BIL694-Lecture 5: Planar Graphs and Hamiltonian Graphs

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Resources for the presentation:

“Introduction to Graph Theory” by Douglas B. West
1 Planar Graphs

2 Hamiltonian Graphs
1. Planar Graphs

2. Hamiltonian Graphs
A graph $G$ is planar if it has a drawing without crossings. Such a drawing is a planar embedding of $G$.

**planar graph**: A graph, that has a planar embedding.

**Proposition**

$K_5$ and $K_{3,3}$ cannot be drawn without crossings.

**Proof**: Consider a spanning cycle for both graphs. Such a cycle should be drawn without any crossings.
faces of a planar graph: The maximal regions of the plane that contain no point used in the embedding.

Consider a planar embedding of a planar graph $G$. The dual graph of a planar graph $G$, denoted by $G^*$, is a graph, where $V(G^*)$ consist of the faces of $G$ and $E(G^*)$ defined as follows.

If $e$ is an edge of $G$ with the faces $X$ and $Y$ in the planar embedding of $G$, then the dual graph has an edge between the vertices $X$ and $Y$.

Proposition

Let $\ell(F_i)$ denote the length of face $F_i$ in a planar graph $G$. Then $2e(G) = \sum \ell(F_i)$. (implied by the degree-sum formula for the dual graph of $G$)
Theorem

The following are equivalent for a planar graph $G$.

1. $G$ is bipartite.
2. Every face of $G$ has even length.
3. The dual graph $G^*$ is Eulerian.

(1) $\implies$ (2): Trivial, since bipartite graphs have no odd cycle.

(2) $\implies$ (1): Every cycle $C$ is consist of the edges of one face or of a collection of faces $\mathcal{F}$ in the region surrounded by $C$. Thus, $C$ has even length (the sum of the face-lengths in $\mathcal{F}$ minus twice the edges not in $C$).

(2) $\iff$ (3): Having all degrees in $G^*$ is equivalent to being Eulerian.
A planar graph is called outerplanar if it has a planar embedding with every vertex on the boundary of the unbounded face. Examples: $K_4$ and $K_{2,3}$ are planar but not outerplanar. To show that, observe that the boundary of the outer face of a 2-connected outerplanar graph is a spanning cycle.

**Proposition**

*Every simple outerplanar graph has a vertex of degree at most 2.*
Euler’s Formula

Theorem (Euler, 1758)

If a connected planar graph \( G \) has exactly \( n \) vertices, \( e \) edges and \( f \) faces, then \( n - e + f = 2 \).

- Let \( P(i) \) be the proposition that the Euler formula holds for every planar graph on \( i \) vertices.
- Use induction on \( n \) to show that \( P(n) \) is true for all \( n \geq 1 \).
  Base step \((n=1)\): \( G \) is a “bouquet” of loops, \( P(1) \) is true. (If \( e = 0 \), then \( f = 1 \), the statement is true.)
- Induction step \((n > 1)\): Because \( G \) is connected, there is an edge that is not a loop, call it \( e \). Contract the edge \( e \). Let \( n', e', f' \) be the parameters of this new graph \( G' \).
  - By inductive hypothesis, \( P(n') \) is true, thus \( n' - e' + f' = 2 \).
  - Substituting \( n' = n - 1 \), \( e' = e - 1 \), \( f' = f \) shows \( P(n) \) is also true.
Corollaries of Euler’s Theorem

**Corollary**

If $G$ is a **simple** planar graph with at least three vertices, then $e(G) \leq 3n(G) - 6$. If $G$ is also triangle-free, then $e(G) \leq 2n(G) - 4$.

- Note that $2e = \sum \ell(f_i) \geq 3f$. ($\geq 4f$, for triangle-free $G$)
- This together with Euler's formula yields the corollary.

**Exercise:** Use Euler’s formula, to show $K_5$ and $K_{3,3}$ are not planar.

**Proposition**

For a simple planar graph on $n$ vertices, TFAE.

- $G$ has $3n - 6$ edges.
- $G$ is a triangulation.
- $G$ is a maximal planar graph (no more edges can be added without making $G$ non-planar or non-simple).
Is there a way to check if a graph $G$ is planar, without searching for a planar embedding of $G$?

Yes. Observe that not only $K_5$ and $K_{3,3}$ but also their subdivisions are non-planar graphs.

**Theorem (Kuratowski, 1930)**

A graph is planar if and only if it does not contain a subdivision of $K_5$ or $K_{3,3}$.

(Proof, skipped)

Theorem (Five Color Theorem, Heawood, 1890)

Every planar graph is 5-colorable.

MUCH LATER...

Theorem (Four Color Theorem, Appel-Haken-Koch, 1977)

Every planar graph is 4-colorable.
Outline

1 Planar Graphs

2 Hamiltonian Graphs
A Hamiltonian graph is a graph with a spanning cycle, also called a Hamiltonian cycle. The problem on deciding whether a graph is hamiltonian or not is an NP-complete problem (no algorithm exists that runs in polynomial time).

So, there are known necessary conditions needed for a graph to be hamiltonian. Also, we know some sufficient conditions.

But, no “necessary and sufficient (if and only if)” is known.

**Proposition (A necessary condition)**

If $G$ has a Hamilton cycle, then for each nonempty set $S \subset V$, the graph $G - S$ has at most $|S|$ components.

See Example 7.2.5 in West.
Sufficient Conditions for being Hamiltonian

Example: Two cliques or order $\lceil(n + 1)/2\rceil$ and $\lceil(n + 1)/2\rceil$ merged at one vertex. This graph has a very high minimum degree, but it is not hamiltonian.

**Theorem (Dirac, 1952)**

If $G$ is a simple graph with at least three vertices and $\delta(G) \geq n(G)/2$, then $G$ is Hamiltonian.

- Assume on the contrary that $G$ is a maximal non-Hamiltonian graph that satisfies the minimum degree condition.
- By the maximality of $G$, adding any other edge to $G$ would create a Hamiltonian cycle. So, let $uv \notin E(G)$. There is a Ham. path $v_1, v_2, \ldots, v_n$ with ends $u = v_1$ and $v = v_n$.
- **Fact:** If $v_i \in N(v)$ and $v_{i+1} \in N(u)$ for some $1 < i < n - 1$, done.
- We claim that there is such an $i$, let $S = \{i : v_{i+1} \in N(u)\}$ and $T = \{i : v_i \in N(v)\}$.

\[ |S \cup T| + |S \cap T| = |S| + |T| = \deg(u) + \deg(v) \geq n. \]

Since $n \notin S \cup T$, $|S \cup T| \leq n - 1$, done.
Theorem (Ore, 1960)

Let $G$ be a simple graph. If $u$ and $v$ are distinct non-adjacent vertices such that $\text{deg}(u) + \text{deg}(v) \geq n(G)$, then $G$ is Hamiltonian iff $G + uv$ is Hamiltonian.

The closure of a graph $G$, denoted by $C(G)$, is the graph with the same vertex set as $G$ that is obtained by iteratively adding the edges to $G$ whose endvertices are a non-adjacent pair with degree sum at least $n$.

Theorem (Bondy-Chvátal, 1976)

A simple graph on $n$ vertices is Hamiltonian iff its closure is Hamiltonian.

Theorem (Chvatal's condition, 1972)

Let $G$ be a simple graph with vertex degrees $d_1 \leq \ldots d_n$, where $n \geq 3$. If for each $i < n/2$, $d_i > i$ or $d_{n-i} \geq n - i$, then $G$ is Hamiltonian.
Chvátal’s Condition

Theorem (Chvatal’s condition, 1972)

Let $G$ be a simple graph with vertex degrees $d_1 \leq \ldots d_n$, where $n \geq 3$. If for each $i < n/2$, $d_i > i$ or $d_{n-i} \geq n - i$, then $G$ is Hamiltonian.

- By using Bondy-Chvátal condition (BCC), we will show that $C(G)$ is Hamiltonian under these assumptions and thus $G$ is Ham.

- **Claim:** $C(G) = K_n$.
  
  To prove this, again assume on the contrary that $C(G) \neq K_n$. We will show that there is an $i$ for which BCC does not hold, i.e. for some $i$, at least $i$ vertices have degree at most $i$ and at least $n - i$ vertices have degree less than $n - i$.

- Details left for reading.

**Example:** The graph $K_i \lor (\bar{K}_i + K_{n-2i})$ is an example where Chvátal’s condition is not satisfied, but still the degrees are high.