BBM 205 Discrete Mathematics
Hacettepe University
http://web.cs.hacettepe.edu.tr/~bbm205

Lecture 4: Probability, Proof Techniques, Method of Induction

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Resources:
Kenneth Rosen, "Discrete Mathematics and App."
cs.colostate.edu/ cs122/.Spring15/home_resources.php
inf.ed.ac.uk/teaching/courses/dmmr/slides/14-15/newprob.pdf

The "sample space" of a probabilistic experiment

Consider the following probabilistic (random) experiment:

> *"Flip a fair coin 7 times in a row, and see what happens"*

Question: What are the possible outcomes of this experiment?

Answer: The possible outcomes are all the sequences of "Heads" and "Tails", of length 7. In other words, they are the set of strings $\Omega = \{H, T\}^7$.

The set $\Omega = \{H, T\}^7$ of possible outcomes is called the sample space associated with this probabilistic experiment.

Sample Spaces

For any probabilistic experiment or process, the set Ω of all its possible outcomes is called its **sample space**.

In general, sample spaces need not be finite, and they need not even be countable. In "Discrete Probability", we focus on finite and countable sample spaces. This simplifies the axiomatic treatment needed to do probability theory. We only consider discrete probability (and mainly finite sample spaces).

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Question: What is the sample space, Ω, for the following probabilistic experiment:

"Flip a fair coin repeatedly until it comes up heads."

Answer: $\Omega = \{H, TH, TTH, TTTH, TTTTH, \ldots\} = T^*H$.

Note: This set is not finite. So, even for simple random experiments we do have to consider countable sample spaces.

Probability distributions

A probability distribution over a finite or countable set Ω, is a function:

$$P : \Omega \to [0, 1]$$

such that $\sum_{s \in \Omega} P(s) = 1$.

In other words, to each outcome $s \in \Omega$, $P(s)$ assigns a probability, such that $0 \leq P(s) \leq 1$, and of course such that the probabilities of all outcomes sum to 1, so $\sum_{s \in \Omega} P(s) = 1$.

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Example 2: Suppose a fair coin is tossed repeatedly until it lands heads. This random experiment defines a probability distribution $P : \Omega \to [0, 1]$, on $\Omega = T^*H$, such that, for all $k \geq 0$,

$$P(T^k H) = \frac{1}{2^{k+1}}$$

Note that
$\sum_{s \in \Omega} P(s) = P(H) + P(TH) + P(TTH) + \ldots = \sum_{k=1}^{\infty} \frac{1}{2^k} = 1.$

Events

For a countable sample space Ω, an **event**, E, is simply a subset $E \subseteq \Omega$ of the set of possible outcomes.
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- "The fourth and fifth coin tosses did not both come up tails".
  This is $E_2 = \Omega - \{H, T\}^3 TT \{H, T\}^2$; $P(E_2) = 1 - 1/4 = 3/4$.

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Example: For $\Omega = T^* H$, the following is an event:

- "The first time the coin comes up heads is after an even number of coin tosses."
  This is $E_3 = \{T^k H \mid k \text{ is odd}\}$; $P(E_3) = \sum_{k=1}^{\infty} (1/2^{2k}) = 1/3$.

Basic facts about probabilities of events

For event $E \subseteq \Omega$, define the complement event to be $\overline{E} \doteq \Omega - E$.

Theorem: Suppose $E_0, E_1, E_2, \ldots$ are a (finite or countable) sequence of pairwise disjoint events from the sample space Ω. In other words, $E_i \in \Omega$, and $E_i \cap E_j = \emptyset$ for all $i, j \in \mathbb{N}$. Then

$$P(\bigcup_i E_i) = \sum_i P(E_i)$$

Furthermore, for each event $E \subseteq \Omega$, $P(\overline{E}) = 1 - P(E)$.

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Proof: Follows easily from definitions:
for each E_i, $P(E_i) = \sum_{s \in E_i} P(s)$, thus, since the sets E_i are disjoint, $P(\bigcup_i E_i) = \sum_{s \in \bigcup_i E_i} P(s) = \sum_i \sum_{s \in E_i} P(s) = \sum_i P(E_i)$.

Likewise, since $P(\Omega) = \sum_{s \in \Omega} P(s) = 1$, $P(\overline{E}) = P(\Omega - E) = \sum_{s \in \Omega - E} P(s) = \sum_{s \in \Omega} P(s) - \sum_{s \in E} P(s) = 1 - P(E)$.

Brief comment about non-discrete probability theory

In general (non-discrete) probability theory, with uncountable sample space Ω, the conditions of the prior theorem are actually taken as axioms about a "probability measure", P, that maps events to probabilities, and events are not arbitrary subsets of Ω. Rather, the axioms say: Ω is an event; If $E_0, E_1, \ldots,$ are events, then so is $\bigcup_i E_i$; and If E is an event, then so is $\overline{E} = \Omega - E$.

A set of events $\mathcal{F} \subseteq 2^\Omega$ with these properties is called a σ-algebra. General probability theory studies probability spaces consisting of a triple $(\Omega, \mathcal{F}, P)$, where Ω is a set, $\mathcal{F} \subseteq 2^\Omega$ is a σ-algebra of events over Ω, and $P : \mathcal{F} \to [0, 1]$ is a probability measure, defined to have the properties in the prior theorem.

We only discuss discrete probabability, and will not assume you know definitions for general (non-discrete) probability.

Conditional probability

Definition: Let $P : \Omega \to [0, 1]$ be a probability distribution, and let $E, F \subseteq \Omega$ be two events, such that $P(F) > 0$.

The **conditional probability** of E given F, denoted $P(E \mid F)$, is defined by:

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)}$$

Example: A fair coin is flipped three times. Suppose we know that the event $F =$ "heads came up exactly once" occurs. what is the probability then of the event $E =$ "the first coin flip came up heads" occurs?

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Answer: There are 8 flip sequences $\{H, T\}^3$, all with probability $1/8$. The event that "heads came up exactly once" is $F = \{HTT, THT, TTH\}$. The event $E \cap F = \{HTT\}$.
So, $P(E \mid F) = \frac{P(E \cap F)}{P(F)} = \frac{1/8}{3/8} = \frac{1}{3}$.

Independence of two events

Intuitively, two events are *independent* if knowing whether one occurred does not alter the probability of the other. Formally:

Definition: Events A and B are called **independent** if $P(A \cap B) = P(A)P(B)$.

Note that if $P(B) > 0$ then A and B are independent if and only if

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = P(A)$$

Thus, the probability of A is not altered by knowing B occurs.

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Example: A fair coin is flipped three times. Are the events $A =$ "the first coin toss came up heads" and $B =$ "an even number of coin tosses came up head", independent?
Answer: Yes. $P(A \cap B) = 1/4$, $P(A) = 1/2$, and $P(B) = 1/2$, so $P(A \cap B) = P(A)P(B)$.

Pairwise and mutual independence

What if we have more than two events: $E_1, E_2, \ldots, E_n$.
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When should we consider them "independent"?

Definition: Events $E_1, \ldots, E_n$ are called **pairwise independent**, if for every pair $i, j \in \{1, \ldots, n\}$, $i \neq j$, E_i and E_j are independent (i.e., $P(E_i \cap E_j) = P(E_i)P(E_j)$).

Events $E_1, \ldots, E_n$ are called **mutually independent**, if for every subset $J \subseteq \{1, \ldots, n\}$, $P(\bigcap_{j \in J} E_j) = \prod_{j \in J} P(E_j)$.

Clearly, mutual independence implies pairwise independent.
But... **Warning:** pairwise independence does not imply mutual independence.
Typically, when we refer to > 2 events as "independent", we mean they are "mutually independent".

Biased coins and Bernoulli trials

In probability theory there are a number of fundamental probability distributions that one should study and understand in detail.
One of these distributions arises from (repeatedly) flipping a biased coin.
A **Bernoulli trial** is a probabilistic experiment that has two outcomes: success or failure (e.g., heads or tails).
We suppose that p is the probability of success, and $q = (1 - p)$ is the probability of failure.

We can of course have repeated Bernoulli trials. We typically assume the different trials are mutually independent.
Question: A biased coin, which comes up heads with probability $p = 2/3$, is flipped 7 times consecutively. What is the probability that it comes up heads exactly 4 times?

The Binomial Distribution

Theorem: The probability of exactly k successes in n (mutually) independent Bernoulli trials, with probability p of success and $q = (1 - p)$ of failure in each trial, is

$$\binom{n}{k} p^k q^{n-k}$$

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Proof: We can associate n Bernoulli trials with outcomes $\Omega = \{H, T\}^n$. Each sequence $s = (s_1, \ldots, s_n)$ with exactly k heads and $n - k$ tails occurs with probability $p^k q^{n-k}$. There are $\binom{n}{k}$ such sequences with exactly k heads. □

Definition: The **binomial distribution**, with parameters n and p, denoted $b(k; n, p)$, defines a probability distribution on $k \in \{0, \ldots, n\}$, given by

$$b(k; n, p) \doteq \binom{n}{k} \cdot p^k q^{n-k}$$

Random variables

Definition: A **random variable**, is a function $X : \Omega \to \mathbb{R}$, that assigns a real value to each outcome in a sample space Ω.

Example: Suppose a biased coin is flipped n times. The sample space is $\Omega = \{H, T\}^n$. The function $X : \Omega \to \mathbb{N}$ that assigns to each outcome $s \in \Omega$ the number $X(s) \in \mathbb{N}$ of coin tosses that came up heads is one random variable.

For a random variable $X : \Omega \to \mathbb{R}$, we write $P(X = r)$ as shorthand for the probability $P(\{s \in \Omega \mid X(s) = r\})$. The **distribution** of a random variable X is given by the set of pairs $\{(r, P(X = r)) \mid r \text{ is in the range of } X\}$.

Note: These definitions of a random variable and its distribution are only adequate in the context of discrete probability distributions. For general probability theory we need more elaborate definitions.

Biased coins and the Geometric Distribution

Question: Suppose a biased coin, comes up heads with probability p, $0 < p < 1$, each time it is tossed. Suppose we repeatedly flip this coin until it comes up heads.
What is the probability that we flip the coin k times, for $k \geq 1$?

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What is the probability that we flip the coin k times, for $k \geq 1$?

Answer: The sample space is $\Omega = \{H, TH, TTH, \ldots\}$.
Assuming mutual independence of coin flips, the probability of $T^{k-1}H$ is $(1-p)^{k-1}p$. Note: this does define a probability distribution on $k \geq 1$, because
$\sum_{k=1}^{\infty}(1-p)^{k-1}p = p\sum_{k=0}^{\infty}(1-p)^k = p(1/p) = 1$. □

A random variable $X : \Omega \to \mathbb{N}$, is said to have a **geometric distribution** with parameter p, $0 \leq p \leq 1$, if for all positive integers $k \geq 1$, $P(X = k) = (1-p)^{k-1}p$.

Proof Terminology

Theorem: statement that can be shown to be true

Proof: a valid argument that establishes the truth of a theorem

Axioms: statements we assume to be true

Lemma: a less important theorem that is helpful in the proof of other results

Corollary: theorem that can be established directly from a theorem that has been proved

Conjecture: statement that is being *proposed* to be a true statement

Learning objectives

- Direct proofs
- Proof by contrapositive
- Proof by contradiction
- Proof by cases

Technique #1: Direct Proof

- Direct Proof:
  - First step is a premise
  - Subsequent steps use rules of inference or other premises
  - Last step proves the conclusion

Direct Proof Example

- Prove "If n is an odd integer, then n^2 is odd."
  - If n is odd, then $n = 2k+1$ for some integer k.
  - $n^2 = (2k+1)^2 = 4k^2 + 4k + 1$
  - Therefore, $n^2 = 2(2k^2 + 2k) + 1$, which is odd.

2 * any # → even

Add 1 to any even # → odd #

More formal version…

	Step	**Reason**
1.	n is odd	Premise
2.	$\exists k \in Z \;\; n = 2k+1$	Def of odd integer in (1)
3.	$n^2 = (2k+1)^2$	Squaring (2)
4.	$= 4k^2 + 4k + 1$	Algebra on (3)
5.	$= 2(2k^2 + 2k) + 1$	Algebra on (4)
6.	$\therefore$ n^2 is odd	Def odd int, from (5)

Class Exercise

- Prove: If n is an even integer, then n^2 is even.
  - If $\boldsymbol{n}$ is even, then $\boldsymbol{n = 2k}$ for some integer $\boldsymbol{k}$.
  - $\boldsymbol{n^2 = (2k)^2 = 4k^2}$
  - Therefore, $\boldsymbol{n = 2(2k^2)}$, which is even.

Can you do the formal version?

	Step	**Reason**
1.	n is even	Premise
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3.	$n^2 = (2k)^2$	Squaring (2)
4.	$= 4k^2$	Algebra on (3)
5.	$= 2(2k^2)$	Algebra on (4)
6.	$\therefore$ n^2 is even	Def even int, from (5)

Technique #2: Proof by Contrapositive

- A direct proof, but starting with the contrapositive equivalence:
  - $p \rightarrow q \equiv \neg q \rightarrow \neg p$
- If you are asked to prove $p \rightarrow q$
- you instead prove $\neg q \rightarrow \neg p$
- Why? Sometimes, it may be easier to directly prove $\neg q \rightarrow \neg p$ than $p \rightarrow q$

Proof by contrapositive

Prove: If n^2 is an even integer, then n is even.

$(n^2 \text{ even}) \rightarrow (n \text{ even})$

By the contrapositive: This is the same as showing that

- $\neg(n \text{ even}) \rightarrow \neg(n^2 \text{ even})$
- If ***n*** is odd, then n^2 is odd.
- We already proved this on slides 4 and 5.

Since we have proved the contrapositive:

$\neg(n \text{ even}) \rightarrow \neg(n^2 \text{ even})$

We have also proved the original hypothesis:

$(n^2 \text{ even}) \rightarrow (n \text{ even})$

Technique #3: Proof by contradiction

Prove: If p then q.

Proof strategy:

- Assume the negation of q.
- In other words, assume that $p \wedge \neg q$ is true.
- Then arrive at a contradiction $p \wedge \neg$p (or something that contradicts a known fact).
- Since this cannot happen, our assumption must be wrong.
- Thus, $\neg q$ is false. q is true.

Proof by contradiction example

Prove: If *(3n+2) is odd,* then *n is odd*.

Proof:

- Given: (3n+2) is odd.
- Assume that n is not odd, that is n is even.
- If n is even, there is some integer k such that n=2k.
- (3n+2) = (3(2k)+2)=6k+2 = 2(3k+1), which is 2 times a number.
- Thus 3n+2 turned out to be even, but we know it's odd.
- This is a contradiction. Our assumption was wrong.
- Thus, n must be odd.

Proof by Contradiction Example

Prove that the $\sqrt{2}$ is irrational.

Assume that "$\sqrt{2}$ is irrational" is false, that is, $\sqrt{2}$ is rational.

Hence, $\sqrt{2} = \frac{a}{b}$ and a and b have no common factors. The fraction is in its lowest terms.

So $a^2 = 2b^2$ which means a must be even,

Hence, $a = 2c$

Therefore, $b^2 = 2c^2$ then b must be even, which means a and b must have common factors.

Contradiction.

Technique #4: Proof by cases

- Given a problem of the form:
  - $(p_1 \vee p_2 \vee \ \ldots \vee p_n) \rightarrow q$
  - where $p_1, p_2, \ldots p_n$ are the cases

- This is equivalent to the following:
  - $[(p_1 \rightarrow q) \wedge (p_2 \rightarrow q) \wedge \ldots \wedge (p_n \rightarrow q)]$
- So prove all the clauses are true.

Proof by cases (example)

- Prove: If n is an integer, then $n^2 \geq n$
  - $(n = 0 \vee n \geq 1 \vee n \leq -1) \rightarrow n^2 \geq n$
- Show for all the three cases, i.e.,
  - $(n = 0 \rightarrow n^2 \geq n) \wedge (n \geq 1 \rightarrow n^2 \geq n)$
    $\wedge$ $(n \leq -1 \rightarrow n^2 \geq n)$
- Case 1: Show that $n = 0 \rightarrow n^2 \geq n$
  - When n=0, $n^2 = 0$.
  - 0=0 ☺

Proof by cases (example contd)

- Case 2: Show that $n \geq 1 \rightarrow n^2 \geq n$
  - Multiply both sides of the inequality $n \geq 1$ by n
  - We get $n^2 \geq n$

Proof by cases (example contd)

- Case 3: Show that $n \leq -1 \rightarrow n^2 \geq n$
  - Given $n \leq -1$,
  - We know that n^2 cannot be negative, i.e., $n^2 > 0$
  - We know that $0 > -1$
  - Thus, $n^2 > -1$. We also know that $-1 \geq n$ (given)
  - Therefore, $n^2 \geq n$

Proof by Cases Example

Theorem: Given two real numbers x and y, $abs(x*y)=abs(x)*abs(y)$

Exhaustively determine the premises

Case p1: x>=0, y>=0, so x*y>=0 so abs(x*y)=x*y and abs(x)=x and abs(y)=y so abs(x)*abs(y)=x*y

Case p2: x<0, y>=0

Case p3: x>=0, y<0

Case p4: x<0, y<0

The principle of (ordinary) induction

Let $P(n)$ be a predicate. If

1. $P(0)$ is true, and
2. $P(n)$ IMPLIES $P(n+1)$ for all non-negative integers n

then

- $P(m)$ is true for all non-negative integers m

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then

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1. The first item says that $P(0)$ holds
2. The second item says that $P(0) \to P(1)$, and $P(1) \to P(2)$, and $P(2) \to P(3)$, *etc.*

- Intuitively, there is a domino effect that eventually shows that $\forall n \in \mathbb{N}.\ P(n)$

Proof by induction

To prove by induction $\forall k \in \mathbb{N}.\ P(k)$ is true, follow these three steps:

Base Case: Prove that $P(0)$ is true

Inductive Hypothesis: Let $k \geq 0$. We assume that $P(k)$ is true

Inductive Step: Prove that $P(k+1)$ is true

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Inductive Step: Prove that $P(k+1)$ is true

Remark

Proofs by mathematical induction do not always start at the integer 0. In such a case, the base case begins at a starting point $b \in \mathbb{Z}$. In this case we prove the property only for integers $\geq b$ instead of for all $n \in \mathbb{N}$

$$\forall k \in . \ \sum_{i=1}^{k} i = \frac{k(k+1)}{2}$$

$\forall k \in . \sum_{i=1}^{k} i = \frac{k(k+1)}{2}$

(By induction) Let $P(k)$ be the predicate "$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$"

Base Case: $\sum_{i=1}^{0} i = 0 = \frac{0(0+1)}{2}$, thus $P(0)$ is true

Inductive Hypothesis: Let $k \geq 0$. We assume that $P(k)$ is true, *i.e.* $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$

Inductive Step:

$$\begin{aligned} \sum_{i=1}^{k+1} i &= \left[\sum_{i=1}^{k} i\right] + (k+1) \\ &= \frac{k(k+1)}{2} + (k+1) \qquad \text{(by I.H.)} \\ &= \frac{k(k+1)+2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2} \end{aligned}$$

Thus $P(k+1)$ is true □

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(By induction) Let $P(k)$ be the predicate "$k^3 - k$ is divisible by 3"

Base Case: Since $0 = 3 \cdot 0$, it is the case that 3 divides $0 = 0^3 - 0$, thus $P(0)$ is true

Inductive Hypothesis: Let $k \geq 0$. We assume that $P(k)$ is true, *i.e.* $k^3 - k$ is divisible by 3

Inductive Step:

$$\begin{aligned}
(k+1)^3 - (k+1) &= (k^3 + 3k^2 + 3k + 1) - (k+1) \\
&= k^3 + 3k^2 + 2k \\
&= (k^3 - k) + 3k^2 + 3k \\
&= 3(\ell + k^2 + k) \text{ for some } \ell \qquad \text{(by I.H.)}
\end{aligned}$$

Thus $(k+1)^3 - (k+1)$ is divisible by 3. So we can conclude that $P(k+1)$ is true $\square$

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(By induction) Let $P(k)$ be the predicate "$2^k < k!$"

Base Case: $2^4 = 16 < 24 = 4!$, thus $P(4)$ is true

Inductive Hypothesis: Let $k \geq 4$. We assume that $P(k)$ is true, *i.e.* $2^k < k!$

Inductive Step:

$$\begin{aligned} 2^{k+1} &= 2 \cdot 2^k \\ &< 2 \cdot k! && \text{(by I.H.)} \\ &< (k+1) \cdot k! && (k \geq 4) \\ &= (k+1)! \end{aligned}$$

Thus $P(k+1)$ is true $\square$

All horses are of the same color

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(By induction) Let $P(k)$ be the predicate "in any set of k horses, all the horses are of the same color"

Base Case: In any set of just one horse, all horses obviously have the same color, thus $P(1)$ is true

Inductive Hypothesis: Let $k \geq 1$. We assume that $P(k)$ is true, *i.e.* "in any set of k horses, all the horses are of the same color"

Inductive Step: Let $\{H_1, H_2, \ldots, H_{k+1}\}$ be a set of $k+1$ horses. Then, by I.H., all the horses in $\{H_1, H_2, \ldots, H_k\}$ have the same color. Similarly, by I.H., all the horses in $\{H_2, \ldots, H_{k+1}\}$ have the same color. Thus $col(H_1) = col(H_2) = col(H_{k+1})$. But this implies that all the $k+1$ horses are of the same color. Thus, $P(k+1)$ is true. $\square$

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!!!The inductive step is not true for k=1!!!

The principle of strong induction

Let $P(n)$ be a predicate. If

1. $P(0)$ is true, and
2. $P(0) \wedge \cdots \wedge P(n)$ IMPLIES $P(n+1)$ for all non-negative integers n

then

- $P(m)$ is true for all non-negative integers m

- Intuitively, there is a domino effect that eventually shows that $\forall n \in \mathbb{N}.\ P(n)$
- Strong induction sometimes makes the proof of the inductive step much easier since we assume a stronger statement

Every natural number $k > 1$ can be written as a product of primes

(By induction) Let $P(k)$ be the predicate "k can be written as a product of primes"

Base Case: Since 2 is a prime number, $P(2)$ is true

Inductive Hypothesis: Let $k \geq 1$. We assume that $P(k)$ is true, *i.e.* "k can be written as a product of primes"

Inductive Step: We distinguish two cases: *(i)* Case $k+1$ is a prime, then $P(k+1)$ is true; *(ii)* Case $k+1$ is not a prime. Then by definition of primality, there must exist $1 < n, m < k+1$ such that $k+1 = n \cdot m$. But then we know by I.H. that n and m can be written as a product of primes (since $n, m \leq k$). Therefore, $k+1$ can also be written as a product of primes. Thus, $P(k+1)$ is true

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□

$\longrightarrow$ If we had only assumed $P(k)$ to be true, then we could not apply our I.H. to n and m

Exercises

Use induction to show that

1. if $S(n)$ is the sum of integers $1, \ldots, n$, then $S(n) = n(n+1)/2$.
2. $n! \geq 2^{n-1}$ for $n \geq 1$.
3. if $r \neq 1$, then

$$a + ar^1 + ar^2 + \cdots + ar^n = \frac{a(r^{n+1} - 1)}{r - 1} \qquad \text{for } n \geq 1.$$

4. $5^n - 1$ is divisible by 4 for $n \geq 1$;
5. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$;
6. $\frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$;
7. $2^n \geq n^2$ for $n \geq 4$;
8. $7^n - 1$ is divisible by 6;
9. n straight lines in the plane divide the plane into $(n^2 + n + 2)/2$ regions. Assume that no two lines are parallel and no three lines have a common point.
10. the regions in the question above can be colored red and green so that no two regions that share an edge have the same color.
11. postage of 6 kuruş or more can be achieved by using only 2-kuruş and 7-kuruş stamps.