BIL694-Lecture 1: Introduction to Graphs

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Resources for the presentation:
http://www.math.ucsd.edu/~gptesler/184a/calendar.html
http://www.inf.ed.ac.uk/teaching/courses/dmmr/
Outline

1 Simple Graph, Multigraphs and Directed Graphs

2 Graph Isomorphism

3 Special Families of Graphs

4 Paths, Cycles, Trails, Trees
Outline

1. Simple Graph, Multigraphs and Directed Graphs
2. Graph Isomorphism
3. Special Families of Graphs
4. Paths, Cycles, Trails, Trees
We have a network of items and connections between them. Examples:

- Telephone networks, computer networks
- Transportation networks (bus/subway/train/plane)
- Social networks
- Family trees, evolutionary trees
- Molecular graphs (atoms and chemical bonds)
- Various data structures in Computer Science
The dots are called **vertices** or **nodes** (singular: vertex, node)

\[ V = \text{set of vertices} = \{1, 2, 3, 4, 5\} \]

- The connections between vertices are called **edges**.
- Represent an edge as a set \( \{i, j\} \) of two vertices.
  - E.g., the edge between 2 and 5 is \( \{2, 5\} = \{5, 2\} \).

\[ E = \text{set of edges} = \{\{1, 2\}, \{2, 3\}, \{2, 5\}, \{3, 4\}, \{3, 5\}, \{4, 5\}\} \]
A **simple graph** is $G = (V, E)$:

- $V$ is the set of vertices. It can be any set; $\{1, \ldots, n\}$ is just an example.

- $E$ is the set of edges, of form $\{u, v\}$, where $u, v \in V$ and $u \neq v$. Every pair of vertices has either 0 or 1 edges between them.

- The drawings above represent the same abstract graph since they have the same $V$ and $E$, even though the drawings look different.
The *degree* of a vertex is the number of edges on it.

\[
d(1) = 1 \quad d(2) = 3 \quad d(3) = 3 \quad d(4) = 2 \quad d(5) = 3
\]

Sum of degrees \(= 1 + 3 + 3 + 2 + 3 = 12\)
Number of edges \(= 6\)
The sum of degrees of all vertices is twice the number of edges:

\[ \sum_{v \in V} d(v) = 2 |E| \]

Proof.

- Let \( S = \{(v, e) : v \in V, e \in E, \text{ vertex } v \text{ is in edge } e\} \)

- **Count \(|S| \) by vertices:** Each vertex \( v \) is contained in \( d(v) \) edges, so
  \[ |S| = \sum_{v \in V} d(v). \]

- **Count \(|S| \) by edges:** Each edge has two vertices, so
  \[ |S| = \sum_{e \in E} 2 = 2 |E| . \]
Some networks have *multiple edges* between two vertices. Notation \( \{3, 4\} \) is ambiguous, so write labels on the edges: \( c, d, e \).

There can be an edge from a vertex to itself, called a *loop* (such as \( h \) above). A loop has one vertex, so \( \{2, 2\} = \{2\} \).

A simple graph does not have multiple edges or loops.
Multigraphs

- Computer network with multiple connections between machines.
- Transportation network with multiple routes between stations.
- **But:** A graph of Facebook friends is a simple graph. It does not have multiple edges, since you’re either friends or you’re not. Also, you cannot be your own Facebook friend, so no loops.
Multigraphs

A **multigraph** is $G = (V, E, \phi)$, where:

- $V$ is the set of vertices. It can be any set.
- $E$ is the set of edge labels (with a unique label for each edge).
- $\phi : E \rightarrow \{\{u, v\} : u, v \in V\}$
  is a function from the edge labels to the pairs of vertices.
  $\phi(L) = \{u, v\}$ means the edge with label $L$ connects $u$ and $v$.

$V = \{1, 2, 3, 4\}$

$E = \{a, b, c, d, e, f, g, h\}$

$\phi(a) = \{1, 2\}$
$\phi(b) = \{2, 3\}$
$\phi(c) = \phi(d) = \phi(e) = \{3, 4\}$
$\phi(f) = \phi(g) = \{1, 4\}$
$\phi(h) = \{2\}$
Adjacency matrix of a multigraph

- Let $n = |V|$
- The **adjacency matrix** of a multigraph is an $n \times n$ matrix $A = (a_{uv})$. Entry $a_{uv}$ is the number of edges between vertices $u, v \in V$.

- $a_{uv} = a_{vu}$ for all vertices $u, v$. Thus, $A$ is a symmetric matrix ($A = A^T$).
- The sum of entries in row $u$ is the degree of $u$.
- **Technicallity:** A loop on vertex $v$ counts as
  - 1 edge in $E$,
  - degree 2 in $d(v)$ and in $a_{vv}$ (it touches vertex $v$ twice),

With these rules, graphs with loops also satisfy $\sum_{v \in V} d(v) = 2 |E|$. 

```
A =
\begin{bmatrix}
1 & 2 & 3 & 4 \\
1 & 0 & 1 & 0 & 2 \\
2 & 1 & 2 & 1 & 0 \\
3 & 0 & 1 & 0 & 3 \\
4 & 2 & 0 & 3 & 0 \\
\end{bmatrix}
```
In a simple graph:

- All entries of the adjacency matrix are 0 or 1 (since there either is or is not an edge between each pair of vertices).
- The diagonal is all 0’s (since there are no loops).

\[ A = \begin{bmatrix}
1 & 2 & 3 & 4 & 5 \\
1 & 0 & 1 & 0 & 0 \\
2 & 1 & 0 & 1 & 0 \\
3 & 0 & 1 & 0 & 1 \\
4 & 0 & 0 & 1 & 0 \\
5 & 0 & 1 & 1 & 1 & 0
\end{bmatrix} \]
A *directed edge* is a connection with a direction.

One-way transportation routes.

Broadcast TV and satellite TV are one-way connections from the broadcaster to your antenna.

Family tree: parent → child

An unrequited Facebook friend request.
Directed graph (a.k.a. digraph)

Let's represent a directed edge $u \to v$ by an ordered pair $(u, v)$. For example, $3 \to 2$ is $(3, 2)$, but we do not have $2 \to 3$, which is $(2, 3)$.

A directed graph is **simple** if each $(u, v)$ occurs at most once, and there are no loops.

- Represent it as $G = (V, E)$.
- $V$ is a set of vertices. It can be any set.
- $E$ is the set of edges. Each edge has form $(u, v)$ with $u, v \in V$, $u \neq v$.
- It is permissible to have both $(4, 5)$ and $(5, 4)$, since they are distinct.

$V = \{1, 2, 3, 4, 5\}$

$E = \{(1, 5), (2, 1), (3, 2), (3, 4), (4, 5), (5, 2), (5, 4)\}$
For a vertex $v$, the **indegree** is the number of edges going into $v$, and the **outdegree** is the number of edges going out from $v$.

<table>
<thead>
<tr>
<th>$v$</th>
<th>indegree($v$)</th>
<th>outdegree($v$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Total</td>
<td>7</td>
<td>7</td>
</tr>
</tbody>
</table>

The sum of indegrees is $|E|$ and the sum of outdegrees is $|E|$. 
Let \( n = |V| \)

The **adjacency matrix** of a directed graph is an \( n \times n \) matrix \( A = (a_{uv}) \) with \( u, v \in V \).

Entry \( a_{uv} \) is the number of edges directed from \( u \) to \( v \).

\( a_{uv} \) and \( a_{vu} \) are not necessarily equal, so \( A \) is usually not symmetric.

The sum of entries in row \( u \) is the outdegree of \( u \).
The sum of entries in column \( v \) is the indegree of \( v \).
A directed multigraph may have loops and multiple edges.

- Represent it as $G = (V, E, \phi)$.
- Name the edges with labels. Let $E$ be the set of the labels.
- $\phi(L) = (u, v)$ means the edge with label $L$ goes from $u$ to $v$.
- **Technicality:** A loop counts once in indegree, outdegree, and $a_{vv}$.

### Example

- $V = \{1, \ldots, 5\}$
- $E = \{a, \ldots, i\}$
- $\phi(a) = (2, 1)$
- $\phi(d) = (3, 2)$
- $\phi(g) = (3, 4)$
- $\phi(b) = (1, 5)$
- $\phi(e) = (5, 2)$
- $\phi(h) = (4, 5)$
- $\phi(c) = (1, 1)$
- $\phi(f) = (5, 2)$
- $\phi(i) = (5, 4)$

### Matrix Representation

$$A = \begin{bmatrix}
1 & 2 & 3 & 4 & 5 \\
1 & 0 & 0 & 0 & 1 \\
2 & 1 & 0 & 0 & 0 \\
3 & 0 & 1 & 0 & 1 & 0 \\
4 & 0 & 0 & 0 & 0 & 1 \\
5 & 0 & 2 & 0 & 1 & 0 \\
\end{bmatrix}$$
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Isomorphism of Graphs

**Definition**: Two (undirected) graphs \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \) are **isomorphic** if there is a bijection, \( f : V_1 \rightarrow V_2 \), with the property that for all vertices \( a, b \in V_1 \)

\[
\{a, b\} \in E_1 \quad \text{if and only if} \quad \{f(a), f(b)\} \in E_2
\]

Such a function \( f \) is called an **isomorphism**.

Intuitively, isomorphic graphs are “THE SAME”, except for “renamed” vertices.
Isomorphism of Graphs (cont.)

**Example:** Show that the graphs $G = (V, E)$ and $H = (W, F)$ are isomorphic.

**Solution:** The function $f$ with $f(u_1) = v_1$, $f(u_2) = v_4$, $f(u_3) = v_3$, and $f(u_4) = v_2$ is a one-to-one correspondence between $V$ and $W$. 
Isomorphism of Graphs (cont.)

It is difficult to determine whether two graphs are isomorphic by brute force: there are $n!$ bijections between vertices of two $n$-vertex graphs.

Often, we can show two graphs are not isomorphic by finding a property that only one of the two graphs has. Such a property is called graph invariant:

- e.g., number of vertices of given degree, the degree sequence (list of the degrees), .....
Isomorphism of Graphs (cont.)

Example: Are these graphs are isomorphic?

Solution: No! Since $\text{deg}(a) = 2$ in $G$, $a$ must correspond to $t$, $u$, $x$, or $y$, since these are the vertices of degree 2 in $H$. But each of these vertices is adjacent to another vertex of degree 2 in $H$, which is not true for $a$ in $G$. So, $G$ and $H$ can not be isomorphic.
Example: Determine whether these two graphs are isomorphic.

Solution: The function $f$ is defined by: $f(u_1) = v_6$, $f(u_2) = v_3$, $f(u_3) = v_4$, $f(u_4) = v_5$, $f(u_5) = v_1$, and $f(u_6) = v_2$ is a bijection.
Algorithms for Graph Isomorphism

- The best algorithms known for determining whether two graphs are isomorphic have exponential worst-case time complexity (in the number of vertices of the graphs).
- However, there are algorithms with good time complexity in many practical cases.
- See, e.g., a publicly available software called NAUTY for graph isomorphism.
Applications of Graph Isomorphism

The question whether graphs are isomorphic plays an important role in applications of graph theory. For example:

Chemists use molecular graphs to model chemical compounds. Vertices represent atoms and edges represent chemical bonds. When a new compound is synthesized, a database of molecular graphs is checked to determine whether the new compound is isomorphic to the graph of an already known one.
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A complete graph on $n$ vertices, denoted by $K_n$, is the simple graph that contains exactly one edge between each pair of distinct vertices.
Special Types of Graphs: Cycles

A cycle $C_n$ for $n \geq 3$ consists of $n$ vertices $v_1, v_2, \ldots, v_n$, and edges $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\}$.
Special Types of Simple Graphs: \( n \)-Cubes

An *\( n \)-dimensional hypercube*, or *\( n \)-cube*, is a graph with \( 2^n \) vertices representing all bit strings of length \( n \), where there is an edge between two vertices if and only if they differ in exactly one bit position.

\begin{align*}
\text{\( \varrho_1 \)} & \\
\text{\( \varrho_2 \)} & \\
\text{\( \varrho_3 \)} &
\end{align*}
Bipartite Graphs

**Definition:**
An equivalent definition of a bipartite graph is one where it is possible to color the vertices either red or blue so that no two adjacent vertices are the same color.

$G$ is bipartite

$H$ is not bipartite: if we color $a$ red, then its neighbors $f$ and $b$ must be blue. But $f$ and $b$ are adjacent.
Example: Show that $C_6$ is bipartite.
Solution: Partition the vertex set into $V_1 = \{v_1, v_3, v_5\}$ and $V_2 = \{v_2, v_4, v_6\}$:

Example: Show that $C_3$ is not bipartite.
Solution: If we partition vertices of $C_3$ into two nonempty sets, one set must contain two vertices. But every vertex is connected to every other. So, the two vertices in the same partition are connected. Hence, $C_3$ is not bipartite.
Complete Bipartite Graphs

**Definition:** A complete bipartite graph is a graph that has its vertex set partitioned into two subsets $V_1$ of size $m$ and $V_2$ of size $n$ such that there is an edge from every vertex in $V_1$ to every vertex in $V_2$.

**Examples:**
- $K_{2,3}$
- $K_{3,3}$
- $K_{3,5}$
- $K_{2,6}$
Subgraphs

Definition: A subgraph of a graph \( G = (V,E) \) is a graph \((W,F)\), where \( W \subseteq V \) and \( F \subseteq E \). A subgraph \( H \) of \( G \) is a proper subgraph of \( G \) if \( H \neq G \).

Example: here is \( K_5 \) and one of its (proper) subgraphs:
Induced Subgraphs

**Definition:** Let $G = (V, E)$ be a graph. The *subgraph induced* by a subset $W$ of the vertex set $V$ is the graph $H = (W, F)$, whose edge set $F$ contains an edge in $E$ if and only if both endpoints are in $W$.

**Example:** Here is $K_5$ and its induced subgraph induced by $W = \{a, b, c, e\}$.
Bipartite Graphs

Theorem (König, 1936)

A graph is bipartite if and only if it has no odd cycle.

Proof

Necessity: If G is bipartite, clearly every cycle has even length.

Sufficiency:

- Assume that G has no odd cycle, show that G is bipartite.
- Let $u \in V(G)$ and for each $v \in V(G)$, let $f(v)$ be the minimum length of a $u, v$-path (finite, assuming G is connected).
- $X := \{v \in V(G) : f(v)\text{ is even}\}$, $Y := \{v \in V(G) : f(v)\text{ is odd}\}$. 

Any edge in $vw \in G[X]$, together with the shortest $u, v$- and $u, w$-paths create an odd walk, call it $W$. (Same if $vw$ is an edge in $G[Y]$)

And any odd walk contains an odd cycle (by Lemma 1.2.15 in the book). Contradiction!

Therefore, $X$ and $Y$ are independent sets.

Same ideas can be applied for each component, if $G$ had more than one component.
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• Therefore, X and Y are independent sets.
**Theorem (König, 1936)**

A graph is bipartite if and only if it has no odd cycle.

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Connectness in undirected graphs

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This graph is connected.
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This graph is connected

This graph is **not** connected
Walks

- Trace along edges from vertex \( x \) to \( y \), without lifting your pen.

- The walk in yellow is represented as a sequence of edges \( c, f, g, i \), or a sequence of vertices \( 1, 2, 5, 4, 3 \).

- A **walk** from vertex \( x \) to \( y \) is a sequence of edges, each connected to the next by a vertex:
  
  \[
  e_1 = \{x, v_1\} \quad e_2 = \{v_1, v_2\} \quad e_3 = \{v_2, v_3\} \quad \cdots \quad e_k = \{v_{k-1}, y\}
  \]

  In a directed graph, edge directions must be respected:
  
  \[
  e_1 = (x, v_1) \quad e_2 = (v_1, v_2) \quad e_3 = (v_2, v_3) \quad \cdots \quad e_k = (v_{k-1}, y)
  \]
In a walk, edges and vertices may be re-used.

A trail is a walk with all edges distinct.

A path is a walk with all vertices and edges distinct.

A walk/trail/path is open if the start and end vertices are different, and closed if they are the same (this is allowed in a closed path, but no other vertices may be repeated).

A cycle is a closed path.
A **tree** is a simple connected graph that contains no cycle.

**Theorem**

The following statements are equivalent for a graph $T$:

1. $T$ is a tree;
2. Any two vertices of $T$ are connected by a unique path in $T$;
3. $T$ is minimally connected, that is, $T$ is connected but $T - e$ is disconnected for every edge $e$ in $T$;
4. $T$ is maximally acyclic, that is $T$ contains no cycle, but $T + xy$ does, for any two non-adjacent vertices $x, y$ in $T$. 
Corollary: Every tree has at least two vertices with degree 1.
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Proof: Take any longest simple path $x_0, \ldots, x_m$ in $T$. Both $x_0$ and $x_m$ must have degree 1: otherwise there is a longer path in $T$.

Corollary: The vertices of a tree can always be enumerated, say as $v_1, \ldots, v_n$ so that every $v_i$ with $i \geq 2$ has a unique neighbor in $\{v_1, \ldots, v_{i-1}\}$. 
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Corollary: A connected graph with $n$ vertices is a tree if and only if it has $n - 1$ edges.
Proof: Exercise, use induction on $n$
**Corollary:** Every tree has at least two vertices with degree 1.

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**Proof:** Exercise, use induction on $n$.

**Corollary:** If $T$ is a tree and $G$ is any graph with $\delta(G) \geq |T|−1$, then $T \subset G$, that is $G$ has a subgraph isomorphic to $T$.

**Proof** Construct $T$ by using a greedy algorithm.
The Königsberg Bridge Problem
Leonard Euler (1707-1783) was asked to solve the following:

**Question:** Can you start a walk somewhere in Königsberg, walk across each of the 7 bridges *exactly once*, and end up back where you started from?

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Euler (in 1736) used “graph theory” to answer this question.
Theorem

A connected graph $G$ is Eulerian if and only if all of its vertices have even degree.

Proof

**Necessity:** Add direction to each edge as it is visited along the Eulerian circuit. Since each vertex has indegree equal to its outdegree, the degree (undirected) of each vertex is even.
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**Sufficiency:** Proof by induction on the number of edges, \( m \).

- Basis step: \( m = 0 \), true.
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- **Basis step:** $m = 0$, true.
- **Key Lemma:** (Lemma 1.2.25): If every vertex of a graph $G$ has degree at least 2, then $G$ contains a cycle.
- Since every vertex in $G$ has even degree, every vertex has degree at least 2 in every component of $G$. By the key lemma, there is a cycle in each component of $G$. 
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- Since every vertex in $G$ has even degree, every vertex has degree at least 2 in every component of $G$. By the key lemma, there is a cycle in each component of $G$.
- So, pick a cycle in one component, call it $C$. Let $G' := G - C$. 
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- Since every vertex in $G$ has even degree, every vertex has degree at least 2 in every component of $G$. By the key lemma, there is a cycle in each component of $G$.
- So, pick a cycle in one component, call it $C$. Let $G' := G - C$.
- By inductive hypothesis, every component of $G'$ has an Euler circuit $D$.
- $D$, together with $C$ is an Euler circuit of $G$. 
Conclusions

Corollary

Every even graph (all degree even) decomposes into cycles.
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Proposition

If for a graph $G$, $\delta(G) \geq k$, then $G$ contains a $P_{k+1}$. If $k \geq 2$, then $G$ also contains a cycle of length at least $k + 1$. 