BBM401-Lecture 2: DFA's and Closure Properties

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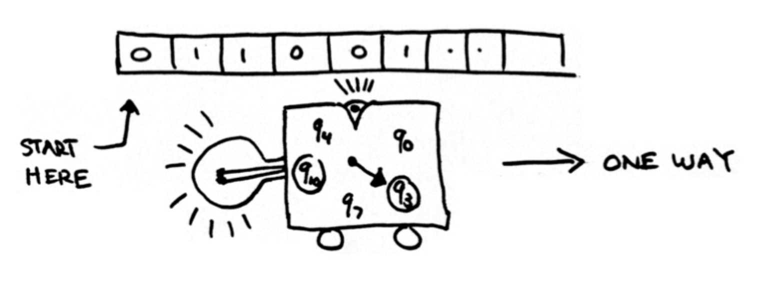
Resources for the presentation:
http://ocw.mit.edu/courses/electrical-engineering-and-computer-science/6-045j-automata-computability-and-complexity-spring-2011/Syllabus/
https://courses.engr.illinois.edu/cs498374/lectures.html

DFAs *(also called FSMs)*

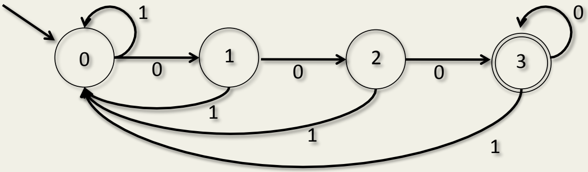
- A simple(st?) model of what a computer is
- Many devices modeled, programmed as DFAs
    - Vending machines
    - Elevators
    - Digital watch logic
    - Calculators
    - Lexical analysis part of program compilation
- Very limited, but observable universe is finite...

Typical DFA



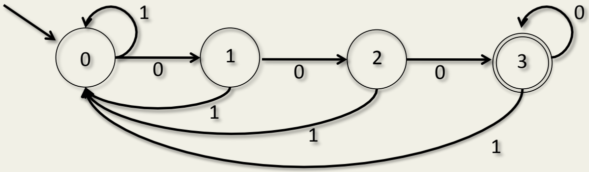
- Start state q_0
- Start at left, scan symbol, change state, move right.
- Rules of form "if in state *q* scanning symbol *s* then go to state *p* and move right."
- Some states (circled) are *accepting*.
- *M accepts* the input string if a circled state is reached after scanning the last symbol.

Graphical Representation



- Directed graph with edges labeled with chars in Σ
- For each state (vertex) *q* and symbol *a* in Σ there is *exactly* one edge leaving *q* labeled with *a*. $q \xrightarrow{a} p$
- Accepting state(s) are double-circled
- Initial state has pointer, or is obviously labeled (0, q_0, *"start"*...)

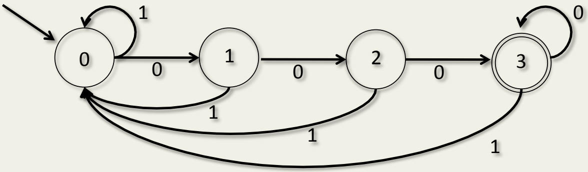
Graphical Representation



- Where does 001 lead?   10010?
- Which strings end up in accepting state?
- Prove it
- *Every string* has *one path* that it follows

$q \xrightarrow{a} p$ versus $q \overset{w}{\rightsquigarrow} p$

Graphical Representation



Definition

- A DFA *M* *accepts a string* *w* iff the unique path starting at the initial state and spelling out *w* ends at an accepting state.

- The *language accepted* *(or "recognized")* by a DFA *M* is denoted *L(M)* and defined by

$$L(M) = \{\, w \mid M \text{ accepts } w \}$$

Warning

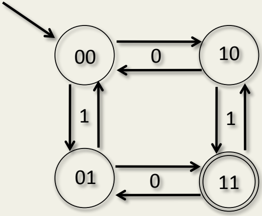
- "*M* accepts language *L*" does not mean simply that *M* accepts each string in *L.*

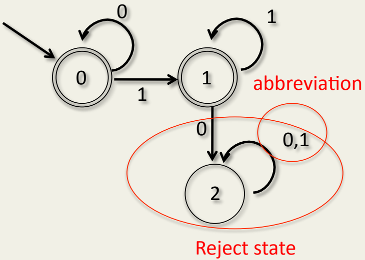
- "*M* accepts language *L*" means

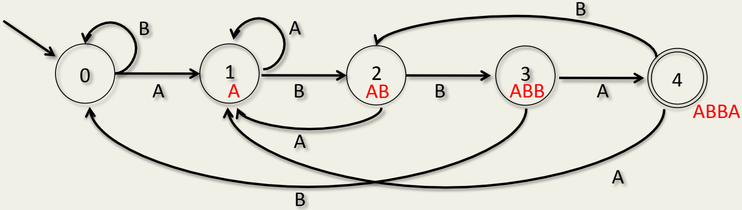
  M accepts each string in *L* *and no others!*

- *M* "recognizes" *L* is a better term, but "accepts" is widely accepted (and recognized).

Examples: What is L(M) ?





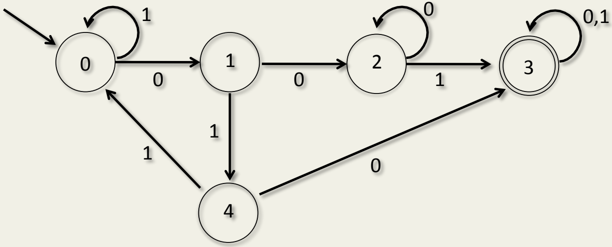


State = Memory

- The state of a DFA is its entire memory of what has come before
- The state must capture enough information to complete the computation on the suffix to come
- When designing a DFA, think "what do I need to know at this moment?" *That* is your state.

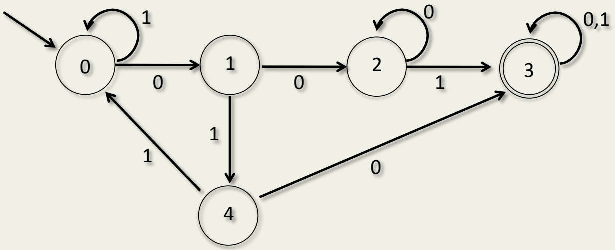
Construction Challenge

- *L(M)* = {*w* | *w* contains 001 or 010}



Construction Challenge

- $L(M) = \{w \mid w$ ~~contains~~ ends with 001 or 010$\}$



Construction Challenge

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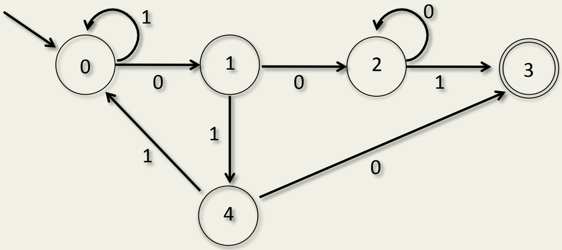

Construction Challenge

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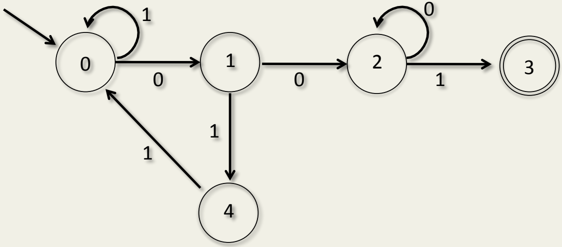

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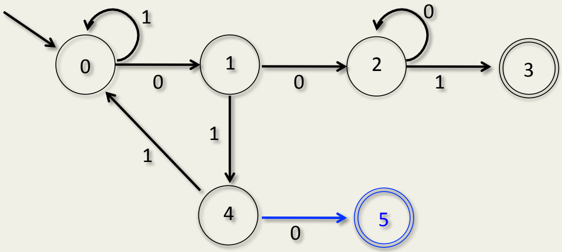

Construction Challenge

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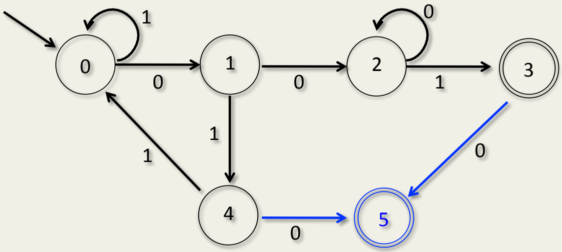

Construction Challenge

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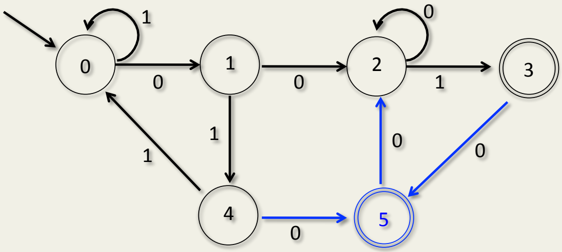

Construction Challenge

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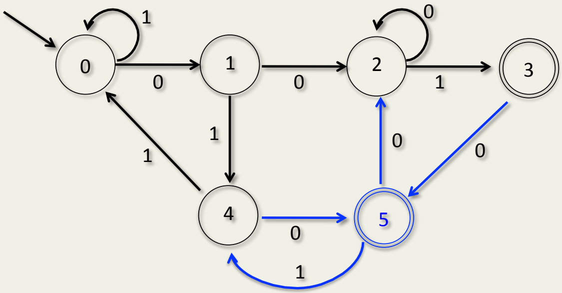

Construction Challenge

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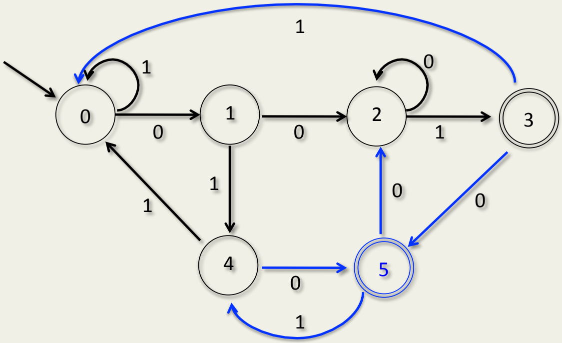
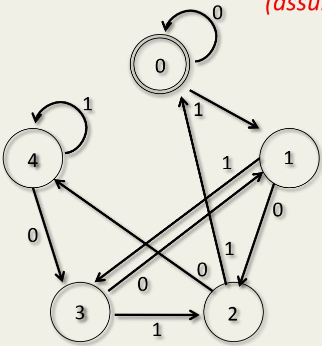


Binary #s congruent to 0 mod 5

(assume no leading 0s)



Key Idea

If *w mod 5 = a, then:*

- *w0 mod 5 = 2a mod 5*
- *w1 mod 5 = 2a+1 mod 5*

Test: 1101011 = 107 = 2 mod 5

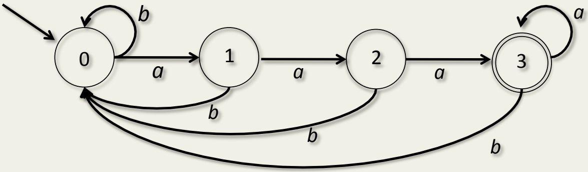
Formal (tuple) Representation

Sometimes, it is easier to specify the DFA using this formalism, instead of drawing a graph

A DFA is a quintuple $M=(Q,\Sigma,\delta,q_0,F)$, *where:*

- Q is a finite set of *states*
- Σ is a finite *alphabet* of symbols
- $\delta: Q \times \Sigma \rightarrow Q$ is a *transition function*
- q_0 is the *initial state*
- $F \subseteq Q$ is the set of *accepting states*

Example



- $Q = \{0,1,2,3\}$
- $\Sigma = \{a,b\}$
- δ specified at right
- $q_0 = 0$
- $F = \{3\}$

state \ input	*a*	*b*
0	1	0
1	2	0
2	3	0
3	3	0

Extending δ

- $\delta(q,a) = p$ means in graph that $q \xrightarrow{a} p$
- But how can we define $\delta(q,w)$ to express $q \overset{w}{\rightsquigarrow} p$
- Must extend $\delta: Q \times \Sigma^* \to Q$
  - $\delta(q,\varepsilon) = q$ for every q; $\delta(q,a)$ already defined
  - $\delta(q,au) = \delta(\delta(q,a),u)$ for $|u| \geq 1$, all q, a

take first step according to δ

take rest of steps inductively according to δ

$\delta(q,w) = p$ corresponds to $q \overset{w}{\rightsquigarrow} p$

Formal definition of L(M)

Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA

Then $L(M) = \{w \mid \delta(q_0,w) \in F\}$

We will show later that:

Theorem

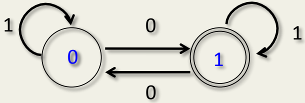
L is regular if and only if L = L(M) for some DFA M

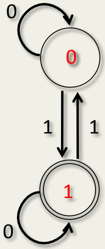
Example use

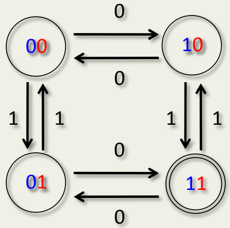
$L(M) = \{w \mid w$ in base b is congruent to k mod m

- $Q = \{0,1,\ldots,m-1\}$
- $\Sigma = \{0,1,\ldots,b-1\}$
- $q_0 = 0$
- $\delta(n,a) = bn+a \bmod m$
- $F = \{k\}$

M simulating both M_1 and M_2







Cross-product machine

M accepting $L(M_1) \cap L(M_2)$

$Q = Q_1 \times Q_2$

$q_0 = (q_0^{(1)}, q_0^{(2)})$

$F = F_1 \times F_2 = \{ (q_1, q_2) \mid q_1 \text{ in } F_1 \text{ and } q_2 \text{ in } F_2 \}$

Transition function:

$\delta((q_1, q_2), a) = (\delta_1(q_1, a), \delta_2(q_2, a))$

$(q_1, q_2) \xrightarrow{a} (p_1, p_2)$ if and only if

- $q_1 \xrightarrow[1]{a} p_1$ in M_1
- $q_2 \xrightarrow[2]{a} p_2$ in M_2

Proof that simulation is correct

- Induction *on what?* that *what?*
- Will need to prove that action of machine is correct starting from any states.
- We know that:

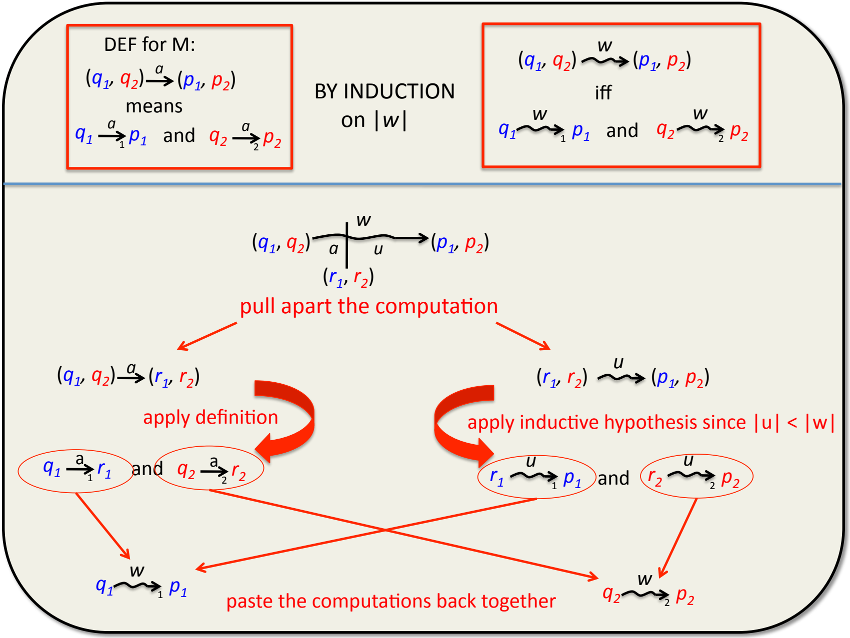
$(q_1, q_2) \xrightarrow{a} (p_1, p_2)$ iff
- $q_1 \xrightarrow{a}_1 p_1$ in M_1
- $q_2 \xrightarrow{a}_2 p_2$ in M_2

By definition

Show that:

$(q_1, q_2) \overset{w}{\leadsto} (p_1, p_2)$ iff
- $q_1 \overset{w}{\leadsto}_1 p_1$ in M_1
- $q_2 \overset{w}{\leadsto}_2 p_2$ in M_2

Just like definition of δ, but with *w* instead of *a*


DEF for M:
(q1, q2) —a→ (p1, p2)
means
q1 —a→1 p1 and q2 —a→2 p2
BY INDUCTION
on |w|
(q1, q2) ~w~> (p1, p2)
iff
q1 ~w~>1 p1 and q2 ~w~>2 p2
(q1, q2) —a— (r1, r2) —u→ (p1, p2)
w
pull apart the computation
(q1, q2) —a→ (r1, r2)
(r1, r2) ~u~> (p1, p2)
apply definition
apply inductive hypothesis since |u| < |w|
q1 —a→1 r1 and q2 —a→2 r2
r1 ~u~>1 p1 and r2 ~u~>2 p2
q1 ~w~>1 p1
paste the computations back together
q2 ~w~>2 p2

Finishing up...

- We proved:

$$(q_1, q_2) \overset{w}{\rightsquigarrow} (p_1, p_2)$$
iff
$$q_1 \overset{w}{\rightsquigarrow}_1 p_1 \text{ and } q_2 \overset{w}{\rightsquigarrow}_2 p_2$$

By definition, *w* accepted by *M*

iff $(q_0^{(1)}, q_0^{(2)}) \overset{w}{\rightsquigarrow} (f_1, f_2)$ in $F_1 \times F_2$

iff $q_0^{(1)} \overset{w}{\rightsquigarrow} f_1$ *in* F_1 *AND* $q_0^{(2)} \overset{w}{\rightsquigarrow} f_2$ *in* F_2

iff *w in* $L(M_1)$ *AND* *w in* $L(M_2)$

Formal proof that simulation is correct

- We know *by* definition that:
  - for all q_1 *in* Q_1, for all q_2 *in* Q_2
  - for all characters a

  $\delta((q_1, q_2), a) = (\delta_1(q_1,a),\delta_2(q_2,a))$

- We prove by induction on $|w|$ that:
  - for all q_1 *in* Q_1, for all q_2 *in* Q_2
  - for all strings *w*

  $\delta((q_1, q_2), w) = (\delta_1(q_1,w),\delta_2(q_2,w))$

Looks just like definition of δ, but with *w* instead of *a*

To prove: $\delta((q_1,q_2), w) = (\delta_1(q_1,w),\delta_2(q_2,w))$

Induction on $|w|$

- Base Case: $|w| = 0$, so $w = \varepsilon$.

  $\delta((q_1, q_2), \varepsilon) = (q_1, q_2) = (\delta_1(q_1,\varepsilon),\delta_2(q_2,\varepsilon))$
- Assume true for strings u of length $< n$.
- Let $w = au$ be an arbitrary string of length n.
- $\delta((q_1, q_2), au)$

$= \delta(\delta((q_1, q_2),a),u)$	defn of δ extension
$= \delta((\delta_1(q_1,a),\delta_2(q_2,a)), u)$	by defn of δ
$= \delta((r_1,r_2), u)$	define r's to simplify
$= (\delta_1(r_1,\mathrm{u}), \delta_2(r_2,u))$	by induction ($|u| < n$)
$= (\delta_1(\delta_1(q_1,a),u), \delta_2(\delta_2(q_2,a),u))$	get rid of r's
$= (\delta_1(q_1,au), \delta_2(q_2,au))$	unsplitting
$= (\delta_1(q_1,w), \delta_2(q_2,w))$	

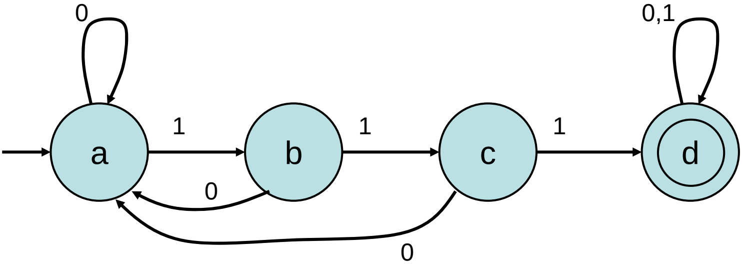
Properties of Regular languages

- We've shown how to accept intersection of two regular languages
- What about union?
- If L is accepted by a DFA, what about $\overline{L}$?
- What about concatenation, and Kleene * ?
- Is there a DFA for $L_1 - L_2$ given M_1 and M_2 ?

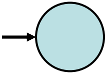
The answer to all of these questions, and more, is "Yes."

Example 1

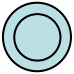
- An FA diagram, machine M



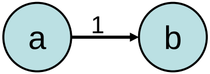
- Conventions:



Start state

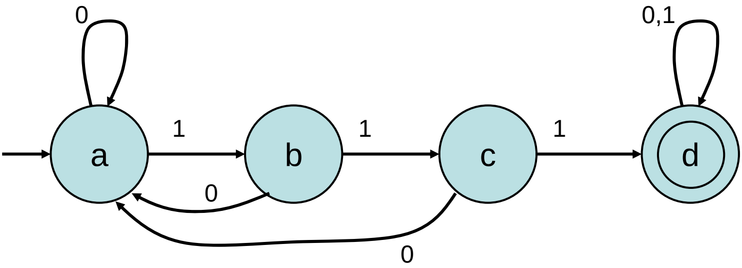


Accept state



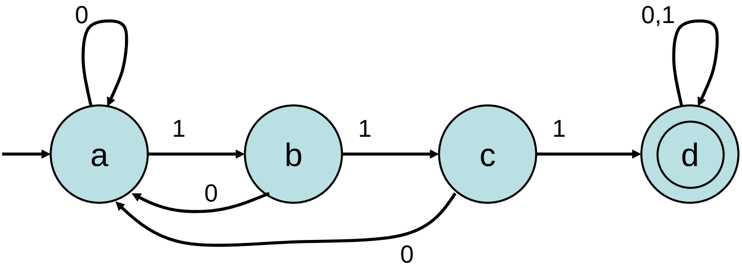
Transition from a to b on input symbol 1.
Allow self-loops

Example 1



- Example computation:
  - Input word w: 1 0 1 1 0 1 1 1 0
  - States: a b a b c a b c d d
- We say that M accepts w, since w leads to d, an accepting state.

Example 1



- What is L(M) for Example 1?
- { w ∈ { 0,1 }* | w contains 111 as a substring }
- Note: Substring refers to consecutive symbols.

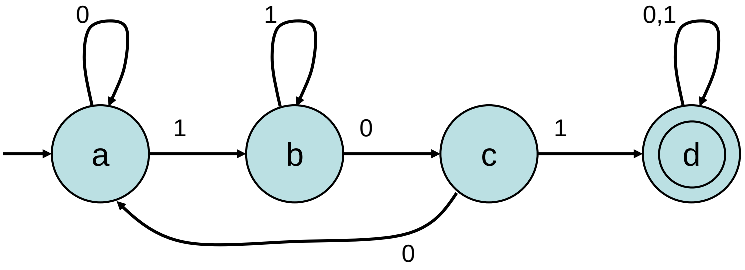
Example 1

- What is the 5-tuple $(Q, \Sigma, \delta, q_0, F)$?
- $Q = \{ a, b, c, d \}$
- $\Sigma = \{ 0, 1 \}$
- δ is given by the state diagram, or alternatively, by a table:
- $q_0 = a$
- $F = \{ d \}$

	0	1
a	a	b
b	a	c
c	a	d
d	d	d

Example 2

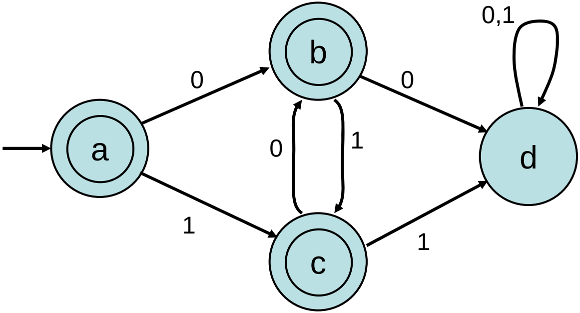
- Design an FA M with L(M) = { w ∈ { 0,1 }* | w contains 101 as a substring }.



- Failure from state b causes the machine to remain in state b.

Example 3

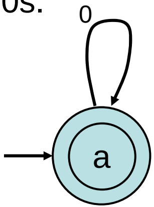
- L = { w ∈ { 0,1 }* | w doesn't contain either 00 or 11 as a substring }.



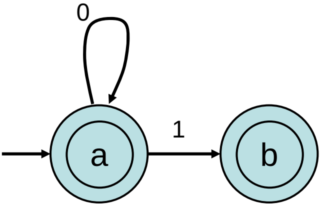
- State d is a trap state = a nonaccepting state that you can't leave.
- Sometimes we'll omit some arrows; by convention, they go to a trap state.

Example 4

- L = { w | all nonempty blocks of 1s in w have odd length }.
- E.g., ε, or 100111000011111, or any number of 0s.
- Initial 0s don't matter, so start with:



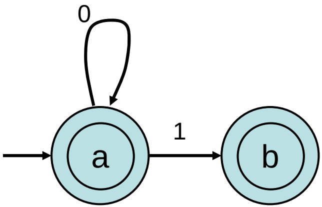
- Then 1 also leads to an accepting state, but it should be a different one, to "remember" that the string ends in one 1.

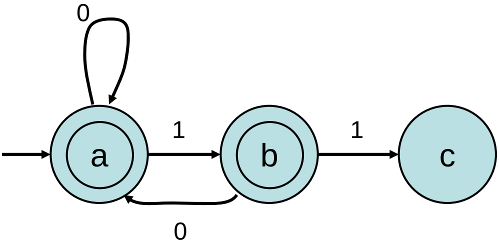


Example 4

- L = { w | all nonempty blocks of 1s in w have odd length }.

- From b:
  - 0 can return to a, which can represent either ε, or any string that is OK so far and ends with 0.
  - 1 should go to a new nonaccepting state, meaning "the string ends with two 1s".

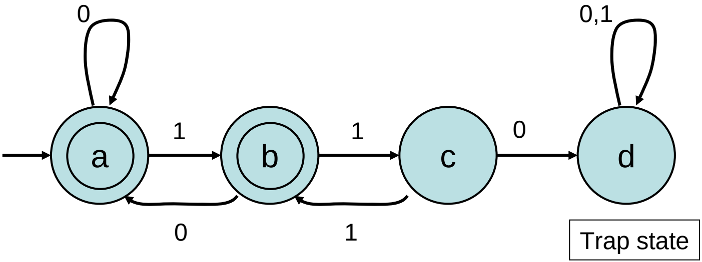




- Note: c isn't a trap state---we can accept some extensions.

Example 4

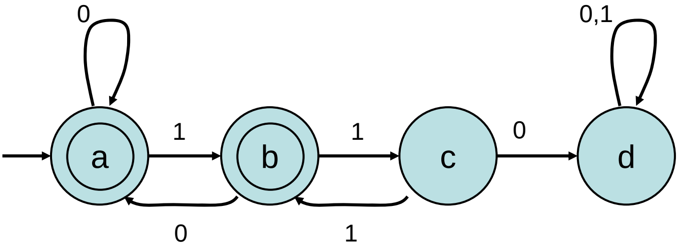
- L = { w | all nonempty blocks of 1s in w have odd length }.



- From c:
  - 1 can lead back to b, since future acceptance decisions are the same if the string so far ends with any odd number of 1s.
    - Reinterpret b as meaning “ends with an odd number of 1s”.
    - Reinterpret c as “ends with an even number of 1s”.
  - 0 means we must reject the current string and all extensions.

Example 4

- L = { w | all nonempty blocks of 1s in w have odd length }.



- Meanings of states (more precisely):

  a: Either ε, or contains no bad block (even block of 1s followed by 0) so far and ends with 0.

  b: No bad block so far, and ends with odd number of 1s.

  c: No bad block so far, and ends with even number of 1s.

  d: Contains a bad block.

Example 5

- L = EQ = { w | w contains an equal number of 0s and 1s }.
- No FA recognizes this language.
- Idea (not a proof):
  - Machine must “remember” how many 0s and 1s it has seen, or at least the difference between these numbers.
  - Since these numbers (and the difference) could be anything, there can’t be enough states to keep track.
  - So the machine will sometimes get confused and give a wrong answer.
- We’ll turn this into an actual proof next week.

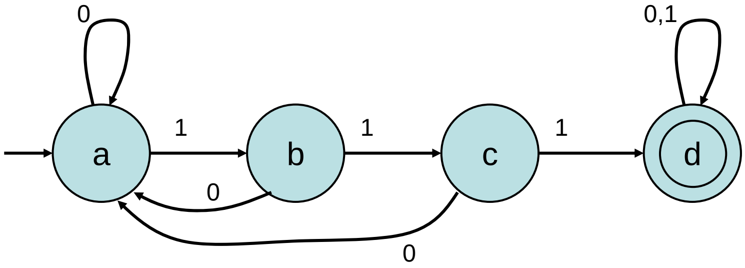
Closure under operations

- The set of FA-recognizable languages is closed under all six operations (union, intersection, complement, set difference, concatenation, star).
- This means: If we start with FA-recognizable languages and apply any of these operations, we get another FA-recognizable language (for a different FA).

- Theorem 1: FA-recognizable languages are closed under complement.
- Proof:
  - Start with a language L_1 over alphabet Σ, recognized by some FA, M_1.
  - Produce another FA, M_2, with $L(M_2) = \Sigma^* - L(M_1)$.
  - Just interchange accepting and non-accepting states.

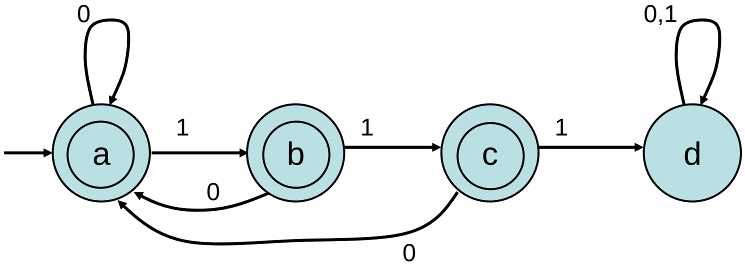
Closure under complement

- Theorem 1: FA-recognizable languages are closed under complement.
- Proof: Interchange accepting and non-accepting states.
- Example: FA for { w | w does not contain 111 }
  - Start with FA for { w | w contains 111 }:



Closure under complement

- Theorem 1: FA-recognizable languages are closed under complement.
- Proof: Interchange accepting and non-accepting states.
- Example: FA for { w | w does not contain 111 }
  - Interchange accepting and non-accepting states:



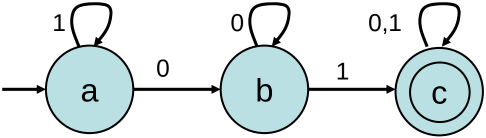
Closure under intersection

- Theorem 2: FA-recognizable languages are closed under intersection.
- Proof:
  - Start with FAs M_1 and M_2 for the same alphabet Σ.
  - Get another FA, M_3, with $L(M_3) = L(M_1) \cap L(M_2)$.
  - Idea: Run M_1 and M_2 "in parallel" on the same input. If both reach accepting states, accept.
  - Example:
    - $L(M_1)$: Contains substring 01.
    - $L(M_2)$: Odd number of 1s.
    - $L(M_3)$: Contains 01 and has an odd number of 1s.

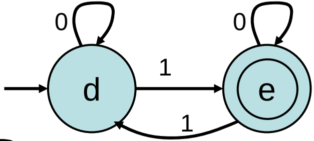
Closure under intersection

- Example:

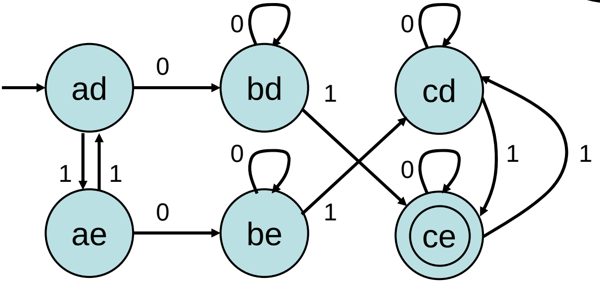
  M_1: Substring 01



  M_2: Odd number of 1s



  M_3:



Closure under intersection, general rule

- Assume:
  - $M_1 = (Q_1, \Sigma, \delta_1, q_{01}, F_1)$
  - $M_2 = (Q_2, \Sigma, \delta_2, q_{02}, F_2)$
- Define $M_3 = (Q_3, \Sigma, \delta_3, q_{03}, F_3)$, where
  - $Q_3 = Q_1 \times Q_2$
    - Cartesian product, $\{(q_1,q_2) \mid q_1 \in Q_1$ and $q_2 \in Q_2 \}$
  - $\delta_3 ((q_1,q_2), a) = (\delta_1(q_1, a), \delta_2(q_2, a))$
  - $q_{03} = (q_{01}, q_{02})$
  - $F_3 = F_1 \times F_2 = \{ (q_1,q_2) \mid q_1 \in F_1$ and $q_2 \in F_2 \}$

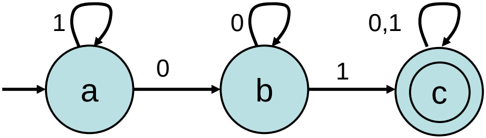
Closure under union

- Theorem 3: FA-recognizable languages are closed under union.
- Proof:
  - Similar to intersection.
  - Start with FAs M_1 and M_2 for the same alphabet Σ.
  - Get another FA, M_3, with $L(M_3) = L(M_1) \cup L(M_2)$.
  - Idea: Run M_1 and M_2 "in parallel" on the same input. If either reaches an accepting state, accept.
  - Example:
    - $L(M_1)$: Contains substring 01.
    - $L(M_2)$: Odd number of 1s.
    - $L(M_3)$: Contains 01 or has an odd number of 1s.

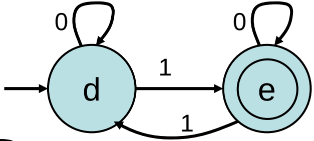
Closure under union

- Example:

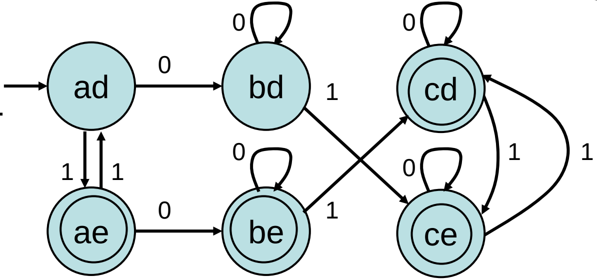
  M_1: Substring 01



  M_2: Odd number of 1s



  M_3: 1



Closure under union, general rule

- Assume:
  - $M_1 = (Q_1, \Sigma, \delta_1, q_{01}, F_1)$
  - $M_2 = (Q_2, \Sigma, \delta_2, q_{02}, F_2)$
- Define $M_3 = (Q_3, \Sigma, \delta_3, q_{03}, F_3)$, where
  - $Q_3 = Q_1 \times Q_2$
    - Cartesian product, $\{(q_1,q_2) \mid q_1 \in Q_1 \text{ and } q_2 \in Q_2 \}$
  - $\delta_3 ((q_1,q_2), a) = (\delta_1(q_1, a), \delta_2(q_2, a))$
  - $q_{03} = (q_{01}, q_{02})$
  - $F_3 = \{ (q_1,q_2) \mid q_1 \in F_1 \text{ or } q_2 \in F_2 \}$

Closure under set difference

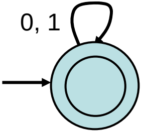
- Theorem 4: FA-recognizable languages are closed under set difference.
- Proof:
  - Similar proof to those for union and intersection.
  - Alternatively, since $L_1 - L_2$ is the same as $L_1 \cap (L_2)^c$, we can just apply Theorems 2 and 3.

Closure under concatenation

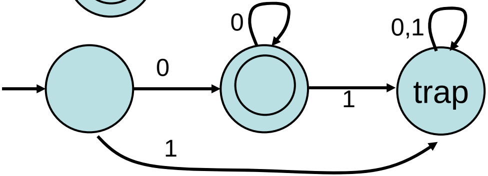
- Theorem 5: FA-recognizable languages are closed under concatenation.
- Proof:
  - Start with FAs M_1 and M_2 for the same alphabet Σ.
  - Get another FA, M_3, with $L(M_3) = L(M_1) \circ L(M_2)$, which is $\{ x_1 x_2 \mid x_1 \in L(M_1) \text{ and } x_2 \in L(M_2) \}$
  - Idea: ???
    - Attach accepting states of M_1 somehow to the start state of M_2.
    - But we have to be careful, since we don't know when we're done with the part of the string in $L(M_1)$---the string could go through accepting states of M_1 several times.

Closure under concatenation

- Theorem 5: FA-recognizable languages are closed under concatenation.
- Example:
  - Σ = { 0, 1}, $L_1 = \Sigma^*$, L_2 = {0} {0}* (just 0s, at least one).
  - $L_1 L_2$ = strings that end with a block of at least one 0
  - M_1:



  - M_2:



  - How to combine?
  - We seem to need to “guess” when to shift to M_2.
  - Leads to our next model, NFAs, which are FAs that can guess.

Closure under star

- Theorem 6: FA-recognizable languages are closed under star.
- Proof:
  - Start with FA M_1.
  - Get another FA, M_2, with $L(M_2) = L(M_1)^*$.
  - Same problems as for concatenation---need guessing.
  - …
  - We'll define NFAs next, then return to complete the proofs of Theorems 5 and 6.