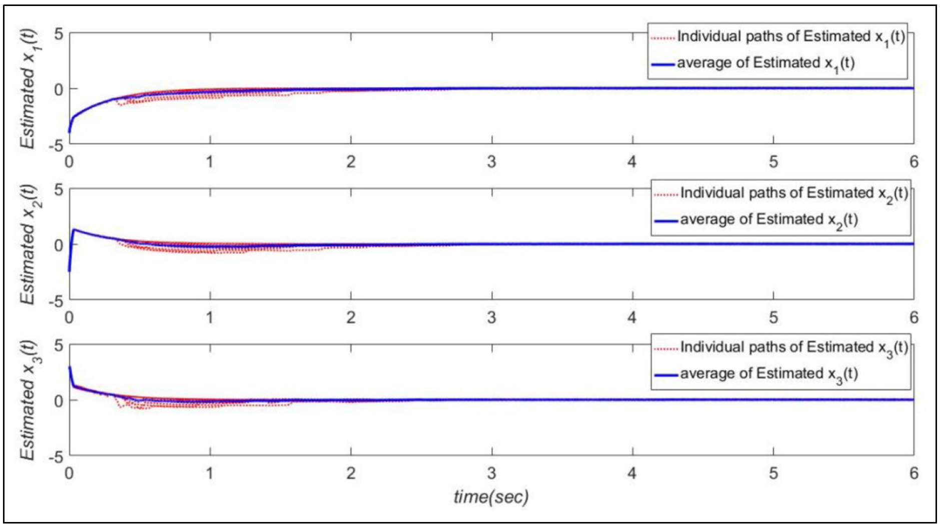

Figure 6. Trajectories of the state estimates $\hat{x}(t)$ for $H = 0.85$.

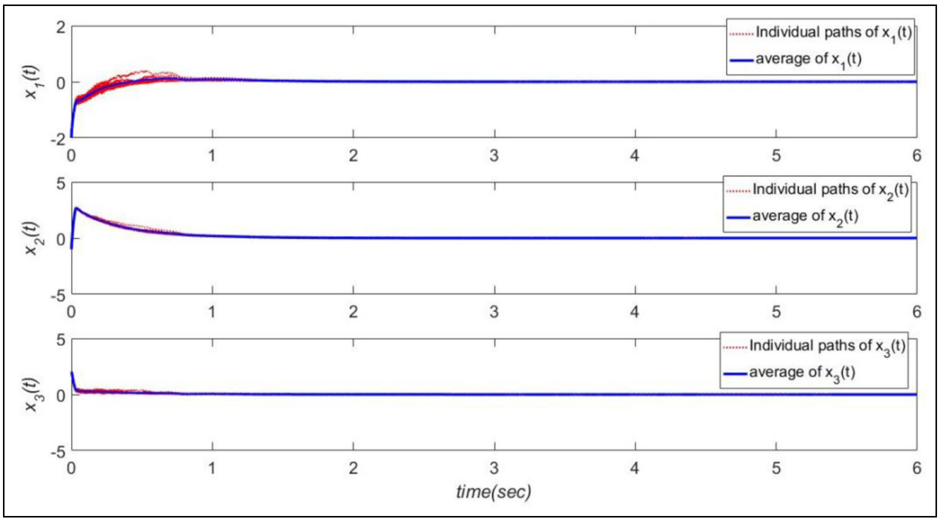

Figure 7. Trajectories of the states $x(t)$ for $H = 0.5$.

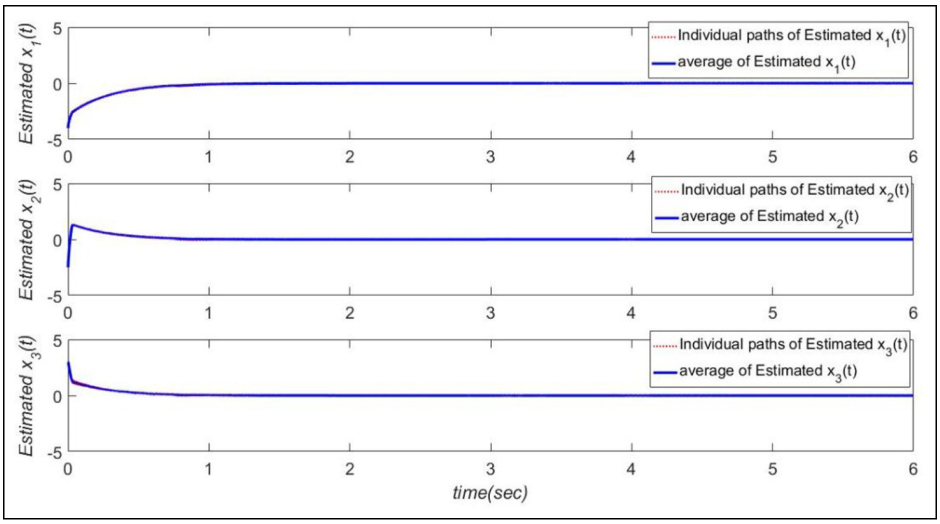

Figure 8. Trajectories of the state estimates $\hat{x}(t)$ for $H = 0.5$.

indicate that the convergence of the state trajectories and the state estimates take place more slowly.

Since the proposed method considers a generalized class of stochastic systems, it can be applied to non-fractional stochastic systems driven by Brownian motion by setting the Hurst parameter $H = 0.5$. As a matter of fact, classical stochastic systems are a special case of fractional stochastic systems investigated in this article. Figures 7 and 8 illustrate this fact where the trajectories of the states and the estimated states have been depicted for a classical stochastic system. Obviously, the strategy works effectively in this case as well.

Conclusion

In this article, a novel observer has been designed for fractional Itô stochastic systems with uncertain parameters, input nonlinearity and time-varying delay. SMC design framework has been employed to derive the results. The proposed non-fragile observer guarantees that the estimation error system is stable in probability. To this end, a sliding surface has been introduced and it is proven that this surface is reachable in finite time. In addition, the stochastic stability of the sliding motion has been proven via a set of linear matrix inequalities. Finally, the SMC design for the fractional Itô stochastic systems with time-varying delay and input nonlinearity with known states has been presented. The simulation example supports the theoretical claims.

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