BBM401-Lecture 12: Non-Context-free Languages

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Resources for the presentation:
https://courses.engr.illinois.edu/cs373/fa2010/

Non-Context Free Languages

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Answer

L is not context-free, because

- Recognizing if $w \in L$ requires remembering the number of *a*s seen, *b*s seen and *c*s seen
- We can remember one of them on the stack (say *a*s) , and compare them to another (say *b*s) by popping, but not to both *b*s and *c*s

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The precise way to capture this intuition is through the pumping lemma

Pumping Lemma for CFLs

Informal Statement

For all sufficiently long strings z in a context free language L, it is possible to find two substrings, not too far apart, that can be simultaneously pumped to obtain more words in L.

Pumping Lemma for CFLs

Formal Statement

Lemma

If L is a CFL, then $\exists p$ (pumping length) such that $\forall z \in L$, if $|z| \geq p$ then $\exists u, v, w, x, y$ such that $z = uvwxy$

Pumping Lemma for CFLs

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1. $|vwx| \leq p$

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If L is a CFL, then $\exists p$ *(pumping length) such that* $\forall z \in L$*, if* $|z| \geq p$ *then* $\exists u, v, w, x, y$ *such that* $z = uvwxy$

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1. $|vwx| \leq p$
2. $|vx| > 0$
3. $\forall i \geq 0.\ uv^i wx^i y \in L$

Two Pumping Lemmas side-by-side

Context-Free Languages

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Regular Languages

If L is a regular language, then $\exists p$ (pumping length) such that $\forall z \in L$, if $|z| \geq p$ then $\exists u, v, w$ such that $z = uvw$

1. $|uv| \leq p$
2. $|v| > 0$
3. $\forall i \geq 0.\ uv^i w \in L$

Pumping Lemma for CFLs

Game View

Game between Defender, who claims L satisfies the pumping condition, and Challenger, who claims L does not.

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Pick pumping length p	$\xrightarrow{p}$	
	$\xleftarrow{z}$	Pick $z \in L$ s.t. $\lvert z\rvert \geq p$
Divide z into u, v, w, x, y s.t. $\lvert vwx\rvert \leq p$, and $\lvert vx\rvert > 0$		

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		Pick i, s.t. $uv^iwx^iy \notin L$

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Pumping Lemma: If L is CFL, then there is always a winning strategy for the defender (i.e., challenger will get stuck).

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Pumping Lemma: If L is CFL, then there is always a winning strategy for the defender (i.e., challenger will get stuck).
Pumping Lemma (in contrapositive): If there is a winning strategy for the challenger, then L is not CFL.

Consequences of Pumping Lemma

- If L is context-free then L satisfies the pumping lemma.

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- If L is context-free then L satisfies the pumping lemma.
- If L satisfies the pumping lemma that does not mean L is context-free
- If L does not satisfy the pumping lemma (i.e., challenger can win the game, *no matter* what the defender does) then L is not context-free.

Example I

Proposition

$L_{anbncn} = \{a^n b^n c^n \mid n \geq 0\}$ *is not a CFL.*

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- Consider $z = a^p b^p c^p \in L_{anbncn}$.
- Since $|z| > p$, there are u, v, w, x, y such that $z = uvwxy$, $|vwx| \leq p$, $|vx| > 0$ and $uv^i wx^i y \in L$ for all $i \geq 0$.

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- Since $|vwx| \leq p$, vwx cannot contain all three of the symbols a, b, c, because there are p bs. So vwx either does not have any as or does not have any bs or does not have any cs.

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- Since $|vwx| \leq p$, vwx cannot contain all three of the symbols a, b, c, because there are p bs. So vwx either does not have any as or does not have any bs or does not have any cs. Suppose, (wlog) vwx does have any as. Then $uv^0wx^0y = uwy$ contains more as than either bs or cs. Hence $uwy \notin L$. □

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- Since $|vwx| \leq p$, v, x cannot contain both as and cs, nor can it contain both bs and ds. Further $|vx| > 0$. Now $uv^0 wx^0 y = uwy \notin L$, because it either contains fewer as than cs, or fewer cs than as, or fewer bs than ds, or fewer ds than bs. □

Example III

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$E = \{ww \mid w \in \{0,1\}^*\}$ *is not a CFL.*

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Wrong Proof

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Suppose E is context-free. Let p be the pumping length.

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Suppose E is context-free. Let p be the pumping length.

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- z can be pumped if we make the following division.

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- vwx must straddle the midpoint of z.

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  - Suppose vwx is only in the first half. Then in uv^2wx^2y the second half starts with 1. Thus, it is not of the form ww.

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- vwx must straddle the midpoint of z.
  - Suppose vwx is only in the first half. Then in uv^2wx^2y the second half starts with 1. Thus, it is not of the form ww.
  - Case when vwx is only in the second half. Then in uv^2wx^2y the first half ends in a 0. Thus, it is not of the form ww. $\cdots\rightarrow$

Example III

Corrected Proof

Proof (contd).

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Proof (contd).

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Proof (contd).

- Suppose vwx straddles the middle. Then uv^0wx^0y must be of the form $0^p1^i0^j1^p$, where either i or j is not p.

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Corrected Proof

Proof (contd).

- Suppose vwx straddles the middle. Then uv^0wx^0y must be of the form $0^p1^i0^j1^p$, where either i or j is not p. Thus, $uv^0wx^0y \notin E$. □

Proof of Pumping Lemma

Recall ...

Lemma

If L is a CFL, then $\exists p$ (pumping length) such that $\forall z \in L$, if $|z| \geq p$ then $\exists u, v, w, x, y$ such that $z = uvwxy$

1. $|vwx| \leq p$
2. $|vx| > 0$
3. $\forall i \geq 0.\ uv^i wx^i y \in L$

Proof Idea

Let G be a CFG in Chomsky Normal Form such that $L(G) = L$.
Let z be a "very long" string in L ("very long" made precise later).

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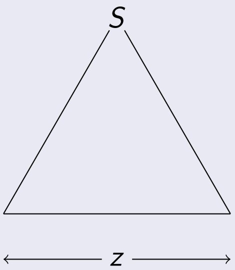

Parse Tree for z

- Since $z \in L$ there is a parse tree for z

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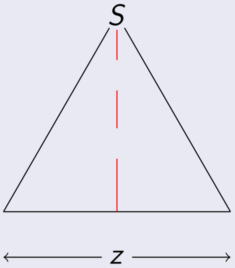

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- Since $z \in L$ there is a parse tree for z
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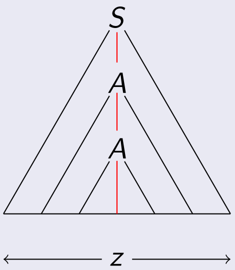

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- Since $z \in L$ there is a parse tree for z
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- The longest path in the tree, by pigeon hole principle, must have some variable (say) A repeat.

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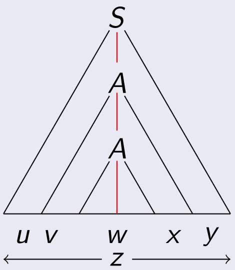

Parse Tree for z

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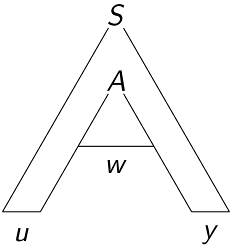
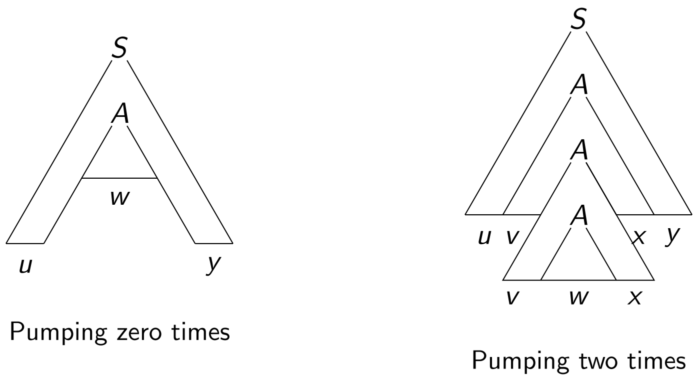
- Since z is very long, the parse tree (which is a binary tree) must be "very tall"
- The longest path in the tree, by pigeon hole principle, must have some variable (say) A repeat. Let u, v, w, x, y be as shown.

Pumping down



Pumping zero times

Pumping down and up



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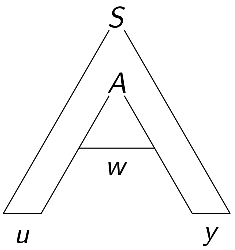

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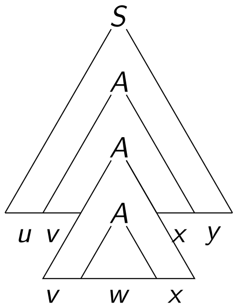

Pumping two times

- Thus, uv^iwx^iy has a parse tree, for any i.

Proof of Pumping Lemma

Existence of tall parse trees

Proof.

Let G be a grammar in Chomsky Normal Form with k variables such that $L(G) = L$. Take $p = 2^k$. Consider $z \in L$ such that $|z| \geq p = 2^k$.

Proof of Pumping Lemma

Existence of tall parse trees

Proof.

Let G be a grammar in Chomsky Normal Form with k variables such that $L(G) = L$. Take $p = 2^k$. Consider $z \in L$ such that $|z| \geq p = 2^k$.

- Consider a parse tree for z. Height of this tree is at least $k + 1$

Proof of Pumping Lemma

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  - $|z|$ =Number of leaves in parse tree of $z = 2^{h-1} \geq 2^k$. Thus, $h \geq k + 1$.

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Repeated Variables

Proof (contd).

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- Thus, there must be two vertices n_1 and n_2 on this path such that n_1 and n_2 have the same label (say A) and n_1 is an ancestor of n_2.

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- Let the yield of tree rooted at n_2 be w, and yield of n_1 be vwx.

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- Let the yield of tree rooted at n_2 be w, and yield of n_1 be vwx. Yield of the root $= z$ is say $uvwxy$.

Proof of Pumping Lemma

Properties of u, v, w, x, y

Proof (contd).

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Proof (contd).

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Proof (contd).

- Height of n_1 can be assumed to be at most $k+1$; thus, the yield of n_1 (vwx) is at most $2^k = p$.

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- Height of n_1 can be assumed to be at most $k+1$; thus, the yield of n_1 (vwx) is at most $2^k = p$.
- $n_1 \neq n_2$.

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Proof (contd).

- Height of n_1 can be assumed to be at most $k+1$; thus, the yield of n_1 (vwx) is at most $2^k = p$.
- $n_1 \neq n_2$. Since the grammar has no ϵ-productions and no unit-productions, $vwx \neq w$. i.e., $|vx| > 0$.

Proof of Pumping Lemma

Pumping the strings

Proof (contd).

Based on the parse tree for z, and definitions of u, v, w, x, y, we have

Proof of Pumping Lemma

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Based on the parse tree for z, and definitions of u, v, w, x, y, we have

- There is a parse tree with yield uAy and root S, obtained by not expanding n_1. Thus, $S \stackrel{*}{\Rightarrow} uAy$.

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- There is a parse tree with yield uAy and root S, obtained by not expanding n_1. Thus, $S \stackrel{*}{\Rightarrow} uAy$.
- There is a parse tree with yield vAx and root A, obtained from n_1 and not expanding n_2. Thus, $A \stackrel{*}{\Rightarrow} vAx$.

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Pumping the strings

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Based on the parse tree for z, and definitions of u, v, w, x, y, we have

- There is a parse tree with yield uAy and root S, obtained by not expanding n_1. Thus, $S \stackrel{*}{\Rightarrow} uAy$.
- There is a parse tree with yield vAx and root A, obtained from n_1 and not expanding n_2. Thus, $A \stackrel{*}{\Rightarrow} vAx$.
- There is a parse tree with yield w and root A; this is the tree rooted at n_2. Thus, $A \stackrel{*}{\Rightarrow} w$.

Proof of Pumping Lemma

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Proof (contd).

Based on the parse tree for z, and definitions of u, v, w, x, y, we have

- There is a parse tree with yield uAy and root S, obtained by not expanding n_1. Thus, $S \overset{*}{\Rightarrow} uAy$.
- There is a parse tree with yield vAx and root A, obtained from n_1 and not expanding n_2. Thus, $A \overset{*}{\Rightarrow} vAx$.
- There is a parse tree with yield w and root A; this is the tree rooted at n_2. Thus, $A \overset{*}{\Rightarrow} w$.

Putting it together, we have

$$S \overset{*}{\Rightarrow} uAy \overset{*}{\Rightarrow} uvAxy \overset{*}{\Rightarrow} uvvAxxy \overset{*}{\Rightarrow} \cdots \overset{*}{\Rightarrow} uv^iAx^iy \overset{*}{\Rightarrow} uv^iwx^iy \quad \square$$