On a covering problem in the hypercube

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Abstract

In this paper, we address a particular variation of the Turán problem for the hypercube. Alon, Krech and Szabó (2007) asked "In an n-dimensional hypercube, Q_n, and for $\ell < d < n$, what is the size of a smallest set, S, of Q_ℓ's so that every Q_d contains at least one member of S?" Likewise, they asked a similar Ramsey type question: "What is the largest number of colors that we can use to color the copies of Q_ℓ in Q_n such that each Q_d contains a Q_ℓ of each color?" We give upper and lower bounds for each of these questions and provide constructions of the set S above for some specific cases.

1 Introduction

For graphs Q and P, let $\mathrm{ex}(Q, P)$ denote the *generalized Turán number*, i.e., the maximum number of edges in a P-free subgraph of Q. The *n-dimensional hypercube*, Q_n, is the graph whose vertex set is $\{0, 1\}^n$ and whose edge set is the set of pairs that differ in exactly one coordinate. For a graph G, we use $n(G)$ and $e(G)$ to denote the number of vertices and the number of edges of G, respectively.

In 1984, Erdős [9] conjectured that

$$\lim_{n\to\infty} \frac{\mathrm{ex}(Q_n, C_4)}{e(Q_n)} = \frac{1}{2}.$$

Note that this limit exists, because the function above is non-increasing for n and bounded. The best upper bound $\mathrm{ex}(Q_n, C_4)/e(Q_n) \le 0.6068$ was recently obtained by Balogh, Hu, Lidický and Liu [2] by improving the bound 0.62256 given by Thomason and Wagner [20] and 0.62284 given by Chung [5]. Brass, Harborth and Nienborg [4] showed that the lower bound is $\frac{1}{2}(1 + 1/\sqrt{n})$, when $n = 4^r$ for integer r, and $\frac{1}{2}(1 + 0.9/\sqrt{n})$, when $n \ge 9$.

Erdős [9] also asked whether $o(e(Q_n))$ edges in a subgraph of Q_n would be sufficient for the existence of a cycle C_{2k} for $k > 2$. The value of $\mathrm{ex}(Q_n, C_6)/e(Q_n)$ is between 1/3 and 0.3755 (improving the bound 0.3941 by Lu [16]) given by Conder [7] and Balogh, Hu, Lidický and Liu [2], respectively. On the other hand, nothing is known for the cycle of length 10. Except C_{10}, the question of Erdős is answered positively by showing that $\mathrm{ex}(Q_n, C_{2k}) = o(e(Q_n))$ for $k \geq 4$ in [5], [8] and [11].

A generalization of Erdős' conjecture above is the problem of determining $\mathrm{ex}(Q_n, Q_d)$ for $d \geq 3$. As for $d = 2$, the exact value of $\mathrm{ex}(Q_n, Q_3)$ is still not known. The best upper bound for $\mathrm{ex}(Q_n, Q_3)/e(Q_n)$ has been $(5/8)^{0.25} \approx 0.88914$ due to Graham, Harary, Livingston and Stout [12] until recently Offner [18] improved it to 0.8835. The best lower bound is $\mathrm{ex}(Q_n, Q_3)/e(Q_n) \geq 0.75$ due to Alon, Krech and Szabó [1]. They also gave the best bounds for $\mathrm{ex}(Q_n, Q_d)$, $d \geq 4$, as

$$\Omega\left(\frac{\log d}{d2^d}\right) = 1 - \frac{\mathrm{ex}(Q_n, Q_d)}{e(Q_n)} \leq \begin{array}{ll} \frac{4}{(d+1)^2} & \text{if } d \text{ is odd,} \\ \frac{4}{d(d+2)} & \text{if } d \text{ is even.} \end{array} \tag{1}$$

These Turán problems are also asked when vertices are removed instead of edges and most of these problems are also still open (see [13], [14] and [15]). In a very recent paper, Bollobás, Leader and Malvenuto [3] discuss open problems on the vertex-version and their relation to Turán problems on hypergraphs.

Here, we present results on a similar dual version of the hypercube Turán problem that is asked by Alon, Krech and Szabó in [1]. Let $\mathcal{H}_n^i$ denote the collection of Q_i's in Q_n for $0 \leq i \leq n-1$. Call a subset of $\mathcal{H}_n^\ell$ a (d,ℓ)-*covering set* if each member of $\mathcal{H}_n^d$ contains some member of this set, i.e., $\mathcal{H}_n^d$ is *covered* by this set. A smallest (d,ℓ)-covering set is called *optimal*. Alon, Krech and Szabó [1] asked what the size of the optimal (d,ℓ)-covering set of Q_n is for fixed $\ell < d$. Call this function $f^{(\ell)}(n,d)$. Determining this function when $\ell = 1$ is equivalent to the determination of $\mathrm{ex}(Q_n, Q_d)$, since $\mathrm{ex}(Q_n, Q_d) + f^{(1)}(n,d) = e(Q_n)$ and the best bounds for $f^{(1)}(n,d)$ are given in [1] as (1). In [1], also the Ramsey version of this problem is asked as follows. A coloring of $\mathcal{H}_n^\ell$ is d,ℓ-*polychromatic* if all colors appear on each copy of Q_d's. Let $pc^{(\ell)}(n,d)$ be the largest number of colors for which there exists a d,ℓ-polychromatic coloring of $\mathcal{H}_n^\ell$.

Let $c^{(\ell)}(n,d)$ be the ratio of $f^{(\ell)}(n,d)$ to the size of $\mathcal{H}_n^\ell$, i.e.,

$$c^{(\ell)}(n,d) = \frac{f^{(\ell)}(n,d)}{2^{n-\ell}\binom{n}{\ell}}. \tag{2}$$

One can observe that

$$c^{(\ell)}(n,d) \leq \frac{1}{pc^{(\ell)}(n,d)}, \tag{3}$$

since any color class used in a d,ℓ-polychromatic coloring is a (d,ℓ)-covering set of Q_n. Note that the following limits exist, since $c^{(\ell)}(n,d)$ is non-decreasing, $pc^{(\ell)}(n,d)$ is non-increasing and both are bounded.

$$c_d^{(\ell)} = \lim_{n\to\infty} c^{(\ell)}(n,d), \quad p_d^{(\ell)} = \lim_{n\to\infty} pc^{(\ell)}(n,d).$$

In Section 2, we obtain bounds on $p_d^{(\ell)}$.

Theorem 1. *For integers $n > d > \ell$, let $0 < r \le \ell+1$ such that $r = d+1 \pmod{\ell+1}$. Then*

$$e^{\ell+1}\left(\frac{d+1}{\ell+1}\right)^{\ell+1} \ge \binom{d+1}{\ell+1} \ge p_d^{(\ell)} \ge \left\lceil\frac{d+1}{\ell+1}\right\rceil^r \left\lfloor\frac{d+1}{\ell+1}\right\rfloor^{\ell+1-r} \approx \left(\frac{d+1}{\ell+1}\right)^{\ell+1}.$$

In Section 3, we present the following bounds on $c_d^{(\ell)}$ and $c^{(\ell)}(n,d)$.

Theorem 2. *For integers $n > d > \ell$ and $r = d-\ell \pmod{\ell+1}$,*

$$\left(2^{d-\ell}\binom{d}{\ell}\right)^{-1} \le c_d^{(\ell)} \le \left\lceil\frac{d+1}{\ell+1}\right\rceil^{-r} \left\lfloor\frac{d+1}{\ell+1}\right\rfloor^{-(\ell+1-r)}.$$

The determination of the exact values of p_d^ℓ and c_d^ℓ remains open. The lower and upper bounds on $c^{(\ell)}(n,d)$ provided in Theorem 2 and Theorem 3, respectively, are a constant factor of each other when d and ℓ have a bounded difference from n.

Theorem 3. *Let $n-d$ and $n-\ell$ be fixed finite integers, where $d > \ell$. Then, for sufficiently large n,*

$$c^{(\ell)}(n,d) \le \left\lceil \frac{r\log(n-\ell)}{\log\left(\frac{r^r}{r^r - r!}\right)} \right\rceil \frac{1+o(1)}{2^{d-\ell}\binom{d}{l}},$$

where $r = n-d$.

Finally, we show an exact result for $c^{(\ell)}(n,d)$ when $d = n-1$.

Theorem 4. *For integers $n-1 > \ell$,*

$$c^{(\ell)}(n,n-1) = \frac{\left\lceil\frac{2n}{n-\ell}\right\rceil}{2^{n-\ell}\binom{n}{\ell}}.$$

In our proofs, we make use of the following terminology. The collection of i-subsets of $[n] = \{1,\dots,n\}$, $1 \le i \le n$, is denoted by $\binom{[n]}{i}$. For an edge $e \in E(Q_n)$, $\mathrm{star}(e)$ denotes the coordinate that is different at endpoints of e. The set of coordinates whose values are 0 (or 1, resp.) at both endpoints of e are denoted by $\mathrm{zero}(e)$ (or $\mathrm{one}(e)$, resp.). For a subcube $F \subset Q_n$, $\mathrm{star}(F) := \cup_{e\subseteq E(F)} \mathrm{star}(e)$, $\mathrm{one}(F) := \cap_{e\subseteq E(F)} \mathrm{one}(e)$ and $\mathrm{zero}(F) := \cap_{e\subseteq E(F)} \mathrm{zero}(e)$. Note that E_1 *covers* E_2 for $E_1 \in \mathcal{H}_n^\ell$ and $E_2 \in \mathcal{H}_n^d$ $(d > \ell)$ if and only if $\mathrm{zero}(E_2) \subset \mathrm{zero}(E_1)$ and $\mathrm{one}(E_2) \subset \mathrm{one}(E_1)$.

Definition 5. *For any* $Q \in \mathcal{H}_n^\ell$ *and* $\mathrm{star}(Q)$ *with coordinates* $s_1 < s_2 < \ldots < s_\ell$*, we define an* $(\ell+1)$*-tuple* $w(Q) = (w_1, w_2, \ldots, w_{\ell+1})$ *as*
- $w_1 = |\{x \in \mathrm{one}(Q) : x < s_1\}|$,
- $w_j = |\{x \in \mathrm{one}(Q) : s_{j-1} < x < s_j\}|$*, for* $2 \le j \le \ell$,
- $w_{\ell+1} = |\{x \in \mathrm{one}(Q) : x > s_\ell\}|$.

2 Polychromatic Coloring of Subcubes

Proof of Theorem 1. *The lower bound:*
For any $Q \in \mathcal{H}_n^\ell$ with $w(Q) = (w_1, w_2, \ldots, w_{\ell+1})$, we define the color of each $Q \in \mathcal{H}_n^\ell$ as the $(\ell+1)$-tuple $c(Q) = (c_1, \ldots, c_{\ell+1})$ such that

$$\begin{aligned} c_i &= w_i \pmod{k} \quad \text{if } 1 \le i \le r \text{ and} \\ c_i &= w_i \pmod{k'} \quad \text{if } r+1 \le i \le \ell+1, \end{aligned} \tag{4}$$

where $k = \lceil (d+1)/(\ell+1) \rceil$ and $k' = \lfloor (d+1)/(\ell+1) \rfloor$. We show that this coloring is d,ℓ-polychromatic.

Let $C \in \mathcal{H}_n^d$, where $\mathrm{star}(C)$ consists of the coordinates $a_1 < a_2 < \cdots < a_d$. We choose a color $(c_1, \ldots, c_{\ell+1})$ arbitrarily and show that C contains a copy of Q_ℓ, call it Q, with this color.

Since Q must be a subgraph of C, $\mathrm{zero}(C) \subset \mathrm{zero}(Q)$ and $\mathrm{one}(C) \subset \mathrm{one}(Q)$. We define $\mathrm{star}(Q) = \{s_1, \ldots, s_\ell\}$ such that

$$s_i = \begin{cases} a_{ik} & \text{if } 1 \le i \le r, \\ a_{rk+(i-r)k'} & \text{if } r+1 \le i \le \ell. \end{cases}$$

We include the remaining $d - \ell$ positions of $\mathrm{star}(C)$ to $\mathrm{one}(Q)$ or $\mathrm{zero}(Q)$ such that $w(Q) = (w_1, w_2, \ldots, w_{\ell+1})$ satisfies (4). This is possible since by the definition of r, we have $d - \ell = r(k-1) + (\ell + 1 - r)(k' - 1)$.

The upper bound:
Since $pc^{(\ell)}(n, d)$ is a non-increasing function of n, we provide an upper bound for this function when n is sufficiently large which is also an upper bound for $p_d^{(\ell)}$.

For a subset S of $[n]$, we define $\mathrm{cube}(S)$ as the subcube Q of Q_n such that $\mathrm{star}(Q) = S$ and $\mathrm{zero}(Q) = [n] \setminus S$. Let $\mathcal{G}$ be a subfamily of $\mathcal{H}_n^d$ such that $\mathcal{G} = \{\mathrm{cube}(S) : S \in \binom{[n]}{d}\}$. We define a coloring of the members of $\mathcal{G}$ as follows.

Consider a d,ℓ-polychromatic coloring of $\mathcal{H}_n^\ell$ using p colors, call this coloring P. Fix an arbitrary ordering of the copies of Q_ℓ's in Q_d. We define a coloring of $\mathcal{H}_n^d$ such that the color of a copy of Q_d is the list of colors of each Q_ℓ under P in this fixed order. By using this coloring on the members of $\mathcal{G}$, we obtain a coloring of $\mathcal{G}$ using $p^{\binom{d}{\ell} 2^{d-\ell}}$ colors.

Now, consider the auxiliary d-uniform hypergraph $\mathcal{G}'$ whose vertex set is the set of coordinates $[n]$ and whose edge set is defined as the collection of star(E)'s for each E in $\mathcal{G}$, i.e., $\mathcal{G}'$ is a complete d-uniform hypergraph on the vertex set $[n]$. Also we define a coloring of the edges of $\mathcal{G}'$ by using the colors on the corresponding members of $\mathcal{G}$ as described above. Ramsey's theorem on hypergraphs implies that there is a sufficiently large value of n such that there exists a complete monochromatic subgraph on d^2+d-1 vertices in any edge coloring of $\mathcal{G}'$ with $p^{\binom{d}{\ell}2^{d-\ell}}$ colors. Let $K \subset [n]$ be the vertex set of a monochromatic complete subgraph of $\mathcal{G}'$ on d^2+d-1 vertices. We define S as the collection of id^{th} coordinates in K, $1 \leq i \leq d$, so that there are at least $d-1$ coordinates between elements of S.

Claim 6. *If Q is a copy of Q_ℓ in* cube(S)*, then the color of Q under P depends only on $w(Q)$.*

Proof. Let E_1 and E_2 be two different copies of Q_ℓ in cube(S) such that $w(E_1) = w(E_2)$ according to Definition 5. There exists a subset $S' \subset K$ with $|S'| = d$ such that
- (one$(E_2) \cup$ star$(E_2)) \subset S'$, i.e., E_2 is contained in cube(S') and
- the restriction of E_2 on S' gives the same vector as the restriction of E_1 on S.
Clearly, one can find S' that satisfies the first condition. It is also possible that S' fulfills the second condition, since we can remove or add up to $d-1$ coordinates from K between consecutive coordinates of ones and stars in E_2 to define S'. This implies that the colors of E_1 and E_2 are the same under P, since cube(S) and cube(S') have the same colors. □

Hence, the number of colors used in any d, ℓ-polychromatic coloring of $\mathcal{H}_n^\ell$ is at most the number of possible vectors $w(Q)$ for any $Q \in \mathcal{H}_n^\ell$. The number of possible $(\ell+1)$-tuples $w(Q)$ for any $Q \in \mathcal{G}$ is given by the number of partitions of at most $d-\ell$ ones into $\ell+1$ parts and therefore it is at most $\binom{d+1}{\ell+1}$. □

3 The Covering Problem

Proof of Theorem 2. Note that a trivial lower bound on $f^{(\ell)}(n,d)$ is given by the ratio of $|\mathcal{H}_n^d|$ to the exact number of Q_d's that a single Q_ℓ covers in Q_n. Thus, by (2), for all n,

$$c^{(\ell)}(n,d) \geq \left\lceil \frac{2^{n-d}\binom{n}{d}}{\binom{n-\ell}{n-d}} \right\rceil \cdot \frac{1}{2^{n-\ell}\binom{n}{\ell}}. \tag{5}$$

By using the equality $\binom{n}{d}\binom{d}{d-\ell} = \binom{n}{\ell}\binom{n-\ell}{d-\ell}$, we are done.

The upper bound is implied together by (3) and Theorem 1. □

We define a $(0,1)$-*labelling* of a set as an assignment of labels 0 or 1 to its elements.

Observation 7. *Since any subcube $Q \subset Q_n$ is defined by* zero(Q) *and* one(Q)*, a (d,ℓ)-covering set of Q_n can be defined as a collection of $(0,1)$-labellings of sets chosen from $\binom{[n]}{n-\ell}$ such that any $(0,1)$-labelling of sets in $\binom{[n]}{n-d}$ is contained in at least one of the labelled $(n-\ell)$-sets.*

When providing constructions for the upper bounds in Theorems 3 and 4, we provide constructions for the equivalent covering problem in Observation 7.

Proof of Theorem 3.

We construct a (d,ℓ)-covering of Q_n by providing a construction for the equivalent problem as stated in Observation 7. In the following, we describe this construction in two steps. First, we choose the $(n-\ell)$-subsets of $[n]$ to label and then, we describe an efficient way to $(0,1)$-label these sets.

Step 1: We make use of the following well-known result on the general covering problem. An (n,k,t)-*covering* is defined as a collection of k-subsets of n elements such that every t-set is contained in at least one k-set. Let $C(n,k,t)$ be the minimum number of k-sets in an (n,k,t)-covering. Rödl proved the following result by also settling a long-standing conjecture of Erdős and Hanani [10]. For any fixed integers k and t with $2 \le t < k < n$,

$$\lim_{n\to\infty} \frac{C(n,k,t)}{\binom{n}{t}/\binom{k}{t}} = 1. \tag{6}$$

By our assumption, $n-d$ and $n-\ell$ are fixed integers where $n-d < n-\ell$. By (6), there exists a $(n, n-\ell, n-d)$-covering $\mathcal{F}$ for sufficiently large n such that $|\mathcal{F}| = (1+o(1))\binom{n}{n-d}/\binom{n-\ell}{n-d}$.

Step 2: We obtain a collection of $(0,1)$-labellings for each edge $e \in \mathcal{F}$ so that all $(0,1)$-labellings of $(n-d)$-subsets of e are covered. The union of these $(0,1)$-labellings is a covering set.

An r-*cut* of an r-uniform hypergraph is obtained by partitioning its vertex set into r parts and taking all edges that meet every part in exactly one vertex. An r-*cut cover* of a hypergraph is a collection of r-cuts such that each edge is in at least one of the cuts. An upper bound on the minimum size of an r-cut cover is shown by Cioabă, Kündgen, Timmons and Vysotsky in [6] using a probabilistic proof.

Theorem 8 ([6]). *For every r, an r-uniform complete hypergraph on n vertices can be covered with $\lceil c \log n \rceil$ r-cuts if*

$$c > \frac{-r}{\log\left(\frac{r^r - r!}{r^r}\right)}.$$

For a fixed edge e of $\mathcal{F}$, let $\mathcal{G}_e$ be the complete $(n-d)$-uniform hypergraph on the vertex set of e. Let $C = \lceil c \log(n-\ell) \rceil$ be the size of a minimum $(n-d)$-cut cover of $\mathcal{G}_e$ as given by Theorem 8. We obtain a collection of $(0,1)$-labellings of e by labelling each cut in this cover such that the vertices in each part are labelled identically with 0 or 1. Thus, the total number of $(0,1)$-labellings of e is $2^{n-d}C$. (If some labelling of an edge is used more than once, then we count this labelling only once.) Finally, we use similarly labellings for each edge of $\mathcal{F}$ in the covering set. This yields that

$$c^{(\ell)}(n,d) \le \frac{1}{2^{n-\ell}\binom{n}{\ell}} (C(1+o(1))2^{n-d} \frac{\binom{n}{n-d}}{\binom{n-\ell}{n-d}} = C(1+o(1)) \frac{1}{2^{d-\ell}\binom{d}{l}},$$

where the last equality is obtained by using the relation $\binom{n}{d}\binom{d}{d-\ell} = \binom{n}{\ell}\binom{n-\ell}{d-\ell}$.

$\square$

Proof of Theorem 4.

The lower bound follows from (5).

For the upper bound, we construct a collection of $(0,1)$-labellings of sets chosen from $\binom{[n]}{n-\ell}$, where singletons in $[n]$ have both 0 and 1 in some labelling. Let $k = \lceil n/(n-\ell) \rceil$. We choose a partition $[n] = (P_1, \ldots, P_k)$ such that $|P_i| = n - \ell$ for $i < k$. Let $P \in \binom{[n]}{n-\ell}$ such that $P_k \subset P$. In the covering set, we include two labellings of each of $P_1, \ldots, P_{k-1}, P$, where all labels are the same, either 0 or 1.

$\square$

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