Degree-Splittability of Multigraphs and Caterpillars

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Abstract

A multigraph is *degree-splittable* if it decomposes into two submultigraphs whose degree lists are the same, plus one leftover edge when the total number of edges is odd. We prove that a connected multigraph is degree-splittable if it has an even number of vertices of each odd degree. For caterpillars, we characterize the splittable caterpillars with diameter at most 4, provide a general sufficient condition for splittability of caterpillars, and prove that the smallest maximum degree of a non-splittable caterpillar is 5.

Keywords: graph decomposition, degree list, eulerian multigraph, caterpillar.

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1 Introduction

A *decomposition* of a graph G is a set of pairwise edge-disjoint subgraphs whose union is G. The problem of decomposing a graph into isomorphic copies of a fixed graph has a long history. MathSciNet lists more than 60 papers on the subject; see also the survey paper by Plummer [3] and the book by Bosák [1].

We study a relaxed problem. A [multi]graph is *degree-splittable* if it decomposes into two graphs having identical degree lists, plus one extra edge when the total number of edges is odd. Ignoring the extra edge echoes the convention that a k-regular graph is *Hamiltonian-decomposable* if it has $\lfloor k/2 \rfloor$ pairwise edge-disjoint spanning cycles, leaving a perfect matching when k is odd. We henceforth abbreviate "degree-splittable" to *splittable*.

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Analogous problems for isomorphic decomposition motivate our study. Decomposing complete graphs into isomorphic complete subgraphs is the fundamental problem of design theory. Other classical problems include decomposition of complete graphs into cycles of fixed lengths and the famous conjecture of Ringel that the complete graph K_{2m+1} decomposes into isomorphic copies of any tree with m edges.

One can also ask whether G decomposes into t isomorphic copies of any one (unspecified) subgraph; if so, then t *divides* G. Ellingham and Wormald [2] showed that if G is a t-edge-colorable multigraph and t divides $|E(G)|$, then t divides G. Wormald [4] proved that for fixed r and t with $t \geq 2$ and $r \geq 2t+1$, almost all r-regular graphs are not divisible by t. Nevertheless, small examples with $t = 2$ are not so prevalent. In Section 2, we exhibit a 5-regular multigraph that has no such decomposition and yet is splittable. We also prove that every connected multigraph having an even number of vertices of each odd degree is splittable. Thus all regular (connected) multigraphs are splittable.

Not all graphs are splittable. The components of graphs with maximum degree 2 are paths and cycles, which are splittable, so all graphs with maximum degree 2 are splittable. Already with maximum degree 3 there are non-splittable graphs.

Example 1.1. Let G be the 7-vertex tree obtained by subdividing each edge of the claw $K_{1,3}$. Since G has only one vertex of degree 3, each subgraph in a splitting must have a vertex of degree 2 and none of degree 3. One graph has two edges at the central vertex, so the other has a path of length 2 starting there. Each subgraph has one of the two remaining edges, but this puts a second vertex of degree 2 into only one of them. $\square$

As another example, consider a *double-star* (a tree having only two non-leaf vertices). When decomposing the double-star with central vertices of degrees 5 and 2 into two subgraphs, one subgraph must have a vertex of degree at least 3, and the other cannot. Indeed, whenever the two central vertices have degrees of opposite parity that differ by more than 1, the double-star is not splittable.

Double-stars belong to the class of trees we study in Section 3. A *caterpillar* is a tree whose non-leaf vertices form a path called the *spine*. We determine which caterpillars with at most three spine vertices are splittable. We also prove a general sufficient condition for splittability of caterpillars with an odd number of spine vertices. As a corollary, all caterpillars with an odd number of spine vertices and maximum degree 3 are splittable. We also prove constructively that all caterpillars with maximum degree at most 4 are splittable. The double-star mentioned above with central vertices of degrees 5 and 2 is a caterpillar with maximum degree 5 that is not splittable.

The non-splittable tree in Example 1.1 is not a caterpillar but has maximum degree only 3. This suggests an extremal problem: *What is the smallest maximum degree of a non-splittable graph with minimum degree k?* Upper bounds are available.

Example 1.2. *For each positive integer k, there is a non-splittable graph with minimum degree k and maximum degree at most $2k+3$.*

When $k \equiv 3 \mod 4$, form G with maximum degree $2k+1$ by adding one vertex adjacent to all vertices of a $(k-1)$-regular graph with $2k+1$ vertices. Here $|E(G)| = (2k+1)(k+1)/2$, which is even, but G is not splittable, because the degree of the central vertex is too high.

In other congruence classes, $2k+3$ is an upper bound, shown by joining one vertex to all vertices of a $(2k+3)$-vertex graph that has vertices of degrees $k-1$ and k and an odd number of edges. Again the larger "half" of the maximum degree vertex cannot be matched in the other graph.

When $k \equiv 3 \mod 4$, the upper bound improves from $2k+1$ to $2k-1$. Join one vertex to all vertices of a $(k-1)$-regular graph with $2k-1$ vertices; again the number of edges is even. In a splitting, isolated vertices arise only by splitting vertices of degree k into k and 0. Hence each subgraph in the splitting has the same number of vertices of degree k retaining their full degree. Now the large half of the high-degree vertex cannot be matched by the other vertices, even if it splits into degrees k and $k-1$.

The upper bounds here may not be sharp. □

Another question that seems to be open is the complexity of determining whether an input graph is splittable, even for trees.

2 Splittability of Multigraphs

We treat a decomposition of a multigraph G into two subgraphs as a labeling of $E(G)$ using A and B. The two subgraphs, denoted G_A and G_B respectively, are viewed as spanning subgraphs, so the degree lists have the same length. We write $d_A(v)$ and $d_B(v)$ for the degrees of vertex v in G_A and G_B. When the degree lists are identical (under reordering), the decomposition is a *splitting* of G.

To illustrate the distinction between divisibility by 2 and splitting, we present a small 5-regular multigraph without loops that has no decomposition into two isomorphic subgraphs.

Example 2.1. Let T be the loopless multigraph with vertex set $\{x, y, z\}$ and seven edges in which the vertex degrees are $4, 5, 5$, respectively. Form G from five disjoint copies of T by adding one central vertex w adjacent to the five copies of x (see Figure 1); G is 5-regular with 16 vertices and 40 edges. Let $T_1, \ldots, T_5$ be the copies of T, with $V(T_i) = \{x_i, y_i, z_i\}$.

Suppose that G decomposes into isomorphic subgraphs G_A and G_B. We may assume by symmetry that $d_A(w) = 3$, since no other vertex has four distinct neighbors. To match w, we may assume that x_1 is the center of an induced claw ($K_{1,3}$) in G_B. This leaves the other five edges of T_1 in G_A.

Now G_B must have a component that is a triangle with edges of multiplicities 3, 1, and 1. Being a component, it lies in some T_i such that $x_i w$ is in G_A. Now x_i has three distinct neighbors in G_A. We now have forced a 7-vertex double-star into G_A. All such subgraphs use w as a central vertex, so we cannot have such subgraphs in both G_A and G_V.

Nevertheless, G is splittable, with each submultigraph having eight 2-valent and eight 3-valent vertices. Take one triangle from each copy of T, add the edges from w to three of the triangles, and add one more copy of yz from each of the other two triangles. This multigraph has the desired degree list, as does the one obtained by deleting these edges. □

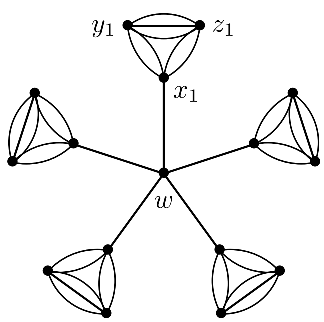

Figure 1: 5-regular multigraph with no isomorphic decomposition

Our positive result on splittability of general multigraphs is easy to prove. Our proof extends a short proof given by A. Kostochka for the special case of regular multigraphs with even degree. We restrict to connected multigraphs because rounding difficulties could arise for multigraphs with multiple components that each have an odd number of edges.

Theorem 2.2. *Every connected multigraph having an even number of vertices with each odd degree is splittable.*

Proof. To such a graph G, add a matching M covering the vertices of odd degree, each edge joining vertices of the same degree (this may increase multiplicity of edges). The resulting graph G' is connected and has even degree at each vertex, so it has an Eulerian circuit C.

Traverse C in order, alternately using labels A and B on edges, but skip labeling edges of M. Each traversal through a vertex that does not involve M puts one incident edge into each of G_A and G_B. When an edge of M is traversed, one incident vertex gains an extra A and the other gains an extra B, and this occurs at most once at each vertex.

Thus from the set of vertices of each degree in G, the contribution to the degree lists of G_A and G_B is the same. □

3 Splittability of Caterpillars

When a multigraph has odd maximum degree, splittability requires a second vertex with degree greater than half the maximum degree. We will see that this necessary condition is not sufficient, not even for caterpillars with small spines. We focus primarily on caterpillars with even size (number of edges), since a caterpillar with odd size is splittable if and only if deleting some edge leaves a splittable forest of caterpillars with total size even.

First we note a necessary condition for splittings of trees with even size. A graph is *nontrivial* if it has at least one edge.

Lemma 3.1. *In every splitting of a tree with even size, the two decomposing subgraphs have the same number of nontrivial components.*

Proof. Each of the two subgraphs in the splitting is a spanning forest. A forest with n vertices and k edges has $n - k$ components. Since the two subgraphs have the same number of edges, they have the same number of components. Since the vertices of degree 0 are the isolated vertices, and they have the same number of vertices of degree 0, they also have the same number of nontrivial components. □

To specify a caterpillar, let $v_1, \dots, v_t$ be the vertices along the spine (the non-leaves), and let $d_1, \dots, d_t$ be their degrees in order. Note that $d_i \geq 2$ for all i. Henceforth fix this notation. We begin by determining the splittable caterpillars that have small spines. Those having one spine vertex are stars and are splittable. Those having two spine vertices are double-stars and generally are not splittable.

Theorem 3.2. *A double-star of even size is splittable if and only if the degrees of the two spine vertices differ by exactly 1. A double-star of odd size is splittable if and only if the degrees of the two spine vertices are both odd or differ by at most 2.*

Proof. Let G be a double-star of even size. By symmetry, we may assume that $d_1 \geq d_2$ and that the subgraph G_A in some splitting contains the edge of the spine. Now G_A can have only one nontrivial component. By Lemma 3.1, also G_B has only one nontrivial component. Hence G_A and G_B each is one star plus isolated vertices. Thus G_A contains all the pendant edges at one spine vertex, and G_B contains all those at the other. Splittability now requires $d_1 = d_2 + 1$, and both stars have d_2 edges.

When G is a double-star of odd size, $d_1 + d_2$ is even. If d_1 and d_2 are odd, then deleting the spine edge leaves two stars of even size; put half the edges of each star in G_A. If d_1 and d_2 are both even, then deleting the central edge does not leave a splittable forest unless $d_1 = d_2$. When $d_1 \neq d_2$, therefore, splittability requires obtaining a splittable double-star by deleting one edge. By the preceding paragraph, this requires $|d_1 - d_2| = 2$. □

Before determining the splittable caterpillars with three-vertex spines, we develop some more general machinery.

Lemma 3.3. *In a splittable caterpillar with even size, the number of spine vertices has opposite parity from the sum of the degrees of the spine vertices.*

Proof. The total number of edges in the caterpillar is $(\sum_{i=1}^{t} d_i) - (t-1)$. □

For caterpillars of even size, an arithmetic condition is sufficient when the spine has an odd number of vertices.

Lemma 3.4. *Let G be a caterpillar with even size and degrees $d_1, \ldots, d_t$ along the spine. If t is odd and the system $\{\alpha_i + \beta_i = d_i \colon 1 \leq i \leq t\}$ has an integer solution with $\{\alpha_i\}_{i=1}^{t} = \{\beta_i\}_{i=1}^{t}$ and all values positive except possibly $\alpha_t = \beta_1 = 0$, then G is splittable.*

Proof. We construct a splitting. Since t is odd, the number of edges on the spine is even. Let M_A and M_B be the matchings that arise by alternating labels A and B along the spine. We may assume that M_A covers v_1 and M_B covers v_t. Put $\alpha_i - 1$ pendant edges at v_i into G_A and $\beta_i - 1$ pendant edges at v_i into G_B, except for putting α_t pendant edges at v_t into G_A and β_1 pendant edges at v_1 into G_B.

Since $\alpha_i + \beta_i = d_i$, the edges in this construction all exist. The number of nontrivial components in both G_A and G_B is $(t-1)/2$ if $\alpha_t = \beta_1 = 0$ and is $(t+1)/2$ otherwise. Degrees $\alpha_1, \ldots, \alpha_t$ and $\beta_1, \ldots, \beta_t$ are enforced at the spine vertices; by hypothesis these are the same numbers. The remaining degrees are 0 or 1 in each of G_A and G_B. Each subgraph has $\alpha_t + \sum_{i=1}^{t-1}(\alpha_i - 1)$ vertices of degree 1 among the leaves of G. The other leaves of G are isolated vertices in that subgraph. Hence indeed G_A and G_B form a splitting of G. □

A corresponding sufficient condition for splittability of caterpillars of odd size is that reducing some d_i by 1 yields a list that is solvable as in Lemma 3.4. The condition of Lemma 3.4 produces splittings for all splittable caterpillars of even size that have three spine vertices. Unlike double-stars, most caterpillars with three spine vertices are splittable, including those whose spine degrees are all even or satisfy the strict triangle inequality.

Theorem 3.5. *Let G be a caterpillar with three spine vertices. Let the set of degrees of the spine vertices be $\{a, b, c\}$, labeled so that $a \geq b \geq c$. If G has even size, then G is splittable if and only if at least one of the following conditions holds:*
(1) a, b, c are all even,
(2) $a < b + c$,
(3) $a = b + c$ and $a = d_2$, or
(4) $a > b + c$ and $b = c$.

Proof. By Lemma 3.3, $a+b+c$ is even. In each case, we construct a splitting with v_1v_2 and v_2v_3 in opposite subgraphs. These are obtained by applying Lemma 3.4. In each case, we provide a solution to the system of equations in Lemma 3.4 that satisfies all the requirements. We then complete the proof by showing that when the sufficient conditions all fail, there is no splitting. There are only two ways for the conditions to all fail: $a = b + c$ with $a = d_1$ (or similarly $a = d_3$), or $a > b + c$ with $b > c$.

Case 1: *a, b, and c are all even.* In this case, set $\alpha_i = \beta_i = d_i/2$.

Case 2: *$a < b+c$.* Since the sum is even, the triangle inequality implies that each quantity of the form $(d_i + d_j - d_k)/2$ is a positive integer. Also, $\frac{d_i+d_j-d_k}{2} + \frac{d_i+d_k-d_j}{2} = d_i$. For $i \in \{1, 2, 3\}$, set α_i and β_i to be these two summands, where always $k \equiv j + 1 \mod 3$. Now $\alpha_1 = \beta_2$, $\alpha_2 = \beta_3$, and $\alpha_3 = \beta_1$.

Case 3: *$a = b + c$ and $a = d_2$.* Set $\beta_1 = \alpha_3 = 0$, $\alpha_1 = \beta_2 = d_1$, and $\alpha_2 = \beta_3 = d_3$. Note that $\alpha_2 + \beta_2 = d_1 + d_3 = d_2$. (Each subgraph is a double-star with vertices of degrees d_1 and d_3, plus isolated vertices.)

Case 4: *$a > b+c$ and $b = c$.* Since the sum is even, a is even. If $d_1 = d_3$, then set $\alpha_3 = \beta_1 = 0$, $\alpha_1 = \beta_3 = d_1$, and $\alpha_2 = \beta_2 = a/2$. If $d_2 = d_3$, then set $\alpha_1 = \beta_1 = d_1/2$, $\alpha_2 = \beta_3 = 1$, and $\alpha_3 = \beta_2 = d_2 - 1$. In each case, Lemma 3.4 applies. (The construction does not require $a > b + c$.)

Case 5: *$a \geq b+c$ and $b > c$, with $a = d_1$ if $a = b+c$.* Let x be the vertex with degree a. If x has the same degree in G_A and G_B, then the union of the contributions greater than 1 from the other two spine vertices must be the same in G_A and G_B. Also a is then even, so to avoid Case 1 with even sum, b and c must be odd. Hence the other two vertices each contribute one odd and one even degree, and the number of contributions that are greater than 1 is even. Since $b \neq c$, we cannot complete this allocation.

Hence the contributions from x to the two subgraphs are not equal. Neither contribution can exceed b, because no vertex can provide that much for the other subgraph. Also x cannot contribute more than c to both subgraphs, because only the vertex with degree b can provide more than c for the second copy of each such degree, and it cannot provide a total of a. Therefore, splittability in this case requires that $a = b + c$ and that x contributes degrees b and c to the two subgraphs. Since $b > c$, the degree equalling b can be matched only by the vertex of degree b contributing all its incident edges to the same subgraph, and then the vertex of degree c must contribute all its incident edges to the other subgraph. This is impossible when $a = d_1$, since the vertices of degrees b and c are adjacent. □

A caterpillar of odd size with three spine vertices is splittable if and only if reducing the degree of some spine vertex yields a splittable caterpillar of even size or deleting some spine edge yields a splittable union of a star and a double-star. It is lengthy but not hard to characterize such graphs.

The last part of Case 5 yields a general necessary condition.

Lemma 3.6. *Let a, b, and c be three largest terms in the degree list of a graph G. Suppose that $a > b + c$ and $b \neq c$. If a is odd, then G is not splittable. If a is even, then in every splitting the vertex with degree a has half its edges in both subgraphs.* □

Generalizing the characterization in Theorem 3.5 to longer spines seems difficult. In the proof, we were able to use Lemma 3.4 to construct splittings in all cases where they exist, because Lemma 3.1 guarantees that in all splittings, the spine edges alternate between the decomposing subgraphs. Lemma 3.4 precisely describes all splittings of this type.

For general caterpillars, it may be necessary to find splittings of other types, which correspond to modified versions of the equations. When the spine edges incident to v_i both lie in G_A, the variable β_i is allowed to be 0. Similarly, α_i is allowed to be 0 when those edges both lie in G_B. In addition to having exponentially many versions of the equations, it may be difficult to tell when a solution exists with the appropriate nonnegativity or positivity conditions on the variables.

Note also that the splittability of caterpillars with given degree lists may depend on the order of the degrees along the spine. For example, when $a = b+c$ in Theorem 3.5, the caterpillar is splittable if a is the degree of the middle vertex and is not splittable otherwise.

Next we give another sufficient condition for splittability of caterpillars having even size and an odd number of spine vertices. It does not depend on the order of the vertex degrees along the spine. It is a sufficient condition for Lemma 3.4 to provide a splitting.

Theorem 3.7. *Let G be a caterpillar of even size with t spine vertices, where t is odd. Let $d_1, \ldots, d_t$ be the degrees of the spine vertices, indexed so that $d_1 \geq \cdots \geq d_t$. If $d_t + \sum_{i=1}^{\lfloor t/2 \rfloor} d_{2i} > \sum_{i=1}^{\lfloor t/2 \rfloor} d_{2i-1}$, then G is splittable.*

Proof. For $1 \leq i \leq t$, let S_i be the $\lfloor t/2 \rfloor$-element set in $\{1, \ldots, t\}$ obtained from $\{i + 2j \colon 1 \leq j \leq \lfloor t/2 \rfloor\}$ by reducing each element modulo t. Treating $t + 1$ as 1, for each i we have $S_i \cap S_{i-1} = \varnothing$, and the element missing from their union is i.

Let $D = \sum_{i=1}^{t} d_i$. For $1 \leq i \leq t$, let $\alpha_i = D/2 - \sum_{j \in S_i} d_j$ and $\beta_i = D/2 - \sum_{j \in S_{i-1}} d_j$. By construction, $\alpha_i + \beta_i = d_i$. Also, since t is odd and G has even size, D is even, and hence each α_i and β_i is an integer. Finally, $\beta_i = \alpha_{i-1}$, cyclically, so $\{\beta_i\}_{i=1}^{t} = \{\alpha_i\}_{i=1}^{t}$.

Because Lemma 3.4 allows any permutation of the indices in matching $\{\beta_i\}_{i=1}^{t}$ with $\{\alpha_i\}_{i=1}^{t}$ (when no 0s are used), it suffices to find a solution under any indexing of the spine degrees, such as placing them in nonincreasing order as above. Lemma 3.4 thus completes the proof each value in the solution is positive, which requires $\sum_{j \in S_i} d_j < D/2$ for all i.

We need $d_i + \sum_{j \in S_{i-1}} d_j - \sum_{j \in S_i} d_j > 0$. The numbers of the form d_{i+2j-1} for $1 \leq j \leq (t+1)/2$ are counted positively, and those of the form d_{i+2j} for $1 \leq j \leq (t-1)/2$ are counted negatively. Group the contributions as $(d_{i+1} - d_{i+2}) + (d_{i+3} - d_{i+4}) + \cdots (d_{i-2} - d_{i-1}) + d_i$.

With the degrees indexed in nonincreasing order, all these contributions are positive except one, which is $d_t - d_1$ when i is even. (When i is odd, all the contributions are positive.) Always $\lfloor t/2 \rfloor$ terms have negative signs, and they are largest when $i = t-1$ and the negatives are the odd terms from d_1 through d_{t-2}. The condition in the hypothesis is precisely the condition that the result remains positive in this case. □

Corollary 3.8. *Let G be a caterpillar of even size with an odd number of spine vertices. If the degree of each spine vertex is more than half the maximum degree, then G is splittable.*

Proof. Let the spine degrees be $d_1, \ldots, d_t$ in nonincreasing order. Because the negative contributions are differences of successive entries in a monotone list, the total of the negative contributions in the computation of Theorem 3.7 is at most $d_1 - d_t$. By the hypothesis, this is less than the positive contribution of d_t, so Theorem 3.7 applies. □

The condition of Corollary 3.8 holds for caterpillars with maximum degree 3. The corollary fails to determine whether caterpillars with spine degrees like $(4, 3, 3, 2, 2)$ are splittable. Nevertheless our final result is a constructive proof that all caterpillars of even size with maximum degree at most 4 are splittable (regardless of the number of spine vertices). Since the double-star with vertices of degrees 5 and 2 is not splittable, the smallest maximum degree in a non-splittable caterpillar is thus 5.

Theorem 3.9. *Every caterpillar with even size and maximum degree at most* 4 *is splittable.*

Proof. We construct a splitting by specifying a red/blue coloring of the edges of such a caterpillar G. Let t be the number of spine vertices.

Case 1: t *is odd.* Alternate colors red and blue along the spine edges. This will ensure that both subgraphs have $(t+1)/2$ nontrivial components, since each end-vertex of the spine will get an incident edge of the other color. Hence the two subgraphs will have the same number of isolated vertices (by Lemma 3.1) if they have the same lists of nonzero degrees.

Since t is odd and G has even size, the sum of the spine degrees is even (by Lemma 3.3). Since the spine degrees lie in $\{2,3,4\}$, the number of vertices of degree 3 is even. Arbitrarily pick half to be majority red and half to be majority blue. Whether a vertex of degree 3 is in the middle or at the end of the spine, we can allocate the pendant edge(s) at that vertex to obtain degrees 2 and 1 there in the desired colors.

Each remaining vertex has even degree, with one assigned incident edge of each color (or just one assigned incident edge if at the end of the spine). The pendant edges can now be colored so that each vertex of even degree has half its incident edges in each color.

We have now colored the edges, with half in each color. We ensured that each color has the same number of vertices with degree 2 (half the vertices of degree 3 in G, plus all the vertices of degree 4 in G). Hence the two colors also have the same number of vertices of degree 1 and then degree 0.

Case 2: t *is even.* The degree-sum of the spine vertices is odd, by Lemma 3.3. Hence G has an odd number of vertices of degree 3. Since t is even, there is a vertex v of degree 3 adjacent to a vertex w of degree 2 or 4. Let u and x be the other neighbors of v and w along the spine, respectively, so that u, v, w, x lie along the spine in order (u or x may be a leaf if v or w is the end of the spine).

Form G' by deleting v and w (and their incident edges and resulting isolated vertices) from G and replacing them with one vertex z adjacent to u and x. Now G' is a caterpillar with an even number of edges (four or six edges were replaced with two) and $t-1$ spine vertices. Since $t-1$ is odd, G' is splittable via the argument in Case 1. Furthermore, that splitting alternates colors along the spine, so z has one incident edge of each color; we may assume that uz is red and xz is blue. See Figure 2, where red is shown in bold.

Returning to G, give the edges of G that lie also in G' the same color as in the splitting of G'. Replace the red uz with uv in red; also put red on the remaining edge incident to v other than vw. If w has degree 2, then make both edges at w blue. Now degree 1 in each subgraph at z has been replaced with vertices of degrees $0, 1, 2$ in each subgraph, so G has been split. If w has degree 4, then make wv red and wx blue, and make the two remaining edges at w blue. Now degree 1 in each subgraph at z has been replaced with vertices of degrees $0, 0, 1, 1, 3$ in each subgraph, so again G has been split. □

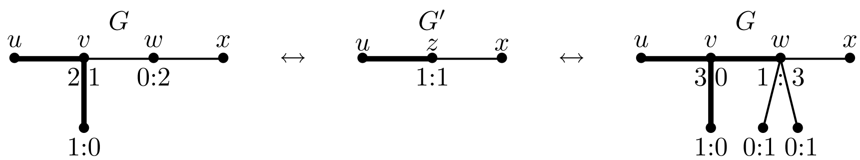

Figure 2: Two cases for splitting caterpillars

It is disappointing in Case 2 to introduce a vertex of degree 2 or 4 whose incident edges are not split equally between the two subgraphs, but it is unavoidable, as shown by the caterpillar obtained by subdividing one edge of the claw $K_{1,3}$.

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References

[1] J. Bosák, *Decompositions of graphs.* Translated from the Slovak. *Mathematics and its Applications (East European Series)*, **47** (Kluwer Academic Publishers Group, Dordrecht, 1990), xviii+248 pp.

[2] M. N. Ellingham, and N. C. Wormald, Isomorphic factorization of regular graphs and 3-regular multigraphs., *J. London Math. Soc.* (2) **37** (1988) no. 1, 14–24.

[3] M. D. Plummer, Graph factors and factorization: 1985–2003: a survey. (English summary), *Discrete Math.* **307** (2007) no. 7-8, 791–821.

[4] N. C. Wormald, Isomorphic factorizations. VII. Regular graphs and tournaments, *J. Graph Theory* **8** (1984) 117–122.