# BBM402-Lecture 9: PSPACE: A Class of Problems beyond NP

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# Review of co-NP (Chapter 8)

$\bar{X}$ : Complementary problem for $X$

$$s \in X \iff s \notin \bar{X}$$

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$\mathcal{NP} = co - \mathcal{NP}?$

# Belief: $\mathcal{NP} \neq co - \mathcal{NP}$

*From Kleinberg-Tardos' book:*

Again, the widespread belief is that $\mathcal{NP} \neq$ co-$\mathcal{NP}$: Just because the "yes" instances of a problem have short proofs, it is not clear why we should believe that the "no" instances have short proofs as well.

Proving $\mathcal{NP} \neq$ co-$\mathcal{NP}$ would be an even bigger step than proving $\mathcal{P} \neq \mathcal{NP}$, for the following reason:

**(8.26)** *If* $\mathcal{NP} \neq$ co-$\mathcal{NP}$, *then* $\mathcal{P} \neq \mathcal{NP}$.

**Proof.** We'll actually prove the contrapositive statement: $\mathcal{P} = \mathcal{NP}$ implies $\mathcal{NP} =$ co-$\mathcal{NP}$. Essentially, the point is that $\mathcal{P}$ is closed under complementation; so if $\mathcal{P} = \mathcal{NP}$, then $\mathcal{NP}$ would be closed under complementation as well. More formally, starting from the assumption $\mathcal{P} = \mathcal{NP}$, we have

$$X \in \mathcal{NP} \Longrightarrow X \in \mathcal{P} \Longrightarrow \overline{X} \in \mathcal{P} \Longrightarrow \overline{X} \in \mathcal{NP} \Longrightarrow X \in \text{co-}\mathcal{NP}$$

and

$$X \in \text{co-}\mathcal{NP} \Longrightarrow \overline{X} \in \mathcal{NP} \Longrightarrow \overline{X} \in \mathcal{P} \Longrightarrow X \in \mathcal{P} \Longrightarrow X \in \mathcal{NP}.$$

Hence it would follow that $\mathcal{NP} \subseteq$ co-$\mathcal{NP}$ and co-$\mathcal{NP} \subseteq \mathcal{NP}$, whence $\mathcal{NP} =$ co-$\mathcal{NP}$. ∎

# A recursive Algorithm solving Q-SAT

*From Kleinberg-Tardos' book:*

```
If the first quantifier is ∃ x_i then
  Set x_i = 0 and recursively evaluate the quantified expression
                    over the remaining variables
  Save the result (0 or 1) and delete all other intermediate work
  Set x_i = 1 and recursively evaluate the quantified expression
                    over the remaining variables
  If either outcome yielded an evaluation of 1, then
     return 1
  Else return 0
  Endif
If the first quantifier is ∀ x_i then
  Set x_i = 0 and recursively evaluate the quantified expression
                    over the remaining variables
  Save the result (0 or 1) and delete all other intermediate work
  Set x_i = 1 and recursively evaluate the quantified expression
                    over the remaining variables
  If both outcomes yielded an evaluation of 1, then
     return 1
  Else return 0
  Endif
Endif
```