BBM402-Lecture 9: Reducibility

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Resources for the presentation:
https://courses.engr.illinois.edu/cs373/fa2010/lectures

Reductions

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- Informal Examples: Measuring the area of rectangle reduces to measuring the length of the sides; Solving a system of linear equations reduces to inverting a matrix
- The problem L_d reduces to the problem A_{TM} as follows: "To see if $w \in L_d$ check if $\langle w, w \rangle \in A_{\mathrm{TM}}$."

Undecidability using Reductions

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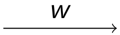
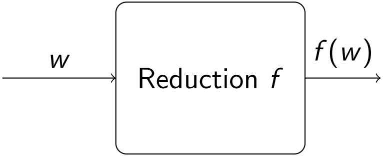
- On input w, apply reduction to transform w into an input w' for problem 2
- Run M on w', and use its answer.

Schematic View



Reductions schematically

Schematic View



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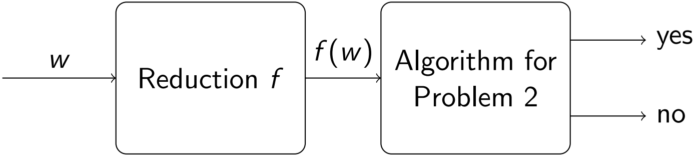

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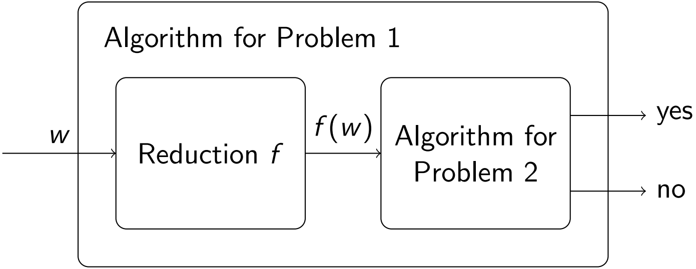

Reductions schematically

The Halting Problem

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Observe that $f(M)$ halts on input w if and only if M accepts w

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Completing the proof

Proof (contd).

Suppose HALT is decidable. Then there is a Turing machine H that always halts and $L(H) = \text{HALT}$.

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T decides A_{TM}. But, A_{TM} is undecidable, which gives us the contradiction. □

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A mapping/many-one reduction from A to B is a computable function $f : \Sigma^* \rightarrow \Sigma^*$ such that

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In this case, we say A is mapping/many-one reducible to B, and we denote it by $A \leq_m B$.

Convention

In this course, we will drop the adjective "mapping" or "many-one", and simply talk about reductions and reducibility.

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On input w
    Compute $f(w)$
    Run M_B on $f(w)$
    Accept if M_B does and reject if M_B rejects

□

Reductions and non-r.e.

Corollary

If $A \leq_m B$ and A is not recursively enumerable then B is not recursively enumerable.

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Corollary

If $A \leq_m B$ and A is undecidable then B is undecidable.

Emptiness of Turing Machines

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L_d is reducible to E_{TM} as follows.

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L_d is reducible to E_{TM} as follows. Let $f(M) = N$ where N is a TM that behaves as follows:

On input x
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Observe that $L(N) = \emptyset$ if and only if M does not accept $\langle M \rangle$ if and only if $\langle M \rangle \in L_d$. □

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We give a reduction f from A_{TM} to REGULAR. Let $f(\langle M, w\rangle) = N$, where N is a TM that works as follows:

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If $w \in L(M)$ then $L(N) = \Sigma^*$. If $w \notin L(M)$ then $L(N) = \{0^n1^n \mid n \geq 0\}$.

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If $w \in L(M)$ then $L(N) = \Sigma^*$. If $w \notin L(M)$ then $L(N) = \{0^n1^n \mid n \geq 0\}$. Thus, $\langle N\rangle \in$ REGULAR if and only if $\langle M, w\rangle \in A_{\text{TM}}$ □

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Observe $M \in E_{\mathrm{TM}}$ iff $L(M) = \emptyset$ iff $L(M) = L(M_1)$ iff $\langle M, M_1\rangle \in \mathrm{EQ}_{\mathrm{TM}}$. □