Anti-Ramsey number of matchings in hypergraphs

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June 13, 2013

Abstract

A k-matching in a hypergraph is a set of k edges such that no two of these edges intersect. The *anti-Ramsey number* of a k-matching in a complete s-uniform hypergraph $\mathcal{H}$ on n vertices, denoted by $\mathrm{ar}(n,s,k)$, is the smallest integer c such that in any coloring of the edges of $\mathcal{H}$ with exactly c colors, there is a k-matching whose edges have distinct colors. The *Turán number*, denoted by $\mathrm{ex}(n,s,k)$, is the maximum number of edges in an s-uniform hypergraph on n vertices with no k-matching. For $k \geq 3$, we conjecture that if $n > sk$, then $\mathrm{ar}(n,s,k) = \mathrm{ex}(n,s,k-1)+2$. Also, if $n = sk$, then $\mathrm{ar}(n,s,k) = \begin{cases} \mathrm{ex}(n,s,k-1)+2 & \text{if } k < c_s \\ \mathrm{ex}(n,s,k-1)+s+1 & \text{if } k \geq c_s \end{cases}$,where c_s is a constant dependent on s. We prove this conjecture for $k=2, k=3$, and sufficiently large n, as well as provide upper and lower bounds.

1 Introduction

A *hypergraph* $\mathcal{H}$ consists of a set $V(\mathcal{H})$ of *vertices* and a family $\mathcal{E}(\mathcal{H})$ of nonempty subsets of $V(\mathcal{H})$ called *edges* of $\mathcal{H}$. If each edge of $\mathcal{H}$ has exactly s vertices then $\mathcal{H}$ is *s-uniform*. A *complete* s-uniform hypergraph is a hypergraph whose edge set is the set of all s-subsets of the vertex set. A *matching* is a set of edges in a (hyper)graph in which no two edges have a common vertex. We call a matching with k edges a *k-matching* and a matching containing all vertices a *perfect matching*. In an edge-coloring of a (hyper)graph $\mathcal{H}$, a sub(hyper)graph $\mathcal{F} \subseteq \mathcal{H}$ is *rainbow* if all edges of $\mathcal{F}$ have distinct colors. The *anti-Ramsey number* of a graph G, denoted by $\mathrm{ar}(G,n)$, is the minimum number of colors needed to color the edges of K_n so that, in any coloring, there exists a rainbow copy of G. The *Turán number* of a graph G, denoted by $\mathrm{ex}(n,G)$, is the the maximum number of edges in a graph on n vertices that does not contain G as a subgraph. The *anti-Ramsey number* of a k-matching, denoted by $\mathrm{ar}(n,s,k)$, is the minimum number of colors

[*]Research supported by DMS 0946431.
Keywords: anti-Ramsey, rainbow, matching, hypergraph.

needed to color the edges of a complete s-uniform hypergraph on n vertices so that there exists a rainbow k-matching in any coloring. The *Turán number* of a k-matching, denoted by $\mathrm{ex}(n, s, k)$, is the maximum number of edges in an s-uniform hypergraph on n vertices that contains no k-matching.

In 1973, Erdős, Simonovits, and Sós [6] showed that $\mathrm{ar}(K_p, n) = \mathrm{ex}(n, K_{p-1}) + 2$ for sufficiently large n. More recently, Montellano-Ballesteros and Neumann-Lara [10] extended this result to all values of n and p with $n > p \geq 3$. A history of results and open problems on this topic was given by Fujita, Magnant, and Ozeki [8]. The Turán number $\mathrm{ex}(n, 2, k)$ was determined by Erdős and Gallai [4] as

$$\mathrm{ex}(n, 2, k) = \max\{\binom{2k-1}{2}, \binom{k-1}{2} + (k-1)(n-k+1)\}$$

for $n \geq 2k$ and $k \geq 1$. Schiermeyer [11] proved that $\mathrm{ar}(n, 2, k) = \mathrm{ex}(n, 2, k-1) + 2$ for $k \geq 2$ and $n \geq 3k + 3$. Later, Chen, Li, and Tu [2] and independently Fujita, Kaneko, Schiermeyer, and Suzuki [7] showed that $\mathrm{ar}(n, 2, k) = \mathrm{ex}(n, 2, k-1) + 2$ for $k \geq 2$ and $n \geq 2k + 1$. The value

$$\mathrm{ar}(n, 2, k) = \begin{cases} \mathrm{ex}(n, 2, k-1) + 2 & \text{if } k < 7 \\ \mathrm{ex}(n, 2, k-1) + 3 & \text{if } k \geq 7 \end{cases}$$

was determined for $n = 2k$ in [2] and by Haas and the second author [9], independently.

The same ideas implying a lower bound for the anti-Ramsey number of graphs given in [6] provide a lower bound for $\mathrm{ar}(n, s, k)$.

Proposition 1. *For all n,* $\mathrm{ar}(n, s, k) \geq \mathrm{ex}(n, s, k-1) + 2$.

Proof. Let $\mathcal{H}$ be a complete s-uniform hypergraph on n vertices. Let $\mathcal{G}$ be a subhypergraph of $\mathcal{H}$ with $\mathrm{ex}(n, s, k-1)$ edges such that $\mathcal{G}$ does not contain a $(k-1)$-matching. Color each edge of $\mathcal{G}$ with distinct colors and color all of the remaining edges of $\mathcal{H}$ the same, using an additional color. If there is a rainbow k-matching in this coloring, then it uses $k-1$ edges from $\mathcal{G}$ which is a contradiction. Therefore, this coloring has no rainbow k-matching. $\square$

For k-matchings the Turán number $\mathrm{ex}(n, s, k)$ is still not known for $k \geq 3$ and $s \geq 3$. Erdős [3] conjectured in 1965 the value of $\mathrm{ex}(n, s, k)$ as follows. Let $g(n, s, k-1)$ be the number of s-sets of $\{1, ..., n\}$ that intersect $\{1, ..., k-1\}$. By definition, $g(n, s, k-1) = \binom{n}{s} - \binom{n-k+1}{s}$.

Conjecture 2 (Erdős [3])**.** *For $n \geq sk$, $s \geq 2$, and $k \geq 2$,*

$$\mathrm{ex}(n, s, k) = \max\{\binom{sk-1}{s}, g(n, s, k-1)\}. \tag{1}$$

Erdős, Ko, and Rado [5] proved that $\mathrm{ex}(n, s, 2) = \binom{n-1}{s-1} = g(n, s, 1)$ for $n \geq 2s$. This conjecture is true for $s = 2$, as shown by Erdős and Gallai [4]. Erdős [3] proved that

$$\mathrm{ex}(n, s, k) = g(n, s, k-1) = \binom{n}{s} - \binom{n-k+1}{s} \tag{2}$$

for sufficiently large n. Later, Bollobás, Daykin, and Erdős [1] sharpened this result by showing that (2) holds for $n > 2s^3(k-1)$.

In Section 2, we provide bounds on $\mathrm{ar}(n, s, k)$ and show that anti-Ramsey number and Turán number of a k-matching differ at most by a constant. In Section 3, we determine the value of $\mathrm{ar}(n, s, k)$ for $k \in \{2, 3\}$ and show that $\mathrm{ar}(n, s, k) = \mathrm{ex}(n, s, k-1) + 2$ for $k \in \{2, 3\}$ and $n > ks$. The claim also holds for $n = ks$ when $k = 3$. We conjecture that this is true for all k.

Conjecture 3. *Let $k \geq 3$. If $n > sk$, then* $\mathrm{ar}(n,s,k) = \mathrm{ex}(n,s,k-1)+2$. *Also, if $n = sk$, then*

$$\mathrm{ar}(n,s,k) = \begin{cases} \mathrm{ex}(n,s,k-1)+2 & \text{if } k < c_s \\ \mathrm{ex}(n,s,k-1)+s+1 & \text{if } k \geq c_s \end{cases}$$

where c_s is a constant dependent on s.

Finally, in Section 4, we give the exact value of $\mathrm{ar}(n,s,k)$ when n is sufficiently large.

We introduce some notation for hypergraphs used in the remaining sections. For a set X, $\binom{X}{s}$ denotes all s-subsets of X. We call a hypergraph an *intersecting family* if every two edges intersect. For a vertex x in a hypergraph $\mathcal{H}$, we call the number of edges of $\mathcal{H}$ containing x the *degree* of x written $\deg_{\mathcal{H}}(x)$.The maximum degree of a hypergraph $\mathcal{H}$ is denoted by $\Delta(\mathcal{H})$.

2 General bounds on the anti-Ramsey number

The following constructions provide a lower bound for $\mathrm{ar}(n,s,k)$ in Theorem 6.

Construction 4.
Let $\mathcal{H}$ be the complete s-uniform hypergraph with vertex set $\{v_1,\dots,v_n\}$, where $n = sk$. Let $A = \{v_1,\dots,v_{s+1}\}$ and $c = \binom{n-s-1}{s}+s$. Define a c-coloring h of $\mathcal{E}(\mathcal{H})$ as follows. For any edge $E \in \mathcal{E}$, if $v_1 \in E$, then let $h(e) = \min\{i : v_i \notin E\}$. If $E \cap A \neq \emptyset$ but $v_1 \notin E$, then let $h(E) = \min\{i : v_i \in E\}$. Assign distinct other colors to the remaining edges.

Assume there is a rainbow perfect matching $\mathcal{M}$ in this coloring. Since $n = sk$, at least two edges of M intersect A. Let E be the edge of $\mathcal{M}$ that contains v_1. Let $j = \min\{i : v_i \notin V(E)\}$ and let E' be the edge of $\mathcal{M}$ that contains v_j. By the above construction, E and E' both have color j.

Construction 5.
Let $\mathcal{H}$ be a complete s-uniform hypergraph on $n \geq sk$ vertices. Let S be a subset of $V(\mathcal{H})$ with $k-2$ vertices and color the edges containing any vertex from S with distinct colors. Color all of the remaining edges the same with an additional color. The number of colors used is $\binom{n}{s}-\binom{n-k+2}{s}+1$.

This construction has no rainbow k-matching, since at least two edges among any k must lie completely outside S. Constructions 4 and 5 establish lower bounds for the anti-Ramsey number:

Corollary 6. *If $n \geq sk$, then* $\mathrm{ar}(n,s,k) \geq \begin{cases} \max\{\binom{n}{s}-\binom{n-k+2}{s}+2, \binom{n-s-1}{s}+s+1\} & \text{if } n = sk, \\ \binom{n}{s}-\binom{n-k+2}{s}+2 & \text{otherwise.} \end{cases}$

Theorem 7. *If $n \geq sk+(s-1)(k-1)$, then* $\mathrm{ar}(n,s,k) \leq \mathrm{ex}(n,s,k-1)+k$.

Proof. Let $\mathcal{H}$ be a complete s-uniform hypergraph on n vertices whose edges are colored with $\mathrm{ex}(n,s,k-1)+k$ colors. Since taking exactly one edge of each color gives a subhypergraph with $\mathrm{ex}(n,s,k-1)+k$ edges, there exists a rainbow $(k-1)$-matching $\mathcal{M}$. Let the colors of the edges in $\mathcal{M}$ be $\alpha_1,\dots,\alpha_{k-1}$. Let $A = V(\mathcal{H}) \setminus V(\mathcal{M})$. Note that every edge induced by A has a color in $\{\alpha_1,\dots,\alpha_{k-1}\}$, otherwise, there is a rainbow k-matching containing the edges of $\mathcal{M}$.

Remove all edges of $\mathcal{H}$ that have color α_i for $1 \leq i \leq k-1$ and let $\mathcal{G}$ be the remaining hypergraph (with colors preserved). In this coloring, there are at least $\mathrm{ex}(n,s,k-1)+1$ colors and therefore a rainbow $(k-1)$-matching exists; call it $\mathcal{M}'$. Since no edge of $\mathcal{G}$ is induced by A, $|V(\mathcal{M}') \cap A| \leq (k-1)(s-1)$. Together with the assumed lower bound on n, this yields $|A \setminus V(\mathcal{M}')| = |V(\mathcal{H}) \setminus (V(\mathcal{M} \cup \mathcal{M}'))| \geq n - s(k-1) - (s-1)(k-1) \geq s$. Hence some edge induced by A intersects no edge in $\mathcal{M}'$ and completes a rainbow k-matching with $\mathcal{M}$ induced by A that does not intersect any edge in $\mathcal{M}'$. The color of e is α_i for some i, $1 \leq i \leq k-1$ and there is a rainbow k-matching using the edges in $\mathcal{M}'$ and e. □

3 Anti-Ramsey numbers for k-matchings, $k \in \{2,3\}$

Theorem 8. *If $n \geq 2s$, then*

$$\operatorname{ar}(n,s,2) = \begin{cases} \frac{1}{2}\binom{n}{s}+1 & n=2s \\ 2 & n>2s. \end{cases}$$

Proof. Let $\mathcal{H}$ be a complete s-uniform hypergraph on n vertices. If $n = 2s$, then by coloring complementary edges with the same color and using distinct colors for all such pairs, we can obtain a coloring without a rainbow 2-matching. If $\mathcal{H}$ is colored by at least $\frac{1}{2}\binom{n}{s}+1$ colors then, by the pigeonhole principle, one of the vertex-disjoint edge pairs has distinct colors.

Now, let $n \geq 2s+1$ and consider a coloring of the edge set of $\mathcal{H}$ with 2 colors such that there is no rainbow 2-matching. This requires disjoint edges to have the same color. Hence in the Kneser graph $K(n,s)$, where the vertices are the edges of $\mathcal{H}$ and two vertices are adjacent when the corresponding edges of $\mathcal{H}$ are disjoint, all edges in the same component must have the same color. It is well known that the Kneser graph is connected when $n \geq 2s+1$, so only one color can be used when avoiding a rainbow 2-matching.

□

Theorem 9. *If $n \geq 3s$, then* $\operatorname{ar}(n,s,3) = \binom{n-1}{s-1}+2 = \operatorname{ex}(n,s,2)+2$.

Proof. Let $\mathcal{H}$ be a complete s-uniform hypergraph on n vertices with edge set $\mathcal{E}$. We consider a coloring of $\mathcal{E}$ using $\binom{n-1}{s-1}+2$ colors, such that there is no rainbow 3-matching. Fix a vertex v and let $E(v)$ denote the set of edges that contain v. Choose Q as a subset of $\mathcal{E}\setminus E(v)$ such that the edges of Q do not have any color in common with the edges of $E(v)$ and each color not used on $E(v)$ is the color of exactly one edge in Q. This implies that $|Q| \geq 2$, since $|E(v)| = \binom{n-1}{s-1}$.

Note that any pair of edges E_1 and E_2 in Q have nonempty intersection, otherwise there is a rainbow 3-matching containing E_1, E_2, and any edge of $E(v)$ that does not intersect E_1 and E_2. Let $A, B \in Q$ and $C, D \in E(v)$ We use (A,B) to denote an unordered pair of edges A and B. We write $(A,B) \diamond (C,D)$ if

$$\begin{aligned} A\cap D=\emptyset,\quad B\cap C=\emptyset,\quad &\text{and } A\cup D=B\cup C \\ &\text{or} \\ A\cap C=\emptyset,\quad B\cap D=\emptyset,\quad &\text{and } A\cup C=B\cup D. \end{aligned} \tag{3}$$

An example of the configuration of A, B, C and D is shown in Figure 1.

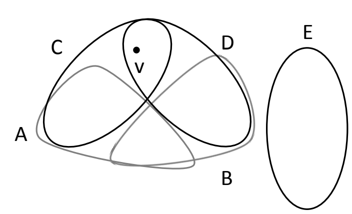

Figure 1: The edges A,B,C, D and E.

We define an auxiliary bipartite graph G with vertex set $V(G) = X \cup Y$, where $X = \binom{Q}{2}$, $Y = \binom{E(v)}{2}$ and the edge set of G is defined as $E(G) = \{(A,B)(C,D) : (A,B)\diamond(C,D),\ (A,B) \in$

X, $(C,D) \in Y\}$. In the proof of Claim 10, we use the following result of Erdős, Ko and Rado [5] which gives an upper bound on the size of an s-uniform intersecting family on n vertices.

$$\mathrm{ex}(n,s,2) = \binom{n-1}{s-1}, \text{ for } n \geq 2s. \tag{4}$$

Claim 10. *There is a matching in G whose vertex set contains all vertices in $X = \binom{Q}{2}$.*

Recall that Q is an intersecting subfamily. The degree $deg_G(A,B)$ is the number of vertices (C,D) in Y that satisfy the relation in (3). Therefore, the number of neighbors of (A,B) are given by the number of choices for the set $(C \cap D) \setminus \{v\}$. Let $\ell = |A \cap B|$, where $1 \leq \ell \leq s-1$. Since $|C \cap D| = \ell$, each vertex in X has the same degree given by

$$deg_G((A,B)) = \binom{n-(2s-\ell)-1}{\ell-1} \tag{5}$$

Now, by the same observations as above, the degree of a vertex (C,D) in Y can be bounded above. Let (A,B) and (A',B'), where $(A',B') \neq (A,B)$, be neighbors of (C,D). By definition of the relation $\diamond$, the edges A, A', B, and B' are all distinct. Since Q is an intersecting family, $A \cap B$ and $A' \cap B'$ cannot be vertex-disjoint. Therefore the collection of $A \cap B$'s that satisfy $(A,B) \diamond (C,D)$ for a fixed vertex (C,D) in Y with $|C \cap D| = \ell$ is an ℓ-uniform intersecting family on the vertex set $V \setminus (C \cup D)$ which has $n-(2s-\ell)$ vertices. By using (4), we obtain an upper bound on the degree of (C,D) as

$$deg_G((C,D)) \leq \binom{n-(2s-\ell)-1}{\ell-1}. \tag{6}$$

Let G' be a connected component of G. A result of the definition of the edge set of G is that if $(U_1,U_2),(V_1,V_2) \in V(G')$ and $|U_1 \cap U_2| = \ell$, then $|V_1 \cap V_2| = \ell$. Let $T \subseteq (V(G') \cap X)$ and $N(T) \subseteq (V(G') \cap Y)$ be the neighborhood of T. Since (5) and (6) also hold for G' we have

$$\begin{aligned} |T|\binom{n-(2s-\ell)-1}{\ell-1} &= \sum_{(A,B)\in T} deg_{G'}((A,B)) \\ &\leq \sum_{(C,D)\in N(T)} deg_{G'}((C,D)) \\ &\leq |N(T)|\binom{n-(2s-\ell)-1}{\ell-1}. \end{aligned}$$

Therefore, $|T| \leq |N(T)|$ for any $T \subseteq (V(G') \cap X)$ and by Hall's Theorem, there is a matching containing each vertex in $G' \cap X$. Applying this to each component of G completes the proof of the claim.

Claim 11. *Let $(A,B) \in \binom{Q}{2}$ and $(C,D) \in \binom{E(v)}{2}$ with $(A,B) \diamond (C,D)$. Then the edges C and D have the same color.*

Let S be the subset of $V(\mathcal{H})$ that is vertex-disjoint from these four edges, thus $|S| = n-2s \geq s$. Let E be an edge induced by S. Let A, B, C and D be related as in (3) such that without loss of generality $\{A,D,E\}$ and $\{B,C,E\}$ are matchings. If E has the same color as A or B then $\{B,C,E\}$ or $\{A,D,E\}$, respectively, must be a rainbow matching. Therefore, E must have the same color as C and D, since there are no rainbow 3-matchings. Hence, C and D have the same color.

We define another auxiliary graph G_v with vertex set $E(v)$ and edge set $\{CD : C,D \in E(v) \text{ and } \deg_G((C,D)) > 0\}$. Let $|Q| = q$ and p be the number of components of G_v. By

Claim 11, each component of G_v corresponds to a subset of $E(v)$ whose members have the same color. Therefore, $p \geq \binom{n-1}{s-1} + 2 - q$.

One can find an injective mapping $f : \binom{Q}{2} \to \binom{E(v)}{2}$ defined by using the adjacencies of vertices in a matching of G given by Claim 10. Therefore there are at least $\binom{q}{2}$ edges in G_v. The maximum number of components of a graph with fixed number of vertices and edges is attained in the case when all edges are in a single component with minimum number of vertices and remaining components are isolated vertices. Thus, $p \leq \binom{n-1}{s-1} - q + 1$. This is a contradiction with the lower bound of p given above. □

4 Anti-Ramsey Number for Large n

By following the same ideas of the proof of (2) in [1] and [3], one can prove Theorem 12. For completeness, we provide its proof here.

Theorem 12. *For fixed s and k and $n \geq 2s^3k$,* $\mathrm{ar}(n,s,k) = \binom{n}{s} - \binom{n-k+2}{s} + 2 = \sum_{i=1}^{k-2}\binom{n-i}{s-1} + 2 = \mathrm{ex}(n,s,k-1) + 2$.

Proof of Theorem 12. Let $\mathcal{H}$ be a complete s-uniform hypergraph on n vertices. The lower bound for $\mathrm{ar}(n,s,k)$ is provided by Construction 5. To prove the upper bound, we proceed by induction on k. Theorem 9 deals with the base case when $k = 3$ and $n \geq 3s$.

For the inductive case, color the edges of $\mathcal{H}$ with exactly $c = \binom{n}{s} - \binom{n-k+2}{s} + 2 = \sum_{i=1}^{k-2}\binom{n-i}{s-1} + 2$ colors. We show that $\mathcal{H}$ has a rainbow k-matching. Let $\mathcal{G}$ be a subgraph of $\mathcal{H}$ with c edges such that each color appears on exactly one edge of $\mathcal{G}$. Let v be a vertex such that $\deg_{\mathcal{G}}(v) = \Delta(\mathcal{G})$.

Note that there are at least $c - \binom{n-1}{s-1}$ colors on the edges of the complete subhypergraph $\mathcal{H}\setminus\{v\}$ and the inductive hypothesis implies that $c - \binom{n-1}{s-1} = \mathrm{ar}(n-1,s,k-1)$ and there is a rainbow $(k-1)$-matching in $\mathcal{H}\setminus\{v\}$. Call this matching $\mathcal{M}$ and modify $\mathcal{G}$ to obtain a new hypergraph $\mathcal{G}'$ such that the edge set of $\mathcal{G}'$ consists of the edges of $\mathcal{M}$ and all edges of $\mathcal{G}$ except the ones that have a color from $\mathcal{M}$. By this definition, $\mathcal{G}$ and $\mathcal{G}'$ have the same number of colors and each color on $\mathcal{H}$ appears exactly once on $\mathcal{G}'$. The only difference is that $\deg_{\mathcal{G}'}(v) \geq \Delta(\mathcal{G}') - (k-1)$ and v may not be a vertex with maximum degree in $\mathcal{G}'$, but its degree is still high enough.

We analyze the two cases depending on the maximum degree in $\mathcal{G}'$. If $\Delta(\mathcal{G}') < c/((k-1)s)$ then the number of edges containing a vertex in $\mathcal{M}$ is less than c and there is an edge of $\mathcal{G}'$ that is vertex-disjoint from $\mathcal{M}$ and we are done. Otherwise, $\Delta(\mathcal{G}') \geq c/((k-1)s)$. The number of edges of $\mathcal{G}'$ containing both v and a vertex of $\mathcal{M}$ is at most $(k-1)s\binom{n-2}{s-2}$. For $n \geq 2s^3k$, we have

$$\deg_{\mathcal{G}'}(v) \geq \Delta(\mathcal{G}') - (k-1) \geq \frac{c}{(k-1)s} - (k-1) = \frac{\binom{n}{s} - \binom{n-k+2}{s} + 2}{(k-1)s} - (k-1) > (k-1)s\binom{n-2}{s-2}, \tag{7}$$

where the last inequality will be proved as Claim 13. Therefore, there is an edge of $\mathcal{G}'$ that contains v and does not intersect any edge of $\mathcal{M}$, which implies that there is a rainbow k-matching.

Claim 13. *For $n \geq 2s^3k$,*

$$\binom{n}{s} - \binom{n-k+2}{s} + 2 > (k-1)^2 s\left(s + \binom{n-2}{s-2}^{-1}\right)\binom{n-2}{s-2}.$$

Below, we first present the observations that will be used later.

Note that for $r \le m \le n$,

$$\binom{m}{r} \ge \left(\frac{m-r+1}{n-r+1}\right)^r \binom{n}{r} = \left(1 - \frac{n-m}{n-r+1}\right)^r \binom{n}{r}$$

By using the fact that $(1-x)^a \ge 1 - ax$ for $0 \le x < 1$, the relation above gives that

$$\binom{m}{r} \ge \left(1 - \frac{r(n-m)}{n-r+1}\right) \binom{n}{r} \tag{8}$$

Observe that

$$\binom{n}{s} - \binom{n-k+2}{s} + 2 = \sum_{i=1}^{k-2} \binom{n-i}{s-1} + 2 > (k-2)\frac{n-k+2}{s-1}\binom{n-k+1}{s-2}.$$

By (8) and the inequality above, we obtain

$$\binom{n}{s} - \binom{n-k+2}{s} + 2 > (k-2)\frac{n-k+2}{s-1}\left(1 - \frac{(s-2)(k-3)}{n-s+1}\right)\binom{n-2}{s-2} \tag{9}$$

Assume that our claim does not hold. Then, (9) implies that

$$(k-1)^2 s\left(s + \binom{n-2}{s-2}^{-1}\right) > (k-2)\frac{n-k+2}{s-1}\left(1 - \frac{(s-2)(k-3)}{n-s+1}\right).$$

One can check that this is a contradiction for $n \ge 2s^3k$ and we are done. □

5 Acknowledgements

We are very thankful to the referees for reading the paper carefully, suggesting helpful clarifications and improving the presentation of the paper. We also thank Ryan Martin for helpful suggestions.

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