

Fuzzy Boundary Control of 2D Burgers Equation with an Observer

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Abstract—Fuzzy boundary control of 2D Burgers Equation is considered in this paper. The Burgers equation has extensively been considered as a benchmark problem by flow control researchers. The flavor of the problem is its multidimensionality and nonlinearity, which are the reasons making the governing equation a good start for developing a well-defined understanding of control of flows. This paper demonstrates that an infinite dimensional observer can be designed, and by the aid of the observer, a simple fuzzy boundary feedback control can be achieved under severe noise and disturbance scenarios. The results are in good compliance with the theoretical claims and the simplicity of the controller is another prominent feature.

I. INTRODUCTION

Due to its nonlinearity and multidimensionality, 2D Burgers equation constitutes a good benchmark among infinite dimensional systems that can enjoy the standard techniques of classical control theory. Various kinds of Burgers equation have been studied in the past. In (Donea and Huerta 2003, McDonough and Huang 2004, Efe 2004, Blender 1991, Boules and Eick 2003, Hataue 1998, Nishinari et al. 2001), a simplified version of Navier-Stokes equations given by the Partial Differential Equation (PDE) set $\underline{u}_t + \epsilon(\underline{u} \cdot \nabla)\underline{u} = \mu \nabla^2 \underline{u}$ with \underline{u} being 2-by-1 vector function is described as the 2D Burgers equation. The 2D Burgers equation is therefore considered as a turbulence free cartoon for Navier-Stokes equations and has been studied in the past for modelling traffic flows, shock waves and acoustic transmission. Blender postulates a method to obtain the solution of the above mentioned PDE set iteratively, (Blender 1991). Efe demonstrates the low dimensional modelling with proper orthogonal decomposition in (Efe 2004), and Boules and Eick (2003) perform the model reduction with Fourier expansions. In (Sirendaoreji 1999, Hietarinta 2000, Zhu 1996), some other variants of 2D Burgers equation have been considered with the goal of finding exact solutions under certain circumstances. These types are $(u_t + uu_x - u_{xx})_x + u_{yy} = 0$ in (Sirendaoreji 1999, Hietarinta 2000), $u_t + uu_x + u_{xx} + u_{xxx} = 0$ in (Zhu 1996). Nishinari et al. (2001) focus on cellular automaton, which is extensively studied for developing models of traffic flow, fluids and immune systems, and therefore a good model to work on is a variant of Burgers equation. In Hataue (1998), the dynamics that arises upon discretization of 2D Burgers equation is analyzed. The effects of chosen time step (Δt) for getting physically reasonable numerical solutions are elaborated. Wescott et al. (2001) present a computational technique to obtain the numerical solutions of PDEs having nonlinear convection terms like

2D Burgers equation and Navier-Stokes equations. The goal of the paper is to reduce the computation time without giving concessions from the accuracy. Boules and Eick (2003) obtain the solution of Burgers equation for a specific boundary regime and initial conditions. Using a truncated Fourier series expansion yields an autonomous ODE set, the solution of which approximates the numerical solution, and the derived model rebuilds the situation implied by the chosen initial and boundary conditions. When the 1D version given by $u_t = -uu_x + u_{xx}$ is taken into consideration, it is seen that a significant amount of research outcome has been reported on modelling and control system design, (Hinze and Volkwein 2002, Burns et al. 2002a-b, Vedantham 2000, Park and Jang 2002, Liu and Krstić 2000-2001, Krstić 1999, Efe and Özbay 2003a-2004, Efe et al. 2004). A majority of the cited references emphasizes the similar difficulties as the motivating factors and focus on the solutions and solvability issues. The current paper, on the other hand, derives an infinite dimensional observer that can be used for boundary control purposes. The contribution of this paper is the to demonstrate that simple yet effective observers can be devised to obtain the information about the infinite dimensional process, and this information can be used for boundary control purposes. The second section discusses the design of the observer, the third section presents the fuzzy controller, simulation conditions and obtained results. The concluding remarks are given at the end of the paper.

II. DESIGN OF THE OBSERVER

Define $\Omega := \{(x, y) | (x, y) \in [0, 1] \times [0, 1]\}$ as the physical domain of the process and the observer described by

$$u_t = -k_1 uu_x - k_2 uu_y + \mu u_{xx} + \mu u_{yy}, \quad (1)$$

$$w_t = -k_1 ww_x - k_2 ww_y + \mu w_{xx} + \mu w_{yy} + K(u - w), \quad (2)$$

respectively. In (1) and (2), $u = u(x, y, t)$, $w = w(x, y, t)$ and k_1, k_2 and μ are positive valued known process parameters, K is the observer gain. The boundary conditions are specified by

$$\begin{aligned}
u_y(x, 0, t) &= \beta_0 u(x, 0, t), \\
w_y(x, 0, t) &= \beta_0 w(x, 0, t), \\
u_x(x, 1, t) &= -\beta_1 u(x, 1, t), \\
w_x(x, 1, t) &= -\beta_1 w(x, 1, t), \\
u_x(0, y, t) &= \zeta_0 u(0, y, t), \\
w_x(0, y, t) &= \zeta_0 w(0, y, t), \\
u_x(1, y, t) &= -\zeta_1 u(1, y, t), \\
w_x(1, y, t) &= -\zeta_1 w(1, y, t),
\end{aligned} \tag{3}$$

where $\zeta_0, \zeta_1, \beta_0$ and β_1 are positive numbers. The problem is to reconstruct $u(x, y, t)$ over Ω by designing an observer, whose variable is $w(x, y, t)$, and to perform observer based feedback control through boundary (corner) excitations.

Theorem 2.1: For the 2D Burgers equation given in (1), an infinite dimensional observer having the structure given by (2) can be designed if and only if there exists a $\lambda > 0$ such that $|k_1 \frac{\partial(u^2 - w^2)}{\partial x} + k_2 \frac{\partial(u^2 - w^2)}{\partial y}| \leq \lambda |u - w|$ is satisfied. Such an observer with a positive $K \in \mathfrak{R}$ results in globally exponentially stable reconstruction over $(x, y, t) \in [0, 1] \times [0, 1] \times [0, \infty)$.

Proof: Consider the Lyapunov function candidate

$$V = \frac{1}{2} \int_0^1 \int_0^1 (u(x, y, t) - w(x, y, t))^2 dx dy. \tag{4}$$

For the sake of the simplicity, we drop the arguments of the variables u and w , and take the time derivative of the Lyapunov function candidate above. This yields

$$\begin{aligned}
\dot{V} &= \int_0^1 \int_0^1 (u - w)(u_t - w_t) dx dy, \\
&= \int_0^1 \int_0^1 (u - w)(-k_1 uu_x - k_2 uu_y + \mu u_{xx} \\
&\quad + \mu u_{yy} + k_1 ww_x + k_2 ww_y - \mu w_{xx} - \mu w_{yy} \\
&\quad - K(u - w)) dx dy, \\
&= -2KV + \int_0^1 \int_0^1 (u - w)(-k_1 uu_x - k_2 uu_y \\
&\quad + \mu u_{xx} + \mu u_{yy} + k_1 ww_x + k_2 ww_y - \mu w_{xx} \\
&\quad - \mu w_{yy}) dx dy.
\end{aligned} \tag{5}$$

Set $s := u - w$, we have

$$\begin{aligned}
\dot{V} &= -2KV + \mu \int_0^1 \int_0^1 (ss_{xx} + ss_{yy}) dx dy - \\
&\quad \frac{1}{2} \int_0^1 \int_0^1 (u - w) \star \\
&\quad \left(k_1 \frac{\partial(u^2 - w^2)}{\partial x} + k_2 \frac{\partial(u^2 - w^2)}{\partial y} \right) dx dy
\end{aligned} \tag{6}$$

Using the boundary conditions in (3), it is straightforward to show that the following equality holds true:

$$\begin{aligned}
\int_0^1 \int_0^1 ss_{xx} dx dy &= \int_0^1 s(1, y) s_x(1, y) \\
&\quad - s(0, y) s_x(0, y) dy - \int_0^1 \int_0^1 s_x^2 dx dy \\
&= - \int_0^1 \zeta_1 s(1, y)^2 + \zeta_0 s(0, y)^2 dy - \int_0^1 \int_0^1 s_x^2 dx dy.
\end{aligned} \tag{7}$$

Likewise we have,

$$\begin{aligned}
\int_0^1 \int_0^1 ss_{yy} dx dy &= \int_0^1 s(x, 1) s_y(x, 1) - \\
&\quad s(x, 0) s_y(x, 0) dx - \int_0^1 \int_0^1 s_y^2 dx dy \\
&= - \int_0^1 \beta_1 s(x, 1)^2 + \beta_0 s(x, 0)^2 dx \\
&\quad - \int_0^1 \int_0^1 s_y^2 dx dy.
\end{aligned} \tag{8}$$

Utilizing these results with the assumption of the theorem lets us rewrite (6) as follows:

$$\begin{aligned}
\dot{V} &= -2KV - \mu \int_0^1 \zeta_1 s(1, y)^2 + \zeta_0 s(0, y)^2 dy \\
&\quad - \mu \int_0^1 \beta_1 s(x, 1)^2 + \beta_0 s(x, 0)^2 dx \\
&\quad - \mu \int_0^1 \int_0^1 (s_x^2 + s_y^2) dx dy - \frac{1}{2} \int_0^1 \int_0^1 (u - w) \star \\
&\quad \left(k_1 \frac{\partial(u^2 - w^2)}{\partial x} + k_2 \frac{\partial(u^2 - w^2)}{\partial y} \right) dx dy \\
&\leq -2KV - \mu \int_0^1 \zeta_1 s(1, y)^2 + \zeta_0 s(0, y)^2 dy \\
&\quad - \mu \int_0^1 \beta_1 s(x, 1)^2 + \beta_0 s(x, 0)^2 dx \\
&\quad - \mu \int_0^1 \int_0^1 (s_x^2 + s_y^2) dx dy + \frac{1}{2} \int_0^1 \int_0^1 |u - w| \star \\
&\quad \left| k_1 \frac{\partial(u^2 - w^2)}{\partial x} + k_2 \frac{\partial(u^2 - w^2)}{\partial y} \right| dx dy
\end{aligned} \tag{9}$$

$$\begin{aligned}
&\leq -2KV - \mu \int_0^1 \zeta_1 s(1, y)^2 + \zeta_0 s(0, y)^2 dy \\
&- \mu \int_0^1 \beta_1 s(x, 1)^2 + \beta_0 s(x, 0)^2 dx \\
&- \mu \int_0^1 \int_0^1 (s_x^2 + s_y^2) dx dy \\
&+ \frac{1}{2} \int_0^1 \int_0^1 \lambda |u - w|^2 dx dy \\
&= -(2K - \lambda)V - \mu \int_0^1 \zeta_1 s(1, y)^2 + \zeta_0 s(0, y)^2 dy \\
&- \mu \int_0^1 \beta_1 s(x, 1)^2 + \beta_0 s(x, 0)^2 dx \\
&- \mu \int_0^1 \int_0^1 (s_x^2 + s_y^2) dx dy \\
&< 0.
\end{aligned} \tag{10}$$

This results shows that the state of the nonlinear process in (1) can be reconstructed by the nonlinear observer in (2). The next section presents the practical implications of the theorem and its proof above.

III. FUZZY BOUNDARY CONTROL OF THE SYSTEM

A. Fuzzy Controller

Fuzzy systems are well known for their rule based structure letting the designer compose local decisions of a possibly complicated map. The map in this context refers to the control law. Although many variants of fuzzy controllers are available, (Jang et al, 1997), we aim to demonstrate how feasible the use of fuzzy controllers in the control of spatially continuous processes. For this purpose, we adopt the following fuzzy model with triangular membership functions

$$\gamma_{00} = \frac{\sum_{i=1}^R y_i \Psi_i}{\sum_{i=1}^R \Psi_i}, \tag{11}$$

where γ_{00} is the controller output, R is the number of rules contained in the rule base, y_i is the defuzzifier parameter associated with the i -th rule, Ψ_i is the firing strength of i -th rule calculated as given below:

$$\Psi_i = \mu_{Z_i^{ie}, e_1} \star \mu_{Z_i^e, e_2} \tag{12}$$

where $e_1(t) = \int_0^t e(\sigma) d\sigma$ and $e_2(t) = e(t)$ with $e(t)$ being an error measure yet to be defined. The generic form of the i -th rule is

IF e_1 is Z_i^{ie} AND e_2 is Z_i^e THEN γ_{00} is y_i

In the above rule structure, Z_i^{ie} and Z_i^e denote the fuzzy sets corresponding to the integral of the error and the error respectively, and y_i is the conclusion value (the defuzzifier parameter) for the i -th rule. The membership functions are described in (13), and the defuzzifier parameters are tabulated in Table 1 with the nine rules.

$$\begin{aligned}
\mu_{\mathcal{N}, e_1} &= \max(0, \min(1, -e_1/L_{ie})); \\
\mu_{\mathcal{Z}, e_1} &= \max(0, \min(e_1/L_{ie} + 1, -e_1/L_{ie} + 1)); \\
\mu_{\mathcal{P}, e_1} &= \max(0, \min(1, e_1/L_{ie})); \\
\mu_{\mathcal{N}, e_2} &= \max(0, \min(1, -e_2/L_e)); \\
\mu_{\mathcal{Z}, e_2} &= \max(0, \min(e_2/L_e + 1, -e_2/L_e + 1)); \\
\mu_{\mathcal{P}, e_2} &= \max(0, \min(1, e_2/L_e));
\end{aligned} \tag{13}$$

where $L_e = 1$ and $L_{ie} = 0.8$ are the user-specified membership function parameters.

TABLE I
THE DEFUZZIFIER PARAMETERS

	e_1 is \mathcal{N}	e is \mathcal{Z}	e is \mathcal{P}
e_2 is \mathcal{N}	-80	-1	78
e_2 is \mathcal{Z}	-79	0	79
e_2 is \mathcal{P}	-78	1	80

B. Use of Fuzzy Controller

In this part, we consider the observer based boundary control of the system in (1). For this purpose, we choose $(x, y) = (0, 0)$ corner as the entry for control excitation denoted by $\gamma_{00}(t)$, and other three corners as the entries for disturbances, i.e. $\gamma_{01}(t)$, $\gamma_{10}(t)$ and $\gamma_{11}(t)$. Once the corner conditions are specified, the numerical solution for the boundaries of the domain Ω are obtained. For example, we solve the equation set

$$u_t = -k_1 u u_x + \mu u_{xx}, \tag{14}$$

$$w_t = -k_1 w w_x + \mu w_{xx} + K(u - w) \tag{15}$$

along $x = 0$ and $x = 1$ segments. A similar strategy is followed for other two segments of the boundary. The numerical solution is obtained by using Crank-Nicholson implicit method, (Farlow 1993). The boundary conditions in (3) are forced and the interior grid solution of the PDE set in (1)-(2) are obtained. The settings of the simulation are tabulated in Table 2.

TABLE II
SIMULATION SETTINGS

k_1	5
k_2	5
μ	10
ζ_0	1
ζ_1	1
β_0	1
β_1	1
Δt	1msec.
T	1sec.
K	10
N_x	30

In order to demonstrate that the observer works well under noisy observations of $u(x, y, t)$, a normally distributed measurement noise corrupts the read quantity, u . The controlled variable is $u(x_m, y_m, t)$ with x_m and y_m being the

coordinates of the arbitrarily chosen measurement location. The control problem is to force $u(x_m, y_m, t)$ towards a reference signal $r(t)$ at $(x, y) = (x_m, y_m)$. For this purpose, we have set $(x_m, y_m) = (0.5172, 0.5172)$. As the reference signal, we utilize $r(t) = \text{sign}(\sin(2\pi 2t))$ and define $e(t) = r(t) - w(x_m, y_m, t)$. This choice of the reference signal is deliberate as it excites the shock response of both the process and observer, and makes it easy to see the steady state behavior.

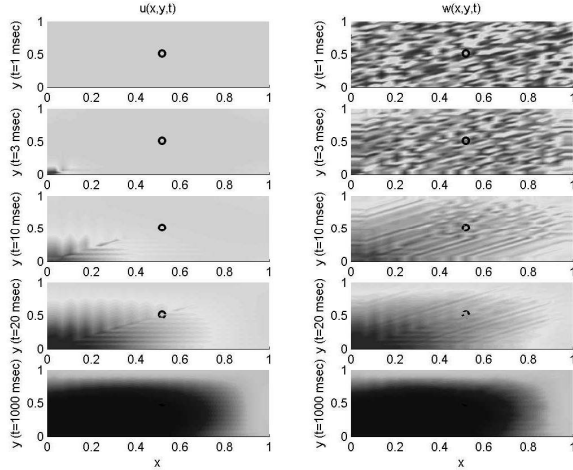


Fig. 1. Simulation results with fuzzy controller

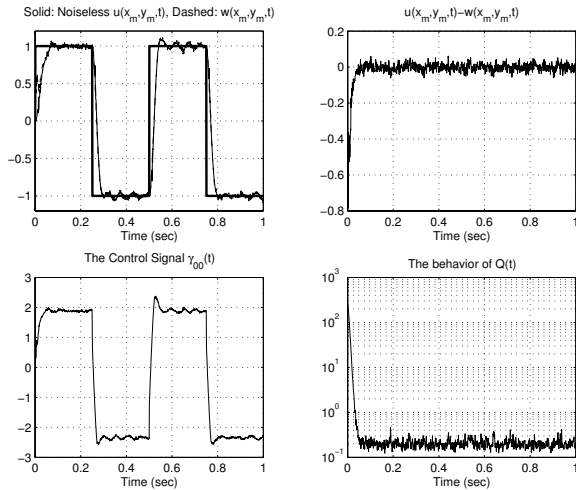


Fig. 2. Simulation results with fuzzy controller

Aside from the reference signal's step-like nature, the disturbance signals are set as

$$\gamma_{01}(t) = 0.1 \text{sign}(\sin(2\pi 20t)), \quad (16)$$

$$\gamma_{10}(t) = 0.1 \text{sign}(\sin(2\pi 17t)), \quad (17)$$

$$\gamma_{11}(t) = 0.1 \text{sign}(\sin(2\pi 15t)), \quad (18)$$

where we inject abruptly changing signals at different frequencies. This kind of a disturbance scenario lets us obtain a rich set of disturbance effect at the fixed location (x_m, y_m) .

A last issue to figure out is the reconstruction performance of the observer. For this purpose, we define the following integral as the measure of performance,

$$Q(t) = \int_0^1 \int_0^1 (u(x, y, t) - w(x, y, t))^2 dx dy. \quad (19)$$

In Figure 1, a matrix of subplots is depicted with a circle indicating the location (x_m, y_m) . The plots on the left column are for $u(x, y, t)$ and those on the right column are for $w(x, y, t)$. The first four rows are the snapshots from both processes at $t = 1, 3, 10$ and 20 msec. The last row is for the final time, $T = 1$ sec. Clearly, the transient phase passes very quickly and a convergence is observed. This fact is further visible in the bottom right subplot of Figure 2, where the convergence is achieved after the first 50 msec. time.

Looking at the results in Figure 2, a good tracking is seen in the top left subplot, where the reference signal $r(t)$ (thick curve), $u(x_m, y_m, t)$ (solid curve) and $w(x_m, y_m, t)$ (dashed curve) are plotted together. The process variable and the observer variable are almost indistinguishable, which is a numerical evidence to the analytical claims and existence of a $\lambda > 0$. In the top right subplot, this difference is shown, where the convergent behavior is noteworthy. The bottom left subplot shows the control signal $(\gamma_{00}(t))$ that is applied from the corner $(x, y) = (0, 0)$. Unsurprisingly, although we impose unit magnitude reference signal at the measurement point, the signal that needs to be applied from the corner is higher in magnitude than the reference. This fact is because the other three corners are driven by small magnitude disturbances, and the control signal magnitude needs to be larger as the measurement point gets away from the control entry, i.e. $(x, y) = (0, 0)$. A natural consequence of this is the sufficiency of relatively small control effort in the close neighborhood of the control entry, and large control effort in the vicinity of locations where the solution is independently specified. For example, in the vicinity of $(x, y) = (1, 1)$, the control signal must overcome the effect of corner condition, and naturally the control effort is expectedly large.

Clearly a last question in this section should ask whether it was difficult to find such a parameter set that works good. The answer is yes, after a few trials, a good parametrization of membership functions, a good resolution parameter (N_x) and a good observer gain (K) can be set. We set these values by trial and error.

IV. CONCLUSIONS

This paper demonstrates that a nonlinear infinite dimensional observer can be designed for a nonlinear infinite dimensional process. Despite the spatial continuity, under mild assumptions, the design problem enjoys the classical approaches of control theory. A simple yet successful fuzzy boundary control is demonstrated. The results indicate that the observer reconstructs the process state in a globally exponentially stable evolution, and the closed loop control performance is satisfactory under severely corrupted

measurements and corner disturbances. The computational advantages of the controller is another prominent feature that deserves emphasis. The contribution of this paper is the method followed to devise a nonlinear observer for a nonlinear infinite dimensional problem.

Future research aims to demonstrate such a strategy for complicated forms of Burgers equation as in (Donea and Huerta 2003, McDonough and Huang 2004, Efe 2004, Blender 1991, Boules and Eick 2003, Hataue 1998, Nishinari et al. 2001). The contribution at large is to underline the design possibility of observers and observer based controllers for flow and fluid problems.

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