Powers of Paths and Cycles
Definitions

Power of a Path:

For $n, h \geq 0$,

the $h$-power of a path, denoted by $P_n^{(h)}$, is a graph with $n$ vertices $v_1, v_2, \ldots, v_n$ such that, for $1 \leq i, j \leq n, i \neq j$, $(v_i, v_j) \in E(P_n^{(h)})$ if and only if $|j - i| \leq h$;

Power of a Cycle:

For $n, h \geq 0$,

the $h$-power of a cycle, denoted by $Q_n^{(h)}$, is a graph with $n$ vertices $v_1, v_2, \ldots, v_n$ such that, for $1 \leq i, j \leq n, i \neq j$, $(v_i, v_j) \in E(Q_n^{(h)})$ if and only if $|j - i| \leq h$ or $|j - i| \geq n - h$. 
Examples for Powers of Paths

The graphs $P_1^{(2)}, \ldots, P_5^{(2)}$

$P_6^3$, $P_6^4$, $P_6^5$
Examples for Powers of Cycles

The graphs $Q_1^{(2)}, \ldots, Q_5^{(2)}$

$C_6^2$  $C_6^3$
Problem

(i) Let $G = P^k_\infty$ be the $k$th power of a 2-way infinite path. For example, let $V(G) = \mathbb{Z}$ and $E(G) = \{ij : i < j \leq i + k\}$. What is the maximal number of edges spanned by $\ell$ vertices?

(ii) Let $G$ be the graph with vertex set $V(G) = \mathbb{Z}_n$ in which vertex $i$ is joined to vertex $j \neq i$ if $-k \leq i - j \leq k$, where $k \geq 1$ and $n \geq 2k + 1$. (Thus, $G = C^k_n$, i.e., $G$ is the $k$th power of the $n$-cycle $C_n$.) Let $V(G) = A \cup B$ be a partition of the vertex set of $G$ into sets with at least $k$ vertices in each. What is the minimal number of edges joining $A$ to $B$?
Solution for Part (i)

Let \( A = \{x_1, x_2, \ldots, x_l\} \), where \( x_1 < \ldots < x_l \) and \( A_0 = \{1, \ldots, l\} \).

If \( x_i < x_j \leq x_i + k \), then \( i < j \leq i + k \).

So, \( A_0 \) spans at least as many edges of \( G \) as \( A \).

The maximal number of edges spanned by \( l \) vertices is:

- \( \frac{l(l - 1)}{2} \), for \( 1 \leq l \leq k+1 \),
- \( kl - k(k+1)/2 \), for \( l \geq k+1 \).
Solution for Part (ii)

- **Claim:** The minimal number of A-B edges, subject to A and B partitioning the vertex set V(G) into sets with at least k vertices each, is \( k(k+1) \).

- Let we denote:
  
  - \( a \) ⇒ # of vertices in A
  - \( b \) ⇒ # of vertices in B
  - \( r \) ⇒ # of edges joining vertices of A
  - \( t \) ⇒ # of edges joining vertices of B
  - \( s \) ⇒ # of A-B edges

- Since G is 2k-regular, \( 2ka = 2r + s \), and so our aim is to show that:
  
  \[ r \leq ka - k(k+1)/2 \Leftrightarrow t \leq kb - k(k+1)/2 \]

- If \( a = k \), it is obvious since \( r \leq k(k-1)/2 \) always holds, which forms a complete subgraph of G.

- If \( a = n - k \), since we always have \( t \leq b(b-1) \), by symmetry, the assertion is also obvious.
Solution for Part (ii) (continued)

Now, we will fix $a$ and $k$, and apply induction on $n$:

- **Base step**: If $n = a + k$, then $b = k$; so, $t \leq k(k-1)/2 = kb - k(k+1)/2$, as required.
- **Ind. Hypothesis**: Suppose that $b > k$ and the assertion holds for $n - 1$.
- **Ind. step**: Pick a vertex $x$ of $B$, and let $G' = G - \{x\}$. Clearly, we can get $H = \binom{k}{n-1}$, i.e, the ground graph with $n-1$ vertices, by adding some edges to $G'$ which are not joining any two vertices of $A$.

Since this operation does not change the edges joining vertices of $A$, by induction we still have $r \leq ka - k(k+1)/2$, which completes the proof of the induction step.
References
