A Brooks-type bound for squares of K_4-minor-free graphs

Alexandr V. Kostochka[a,b,*], Lale Özkahya[a,†], Douglas R. Woodall[c,‡]

[a] Department of Mathematics, University of Illinois, Urbana, IL, 61801, USA

[b] Sobolev Institute of Mathematics, Novosibirsk, Russia

[c] School of Mathematical Sciences, University of Nottingham, Nottingham, NG7 2RD, UK

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Abstract

Refining a bound by Lih, Wang and Zhu, we prove that if the square G^2 of a K_4-minor-free graph G with maximum degree $\Delta \geqslant 6$ does not contain a complete subgraph on $\lfloor \frac{3}{2}\Delta \rfloor + 1$ vertices, then G^2 is $\lfloor \frac{3}{2}\Delta \rfloor$-colorable.

Keywords: Chromatic number; Vertex coloring; Brooks-type bound; Square of a graph; Series-parallel graph

1 Introduction

Problems involving coloring squares of graphs have recently attracted some attention. If G is a graph with maximum degree $\Delta(G) = \Delta$, then the chromatic number $\chi(G^2)$, and even the clique number of G^2, may be of the order of Δ^2. However, this should not happen with graphs of small genus. In particular, Wegner [6] made the following conjecture.

Conjecture 1. *Let G be a planar graph. Then*

$$\chi(G^2) \leqslant \begin{cases} \Delta(G) + 5 & \text{if } 4 \leqslant \Delta(G) \leqslant 7, \\ \lfloor \frac{3}{2}\Delta(G) \rfloor + 1 & \text{if } \Delta(G) \geqslant 8. \end{cases}$$

Recently Havet, van den Heuvel, McDiarmid and Reed [2] proved an approximate upper bound of $\frac{3}{2}\Delta + o(\Delta)$, but the exact result has not been proved.

The bound of Wegner's conjecture, if true, is sharp. Moreover, for every $\Delta \geqslant 4$, there are series-parallel (hence, K_4-minor-free) graphs G with maximum degree Δ such that

*E-mail address: kostochk@math.uiuc.edu. The research of this author is supported in part by NSF grant DMS-06-50784 and by grant 06-01-00694 of the Russian Foundation for Basic Research.

†E-mail address: ozkahya@illinois.edu.

‡E-mail address: douglas.woodall@nottingham.ac.uk.

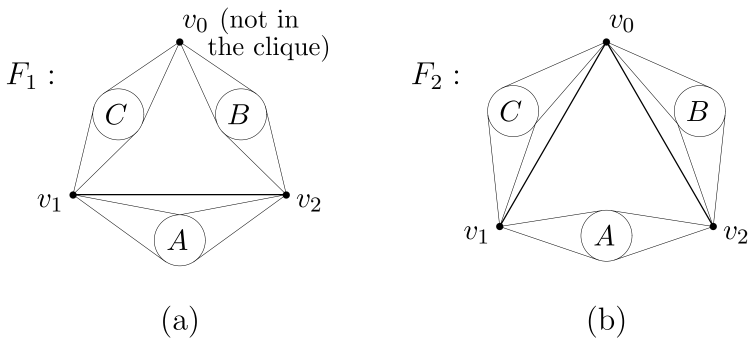

Figure 1. The two possible forms for $G[Q]$.

the chromatic number and clique number of G^2 are both equal to $\lfloor \frac{3}{2}\Delta \rfloor + 1$: see Figure 1, where A, B and C are independent sets of suitable orders, as explained in Section 3. Lih, Wang, and Zhu [5] proved the following theorem, which implies that Wegner's conjecture holds for K_4-minor-free graphs.

Theorem 2. [5] *Let G be a K_4-minor-free graph. Then*

$$\chi(G^2) \leqslant \begin{cases} \Delta(G) + 3 & \text{if } 2 \leqslant \Delta(G) \leqslant 3, \\ \lfloor \frac{3}{2}\Delta(G) \rfloor + 1 & \text{if } \Delta(G) \geqslant 4. \end{cases}$$

Hetherington and Woodall [3] proved that the upper bound in Theorem 2 holds not only for $\chi(G^2)$ but also for the list chromatic number $\mathrm{ch}(G^2)$. They remarked that they "strongly suspect" that the bound $t = \lfloor \frac{3}{2}\Delta \rfloor + 1$ is attained for $\Delta \geqslant 4$ only when G^2 contains a clique of order t. In this paper we show that this suspicion is incorrect for $\Delta \in \{4, 5\}$ but correct for every $\Delta \geqslant 6$, at least for the (ordinary) chromatic number. (We do not see how to prove the analogous result for the list chromatic number. Any counterexample for list colorings would disprove also the conjecture of the first and third authors [4] that $\mathrm{ch}(G^2) = \chi(G^2)$ for every graph G.)

The main result of this paper is the following.

Theorem 3. *Let G be a K_4-minor-free graph with maximum degree at most $\Delta \geqslant 6$. If G^2 does not contain a clique of order $\lfloor \frac{3}{2}\Delta \rfloor + 1$, then $\chi(G^2) \leqslant \lfloor \frac{3}{2}\Delta \rfloor$.*

Our proof uses the approach of Hetherington and Woodall [3]. In the next section we introduce some notation and present examples for $\Delta \in \{4, 5\}$. In Section 3 we discuss the structure of the cliques of order $\lfloor \frac{3}{2}\Delta \rfloor + 1$ in the square of a K_4-minor-free graph G with maximum degree Δ, and in particular we show that if Q is the vertex-set of such a clique in G^2, then Q induces a subgraph of G with one of the forms shown in Figure 1. The proof of Theorem 3 is then given in Sections 4 and 5.

The structure of the proof is as follows. We define G to be a smallest counterexample to Theorem 3 (for a fixed value of Δ). In Section 4 we prove various results about G, culminating in the fact that G must contain an induced subgraph of the form shown in Figure 7. In Section 5 we use this induced subgraph, and the minimality of G, to show

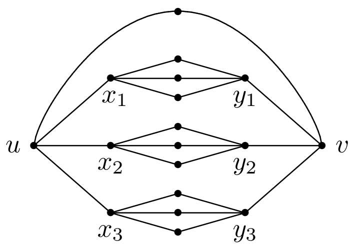

Figure 2. A K_4-minor-free graph G with $\Delta(G) = 4$ such that $\chi(G^2) = 7$ and G^2 does not contain K_7.

that G is $\lfloor \frac{3}{2}\Delta \rfloor$-colorable; this contradicts the choice of G and so proves Theorem 3. In proving the results in Sections 4 and 5, we consider a number of graphs with fewer vertices than G, which are constructed from G in various different ways. We wish to prove that each such graph $\tilde{G}$ is $\lfloor \frac{3}{2}\Delta \rfloor$-colorable, using the fact that G is a minimal counterexample to Theorem 3. To do this, we must verify that $\Delta(\tilde{G}) \leqslant \Delta$, $\tilde{G}$ is K_4-minor-free, and $\tilde{G}^2$ contains no clique of order $\lfloor \frac{3}{2}\Delta \rfloor + 1$. In most cases, verifying the first two of these hypotheses is easy, but the third is much less straightforward. It is here that we repeatedly use the main result of Section 3, which tells us that if Q is the vertex-set of such a clique in $\tilde{G}^2$, then Q induces a subgraph of $\tilde{G}$ of a particular form.

2 Some preliminaries

If G is a graph with vertex-set $V(G)$ and edge-set $E(G)$, and $v \in V(G)$, then the set of neighbors of v in G is denoted by $N_G(v)$ or just $N(v)$, and the degree of v is $d_G(v) = |N_G(v)|$. If $u, v \in V(G)$ then $d_G(u, v)$ denotes the distance between u and v in G, i.e., the length of a shortest u, v-path. If $X \subseteq V(G)$, then $G[X]$ denotes the subgraph of G induced by X. We denote by G^2 the *square* of G: G^2 has the same vertex-set as G, and two vertices are adjacent in G^2 if they are within distance two of each other in G.

Let G be the graph in Figure 2. By inspection, G is a K_4-minor-free graph and G^2 does not contain K_7 as a subgraph. For $i = 1, 2, 3$, let $C_i := \{x_i, y_i\} \cup (N_G(x_i) \cap N_G(y_i))$. Let f be a proper coloring of G^2, and let $\alpha := f(u)$ and $\beta := f(v)$. Since $uv \in E(G^2)$, $\alpha \neq \beta$. Because x_1, x_2 and x_3 all have different colors, at most one of them is colored with β. Similarly, at most one of y_1, y_2 and y_3 is colored with α. Thus, for some $i \in \{1, 2, 3\}$, neither α nor β is used to color any vertex of C_i. But all five vertices of C_i have different colors in f; thus f uses at least seven colors, i.e., $\chi(G^2) \geqslant 7$.

The example for $\Delta = 5$ is very similar, only instead of three copies of $K_{2,3}$ we take three copies of $K_{2,4}$. So the example would need eight colors.

Thus for $\Delta \in \{4, 5\}$ there is a K_4-minor-free graph G with maximum degree Δ such that $\chi(G^2) = \lfloor \frac{3}{2}\Delta \rfloor + 1$ but G^2 contains no clique of order $\chi(G^2)$, contrary to the "strong suspicion" of Hetherington and Woodall [3]. Theorem 3 shows that this cannot happen if $\Delta \geqslant 6$.

Our proof of Theorem 3 depends heavily on the following classic result of Dirac, which is used explicitly in Lemmas 7 and 14.

Lemma 4. [1] *Every K_4-minor-free graph has a vertex with degree at most 2.*

3 Structure of large cliques

Let F denote the configuration F_1 or F_2 in Figure 1, where A, B and C are sets of vertices which initially we do not assume to be independent, and v_0 is adjacent to all vertices in $B \cup C$, v_1 to all vertices in $C \cup A$, and v_2 to all vertices in $A \cup B$. Let $a := |A|$, $b := |B|$ and $c := |C|$. For $F_1^2 - v_0$ or F_2^2 to be a clique of order $\lfloor \frac{3}{2}\Delta \rfloor + 1$, with $\Delta(F) \leqslant \Delta$, we require

$$a + b \leqslant \Delta - 1, \qquad a + c \leqslant \Delta - 1;$$

also, in Figure 1(a),

$$b + c \leqslant \Delta,$$
$$a + b + c = \lfloor \tfrac{3}{2}\Delta \rfloor - 1;$$

and, in Figure 1(b),

$$b + c \leqslant \Delta - 2,$$
$$a + b + c = \lfloor \tfrac{3}{2}\Delta \rfloor - 2.$$

If Δ is even, then there is a unique solution in each case. If Δ is odd, then there are three solutions in each case, depending on which one of the three inequalities is strict; but two of the three solutions are isomorphic (interchanging B and C). Note for future reference that $a, b, c \geqslant \frac{1}{2}(\Delta - 3)$ in each solution, so that each of the sets A, B, C has at least two elements if $\Delta \geqslant 6$. Note also that if A, B, C are independent sets then, in F,

$$\begin{array}{l} \text{if } \Delta \text{ is even then all of } v_0, v_1, v_2 \text{ have degree } \Delta; \\ \text{if } \Delta \text{ is odd then two of } v_0, v_1, v_2 \text{ have degree } \Delta \text{ and one has degree } \Delta - 1; \\ \text{every other vertex of } F \text{ has degree 2.} \end{array} \tag{1}$$

By an *F-path* we mean a path whose endvertices are in F but whose internal vertices (if any) are not in F.

Lemma 5. *Suppose that F ($\cong F_1$ or F_2) is a subgraph of a K_4-minor-free graph G, where each of A, B and C has at least two vertices. Then $A \cup B \cup C$ is an independent set in G, and there is no F-path in G that joins two vertices in $A \cup B \cup C$, or that joins one vertex u in this set to the vertex $v \in \{v_0, v_1, v_2\}$ that is not adjacent to u in F.*

Proof. It is easy to see that if there were an edge or an F-path of the type described, then G would have a K_4 minor. For example, if there is an edge uv or an F-path from u to v, where $u \in A$ and $v \in A \cup B \cup C \cup \{v_0\}$, then there is a K_4 minor with branch vertices u, v, v_1 and v_2. (Note that, since $|A| \geqslant 2$, there is a path from v_1 to v_2 through A that does not use u.) The remaining cases are similar. $\square$

If $Q \subseteq V(G)$ and Q induces a clique of order $t = \lfloor \frac{3}{2}\Delta \rfloor + 1$ in G^2, then we will say that Q, its t-clique, and $G[Q]$, are all of *standard form* if there is a vertex $v \in V(G)$ such that $G[Q \cup \{v\}] \cong F_1$, or if $G[Q] \cong F_2$. We will define

$$F(G,Q) := \begin{cases} G[Q \cup \{v\}] & \text{if } G[Q \cup \{v\}] \cong F_1, \\ G[Q] & \text{if } G[Q] \cong F_2. \end{cases} \tag{2}$$

Lemma 6. *Let G be a 2-connected K_4-minor-free graph with maximum degree at most $\Delta \geqslant 6$, and suppose that G^2 contains a standard-form clique of order $\lfloor \frac{3}{2}\Delta \rfloor + 1$ with vertex-set Q. Let $F := F(G,Q)$. Then either $G \cong F$, or Δ is odd and there is a connected subgraph H of G, and an edge uv of F, where $d_G(u) = \Delta$, $d_F(u) = \Delta - 1$ and $d_F(v) = 2$, such that $G = F \cup H$ and $F \cap H = \{u, uv, v\}$.*

Proof. It follows from Lemma 5 and (1) that F is an induced subgraph of G. Suppose that $G \not\cong F$, and let $C_1, \ldots, C_k$ be the components of $G - V(F)$. Since G is 2-connected, each component C_i has at least two neighbors in F, all of which have F-degree less than Δ (since a vertex with F-degree Δ can have no neighbors outside F). So it follows from Lemma 5 and (1) that if $u, v \in V(F) \cap N(C_i)$, then $uv \in E(F)$, Δ is odd, and one of u and v, say u, is the unique vertex of F-degree $\Delta - 1$, and the other, v, has F-degree 2. Since the one edge between u and C_i raises the degree of u to its maximum possible value Δ, there is therefore exactly one component C_1 of $G - V(F)$, and exactly two edges uu' and vv' between F and C_1, and if we define H to be the union of C_1 and the path $u'uvv'$ then $G = F \cup H$ and $F \cap H = \{u, uv, v\}$ as required. $\square$

The main result of this section is the following.

Lemma 7. *Let G be a K_4-minor-free graph with maximum degree at most $\Delta \geqslant 6$, and let $t := \lfloor \frac{3}{2}\Delta \rfloor + 1$. Then every clique of order t in G^2 is of standard form.*

Proof. Assume this is false, and consider a minor-minimal K_4-minor-free graph G with maximum degree at most Δ such that G^2 contains a t-clique K with $V(K) = Q$ that is not of standard form. By the minimality of G, G has no vertices with degree 0 or 1. Therefore, by Lemma 4, G has a vertex with degree 2. Let v be such a vertex, with neighbors u and w. We consider two cases.

Case 1: $v \notin Q$. If $u \notin Q$ or $w \notin Q$ or $uw \in E(G)$, then $(G - v)^2$ contains the t-clique K. By the minimality of G, $(G - v)[Q]$ is of standard form, which is a contradiction since $G[Q] = (G - v)[Q]$. Therefore $u, w \in Q$ and $uw \notin E(G)$.

Let $H := G - v + uw$. Since H is a minor of G (obtained by contracting the edge uv), H is K_4-minor-free. Since $v \notin Q$, $K \subseteq H^2$. By the minimality of G, $H[Q]$ is of standard form. This implies that uw is one of the edges in Figure 1, and that by subdividing uw we obtain G such that $G^2[Q]$ is the t-clique K. Notice that every edge in Figure 1 is incident with some vertex v_i ($i \in \{0, 1, 2\}$). By symmetry we may assume that $u \in \{v_0, v_1\}$. If $u = v_0$ and $w \in B$ (respectively, $w \in C$) then the distance in G between w and C (respectively, w and B) is greater than two, which contradicts the supposition that $G^2[Q]$ is a clique. If $u = v_1$ we get a similar contradiction with A instead of B. If $uw = v_1v_2$ in F_1, then the distance in G between v_1 and B is greater than two. Finally, if $uw = v_0v_1$

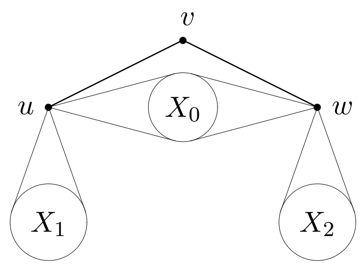

Figure 3. The vertex-sets in Q.

(respectively, v_0v_2) in F_2, then the distance between v_1 and B (respectively, v_2 and C) is greater than two. In each case we have a contradiction; thus Case 1 cannot arise.

Case 2: $v \in Q$. Partition the set of vertices in Q at distance exactly two from v as $X_0 \cup X_1 \cup X_2$, where

$$\begin{aligned}X_0 &:= (N(u) \cap N(w)) \cap Q \setminus \{v\},\\ X_1 &:= (N(u) \setminus N(w)) \cap Q \setminus \{w\},\\ X_2 &:= (N(w) \setminus N(u)) \cap Q \setminus \{u\},\end{aligned}$$

as shown in Figure 3. Let $x_i := |X_i|$ for $i = 0, 1, 2$.

Claim 7.1. *There is a vertex $z_0 \in V(G) \setminus \{u, v, w\}$ such that z_0 is adjacent to all vertices in $(X_1 \cup X_2) - z_0$.*

Proof. Since $X_1 \cup X_2 \subset Q$ by the definition of the sets X_i, and the distance between any two vertices of Q is at most two, every vertex of X_1 is connected to every vertex of X_2 by a path of length at most two. Let H be the subgraph of G induced by the vertices of all paths of length at most two between X_1 and X_2. Note that $u, v, w \notin V(H)$, since there are no edges between u and X_2 or between w and X_1.

Suppose there is no vertex z_0 as in the statement of the claim. Then there is no single vertex whose removal disconnects all paths of H between X_1 and X_2. Thus, by Pym's version of Menger's theorem, there are two vertex-disjoint paths P_1 and P_2 in H between X_1 and X_2. Let P_1 have endvertices $p \in X_1$ and $q \in X_2$, and P_2 have endvertices $r \in X_1$ and $s \in X_2$. Since p and s are in a clique in G^2, there is a path P_3 of length at most two with endvertices p and s. If P_3 is internally disjoint from P_1 and P_2, then G has a K_4 minor with branch vertices p, s, u and w. If P_3 has a central vertex t, and $t \in V(P_1)$, then G has a K_4 minor with branch vertices s, t, u and w. Similarly, if $t \in V(P_2)$, then G has a K_4 minor with branch vertices p, t, u and w. In every case we have a contradiction. □

The argument now splits into two subcases.

Subcase 2.1: $uw \in E(G)$. In this case $x_0 + x_1$, $x_0 + x_2 \leqslant \Delta - 2$ and, since $|Q| = t$, $x_0 + x_1 + x_2 \geqslant \lfloor \frac{3}{2}\Delta \rfloor - 2$. This implies that $x_1, x_2 \geqslant \lfloor \frac{1}{2}\Delta \rfloor \geqslant \frac{1}{2}(\Delta - 1)$.

By Claim 7.1, there is a vertex $z_0 \in V(G)$ such that z_0 is adjacent to every vertex in $(X_1 \cup X_2) - z_0$. Note that z_0 cannot be in X_0 because $|X_1 \cup X_2 \cup \{u, w\}| > \Delta$. If $z_0 \notin X_1 \cup X_2$, then $G[Q \cup \{z_0\}]$ has the form in Figure 1(a), with $A = X_0 \cup \{v\}$, $B = X_1$,

$C = X_2$, and $(v_0, v_1, v_2) = (z_0, w, u)$. If $z_0 \in X_1$ then $G[Q]$ has the form in Figure 1(b) with $A = X_2$, $B = X_0 \cup \{v\}$, $C = X_1 - z_0$, and $(v_0, v_1, v_2) = (u, z_0, w)$. If $z_0 \in X_2$ then the situation is similar, interchanging X_1 and X_2, and u and w. In each case we have a contradiction.

Subcase 2.2: $uw \notin E(G)$. In this case $x_0 + x_1$, $x_0 + x_2 \leqslant \Delta - 1$ and, since $|Q| = t$, $x_0 + x_1 + x_2 \geqslant \lfloor \frac{3}{2}\Delta \rfloor - 2$. This implies that $x_1, x_2 \geqslant \lfloor \frac{1}{2}\Delta \rfloor - 1$, so that $x_1, x_2 \geqslant 2$ since we are assuming that $\Delta \geqslant 6$.

Recall that the distance between any two vertices of Q is at most two. Consider the subgraph induced by the vertices of all paths of length at most two connecting the pairs (u, X_2), (w, X_1) and (X_1, X_2). If all these paths go through the vertex z_0, whose existence was proved in Claim 7.1, then $z_0 \notin X_1 \cup X_2 \cup \{u, w\}$, since u and w are not adjacent to X_2 and X_1 respectively; but z_0 is adjacent to all vertices in $X_1 \cup X_2 \cup \{u, w\}$, so that $z_0 \in Q$. Thus $z_0 \in X_0$, and $G[Q]$ has the form in Figure 1(b) with $A = (X_0 \cup \{v\}) - z_0$, $B = X_1$, $C = X_2$, and $(v_0, v_1, v_2) = (z_0, w, u)$.

This contradiction shows that not all of the paths mentioned go through z_0. By symmetry, interchanging X_1 and X_2 if necessary, we may assume that there is a vertex $q \in X_2$ such that there is a shortest path (of length at most two) from u to q that does not contain z_0, and clearly does not contain w. Then G has a K_4 minor with branch vertices u, w, q and z_0. (This uses the fact that $|X_1| \geqslant 2$ and $|X_2| \geqslant 2$.) This contradiction completes the proof of Lemma 7. $\square$

4 Structure of minimum counterexamples

Let $\Delta \geqslant 6$ and $t := \lfloor \frac{3}{2}\Delta \rfloor + 1$. If Theorem 3 fails for Δ, then there exists a K_4-minor-free graph G, minimum with respect to the total number of edges and vertices, such that $\Delta(G) \leqslant \Delta$, G^2 does not contain a K_t, and $\chi(G^2) \geqslant t$. We will call such a graph a *(Δ, t)-graph*. In this section, we derive a number of properties of (Δ, t)-graphs. We also introduce some terminology that will be used in the proof of Theorem 3 in the final section. Note that

$$t - 1 = \lfloor \tfrac{3}{2}\Delta \rfloor \geqslant \Delta + 3. \tag{3}$$

Lemma 8. *Let G be a (Δ, t)-graph, where $\Delta \geqslant 6$ and $t = \lfloor \frac{3}{2}\Delta \rfloor + 1$. Then G is 2-connected.*

Proof. Clearly G is connected. Suppose G has a cutvertex v, say $G = G' \cup G''$ where $G' \cap G'' = \{v\}$, $|V(G')| > 1$ and $|V(G'')| > 1$. By the minimality of G, there are proper colorings f' and f'' of G'^2 and G''^2 respectively, using colors in $\{1, 2, \ldots, \lfloor \frac{3}{2}\Delta \rfloor\}$. Permute colors in f'' if necessary so that v has color $f'(v)$ and no G''-neighbor of v has the same color as any G'-neighbor of v; this is possible since $|N_G(v) \cup \{v\}| \leqslant \Delta + 1 < \lfloor \frac{3}{2}\Delta \rfloor$. Now the union of the two colorings is a proper $\lfloor \frac{3}{2}\Delta \rfloor$-coloring of G^2, and this contradicts the definition of G. $\square$

For a graph G with $\Delta(G) \geqslant 3$, we follow [3] in denoting by G_1 the graph whose vertices are the vertices of degree at least 3 in G, where two vertices are adjacent in G_1 if and only if they are either adjacent in G or connected in G by a path whose internal vertices all have degree 2. By definition, G_1 is a minor of G.

Lemma 9. *Let G be a graph that does not contain a vertex with degree 0 or 1 or two adjacent vertices with degree 2. Then G_1 exists and has no isolated vertices, and if G is 2-connected then either G_1 is 2-connected or $G_1 \cong K_2$.*

Proof. It is easy to see that G_1 exists and has no isolated vertices. (This fact was stated and used in [3].) Note that G_1 can be obtained from G by contracting some edges, each of which has an endvertex of degree 2 at the time of its contraction, and deleting multiple edges. Neither of these operations can create a cutvertex, and so if G is 2-connected then G_1 is nonseparable, i.e., it is 2-connected or K_2. □

Lemma 10. *Let G be a (Δ, t)-graph, where $\Delta \geqslant 6$ and $t = \lfloor \frac{3}{2}\Delta \rfloor + 1$. Then*
(a) *G does not contain a vertex with degree 0 or 1 or two adjacent vertices with degree 2;*
(b) *G_1 exists and is 2-connected.*

Proof. Suppose first that G contains two adjacent vertices u and w of degree 2. Then $(G - \{u, w\})^2 = G^2 - \{u, w\}$. By the minimality of G, $(G - \{u, w\})^2$ is $\lfloor \frac{3}{2}\Delta \rfloor$-colorable. Since $d_{G^2}(u), d_{G^2}(w) \leqslant \Delta + 2 < \lfloor \frac{3}{2}\Delta \rfloor$, we can extend a $\lfloor \frac{3}{2}\Delta \rfloor$-coloring of $(G - \{u, w\})^2$ to G^2, by coloring u and w with available colors not used on $N_{G^2}(u)$ and $N_{G^2}(w)$, respectively. This contradicts the fact that $\chi(G^2) > \lfloor \frac{3}{2}\Delta \rfloor$. Thus G does not contain two adjacent vertices of degree 2. Also, by the minimality of G, it has no vertex with degree 0 or 1. This proves (a).

Since G is 2-connected by Lemma 8, it follows immediately from (a) and Lemma 9 that G_1 exists and is either 2-connected or K_2. But if $G_1 \cong K_2$, with vertices u, v, say, then every vertex of G other than u, v is adjacent to u and v, and so G^2 is a complete graph; thus G cannot be a (Δ, t)-graph, and this contradiction proves (b). □

For $u, v \in V(G)$, define

$$M_{uv} := \{x \in N_G(u) \cap N_G(v) : d_G(x) = 2\},$$

$$\epsilon_{uv} := \begin{cases} 1 & \text{if } uv \in E(G), \\ 0 & \text{otherwise,} \end{cases}$$

and

$$d_{uv} := |M_{uv}| + \epsilon_{uv}.$$

Lemma 11. *Let G be a (Δ, t)-graph, where $\Delta \geqslant 6$ and $t = \lfloor \frac{3}{2}\Delta \rfloor + 1$. If $v \in V(G)$ and $N_{G_1}(v) = \{u, w\}$, then $d_{uv} \geqslant \lfloor \frac{1}{2}\Delta \rfloor$ and $d_{vw} \geqslant \lfloor \frac{1}{2}\Delta \rfloor$.*

Proof. Since $v \in V(G_1)$, $d_{uv} + d_{vw} = d_G(v) \geqslant 3$. W.l.o.g. we may assume that $d_{uv} \geqslant 2$, so that $M_{uv} \neq \emptyset$. Let $x \in M_{uv}$; then $(G - x)^2 = G^2 - x$. By the minimality of G, $(G - x)^2$ has a $\lfloor \frac{3}{2}\Delta \rfloor$-coloring f. Let

$$N_2(x) := (N(u) \setminus \{x\}) \cup (N(v) \setminus N(u)) \cup \{u, v\},$$

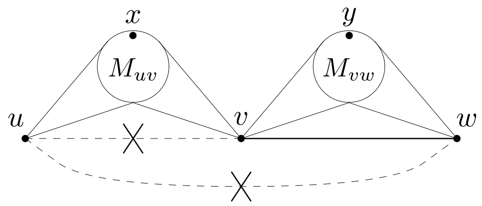

Figure 4. The neighborhood of a vertex v contradicting Lemma 11.

which is the set of G^2-neighbors of x. We may assume that $|N_2(x)| \geqslant \lfloor\frac{3}{2}\Delta\rfloor$, since otherwise we can extend f to G^2 by giving x a color that is not used on any vertex in $N_2(x)$. Since $|N(u)| \leqslant \Delta$ and $|N(v)\setminus(N(u)\cup\{u\})| \leqslant d_{vw}$, it follows that $\Delta-1+d_{vw}+2 \geqslant |N_2(x)| \geqslant \lfloor\frac{3}{2}\Delta\rfloor$, so that

$$d_{vw} \geqslant \lfloor\tfrac{1}{2}\Delta\rfloor - 1 \geqslant 2. \tag{4}$$

By symmetry we may assume also that

$$d_{uv} \geqslant \lfloor\tfrac{1}{2}\Delta\rfloor - 1. \tag{5}$$

Suppose now that the lemma is false, say $d_{vw} < \lfloor\frac{1}{2}\Delta\rfloor$. Then (4) and its derivation imply that

$$d_{vw} = \lfloor\tfrac{1}{2}\Delta\rfloor - 1, \qquad |N_2(x)| = \lfloor\tfrac{3}{2}\Delta\rfloor, \quad \text{and} \quad d_G(u) = \Delta. \tag{6}$$

If $uv \in E(G)$ then $v \in N(u) \setminus \{x\}$ and so we have counted v twice in our estimate for $|N_2(x)|$; thus we may assume that $uv \notin E(G)$. If $vw \notin E(G)$ then the degree of v in G^2 is at most $\Delta + 2 \leqslant \lfloor\frac{3}{2}\Delta\rfloor - 1$, and so we can uncolor v, color x, and then recolor v; thus we may assume that $vw \in E(G)$. If $uw \in E(G)$ then, since $vw \in E(G)$, $|N(v) \setminus (N(u) \cup \{u\})| = d_{vw} - 1$, and so $|N_2(x)| < \lfloor\frac{3}{2}\Delta\rfloor$; thus we may assume that $uw \notin E(G)$. Let y be a vertex in M_{vw}. The picture now is as in Figure 4.

If $d_{uv} < \lfloor\frac{1}{2}\Delta\rfloor$, then by exactly the same argument we can deduce that $uv \in E(G)$ and $vw \notin E(G)$. Since this is not so, we can strengthen (5) to

$$d_{uv} \geqslant \lfloor\tfrac{1}{2}\Delta\rfloor \geqslant 3. \tag{7}$$

Let G' be the graph obtained from G by deleting all vertices in $M_{uv} \cup M_{vw} \cup \{v\}$ and adding an edge between u and w. Then G' is a minor of G, and so G' is K_4-minor-free and connected, since G is.

Suppose that G' has a cutvertex y. If $y \in \{u, w\}$, then y is also a cutvertex in G. Similarly, if $y \notin \{u, w\}$, then since $uw \in E(G')$, vertices u and w are in the same component of $G' - y$, and hence y is a cutvertex in G. But G is 2-connected, by Lemma 8, and so has no cutvertex. It follows that G' also has no cutvertex, and so G' is 2-connected. (Clearly $G' \ncong K_2$, otherwise v is a cutvertex of G.)

Suppose now that G'^2 contains a K_t, with vertex-set Q, say. By Lemma 7, Q is of standard form, and so $F(G', Q)$, defined by (2), is one of the graphs shown in Figure 1. Let $F := F(G', Q)$. Since $G' - uw \subset G$, and G^2 contains no K_t, it follows that $uw \in E(F)$. Now, $d_F(u) \leqslant d_{G'}(u) = \Delta+1-d_{uv} < \Delta-1$ by (6) and(7). By (1), therefore, $d_F(u) = 2$ and $d_F(w) \geqslant \Delta - 1$, with strict inequality if Δ is even. But $d_F(w) \leqslant d_{G'}(w) \leqslant \Delta + 1 - d_{vw} =$

$\Delta+2-\lfloor\frac{1}{2}\Delta\rfloor$, by (6). The only possibility is that $\Delta = 7$, $d_{vw} = 2$, and $d_F(w) = d_{G'}(w) = 6$, so that w is the unique vertex of degree $\Delta - 1$ in F, and it has the same degree in G'. It now follows from Lemma 6 that $F = G'$, so that $d_{G'}(u) = d_F(u) = 2$ and, since $d_G(u) = \Delta = 7$ by (6), $d_{uv} = 6$ and $d_G(v) = d_{uv} + d_{vw} = 8 > \Delta$. This contradiction shows that G'^2 contains no K_t.

By the minimality of G, there is a proper $\lfloor\frac{3}{2}\Delta\rfloor$-coloring f of G'^2. We will use f to give a proper $\lfloor\frac{3}{2}\Delta\rfloor$-coloring of G^2. Since $uw \in E(G')$, color $f(u)$ is not used on vertices in $N_{G'}(w) \setminus \{u\}$. So we can use $f(u)$ to color y. Then we consecutively color vertices in M_{vw}, then v, and then vertices in M_{uv}. We can do this since at the moment of coloring, each vertex in $M_{vw} \cup \{v\}$ has at most $d_G(w)$ colored G^2-neighbors, and (because $f(y) = f(u)$) each vertex in M_{uv} has at most $|N_2(x)| - 1 = \lfloor\frac{3}{2}\Delta\rfloor - 1$ colors on its neighbors.

This contradiction shows that $d_{vw} \geqslant \lfloor\frac{1}{2}\Delta\rfloor$, and it follows by symmetry that $d_{uv} \geqslant \lfloor\frac{1}{2}\Delta\rfloor$. This completes the proof of Lemma 11. $\square$

Lemma 12. *Let G be a (Δ, t)-graph, where $\Delta \geqslant 6$ and $t = \lfloor\frac{3}{2}\Delta\rfloor + 1$. Then the graph G_1 cannot have two adjacent vertices with degree two.*

Proof. Suppose that there are two adjacent vertices $x, y \in V(G_1)$ with degree two. Let w and z, respectively, be the other neighbors of x and y in G_1.

Suppose first that $w = z$. Note that z cannot be a cutvertex of G_1, since G_1 is 2-connected by Lemma 10. Thus z also has degree 2 in G_1, which is a triangle. Let V_0 consist of the vertices in $\{x, y, z\}$ that are not adjacent in G to another vertex of this set, and let $V_1 := \{x, y, z\} \setminus V_0$. Then $M_{xy} \cup M_{xz} \cup M_{yz} \cup V_1$ induces a clique in G^2, with order at most $\lfloor\frac{3}{2}\Delta\rfloor$ since G is a (Δ, t)-graph. Thus these vertices can be colored with at most $\lfloor\frac{3}{2}\Delta\rfloor$ colors, and the vertices in V_0 are now easily colored since each has degree at most $\Delta + 2$ in G^2.

Thus we may assume that $w \neq z$. (See Figure 5, where the broken edges may or may not be present.) By Lemma 11,

$$\lfloor\tfrac{1}{2}\Delta\rfloor \leqslant d_{wx} \leqslant \lceil\tfrac{1}{2}\Delta\rceil \quad \text{and} \quad \lfloor\tfrac{1}{2}\Delta\rfloor \leqslant d_{yz} \leqslant \lceil\tfrac{1}{2}\Delta\rceil, \tag{8}$$

since $d_{wx} = d_G(x) - d_{xy} \leqslant \Delta - d_{xy}$, and similarly for d_{yz}. Without loss of generality we may assume that $d_{wx} \leqslant d_{yz}$. Let $s := d_{wx} - 1$, and note that $s \geqslant 2$ by (8). Also

$$d_{wx} = s + 1 \quad \text{and} \quad d_{yz} \leqslant s + 2 \tag{9}$$

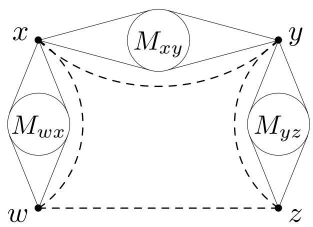

Figure 5. The neighborhood of vertices x and y contradicting Lemma 12.

by (8). Let G' be the graph obtained from G by deleting all vertices in $M_{wx} \cup M_{xy} \cup M_{yz} \cup \{x, y\}$, and adding s vertices, $v_1, \ldots, v_s$, each of which is adjacent to w and z. By the definition of s,

$$d_{G'}(w) \leqslant \Delta - 1 \quad \text{and} \quad d_{G'}(z) \leqslant \Delta - 1; \tag{10}$$

in particular, the maximum degree of G' is at most Δ.

Since G is 2-connected, G' also is 2-connected. Since G' is a minor of G, G' does not have a K_4 minor. If G'^2 contains a K_t, with vertex-set Q, say, then Q is of standard form by Lemma 7, and Q clearly contains at least one of the vertices v_i, and so at least one of w and z has degree Δ in G' by (1); but this contradicts (10). Thus G'^2 contains no K_t. By the minimality of G, G'^2 has a $\lfloor \frac{3}{2}\Delta \rfloor$-coloring f. We will extend f to a $\lfloor \frac{3}{2}\Delta \rfloor$-coloring of G. Color s vertices of M_{wx} and s vertices of M_{yz} with the colors $f(v_i)$ $(1 \leqslant i \leqslant s)$. Then consecutively color the remaining vertices in $M_{wx} \cup M_{yz}$, which is possible since each of these vertices has at most Δ colored G^2-neighbors at the moment of its coloring.

We now color x and y. The number of colored G^2-neighbors of x does not exceed

$$|\{w\} \cup N_G(w) \setminus \{x\}| + d_{yz} \leqslant \Delta + (s + 2) \tag{11}$$

by (9). But s colors are used on both M_{wx} and M_{yz}; thus at most $\Delta + 2 < \lfloor \frac{3}{2}\Delta \rfloor$ colors are forbidden for x, and x can be colored. In coloring y, in the same way as x, we have an extra restriction, that $f(y) \neq f(x)$. But since $d_{wx} = s + 1$, we can replace the term $(s+2)$ by $(s+1)$ on the RHS of (11), which exactly compensates for the extra color $f(x)$ that is forbidden for y. Thus y can be colored.

Finally, note that if $v \in M_{xy}$ then

$$\begin{aligned} d_{G^2}(v) &= (d_G(x) - \epsilon_{xy}) + (d_G(y) - \epsilon_{xy}) - (d_{xy} - \epsilon_{xy} - 1) && (12)\\ &\leqslant d_G(x) + d_G(y) - d_{xy} + 1, && (13) \end{aligned}$$

where the first term in (12) counts x and all its neighbors except v and y, the second term counts y and all its neighbors except v and x, and the third term subtracts the $|M_{xy}| - 1$ vertices of $M_{xy} \setminus \{v\}$ that have been counted twice in the first two terms. The number of distinct colors that cannot be used on v is at most $d_{G^2}(v) - s$, and so if $d_G^2(v) \leqslant \lfloor \frac{3}{2}\Delta \rfloor + 1$ then we can color v, since $s \geqslant 2$. But if $d_{G^2}(v) > \lfloor \frac{3}{2}\Delta \rfloor + 1$ then, by (13) and Lemma 11, Δ is odd, $d_G(x) = d_G(y) = \Delta$, $d_{xy} = \lfloor \frac{1}{2}\Delta \rfloor$, and $d_{G^2}(v) = \lfloor \frac{3}{2}\Delta \rfloor + 2$. But then $d_{wx} = d_{yz} = \Delta - d_{xy} = \lceil \frac{1}{2}\Delta \rceil \geqslant 4$, and so $s \geqslant 3$ and $d_{G^2}(v) - s \leqslant \lfloor \frac{3}{2}\Delta \rfloor - 1$. In every case, $d_{G^2}(v) - s < \lfloor \frac{3}{2}\Delta \rfloor$, and so we can consecutively color all the vertices of M_{xy} to obtain a $\lfloor \frac{3}{2}\Delta \rfloor$-coloring of G^2. This contradicts the definition of G, and this contradiction completes the proof of Lemma 12. □

Lemma 13. *Let G be a (Δ, t)-graph, where $\Delta \geqslant 6$ and $t = \lfloor \frac{3}{2}\Delta \rfloor + 1$. Then the graph G_1 cannot contain a 4-cycle $wxyzw$ such that x and z both have degree 2 in G_1.*

Proof. Suppose there is such a 4-cycle $wxyzw$ in G_1; call it C. By Lemma 12, G_1 does not contain two adjacent vertices with degree 2, and so w and y both have degree at least 3 in G_1. By Lemma 11, Δ is odd and

$$d_{wx} = d_{xy} = d_{yz} = d_{zw} = \lfloor \tfrac{1}{2}\Delta \rfloor, \tag{14}$$

and w and y each have exactly one edge in G that is not counted in (14). Let these edges join w and y to w' and y' respectively. Note that $|M_{wx}| = \lfloor\frac{1}{2}\Delta\rfloor$ if $wx \notin E(G)$ and $|M_{wx}| = \lfloor\frac{1}{2}\Delta\rfloor - 1$ if $wx \in E(G)$, and similarly for the other edges of C.

Suppose first that $wy \in E(G)$, so that $w' = y$, $y' = w$, and

$$V(G) = M_{wx} \cup M_{xy} \cup M_{yz} \cup M_{zw} \cup \{w, x, y, z\}.$$

Then we can color the vertices of G^2 with $\Delta + 3 \leqslant \lfloor\frac{3}{2}\Delta\rfloor$ colors, by coloring the vertices of M_{wx} and those of M_{yz} from the same set of $\lfloor\frac{1}{2}\Delta\rfloor$ colors, coloring the vertices of M_{xy} and M_{zw} from another set of $\lfloor\frac{1}{2}\Delta\rfloor$ colors, and giving the remaining four colors to w, x, y, z.

So we may suppose that $wy \notin E(G)$. Form G' from G by deleting x, z and all their neighbors except for w and y. By the minimality of G, there is a $\lfloor\frac{3}{2}\Delta\rfloor$-coloring f of G'^2. We will extend this coloring to G^2. We may assume that $f(y) \neq f(w)$, since y has at most Δ colored neighbors in G'^2 and so can be recolored if necessary. Choose two disjoint sets A and B of $\lfloor\frac{1}{2}\Delta\rfloor$ colors each, which do not include any of the colors of w, w', y, y'. If there is a color not in $A \cup B \cup f(\{w, w', y, y'\})$ then let γ be such a color and define $\alpha := \gamma$ and $\beta := \gamma$; otherwise, the colors of w, w', y and y' are all distinct (and $\Delta = 7$), and we define $\alpha := f(w')$ and $\beta := f(y')$.

Color all vertices of M_{wx} and M_{yz} with colors from A, and all vertices of M_{xy} and M_{zw} with colors from B, ensuring that if $|M_{wx}| = |M_{yz}| = |A| - 1$ then one color from A is not used at all, and similarly with B. If G contains all four edges of C, then there is a color in A and one in B that we have not used, and we can use these on x and z. If G omits only one edge of C, say the edge wx, then we can color x with α and use a color from B to color z. If G contains edges wx, wz (only) of C, then we can color x with the color from A that is not used on M_{wx}, and z with the color from B that is not used on M_{wz}. If G contains edges wx, xy (only) of C, then we can color x with color γ if it exists; if γ does not exist then let v be the unique vertex in M_{yz} whose color is not used on M_{wx}, color x with $f(v)$, and recolor v with $f(w)$; now z can be colored since it has only $\Delta + 1$ G^2-neighbors. Finally, if G does not contain two adjacent edges of C, say $wx, yz \notin E(G)$, then we can color x with α and z with β. Every other case is similar to one of these, leading to a $\lfloor\frac{3}{2}\Delta\rfloor$-coloring of G, and this contradiction proves Lemma 13. $\square$

Let a 2-*path* in G_1 be a path of length 2 whose central vertex has degree 2 in G_1.

Lemma 14. *Let G be a (Δ, t)-graph, where $\Delta \geqslant 6$ and $t = \lfloor\frac{3}{2}\Delta\rfloor + 1$. Then the graph G_1 has a triangle $xywx$ such that $d_{G_1}(w) = 2$ and $d_{G_1}(y) = 3$.*

Proof. By Lemma 10, G_1 is 2-connected and so does not contain a vertex with degree 0 or 1. By Lemma 12, G_1 does not contain two adjacent vertices with degree 2. Let G_2 be the graph obtained from G_1 by suppressing each vertex v of degree 2 (i.e., contracting one edge incident with v) and removing multiple edges; in other words, $G_2 = (G_1)_1$. It follows from Lemma 9 that G_2 exists and is 2-connected or K_2. But if $G_2 \cong K_2$, with vertices w, y, then, since $d_{G_1}(w) \geqslant 3$, G_1 contains at least two 2-paths wxy and wzy between w and y, and so contains a 4-cycle $wxyzw$ of the sort that was proved impossible in Lemma 13. Thus G_2 is 2-connected and has minimum degree at least 2.

Since G_2 is a minor of G_1, G_2 is K_4-minor-free. So, by Lemma 4, G_2 has a vertex y with degree 2; let its G_2-neighbors be x and z. By Lemma 13, there cannot be two or

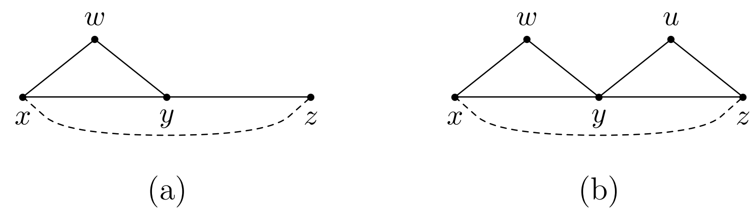

Figure 6. The subgraphs induced by $N_{G_1}(y) \cup \{y\}$ in G_1.

more 2-paths in G_1 between x and y or between y and z, and so y is connected to each of x and z by an edge, or a 2-path, or both. By the definition of G_2, $d_{G_1}(y) > 2$, and so there is no loss of generality in assuming that y is connected to x in G_1 by an edge and a 2-path ywx, forming a triangle $xywx$. If y is connected to z by a 2-path but not by an edge, then redefine z to be the middle vertex of this 2-path. Then y and its neighbors in G_1 induce one of the graphs in Figure 6 (where the broken edges may or may not be present). However, the graph in Figure 6(b) is impossible because, in G, y would have degree at least $d_{uy} + d_{wy} + 2 \geqslant \Delta + 1$, by Lemma 11. Therefore, y and its neighbors in G_1 induce the subgraph in Figure 6(a). $\square$

5 Proof of the main theorem

Let $\Delta \geqslant 6$ and $t := \lfloor \frac{3}{2}\Delta \rfloor + 1$. Suppose that the theorem fails for Δ. Then there exists a (Δ, t)-graph G (defined at the start of Section 4). By Lemma 14, G_1 contains a subgraph of the form depicted in Figure 6(a). In G, this corresponds to the subgraph depicted in Figure 7, where the broken edges may or may not be present. Among all possible subgraphs of this form in G, choose one such that d_{wy} is as small as possible. By Lemma 11,

$$d_{wx} \geqslant \lfloor \tfrac{1}{2}\Delta \rfloor \quad \text{and} \quad d_{wy} \geqslant \lfloor \tfrac{1}{2}\Delta \rfloor. \tag{15}$$

Since $d_{wx} + d_{wy} = d_G(w) \leqslant \Delta$, it follows that equality holds in both parts of (15) if Δ is even, and in at least one part if Δ is odd.

If $v \in M_{wx}$ then

$$d_{G^2}(v) = (d_G(w) - \epsilon_{wx}) + (d_G(x) - \epsilon_{wx}) - (d_{wx} - \epsilon_{wx} - 1) - \epsilon_{wy}\epsilon_{xy}, \tag{16}$$

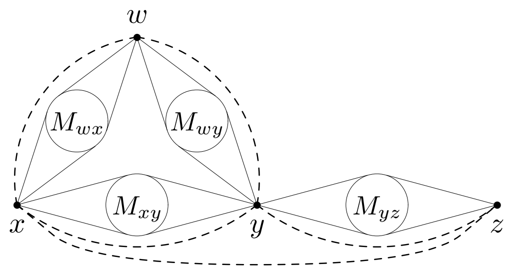

Figure 7. The induced subgraph of G.

where the first term in (16) counts w and all its neighbors except v and x, the second term counts x and all its neighbors except v and w, the third term subtracts the $|M_{wx}|-1$ vertices of $M_{wx} \setminus \{v\}$ that have been counted twice in the first two terms, and the last term accounts for y which is also counted twice if $wy, xy \in E(G)$. Let $p := d_{G^2}(v)$, which is the same for all $v \in M_{wx}$. It follows from (16), using (15) in the third line, that

$$\begin{aligned} p &= d_G(w) + d_G(x) + 1 - d_{wx} - \epsilon_{wx} - \epsilon_{wy}\epsilon_{xy} \\ &\leqslant 2\Delta + 1 - d_{wx} - \epsilon_{wx} - \epsilon_{wy}\epsilon_{xy} \\ &\leqslant \lceil \tfrac{3}{2}\Delta \rceil + 1 \\ &= \begin{cases} \lfloor \tfrac{3}{2}\Delta \rfloor + 2 \text{ if } \Delta \text{ is odd,} \\ \lfloor \tfrac{3}{2}\Delta \rfloor + 1 \text{ if } \Delta \text{ is even.} \end{cases} \end{aligned} \tag{17}$$

Similarly, let $q := d_{G^2}(v)$ for all $v \in M_{wy}$. Then

$$\begin{aligned} q &= d_G(w) + d_G(y) + 1 - d_{wy} - \epsilon_{wy} - \epsilon_{wx}\epsilon_{xy} \\ &\leqslant 2\Delta + 1 - d_{wy} - \epsilon_{wy} - \epsilon_{wx}\epsilon_{xy} \\ &\leqslant \begin{cases} \lfloor \tfrac{3}{2}\Delta \rfloor + 2 \text{ if } \Delta \text{ is odd,} \\ \lfloor \tfrac{3}{2}\Delta \rfloor + 1 \text{ if } \Delta \text{ is even.} \end{cases} \end{aligned} \tag{18}$$

Let G_w denote the graph obtained from G by deleting w and all its neighbors except x and y. Let G^- be obtained from G_w by deleting y and all its neighbors except x and z. Let G^+ be obtained from G^- by adding the edge xz if it is not already present. Let $N'(x) := N_{G^-}(x)$ and $N'(z) := N_{G^-}(z)$. Since, by (15), $d_{wx} + d_{xy}$ and $d_{wy} + d_{xy}$ are both at least $\lfloor \frac{1}{2}\Delta \rfloor + 1$, it follows that

$$|N'(x)| \leqslant \lceil \tfrac{1}{2}\Delta \rceil - 1 \quad \text{and} \quad d_{yz} \leqslant \lceil \tfrac{1}{2}\Delta \rceil - 1. \tag{19}$$

Let $S := N'(x) \cup M_{xy} \cup M_{yz} \cup \{x, y\}$ and $S^+ := S \cup \{z\}$. Note that

$$|S| \leqslant (\lceil \tfrac{1}{2}\Delta \rceil - 1) + (\Delta - d_{wy}) + 2 \leqslant \Delta + 2, \tag{20}$$

by (15) and (19). Recall that $t - 1 = \lfloor \frac{3}{2}\Delta \rfloor \geqslant \Delta + 3$ by (3).

Lemma 15. *Suppose that $z \notin N'(x)$, and either* (i) *or* (ii) *holds, and at least one of* (iii) *and* (iv) *holds*:
(i) $|N'(x)| = 1$;
(ii) $|N'(x)| = 2$ *and* $M_{yz} = \emptyset$;
(iii) *there is a $\lfloor \frac{3}{2}\Delta \rfloor$-coloring f of $(G^-)^2$ such that all vertices in $N'(x) \cup \{x, z\}$ have different colors*;
(iv) *$(G^+)^2$ has no t-cliques.*
Then there is a $\lfloor \frac{3}{2}\Delta \rfloor$-coloring of $(G_w)^2$ such that all vertices in S^+ have different colors.

Proof. We start by proving a claim, which is needed only in one special case, but which cannot be avoided.

Claim 15.1. *Suppose that $d_G(z) = \Delta \in \{6, 7\}$, and* (ii) *and* (iii) *hold, say $N'(x) = \{u_1, u_2\}$. Then f can be chosen so that some vertex in $N'(z)$ has the same color as one of u_1, u_2, x.*

Proof. Let us assume that this is not true for the given f, so that the $\lfloor \frac{3}{2}\Delta \rfloor = \Delta + 3$ distinct colors are those of u_1, u_2, x, z and the $\Delta - 1$ vertices in $N'(z)$ (and $u_1, u_2 \notin N'(z)$). Note that x has degree 2 in G^- and is not a cutvertex, since if it were a cutvertex in G^- then it would be a cutvertex in G, which is 2-connected by Lemma 8. There are two cases.

Case 1: there are two internally disjoint paths between x and z in G^-. Then there is no path between u_1 and u_2 in $G^- - \{x, z\}$, otherwise G contains a K_4 minor. Thus u_1 and u_2 are in different components of $G^- - \{x, z\}$. Choose a vertex $z_1 \in N'(z)$, and let u_1 be in the component not containing z_1. Then interchanging the colors $f(u_1)$ and $f(z_1)$ throughout this component gives a coloring that satisfies the requirements of the claim.

Case 2: there do not exist two paths as in Case 1. Then there is a cutvertex $v \in V(G^-)$ such that x and z are in different components of $G^- - v$. Let $C(x)$ be the component that contains x, and let α be a color not in $f(N(v) \cup \{v, z\})$. If $\alpha \in f(N'(z))$, then interchange colors $f(x)$ and α throughout $C(x)$. Otherwise, $\alpha \in f(\{u_1, u_2, x\})$, by the first sentence of the proof; so choose $z_1 \in N'(z)$ such that $f(z_1) \neq f(v)$, and interchange colors $f(z_1)$ and α throughout $C(x)$. □

We can now prove Lemma 15. Suppose first that (iii) holds. Transfer the given coloring f to $(G_w)^2$, and extend it to all uncolored vertices in $N_G(z)$ by consecutively coloring each of them differently from all colored vertices in the set $T := N'(x) \cup N_G(z) \cup \{x, z\}$. This is possible, because if we are coloring a vertex in T then there are at most $|T| - 1$ vertices in T that are colored already; thus at each stage the number of colored vertices in T is at most $\lfloor \frac{3}{2}\Delta \rfloor - 1$ unless $|N'(x)| = 2$ (so that (ii) holds), and $|N_G(z)| = \Delta$, and $\lfloor \frac{3}{2}\Delta \rfloor = \Delta + 3$, and we have shown in Claim 15.1 that in this case we can choose f so that the colors of the vertices in T are not all distinct.

We can now consecutively color all vertices in M_{xy}, and y if $yz \notin E(G)$, by coloring each of them differently from all colored vertices in S^+, of which there are at most $|S^+| - 1 \leqslant \Delta + 2$ by (20). This gives the required $\lfloor \frac{3}{2}\Delta \rfloor$-coloring of $(G_w)^2$.

This proves the result when (iii) holds. Suppose now that (iv) holds. Since G^+ is a minor of G, G^+ is K_4-minor-free, and by construction its maximum degree is at most Δ. By hypothesis (iv), $(G^+)^2$ has no t-cliques, and so, by the minimality of G, $(G^+)^2$ has a $\lfloor \frac{3}{2}\Delta \rfloor$-coloring f, in which all vertices in $N'(x) \cup \{x, z\}$ necessarily have different colors; thus (iii) holds, and the result follows. □

Lemma 16. *Suppose there is a $\lfloor \frac{3}{2}\Delta \rfloor$-coloring f of $(G_w)^2$ in which all vertices of S have different colors and $f(x) \neq f(z)$. Then f can be chosen so that there exists a vertex $u \in N'(x)$ with $f(u) \neq f(z)$.*

Proof. Suppose this is not true for the given f. Then $|N'(x)| \leqslant 1$. Since G is 2-connected by Lemma 8, z is not a cutvertex, and so $|N'(x)| = 1$ and $N'(x) \neq \{z\}$. Let $N'(x) = \{u\}$. Since $f(u) = f(z)$, $d_G(u, z) \geqslant 3$ and $xz \notin E(G)$.

Suppose, for a contradiction, that $(G^+)^2$ has a t-clique, with vertex-set Q, say. By Lemma 7, Q is of standard form in G^+, and so $F(G^+, Q)$, defined by (2), is one of the graphs shown in Figure 1. Since x has degree 2 in G^+, and G^2 has no t-cliques, it follows that $x \in Q$ and u is connected to z by more than one path of length 2 in G^+. But this is impossible since $d_G(u, z) \geqslant 3$. Thus $(G^+)^2$ has no t-cliques. Thus hypotheses (i) and (iv) of Lemma 15 hold, and the result follows from Lemma 15. $\square$

Lemma 17. *Suppose there is a $\lfloor \frac{3}{2}\Delta \rfloor$-coloring f of $(G_w)^2$ in which all vertices of S have different colors and $f(x) \neq f(z)$. Then there is a $\lfloor \frac{3}{2}\Delta \rfloor$-coloring of G^2.*

Proof. By Lemma 16, we may assume that there is a vertex $u \in N'(x)$ such that $f(u) \neq f(z)$. (We use u in Case 1 only.)

We first color w differently from all the colored vertices in $(N_G(x) \setminus M_{wx}) \cup M_{yz} \cup (N_G(y) \cap \{z\}) \cup \{x, y\}$, of which there are at most $\Delta - d_{wx} + d_{yz} + 2 \leqslant \Delta + 2$ by (15) and (19).

Case 1: $d_{wy} = \lfloor \frac{1}{2}\Delta \rfloor$; then either Δ is even, or $d_G(w) < \Delta$, or $d_{wx} = \lfloor \frac{1}{2}\Delta \rfloor + 1$. In this case we first color consecutively all vertices in M_{wy}, each of them differently from the (at most $\Delta - 1$) colored neighbors of y and from w, x, y, a total of at most $\Delta + 2 \leqslant \lfloor \frac{3}{2}\Delta \rfloor - 1$ by (3). In doing this, we take care to use the color $f(u)$ on one vertex of M_{wy}. We now consecutively color the vertices of M_{wx}, in four subcases.

Subcase 1.1: $wx \in E(G)$. Then $p \leqslant \lfloor \frac{3}{2}\Delta \rfloor$ by the hypothesis of Case 1 and (17) (since $\epsilon_{wx} = 1$). Since every vertex in M_{wx} has two G^2-neighbors with the same color $f(u)$, the vertices in M_{wx} can all be colored.

Subcase 1.2: $wx, wy \notin E(G)$. Then $p \leqslant \lfloor \frac{3}{2}\Delta \rfloor + 1$, and so if we try to color the vertices of M_{wx} as in Subcase 1.1 it is only with the last vertex that we may fail. If this happens, uncolor w, color the last vertex in M_{wx}, then recolor w, which is possible since w has at most $\Delta + 2$ neighbors in G^2.

Subcase 1.3: $wx \notin E(G)$ and $wy, xy \in E(G)$. Then $p \leqslant \lfloor \frac{3}{2}\Delta \rfloor$ by (17), and we color as in Subcase 1.1.

Subcase 1.4: $wx, xy \notin E(G)$ and $wy \in E(G)$. Then $p \leqslant \lfloor \frac{3}{2}\Delta \rfloor + 1$. But now x is not adjacent to the vertices of M_{wy} in G^2, and so when we color M_{wy}, as well as using $f(u)$ on one vertex of M_{wy}, we also use $f(x)$ on another vertex. Then, when we color the vertices of M_{wx} as in Subcase 1.1, each has two pairs of G^2-neighbors with the same color, and the coloring succeeds.

Case 2: $d_{wy} \neq \lfloor \frac{1}{2}\Delta \rfloor$. Then Δ is odd, $d_{wx} = \frac{1}{2}(\Delta - 1)$, and $d_{wy} = \frac{1}{2}(\Delta + 1)$, by (15). We first color consecutively all vertices in M_{wx}, each of them differently from the (at most $\Delta - 1$) colored neighbors of x and from w, x, y, a total of at most $\Delta + 2 \leqslant \lfloor \frac{3}{2}\Delta \rfloor - 1$. Case 2 now divides into Cases 2a, 2b and 2c.

Case 2a: either $M_{yz} \neq \emptyset$, or $M_{yz} = \emptyset$ (so that $yz \in E(G)$) and $f(z)$ is not used on any vertex of $N'(x)$. Choose $v \in M_{yz}$ in the first case and let $v := z$ in the second. When we color M_{wx} we make sure to use $f(v)$ on one vertex of M_{wx}. We can now color the vertices in M_{wy} exactly as in Case 1, interchanging x and y, p and q, and using v instead of u and

(18) instead of (17). Note that, in each subcase, q satisfies the same upper bound as was given for p in the corresponding subcase of Case 1.

Case 2b: $M_{yz} = \emptyset$ and $f(z) \in f(N'(x))$ and $d_G(y) < \Delta$. Then there is no vertex v as in Case 2a, but in each subcase the upper bound for q is one less than in Case 2a, and so the argument works with no need for v.

Case 2c: $M_{yz} = \emptyset$ and $f(z) \in f(N'(x))$ and $d_G(y) = \Delta$. Then $d_{xy} = \frac{1}{2}(\Delta - 3)$ and so $|N'(x)| \leqslant 2$. Let $N'(x) = \{u_1, u_2\}$, where for the moment we allow the possibility that $u_1 = u_2$. We may assume that

$$f(z) \in f(N'(x)) \tag{21}$$

for *every* $\lfloor \frac{3}{2}\Delta \rfloor$-coloring f of $(G_w)^2$ satisfying the hypotheses of the lemma, since otherwise the result follows by Case 2a.

Subcase 2c.1: $z \in \{u_1, u_2\}$, i.e., $xz \in E(G)$. Then we have a t-clique in G^2, a contradiction, unless either *all* of the edges wx, wy, xy are in G, or *none* of these edges are in G. If *all* of these edges are in G, then $q \leqslant \lfloor \frac{3}{2}\Delta \rfloor - 1$ by (18), and so we can color all the vertices in M_{wy}. If *none* of the edges wx, wy, xy are in G, then $q \leqslant \lfloor \frac{3}{2}\Delta \rfloor + 1$, but we can use $f(y)$ on some vertex of M_{wx} and also uncolor w before coloring the last vertex of M_{wy}, after which it easy to recolor w.

Subcase 2c.2: $z \notin \{u_1, u_2\}$. Assume $f(u_1) = f(z)$. This implies that $d_G(u_1, z) \geqslant 3$.

If $(G^+)^2$ has no t-cliques, then hypotheses (i) or (ii), and (iv), of Lemma 15 hold, and the $\lfloor \frac{3}{2}\Delta \rfloor$-coloring of $(G_w)^2$, whose existence was proved in Lemma 15, contradicts (21). So we may assume that $(G^+)^2$ has a t-clique; let its vertex-set be Q. Then by Lemma 7, Q is of standard form in G^+, i.e., $F(G^+, Q)$, defined by (2), is isomorphic to one of the graphs F_1 and F_2 in Figure 1. Let $F := F(G^+, Q)$.

Since G^2 has no t-cliques, and x has degree at most $3 < \Delta - 1$ in G^+, it follows from (1) that x has degree 2 in F and the three vertices of degree at least $\Delta - 1$ in F are z, another neighbor u_i of x, and a third vertex w'. Then u_i and z have common neighbors other than x in F, and hence in G. Since $d_G(u_1, z) \geqslant 3$, it follows that $i \neq 1$, so that $u_1 \neq u_2$ and the 'big' vertices in F are z, u_2 and w'. It follows from this that Q induces the *only* t-clique in $(G^+)^2$.

Since x has a G^+-neighbor u_1 that is not in F, and G is 2-connected by Lemma 8, it follows from Lemma 6 that either $\{u_2, x\}$ or $\{x, z\}$ is a cutset of G, and there is a subgraph H of G^+ such that $G^+ = F \cup H$ where $F \cap H = \{u_2, u_2x, x\}$ or $\{x, xz, z\}$. There are two cases to consider.

Subcase 2c.2i: $F \cap H = \{u_2, u_2x, x\}$. Then u_2 is a cutvertex of G^-. The given coloring f of $(G_w)^2$ induces a coloring of $(G^-)^2$, and we can permute colors on the vertices of $V(F) \setminus V(H)$ in this induced coloring so that z has a different color from both u_1 and x (and, automatically, from u_2). Then hypotheses (ii) and (iii) of Lemma 15 hold, and the $\lfloor \frac{3}{2}\Delta \rfloor$-coloring of $(G_w)^2$, whose existence was proved in Lemma 15, contradicts (21).

Subcase 2c.2ii: $F \cap H = \{x, xz, z\}$. In this case z is the vertex of degree $\Delta - 1$ in F, and both x and z have degree 2 in H. Now, H^2 has no t-cliques, since we have already seen that Q induces the only t-clique in $(G^+)^2$. But H is a minor of G and so is K_4-minor-free. By the minimality of G, H^2 has a $\lfloor \frac{3}{2}\Delta \rfloor$-coloring f'. Let $z_1 \neq x$ be the other neighbor of

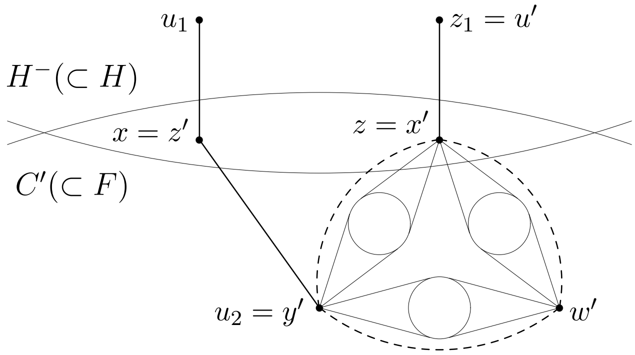

Figure 8. The graph G^- in Subcase 2c.2ii.

z in H. Note that $f'(u_1) \neq f'(z)$ and $f'(x) \neq f'(z_1)$.

Let $C' := F - xz$. Then C' is a configuration of the same type as the configuration C in Figure 7 that we have been working with, with the vertices w', z, u_2, x playing the roles of w, x, y, z respectively; and this configuration exists in G^- with z having exactly one neighbor, z_1, outside C'. Let us emphasize this by writing $x' = z$, $y' = u_2$, $z' = x$, and $u' = z_1$ (see Figure 8). Since $d_{wy} > \lfloor \frac{1}{2}\Delta \rfloor$ by the hypothesis of Case 2, we may assume that $d_{w'y'} > \lfloor \frac{1}{2}\Delta \rfloor$ also, since otherwise we would have chosen to work with C' rather than C at the start of Section 5.

Let $H^- := H - xz$. Then H^- is obtained from G^- by deleting w', y' and all their neighbors other than x' and z'; in other words, C', H^- and G^- are related to each other in exactly the same way that C, G^- and G are. Also, f' is a $\lfloor \frac{3}{2}\Delta \rfloor$-coloring of $(H^-)^2$ in which the vertices u', x', z' (i.e., z_1, z, x) all have different colors. With respect to C', H^- and G^-, therefore, hypotheses (i) and (iii) of Lemma 15 hold, and the proof of that Lemma, and Case 2a of this Lemma, show that f' can be extended to a $\lfloor \frac{3}{2}\Delta \rfloor$-coloring of $(G^-)^2$, in which necessarily $f'(u_2) \neq f'(z)$ since $d_{G^-}(u_2, z) \leqslant 2$.

Note that the color modifications required by Claim 15.1 and Lemma 16 have not been needed here, and the colors of vertices in H have not changed. (This is because Claim 15.1 is needed only when hypothesis (ii) of Lemma 15 holds, not hypothesis (i), and Lemma 16 is not needed if all vertices in S^+ already have different colors, which is guaranteed by Lemma 15.) Thus all vertices in $N'(x) \cup \{x, z\}$ now have different colors. This shows that, with respect to C, hypotheses (ii) and (iii) of Lemma 15 hold, and the $\lfloor \frac{3}{2}\Delta \rfloor$-coloring of $(G_w)^2$, whose existence was proved in Lemma 15, contradicts (21). This completes the proof of Lemma 17. □

Now let $s := d_{yz}$ and let $\hat{G}$ be the graph obtained from G^- by adding s vertices $y_1, \ldots, y_s$ of degree 2, each with neighbors x and z.

Lemma 18. *$\Delta(\hat{G}) \leqslant \Delta$, $\hat{G}$ has no K_4 minor, and $(\hat{G})^2$ has no t-cliques.*

Proof. By (19),

$$d_{\hat{G}}(x) = |N'(x)| + d_{yz} \leqslant 2\lceil \tfrac{1}{2}\Delta \rceil - 2 \leqslant \Delta - 1, \tag{22}$$

and so $\Delta(\hat{G}) \leqslant \Delta$. The graph obtained from G^- by adding just one vertex y_1 adjacent to x and z is a minor of G, and so is K_4-minor-free. Adding $s-1$ further vertices of degree 2 in parallel with y_1 cannot create a K_4 minor, and so $\hat{G}$ is K_4-minor-free. The proof of the final statement of Lemma 18 uses the following claim.

Claim 18.1. *If $(\hat{G})^2$ has a t-clique with vertex-set Q, then:*
(a) *Δ is odd and $|N'(x)| = d_{yz} = s = \frac{1}{2}(\Delta-1)$;*
(b) *$d_{wx} = d_{wy} = \frac{1}{2}(\Delta-1)$ and $d_{xy} = 1$;*
(c) *all vertices $y_1, \dots, y_s$ are in Q.*

Proof. By Lemma 7, Q is of standard form. Since G^2 has no t-clique, Q must contain at least one new vertex y_i. By (1), in $F(\hat{G}, Q)$, x and z both have degree Δ if Δ is even, and if Δ is odd then one of them has degree Δ and the other has degree at least $\Delta - 1$. It follows from (22) that x has degree $\Delta - 1$ (in both $F(\hat{G}, Q)$ and $\hat{G}$), so that Δ is odd; and equality in (22) implies that there is equality in both parts of (19), so that the rest of (a) holds. In proving (19), we used the inequalities $d_{wx} \geqslant \lfloor \frac{1}{2}\Delta \rfloor$, $d_{wy} \geqslant \lfloor \frac{1}{2}\Delta \rfloor$ (by (15)) and $d_{xy} \geqslant 1$, and equality must hold in each case if there is equality in (19); thus (b) holds. (Equality in (19) also implies that $d_G(x) = d_G(y) = \Delta$, but we do not need this here.) Finally, (c) holds because otherwise z, which has degree Δ in $F(\hat{G}, Q)$, would have degree greater than Δ in $\hat{G}$ and hence in G. $\square$

Now suppose, for a contradiction, that $(\hat{G})^2$ has a t-clique, with vertex-set Q, say. Then the graph $G^* := \hat{G} - y_s$ is a K_4-minor-free graph with maximum degree at most Δ whose square does not contain a K_t, by Claim 18.1(c). By the minimality of G, $(G^*)^2$ has a $\lfloor \frac{3}{2}\Delta \rfloor$-coloring f. We will use f to construct a $\lfloor \frac{3}{2}\Delta \rfloor$-coloring of G^2. First we use colors $f(y_1), \dots, f(y_{s-1})$ to color all but one vertex, say v_{yz}, in M_{yz}, and to color y if $yz \in E(G)$. Then we choose a vertex $u \in N'(x)$ such that $f(u) \neq f(z)$ and we color v_{yz} with a color not used on any vertex in $N_G(z) \cup \{u, x, z\}$. There remain at most two uncolored vertices in G_w: possibly y, and, by Claim 18.1(b), at most one vertex in M_{xy}. These vertices (if they exist) can be colored (in this order) differently from all the colored vertices in $N'(x) \cup M_{yz} \cup \{x, y, z\}$, of which by Claim 18.1(a) there are at most $(\Delta - 1) + 3 < \lfloor \frac{3}{2}\Delta \rfloor$.

At this point we have a $\lfloor \frac{3}{2}\Delta \rfloor$-coloring of $(G_w)^2$. It may fail to satisfy the hypotheses of Lemma 17, but only because it is possible that $v_{yz} \in M_{yz}$ may have the same color as some vertex in $N'(x)$. However, we have ensured that $u \in N'(x)$ does not have the same color as any vertex in $M_{yz} \cup \{z\}$, and this is enough to ensure that Case 1 in the proof of Lemma 17 works and gives a $\lfloor \frac{3}{2}\Delta \rfloor$-coloring of G^2. This contradicts the choice of G as a (Δ, t)-graph, and this contradiction shows that $(\hat{G})^2$ has no t-cliques. $\square$

Finally, we prove Theorem 3. By Lemma 18 and the minimality of G, $\hat{G}^2$ has a $\lfloor \frac{3}{2}\Delta \rfloor$-coloring f; clearly $f(x) \neq f(z)$. We will use f to construct a $\lfloor \frac{3}{2}\Delta \rfloor$-coloring of $(G_w)^2$. First, we use $f(y_1), \dots, f(y_s)$ to color all vertices in M_{yz}, and to color y if $yz \in E(G)$. Then we consecutively color all vertices in M_{xy}, and y if $yz \notin E(G)$, differently from all colored vertices in S (see (20)). The result is a $\lfloor \frac{3}{2}\Delta \rfloor$-coloring of $(G_w)^2$ such that all vertices in S have different colors. It now follows from Lemma 17 that there is a $\lfloor \frac{3}{2}\Delta \rfloor$-coloring of G^2, and this contradiction completes the proof of Theorem 3.

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